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A COMMUTATIVE DIAGRAM AND AN APPLICATION TO DIFFERENTIABLE TRANSFORMATION GROUPS

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ABSTRACT. A commutative diagram is presented which relates the groups of concordance classes of diffeomorphisms $\Gamma(S^{2n})$, $\Gamma(CP^n)$ and $\Gamma(S^{2n+1})$. This diagram is applied to show that every equivariant diffeomorphism of S^7 is concordant to the identity. It follows that the exotic 8-sphere, Σ^8, admits no smooth semifree S^1-action with exactly two fixed points.

Introduction. In this paper we shall present a commutative diagram (Theorem 1) involving the stable homotopy group Π_1 and the groups $\Gamma(S^n)$, $\Gamma^+(CP^n)$ defined below. This diagram is then applied to show that every orientation preserving diffeomorphism of S^7 which is equivariant with respect to the standard free action of S^1 on S^7 is concordant to the identity. Finally we show, using a result of R. Lee [3], that the exotic 8-sphere, Σ^8, does not admit a smooth action by S^1 which is semifree with exactly two fixed points. The paper concludes with a brief discussion of possible further applications of Theorem 1 to construction of smooth actions of S^1 on exotic spheres such that the actions are semifree with exactly two fixed points.

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Preliminaries. If f_0 and f_1 are diffeomorphisms of M onto N, M and N C^∞-manifolds, we say f_0 is concordant to f_1 if there is a diffeomorphism $F: M \times I \to N \times I$, $I = [0, 1]$, such that for all $x \in M$ we have $F(x, 0) = (f_0(x), 0)$ and $F(x, 1) = (f_1(x), 1)$. The relation of concordance is an equivalence relation on the set of all diffeomorphisms of M onto N. If $f: M \to N$ is a diffeomorphism its concordance class will be denoted $[f]$. If M is an orientable manifold there is a group, $\Gamma(M)$, of all concordance

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classes of orientation preserving diffeomorphisms of M onto M. We denote by $\Gamma^+(CP^n)$ the subgroup of $\Gamma(CP^n)$ consisting of all $[f]$ such that f^* preserves the generator of $H^2(CP^n; Z)$.

The stable group Π_1 is equal to $\pi_{n+1}(S^n)$ for $n \geqq 3$. It is well known that there is an isomorphism, $J: \pi_1(\mathrm{SO}(n)) \to \pi_{n+1}(S^n)$, for $n \geqq 3$.

Main results. Before stating Theorem 1 we shall describe in some detail each of the four homomorphisms which appear in the diagram of Theorem 1.

$\gamma: \Gamma(S^n) \to \Gamma(M^n)$: Let M^n be a connected, oriented, smooth n-manifold, $\alpha \in \Gamma(S^n)$. We may represent α by a diffeomorphism $f: S^n \to S^n$ which equals the identity on a neighborhood of the lower hemisphere of S^n, $\{(x_0, \cdots, x_n) \in S^n \mid x_n \leqq 0\}$. Let D^n be the closed unit ball in R^n and let $\sigma: D^n \to M^n$ be a smooth imbedding. Define $\tau: \mathrm{Int}\,(D^n) \to S^n$ by

$$\tau(x_1, \cdots, x_n) = (x_1, \cdots, x_n, (1 - x_1^2 - \cdots - x_n^2)^{1/2}).$$

Then define $\gamma(f): M^n \to M^n$ by letting $\gamma(f) \mid M^n - \sigma(D^n)$ be the identity and letting $\gamma(f) \mid \sigma(D^n) = \sigma(\tau^{-1}) f \tau(\sigma^{-1})$. Since f is the identity in a neighborhood of the lower hemisphere of S^n it follows that $\gamma(f)$ is a well-defined, orientation preserving diffeomorphism. Let $\gamma(\alpha) = [\gamma(f)]$ and this is a well-defined group homomorphism.

LEMMA. $\gamma: \Gamma(S^{2n}) \to \Gamma(CP^n)$ *takes values in* $\Gamma^+(CP^n)$.

The above lemma is easily proved using Mayer-Vietoris sequences and the fact that if W is a closed tubular neighborhood of CP^{n-1} in CP^n then $CP^n - \mathrm{Int}\,(W)$ is diffeomorphic to D^{2n}.

$\sigma: \Gamma(S^{2n}) \to \Gamma(S^{2n}) \otimes \Pi_1 : \Pi_1 \cong Z_2$. Let $\sigma(\alpha) = \alpha \otimes \bar{n}$ where $\bar{n}$ is zero or nonzero depending on whether n is even or odd.

