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A compactness result in the gradient theory of phase transitions

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We examine the singularly perturbed variational problem

$$E_\epsilon(\psi) = \int \epsilon^{-1}(1 - |\nabla\psi|^2)^2 + \epsilon|\nabla\nabla\psi|^2$$

in the plane. As $\epsilon \to 0$, this functional favours $|\nabla\psi| = 1$ and penalizes singularities where $|\nabla\nabla\psi|$ concentrates. Our main result is a compactness theorem: if $\{E_\epsilon(\psi_\epsilon)\}_{\epsilon\downarrow 0}$ is uniformly bounded, then $\{\nabla\psi_\epsilon\}_{\epsilon\downarrow 0}$ is compact in L^2. Thus, in the limit $\epsilon \to 0$, ψ solves the eikonal equation $|\nabla\psi| = 1$ almost everywhere. Our analysis uses 'entropy relations' and the 'div-curl lemma,' adopting Tartar's approach to the interaction of linear differential equations and nonlinear algebraic relations.

1. Motivation, statement of the result and idea of the proof

We consider the singularly perturbed functional

$$E_\epsilon(\psi) = \epsilon \int_\Omega |\nabla\nabla\psi|^2 + \frac{1}{\epsilon}\int_\Omega (1 - |\nabla\psi|^2)^2 \tag{1.1}$$

as $\epsilon \downarrow 0$. It arises as a model problem in connection with several physical applications, including smectic liquid crystals [2], thin film blisters [8, 17] and convective pattern formation [7]. Physically, equation (1.1) can be viewed as a simple Landau theory, in which the order parameter is a curl-free vector field $\nabla\psi$ which prefers to be of norm 1.

The functional analysis of (1.1) is still poorly understood, despite considerable attention. A natural goal is to find the 'asymptotic energy' as $\epsilon \downarrow 0$, represented by the Γ-limit of $\{E_\epsilon\}_{\epsilon\downarrow 0}$ (see, for instance, [6]). Aviles and Giga's [2] conjecture for the Γ-limit is

$$E_0(\psi) = \begin{cases} \dfrac{1}{3}\displaystyle\int_{D(\psi)} |[\nabla\psi]|^3 \mathrm{d}s & \text{if } |\nabla\psi| = 1 \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases}$$

where $D(\psi)$ is a suitably defined 'defect set' of ψ (at which $\nabla\psi$ is discontinuous), $[\nabla\psi]$ is the jump in $\nabla\psi$, and ds is arclength. To confirm this conjecture, one would have to show (roughly speaking) the following three assertions.

(a) The Γ-limit is infinite unless $|\nabla\psi| = 1$ almost everywhere. Thus only solutions of the eikonal equation are admissible for the asymptotic functional.

(b) The proposed integrand $\frac{1}{3}|[\nabla\psi]|^3$ is correct. Aviles and Giga derived this formula by assuming that E_ϵ prefers 'locally one-dimensional' transition layers, with $\nabla\psi$ varying rapidly only in the direction normal to the defect set.

(c) The asymptotic energy lives only on a suitably defined one-dimensional defect set $D(\psi)$. Thus, to leading order in ϵ, lower-dimensional singularities carry no energy.

All the analysis to date has been restricted to the case when space is two dimensional. Point (a) is demonstrated in the present paper. Point (b) is substantially confirmed by the work of Jin and Kohn [9, 10] and Aviles and Giga [3]. Point (c) is basically open. After this work was done, but before it was submitted for publication, we learned of related progress by Ambrosio *et al*. [1]. They also demonstrate (a), using a method entirely different from ours, and they show by example that ψ can be surprisingly complex and still have finite asymptotic energy (in particular, $\nabla\psi$ need not have bounded variation).

