

A COMPACT REGULAR OPERATOR WITHOUT MODULUS

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ABSTRACT. We construct an operator with the properties mentioned in the title. Without the regularity assumption such an example was first discovered by U. Krengel.

1. INTRODUCTION

It is a classical result due to Krengel [K1] (see also [AB, Example 16.6]) that a compact operator on a Banach lattice does not necessarily have a modulus. The aim of this article is to strengthen this result by showing that, even under the additional assumption of regularity, a compact operator may still not have a modulus. Recall that an operator is *regular* if it can be written as the difference of two positive operators. It follows from our example that there are compact sets in Banach lattices that are bounded above but fail to possess a supremum (else we could use the Riesz-Kantorovich formula to construct the modulus of a compact regular operator). Since this rather fundamental counterexample seems to have been overlooked in the literature, we present in the next section a direct construction of such a set. This result should be compared with another classical result of Krengel [K2] that each compact subset A of an arbitrary $C(K)$ -space has a supremum, and consequently, each compact operator $T: X \rightarrow C(K)$, where X is any Banach lattice, has a modulus. Some relevant information about the characterizations of compact subsets of AM-spaces may be found in [AB, Ab1, W].

We refer the reader to [AB, S, V, Z] for the basic definitions in this topic.

2. A COMPACT SET THAT IS BOUNDED ABOVE BUT HAS NO SUPREMUM

Before presenting the example, let us establish some notation. If E is a Banach lattice, we shall denote by $c(E)$ the space of all norm convergent sequences (x_n) from E ; when given the usual linear operations, the coordinatewise order and the norm $\|(x_n)\| = \sup\{\|x_n\|_E: n \in \mathbb{N}\}$, this becomes a Banach lattice. If (E_n) is a sequence of Banach lattices, then the space $c_0(E_n)$ of all sequences

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(x_n) such that $x_n \in E_n$ and $\|x_n\| \rightarrow 0$, is also a Banach lattice under the same operations and norm. We will denote by $\mathbf{1}$ the constantly one sequence in c , the space of all convergent real-valued sequences, and by e_n the real sequence with all terms zero except for the n th, which is one. Similarly we will use the notation $\mathbf{1}_{[0,1]}$ to denote the constantly one function on $[0, 1]$. If $(x_n) \in c$ and $\varphi \in E$, then $(x_n) \otimes \varphi$ will denote that element of $c(E)$ with n th term $x_n \times \varphi$, the usual product of the element $\varphi \in E$ by the scalar x_n .

Define a set of elements $A = \{y^{n,k}: 1 \leq k \leq n, n \in \mathbb{N}\} \cup \{0\}$ in $c(L^1[0, 1])$ as follows. Let $y^{n,k}$ be the sequence with all entries zero except for the $2n$ th one, which is the characteristic function of the interval $[(k-1)/n, k/n]$. Then we see that $\|y^{n,k}\| = 1/n$ so that the set A is certainly norm-compact. The sequence $\mathbf{1} \otimes \mathbf{1}_{[0,1]}$ is in $c(L^1[0, 1])$ and is an upper bound for A . It remains only to show that the set A has no supremum.

If some $(z_n) \in c(L^1[0, 1])$ is any upper bound for A , then we have $z_{2n} \geq 1$ on each interval $[(k-1)/n, k/n]$ and, therefore, $z_{2n} \geq 1$ on the whole of $[0, 1]$. Clearly also $z_{2n+1} \geq 0$. Conversely if $z_{2n} \geq 1$ on $[0, 1]$ and $z_{2n+1} \geq 0$, then (z_n) will be an upper bound for A . We show now that such an upper bound (z_n) cannot be a supremum. Since $z_n \rightarrow z_\infty \in L^1[0, 1]$, we see that $z_\infty = \lim z_{2n} \geq 1$ on the whole of $[0, 1]$. Choose n_0 such that $n \geq n_0$ implies that $\|z_n - z_\infty\|_1 < \frac{1}{2}$; then for such n we have $\|z_n\|_1 \geq \|z_\infty\|_1 - \frac{1}{2} \geq \frac{1}{2}$. In particular, if $n > n_0$ then $z_n \neq 0$. But if (z_n) is an upper bound for A , then picking any odd integer $m > n_0$ and replacing z_m by the zero function in $L^1[0, 1]$ gives us an upper bound for A that is strictly less than (z_n) . It follows that A has no supremum.