$\rho: \Gamma^+(CP^n) \to \Gamma(S^{2n+1})$: The circle group, S^1, acts on S^{2n+1} as follows. Let S^{2n+1} be represented as

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in C^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\}.$$

If $t \in S^1$ let $t(z_0, \cdots, z_n) = (tz_0, \cdots, tz_n)$. This action will be called the standard action of S^1 on S^{2n+1}. The orbit space S^{2n+1}/S^1 is just CP^n and the projection $S^{2n+1} \to CP^n$ is the Hopf bundle. $\Gamma^+(CP^n)$ consists of those classes, $[f]$, such that f^* preserves the characteristic class of the above bundle. Thus $\Gamma^+(CP^n)$ consists precisely of those $[f]$ such that f can be covered by an equivariant, orientation preserving diffeomorphism $f': S^{2n+1} \to S^{2n+1}$. The formula $\rho([f]) = [f']$ is a well-defined group homomorphism.

$\rho_{2n+1,1}: \Gamma(S^{2n}) \otimes \Pi_1 \to \Gamma(S^{2n+1})$: We assume $n \geqq 3$. This homomorphism, called the Milnor-Munkres pairing, is discussed by G. Bredon in [**1**].

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Let $[f] \in \Gamma(S^{2n})$, $[\alpha] \in [S^1, SO(2n)] = \Pi_1$. $\rho_{2n+1}([f] \otimes [\alpha]) = [h]$ where $h: S^{2n+1} \to S^{2n+1}$ is defined as follows. There is a standard imbedding $S^1 \times D^{2n} \to S^{2n+1}$ whose image is

$$\{(x_0, \cdots, x_{2n+2}) \in S^{2n+1} \mid x_2^2 + \cdots + x_{2n+2}^2 \leqq \tfrac{1}{2}\}.$$

This gives a representation of S^{2n+1} as $S^1 \times D^{2n} \cup_1 D^2 \times S^{2n-1}$ where $S^1 \times D^{2n}$ and $D^2 \times S^{2n-1}$ are identified along their common boundary, $S^1 \times S^{2n-1}$, by the identity map. Let $h \mid D^2 \times S^{2n-1}$ be the identity and let $h \mid S^1 \times D^{2n}$ be given by $h(z, x) = (z, \alpha(z) \cdot f(\alpha(z)^{-1} \cdot x))$. Here we have chosen $f: S^{2n} \to S^{2n}$ to agree with the identity near the lower hemisphere of S^{2n}. Hence we may regard f as a diffeomorphism $D^{2n} \to D^{2n}$ which equals the identity near S^{2n-1}. Thus h, as given above, is a diffeomorphism.

This is not the definition as given in [**1**] but it is not difficult to check that the two correspondences are equal.

Now our main result may be stated.

THEOREM 1. *For $n \geqq 3$ the following diagram commutes.*

$$\begin{array}{ccc} \Gamma(S^{2n}) & \xrightarrow{\sigma} & \Gamma(S^{2n}) \otimes \Pi_1 \\ \gamma \downarrow & & \downarrow \rho_{2n+1,1} \\ \Gamma^+(CP^n) & \xrightarrow{\rho} & \Gamma(S^{2n+1}) \end{array}$$

PROOF. Let $[f] \in \Gamma(S^{2n})$. As before we may regard f as a diffeomorphism $f: D^{2n} \to D^{2n}$ which equals the identity in a neighborhood of S^{2n-1}. Represent CP^n as $D^{2n} \cup_1 W$ where W is the total space of the 2-disk bundle associated to the Hopf bundle $S^{2n-1} \to CP^{n-1}$. There is a natural action of S^1 on $D^2 \times S^{2n-1}$ by $t(x, y) = (tx, ty)$ where tx is complex multiplication and ty is the action of t on y with respect to the standard action. Now $W = (D^2 \times S^{2n-1})/S^1$ so there is a projection $D^2 \times S^{2n-1} \to W$. Also S^1 acts on $S^1 \times D^{2n}$ by $t(z, x) = (tz, tx)$. The orbit manifold is D^{2n}. The identity map

$$\partial(S^1 \times D^{2n}) \to \partial(D^2 \times S^{2n-1})$$

is equivariant so there is a smooth S^1-action on $S^1 \times D^{2n} \cup_1 D^2 \times S^{2n-1}$. The principal S^1-bundle

$$S^1 \times D^{2n} \cup_1 D^2 \times S^{2n-1} \to D^{2n} \cup_1 W$$

is equivalent to the Hopf bundle. Let $\theta: S^1 \to U(n) \subset SO(2n)$ be given by $\theta(z) = zI$. Define $g: S^1 \times D^{2n} \cup_1 D^2 \times S^{2n-1} \to S^1 \times D^{2n} \cup_1 D^2 \times S^{2n-1}$ by $g \mid D^2 \times S^{2n-1} = \text{identity}$ and, for $(z, x) \in S^1 \times D^{2n}$, $g(z, x) = (z, \theta(z) \cdot f(\theta(z)^{-1}x))$. g is an equivariant diffeomorphism covering $f \cup 1$ Thus $\rho\gamma([f]) = [g]$.