Our functional (1.1) is an obvious generalization to gradient fields of the scalar problem considered by Modica and Mortola [12–14],

$$\tilde{E}_\epsilon(u) = \epsilon\int_\Omega |\nabla u|^2 + \frac{1}{\epsilon}\int_\Omega (1-u^2)^2. \tag{1.2}$$

Let us briefly review the compactness result associated with (1.2). The precise statement is: if, for a sequence $\{u_\epsilon\}_{\epsilon\downarrow 0}$, the energies $\{\tilde{E}_\epsilon(u_\epsilon)\}_{\epsilon\downarrow 0}$ are uniformly bounded, then $\{u_\epsilon\}$ is relatively compact in $L^2(\Omega)$. The essence of the argument is the inequality

$$\begin{aligned} \tfrac{1}{2}\epsilon\int_\Omega |\nabla u_\epsilon|^2 + \frac{1}{2\epsilon}\int_\Omega (1-u_\epsilon^2)^2 &\geqslant \int_\Omega |\nabla u_\epsilon||1-u_\epsilon^2| \\ &= \int_\Omega |\nabla \Phi(u_\epsilon)|, \end{aligned} \tag{1.3}$$

where $\Phi(s) = s(1-\frac{1}{3}s^2)$. The estimate implies the boundedness of $\{\nabla\Phi(u_\epsilon)\}_{\epsilon\downarrow 0}$ in $L^1(\Omega)$, which provides sufficient compactness. It is obvious that the above argument does not generalize to (1.1); there is no analogue of (1.3), since there is no transformation Φ such that $\mathrm{D}[\Phi(\nabla\psi_\epsilon)] = (1-|\nabla\psi_\epsilon|^2)\,\mathrm{D}^2\,\psi_\epsilon$. The difference may also be seen as follows. For (1.2), the favoured values of u form a discrete set $\{-1, 1\}$, while for (1.1), the favoured values of $f = \nabla\psi$ form a continuum $\{|f|^2 = 1\}$. As a result, the fact that $\nabla \times f = 0$, which has no analogue for (1.2), is essential for proving compactness in the context of (1.1). We will have to investigate the combined effect of the linear differential equation $\nabla \times f = 0$ and the nonlinear relation $|f|^2 = 1$.

PROPOSITION 1.1. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Let the sequences $\{\epsilon_\nu\}_{\nu\uparrow\infty} \subset (0,\infty)$ and $\{\psi_\nu\}_{\nu\uparrow\infty} \subset H^2(\Omega)$ be such that*

$$\epsilon_\nu \xrightarrow{\nu\uparrow\infty} 0 \quad \textit{and} \quad \{E_{\epsilon_\nu}(\psi_\nu)\}_{\nu\uparrow\infty} \quad \textit{is bounded.}$$

Then

$$\{\nabla\psi_\nu\}_{\nu\uparrow\infty} \subset L^2(\Omega) \quad \textit{is relatively compact.}$$

Actually, we prove a bit more than proposition 1.1. To state the stronger result, we prefer to work with the divergence-free vector fields $m_\nu = \mathrm{R}\,\nabla\psi_\nu$, where R denotes rotation by $\frac{1}{2}\pi$, that is,

$$\mathrm{R}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}.$$

This shift of perspective entails no loss of generality (our method seems intrinsically limited to two space dimensions). Moreover, it highlights the analogy between (1.1) and the micromagnetic energy of an isotropic thin film, where m is only approximately divergence free, but $|m| = 1$ exactly. In truth, we first found the arguments behind proposition 1.2 while exploring the micromagnetics of thin films. This paper focuses on (1.1) instead of micromagnetics, because that is the more familiar and widely studied problem. Our stronger result is as follows.

PROPOSITION 1.2. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Let the sequence $\{m_\nu\}_{\nu\uparrow\infty} \subset H^1(\Omega)^2$ be such that*

$$\nabla \cdot m_\nu = 0 \quad \textit{a.e. in } \Omega, \tag{1.4}$$

$$\|1 - |m_\nu|^2\|_{L^2(\Omega)} \xrightarrow{\nu\uparrow\infty} 0, \tag{1.5}$$

$$\{\|\nabla m_\nu\|_{L^2(\Omega)}\|1 - |m_\nu|^2\|_{L^2(\Omega)}\}_{\nu\uparrow\infty} \quad \textit{is bounded.} \tag{1.6}$$

Then

$$\{m_\nu\}_{\nu\uparrow\infty} \subset L^2(\Omega) \quad \textit{is relatively compact.} \tag{1.7}$$