Note that a Banach lattice containing a set like A cannot be Dedekind complete. Otherwise, being order bounded, A would have a supremum.

3. A REGULAR COMPACT OPERATOR WITH NO MODULUS

Before presenting this example let us recall from [AW] the following result, which is easily proved by induction.

Lemma. For any elements x_1, x_2, \dots, x_n in a Riesz space

$$\sup \left\{ \sum_{k=1}^n \varepsilon_k x_k : \varepsilon_k \in \{-1, 0, 1\} \right\} = \sum_{k=1}^n |x_k|.$$

Let l_m^∞ denote m -dimensional real space with the pointwise order and the supremum norm $\|\cdot\|_\infty$. It is routine to verify that $c_0(l_{2^n}^\infty)$ is norm and order isomorphic to the familiar sequence space c_0 . Let r_n denote the n th Rademacher function on $[0, 1]$ (see [AB, p. 161]). The only properties that we need of these functions are that they are orthonormal in $L^2[0, 1]$ and all have modulus that is constantly one (almost everywhere). Our example will be an operator from the sequence space c into $c(L^2[0, 1])$.

Define an operator $T_n: l_{2^n}^\infty \rightarrow c(L^2[0, 1])$ by

$$T_n(\alpha_k) = \begin{cases} \frac{1}{2^n} e_n \otimes \sum_{k=1}^{2n} \alpha_k r_k & (n \text{ odd}), \\ 0 & (n \text{ even}). \end{cases}$$

Using the orthogonality of the Rademacher functions we see that if n is odd

then

$$\begin{aligned} \|T_n(\alpha_k)\| &\leq \frac{1}{2^n} \|e_n\| \times \left\| \sum_{k=1}^{2^n} \alpha_k r_k \right\| = \frac{1}{2^n} \sqrt{\sum_{k=1}^{2^n} |\alpha_k|^2} \\ &\leq 2^{-n/2} \|(\alpha_k)\|_\infty, \end{aligned}$$

so that $\|T_n\| \leq 2^{-n/2}$ no matter whether n is odd or even. It follows that we may define a compact operator on $c_0(l_{2^n}^\infty)$ by $T(x_n) = \sum T_n x_n$. If we define $T\mathbf{1} = 0$ and use the identification of $c_0(l_{2^n}^\infty)$ with c_0 , then we have produced a compact operator from c into $c(L^2[0, 1])$.

Define also $S_n: l_{2^n}^\infty \rightarrow c(L^2[0, 1])$ by

$$S_n(\alpha_k) = \frac{1}{2^n} \left(\sum_{k=1}^{2^n} \alpha_k \right) e_n \otimes \mathbf{1}_{[0, 1]}$$

for all $n \in \mathbb{N}$. Note that $\|S_n\| = 1$. Observe now that if F is a finite subset of \mathbb{N} , $x_n \in l_{2^n}^\infty$ and $\|x_n\|_\infty \leq \varepsilon$ for all $n \in F$, then $\|\sum_{n \in F} S_n x_n\| \leq \varepsilon$ as

$$\begin{aligned} \left\| \sum_{n \in F} S_n x_n \right\| &= \sup_{k \in \mathbb{N}} \left\| \left(\sum_{n \in F} S_n x_n \right)_k \right\|_2 \\ &= \sup_{k \in \mathbb{N}} \begin{cases} \|S_k x_k\|_2 & \text{if } k \in F, \\ 0 & \text{if } k \notin F \end{cases} \leq \varepsilon. \end{aligned}$$

It follows that if $x_n \in l_{2^n}^\infty$ and $\|x_n\|_\infty \rightarrow 0$, then $\sum S_n x_n$ converges in norm in $c(L^2[0, 1])$ to an element, which we will denote by $S(x_n)$. Define also $S\mathbf{1} = (2 \times \mathbf{1}) \otimes \mathbf{1}_{[0, 1]}$. Obviously S is a linear operator from c to $c(L^2[0, 1])$. We claim that S is a positive majorant for T .