Now $[\theta] \in \Pi_1 = [S^1, SO(2n)]$. It is known [**1**] that if $\psi: S^1 \to SO(2n)$

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is defined by

$$\psi(e^{it}) = \begin{bmatrix} A^{r_1} & 0 & \cdots & 0 \\ 0 & A^{r_2} & & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ 0 & & & A^{r_n} \end{bmatrix}$$

where

$$A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

then in Π_1, $\psi \equiv r_1 + \cdots + r_n \pmod 2$, i.e., ψ is zero or nonzero depending on whether $r_1 + \cdots + r_n$ is even or odd. θ is of the form ψ with all $r_i = 1$. So θ is zero or nonzero depending on whether n is even or odd. Looking at the definition of $\rho_{2n+1,1}$ we see $\rho_{2n+1,1}(\sigma([f])) = [g] = \rho\gamma([f])$ as asserted. This proves the theorem.

Applications of Theorem 1. We are not able to give many applications of Theorem 1 due to a general lack of knowledge of the maps involved. However, in case $n = 3$, we have some results of G. Bredon **[1]** and R. Lee **[3]**. Specifically,

THEOREM (R. LEE). *$\gamma : \Gamma(S^6) \to \Gamma(CP^3)$ is onto.*

THEOREM (G. BREDON). *$\rho_{7,1} : \Gamma(S^6) \otimes \Pi_1 \to \Gamma(S^7)$ is trivial.*

In case $n = 3$ the diagram of Theorem 1 becomes

$$\begin{array}{ccc} \Gamma(S^6) & \xrightarrow{\sigma} & \Gamma(S^6) \otimes \Pi_1 \\ \gamma \downarrow & & \downarrow \rho_{7,1} \\ \Gamma(CP^3) & \xrightarrow{\rho} & \Gamma(S^7) \end{array}$$

Since $\rho_{7,1}$ is trivial and γ is onto we have:

THEOREM 2. *$\rho : \Gamma(CP^3) \to \Gamma(S^7)$ is trivial, i.e., every orientation preserving diffeomorphism of S^7 which is equivariant with respect to the standard action of S^1 on S^7 is concordant to the identity.*

REMARK. By **[2]**, $\Gamma(S^7) \cong Z_2$ so there do exist orientation preserving diffeomorphisms of S^7 which are not concordant to the identity. Of course by Theorem 2 no such map can be equivariant.

Using a classification theorem of R. Lee **[3]** we now give an application of Theorem 2 to transformation groups.

If a group G acts on a set X the action is called semifree if whenever $g \in G$ and $x \in X$ are such that $gx = x$ then either x is a fixed point or g

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is the identity of G. In other words, if F is the set of fixed points of the action then the action of G on $X - F$ is free.

Let S^1 act on D^{2n} by $t(z_1, \cdots, z_n) = (tz_1, \cdots, tz_n)$ where

$$D^{2n} = \left\{ (z_1, \cdots, z_n) \in C^n \,\middle|\, \sum_{i=1}^{n} |z_1|^2 \leqq 1 \right\}.$$

The restriction of this action to S^{2n-1} is the standard action. Suppose $f: S^{2n-1} \to S^{2n-1}$ is an orientation preserving equivariant diffeomorphism. Then $D^{2n} \cup_f D^{2n}$ is a homotopy $2n$-sphere and it carries a smooth semifree action with exactly two fixed points. R. Lee **[3]** has shown that if (S^1, Σ^{2n}) is any semifree S^1-action on the homotopy sphere Σ^{2n} with exactly two fixed points then there is an f as above such that (S^1, Σ^{2n}) is equivalent to the action of S^1 on $D^{2n} \cup_f D^{2n}$ constructed above. Using this we prove:

THEOREM 3. *Let Σ^8 be the exotic 8-sphere. Then Σ^8 does not admit a smooth, semifree S^1-action with exactly two fixed points.*

PROOF. According to Lee's result, if such an action did exist then Σ^8 would be expressible as $D^8 \cup_f D^8$ where $f: S^7 \to S^7$ is an equivariant, orientation preserving diffeomorphism. But by Theorem 2 such an f is concordant to the identity. Hence Σ^8 would be diffeomorphic to S^8 contrary to assumption.

In closing we make a remark concerning the possibility of further applications of Theorem 1 to transformation groups. Suppose for some *odd n* we knew $\rho_{2n+1,1}$ was nontrivial. Since $\sigma: \Gamma(S^{2n}) \to \Gamma(S^{2n}) \otimes \Pi_1$ is onto for n odd we could conclude that $\rho: \Gamma(CP^n) \to \Gamma(S^{2n+1})$ is nontrivial. Therefore we would have an orientation preserving, equivariant diffeomorphism $f: S^{2n+1} \to S^{2n+1}$ which is not concordant to the identity. Thus $D^{2n+2} \cup_f D^{2n+2}$ is an exotic homotopy sphere. Thus we would have constructed a smooth, semifree action with exactly two fixed points of S^1 on an exotic sphere Σ^{2n+2}. However the author is unaware of any example of an odd n where $\rho_{2n+1,1}$ is nontrivial.

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