The fact that this is a non-trivial issue becomes apparent by the following argument. Assume that (1.7) is true. Then there exists an $m \in L^2(\Omega)$ such that, for a subsequence,

$$m_\nu \xrightarrow{\nu\uparrow\infty} m \quad \text{in } L^2(\Omega).$$

Property (1.4) is conserved in the limit in a weak sense,

$$\nabla \cdot m = 0 \quad \text{in a distributional sense on } \Omega, \tag{1.8}$$

whereas (1.5) sharpens into

$$|m|^2 = 1 \quad \text{a.e. in } \Omega. \tag{1.9}$$

On the level of $L^2(\Omega)$-functions, the combination of the linear partial differential equation (1.8) and the nonlinear relation (1.9) is not enough to ensure compactness in $L^2(\Omega)$. On the level of differentiable functions, it is very rigid. (This can be easily seen by going back to the original description $m = \mathrm{R}\,\nabla\psi$ in which (1.8) is automatically fulfilled and (1.9) turns into the eikonal equation $|\nabla\psi|^2 = 1$.) Hence in our compactness proof we will have to combine the linear partial differential equation (1.4), the increasing penalization of $|m|^2 \neq 1$ through (1.5), and the (fading) control of $\mathrm{D}\,m$ through (1.6).

Let us sketch the basic idea of the proof of proposition 1.2. For this, we reconsider an m which satisfies both the linear partial differential equation (1.8) and the nonlinear relation (1.9). Because of (1.9), we can write

$$m = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

with a function θ so that (1.8) turns into

$$\partial_1(\cos\theta) + \partial_2(\sin\theta) = 0. \tag{1.10}$$

It is enlightening to think of (1.10) as a scalar conservation law for the quantity $s \simeq \cos\theta$ which depends on time $t \simeq x_1$ and a single spatial variable $y \simeq x_2$,

$$\partial_t s + \partial_y f(s) = 0. \tag{1.11}$$

As a scalar conservation law (1.11), equation (1.10) would be highly nonlinear. As can be seen by the method of characteristics, equation (1.11) with a nonlinear flux function f does not admit differentiable solutions to the Cauchy problem for most smooth initial data. On the other hand, there generically are infinitely many distributional solutions to the Cauchy problem. The notion of entropy solution has been introduced; the Cauchy problem is well posed in this framework (see, for instance, [11]).

What is the notion of an entropy solution? If the pair of nonlinear functions (η, q) satisfies $q' = \eta' f'$ (a so-called entropy entropy-flux pair) and if s is a differentiable solution of (1.11), then

$$\partial_t\eta(s) + \partial_y q(s) = 0. \tag{1.12}$$

But if f is nonlinear and s is only a distributional solution of (1.11), then (1.12) is generically not satisfied—even in a distributional sense. An entropy solution s of (1.11) is defined as a distributional solution of (1.11) with the property that

$$\partial_t\eta(s) + \partial_y q(s) \leqslant 0$$

in a distributional sense for all entropy entropy-flux pairs (η, q) such that η is convex. Even if η is not convex, we have, for an entropy solution, that

$$\partial_t\eta(s) + \partial_y q(s) \quad \text{is a measure.}$$

By a lemma of Murat [16], this implies that if $\{s_\nu\}_{\nu\uparrow\infty}$ is a sequence of uniformly bounded entropy solutions, then

$$\{\partial_t\eta(s_\nu) + \partial_y q(s_\nu)\}_{\nu\uparrow\infty} \quad \text{is compact in } H^{-1}.$$

The latter allows for a judicious application of Murat and Tartar's div-curl lemma (a special case of compensated compactness, see [15, 18]). Tartar used this method in [18] to derive restrictions on the Young measure generated by $\{m_\nu\}_{\nu\uparrow\infty}$. In particular, he showed that if f is sufficiently nonlinear, then the set of uniformly bounded entropy solutions is compact. The scope of Tartar's analysis is much more general, however, than this single application. What the paper [18] really explores is how linear partial differential equations (like (1.8)) and nonlinear relations (like (1.9)), taken together, restrict and sometimes rule out oscillations. Tartar's method is perfectly suited to our situation.