If $(x_n) \in c_0(l_{2^n}^\infty)_+$, then $x_n = (\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_{2^n}^{(n)})$, where each $\alpha_k^{(n)} \geq 0$, and hence

$$\begin{aligned} (S - T)(x_n) &= \sum (S_n - T_n)x_n \\ &= \sum_{n=1}^\infty \frac{1}{2^n} e_n \otimes \left[\sum_{k=1}^{2^n} \alpha_k^{(n)} (\mathbf{1}_{[0, 1]} - r_k) \right] \geq 0 \end{aligned}$$

(as $\mathbf{1}_{[0, 1]} \geq r_k \geq -\mathbf{1}_{[0, 1]}$). Also note that

$$S(x_n) = \sum_{n=1}^\infty S_n x_n = \sum_{n=1}^\infty \frac{1}{2^n} e_n \otimes \left(\sum_{k=1}^{2^n} \alpha_k^{(n)} \right) \geq 0.$$

In order to complete a proof that $S \geq T, 0$ we need to consider positive elements of c of the form $\mathbf{1} + (x_n)$, where $(x_n) \in c_0(l_{2^n}^\infty)$. By the positivity of the sum, $x_n \geq (-1, -1, \dots, -1)$, where there are exactly 2^n terms equal to -1 . Thus

$$\begin{aligned} S(\mathbf{1} + (x_n)) &\geq (2 \times \mathbf{1}) \otimes \mathbf{1}_{[0, 1]} + \sum_{n=1}^\infty \frac{1}{2^n} \sum_{k=1}^{2^n} (-1) e_n \otimes \mathbf{1}_{[0, 1]} \\ &= (2 \times \mathbf{1}) \otimes \mathbf{1}_{[0, 1]} + \sum_{n=1}^\infty (-e_n) \otimes \mathbf{1}_{[0, 1]} \geq 0 \end{aligned}$$

and

$$\begin{aligned}(S - T)(\mathbf{1} + (x_n)) &\geq (2 \times \mathbf{1}) \otimes \mathbf{1}_{[0, 1]} + \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \otimes \sum_{k=1}^{2^n} -(|r_k| + 1) \\ &= (2 \times \mathbf{1}) \otimes \mathbf{1}_{[0, 1]} - 2 \sum_{n=1}^{\infty} e_n \otimes \mathbf{1}_{[0, 1]} \geq 0.\end{aligned}$$

It follows that $S \geq T$, 0 and, therefore, $T = S - (S - T)$ is a regular operator.

It remains to prove that T fails to have a modulus. If $U \geq T$, $-T$ then on each $l_{2^n}^{\infty}$, for n odd, we have

$$(1, 1, \dots, 1) \geq (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2^n}) \geq (-1, -1, \dots, -1)$$

for any choice of $\varepsilon_k \in \{-1, 0, 1\}$. Thus

$$U(1, 1, \dots, 1) \geq T_n(\varepsilon_1, \varepsilon_2, \dots) = \frac{1}{2^n} e_n \otimes \sum_{k=1}^{2^n} \varepsilon_k r_k$$

and by the lemma we have

$$U(1, 1, \dots, 1) \geq \frac{1}{2^n} e_n \otimes \sum_{k=1}^{2^n} |r_k| = e_n \otimes \mathbf{1}_{[0, 1]}.$$

This happens for all odd n , so certainly $\lim_{n \rightarrow \infty} (U\mathbf{1})_n \geq \mathbf{1}_{[0, 1]}$. Since each Tx vanishes on even coordinates in $c(L^2[0, 1])$, we see that if $|T|$ existed it would have $(|T|x)_{2n} = 0$ for all n and x . In particular, $(|T|\mathbf{1})_{2n} = 0$ and hence $\lim_{n \rightarrow \infty} (|T|\mathbf{1})_n = 0$. As this contradicts the fact that this limit must be at least $\mathbf{1}_{[0, 1]}$, this shows that $|T|$ cannot exist.

Notice that, as in the previous section, such an example is only possible if the range space Y is not Dedekind complete. For if Y were Dedekind complete, then by the classical Riesz-Kantorovich theorem, the space $L'(X, Y)$ of all regular operators from any vector lattice X into Y would be a vector lattice (even a Dedekind complete vector lattice) and $|T|$ would exist.

4. COMMENTS

(1) The operator S constructed in §3 is easily seen to be noncompact. It is therefore still possible that every compact regular operator majorized by a compact positive operator has a modulus (and even possible that the modulus must be compact).

(2) In spite of the “negative” nature of the results presented in this article, we still hope that it is possible to impose extra assumptions on a compact operator T that would imply some nontrivial information about $|T|$. For example, the assumption in (1) might suffice.

On the other hand, we would like to stress that such additional hypothetical conditions will have to be applied to an individual operator rather than to the whole class of compact operators. Otherwise, as the following result (see [Ab1, Corollary to Theorem 5.2]) says, the nature of the space Y is predetermined.

Let X and Y be Banach lattices, where X is not a KB-space, i.e., X contains a subspace isomorphic to c_0 . Then the following four conditions are equivalent.

(a) Y is an AM-space, (b) for each compact $T: X \rightarrow Y$ there exists $|T|: X \rightarrow Y$,

which is also compact, (c) each compact operator $T: X \rightarrow Y$ is regular, (d) each compact operator $T: X \rightarrow Y$ is regular as an operator from X to Y^{**} .

Each of the implications (b) \Rightarrow (a), (c) \Rightarrow (a), or (d) \Rightarrow (a) shows that a global condition on the whole space $K(X, Y)$ of compact operators implies immediately that Y is a classical AM -space.

(3) Using Khinchin's inequality [LT, Theorem 2.b.3], our example in §3 may be modified to give an example of a regular compact operator taking values in $c(L^p[0, 1])$, with no modulus, for each $p \in [1, \infty)$. Also we can replace the discrete space c by the space $C[0, 1]$ and consider a corresponding analogue of the space $c(L^p[0, 1])$.

(4) The following question was asked in [Ab2] (Question 3, p. 139). Does there exist a pair of Dedekind complete Banach lattices X and Y such that $K(X, Y) \subset L'(X, Y)$ but $L'(X, Y) \neq L(X, Y)$? (Here $L(X, Y)$ denotes, as usual, the space of all continuous operators from X to Y .)

We give below a positive solution to this problem. Let $X = C(Q)$, where Q is an arbitrary (infinite) extremally disconnected compact space. Let q_0 be a fixed nonisolated point in Q and let $Y = \{f \in X: f(q_0) = 0\}$. Both X and Y are Dedekind complete AM -spaces. By the theorem cited in (2) $K(X, Y) \subset L'(X, Y)$. Moreover, $|T| \in K(X, Y)$ for each $T \in K(X, Y)$.

We claim that $L(X, Y) \neq L'(X, Y)$. Let $T \in L(X, Y)$ be defined as $(Tf)(q) = f - f(q_0)\mathbf{1}$. Obviously $T \in L(X, Y)$ and $Tf = f$ for all $f \in Y$. It is easy to see that T is not regular. Indeed, since Y is Dedekind complete, and if T were regular, then there would exist its positive part T^+ , and for each $x \in X_+$ the value $T^+x \in Y$ would be computed by the Riesz-Kantorovich formula. In particular, we would have

$$T^+\mathbf{1} = Y - \sup\{Tf: 0 \leq f \leq \mathbf{1}\}.$$

Observe now that Y is an ideal in X and, therefore, $Y - \sup(D) = X - \sup(D)$ for each order bounded subset D of Y .

On the other hand, $Y - \sup\{Tf: 0 \leq f \leq \mathbf{1}\}$ cannot exist, since otherwise

$$\begin{aligned} T^+\mathbf{1} &= Y - \sup\{Tf: 0 \leq f \leq \mathbf{1}\} \\ &= X - \sup\{Tf: 0 \leq f \leq \mathbf{1}\} \\ &\geq X - \sup\{Tf: 0 \leq f \leq \mathbf{1} \& f(q_0) = 0\} = \mathbf{1}. \end{aligned}$$

Since $T^+\mathbf{1}$ is in the ideal Y , this would imply that $\mathbf{1} \in Y$, which is a contradiction.

However, under the natural additional assumption that at least one of the Banach lattices X or Y be reflexive, Question 3 from [Ab2] remains open.

(5) The first author wants to use this opportunity to improve the phraseology of Theorem 8.12 in [Ab2]. The concluding sentence, "Then S is finite," should have read, "Then $C(S)$ is Dedekind complete." The original version is correct but the modification makes it easier to understand the comment that follows Theorem 8.12.

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