In the first part of § 2 (lemmas 2.2 and 2.3), we will identify all (nonlinear) functions Φ of m with the property that $\Phi(m)$ satisfies a certain linear partial differential equation, provided m satisfies the linear partial differential equation (1.8) and the nonlinear relation (1.9). More precisely, we will identify all Φ such that

$$\text{if } m \text{ is differentiable with } \nabla\cdot m = 0 \text{ and } |m|^2 = 1, \text{ then } \nabla\cdot[\Phi(m)] = 0.$$

This follows a concept of Tartar and mimics the tool of entropy and entropy-flux pairs (η, q). In the second part of § 2 (lemma 2.6), we will show that the class of entropies is rich enough for our purposes. This doesn't come as a surprise, since the set of all entropy and entropy-flux pairs (η, q) is rich enough for a scalar conservation law in one space dimension (1.11). In the first part of § 3, we will show that the control expressed in (1.6) is strong enough to ensure that, for our sequence $\{m_\nu\}_{\nu\uparrow\infty}$,

$$\{\nabla\cdot[\Phi(m_\nu)]\}_{\nu\uparrow 0} \quad \text{is compact in } H^{-1} \text{ for above } \Phi\text{s}.$$

Then, in the second part of § 3, we will apply Tartar's programme.

2. Entropies

DEFINITION 2.1. A $\Phi \in C_0^\infty(\mathbb{R}^2)^2$ is called an entropy if, for all z,

$$z\cdot \mathrm{D}\,\Phi(z)\,\mathrm{R}\,z = 0, \qquad \Phi(0) = 0, \qquad \mathrm{D}\,\Phi(0) = 0, \tag{2.1}$$

where $\mathrm{D}\,\Phi_{i,j} = \partial\Phi_i/\partial x_j$ denotes the Jacobian of Φ and R the rotation by $\frac{1}{2}\pi$, that is,

$$\mathrm{R}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}.$$

LEMMA 2.2. *Let $\Phi \in C_0^\infty(\mathbb{R}^2)^2$ be an entropy. Then there exists a $\Psi \in C_0^\infty(\mathbb{R}^2)^2$ such that, for all $z \in \mathbb{R}^2$,*

$$\mathrm{D}\,\Phi(z) + 2\Psi(z)\otimes z = \alpha\,\mathrm{id} \quad \textit{for some } \alpha, \tag{2.2}$$

where id *denotes the* 2×2 *identity matrix.*

Proof of lemma 2.2. Componentwise, equation (2.2) is equivalent to the three equations

$$\Phi_{1,1}(z) + 2\Psi_1(z)z_1 = \Phi_{2,2}(z) + 2\Psi_2(z)z_2 \tag{2.3}$$

and

$$\Phi_{1,2}(z) + 2\Psi_1(z)z_2 = 0, \qquad \Phi_{2,1}(z) + 2\Psi_2(z)z_1 = 0. \tag{2.4}$$

By continuity, equation (2.3) is equivalent to (2.3) multiplied with $z_1 z_2$, that is,

$$z_1 z_2 \Phi_{1,1}(z) + 2z_1^2 z_2 \Psi_1(z) = z_1 z_2 \Phi_{2,2}(z) + 2z_1 z_2^2 \Psi_2(z).$$

Hence the conjunction of (2.3) and (2.4) is equivalent to the conjunction of

$$z_1 z_2 \Phi_{1,1}(z) - z_1^2 \Phi_{1,2}(z) = z_1 z_2 \Phi_{2,2}(z) - z_2^2 \Phi_{2,1}(z) \tag{2.5}$$

and (2.4). But (2.5) is just (2.1) written in a componentwise fashion and (2.4) can be satisfied by choosing

$$\Psi_1(z) = -\frac{1}{2z_2}\Phi_{1,2}(z) \quad \text{and} \quad \Psi_2(z) = -\frac{1}{2z_1}\Phi_{2,1}(z).$$

We observe that, by definition, we have $\mathrm{D}\,\Phi(0) = 0$, which ensures $\Psi \in C_0^\infty(\mathbb{R}^2)^2$. □

Lemma 2.3. *Let $\Phi \in C_0^\infty(\mathbb{R}^2)^2$ and $\Psi \in C_0^\infty(\mathbb{R}^2)^2$ satisfy (2.2). Let $m \in H^1(\Omega)^2$ satisfy*

$$\nabla \cdot m = 0 \quad \text{a.e. in } \Omega.$$

Then

$$\nabla \cdot [\Phi(m)] = \Psi(m) \cdot \nabla(1 - |m|^2) \quad \text{a.e. in } \Omega.$$

Proof of lemma 2.3. According to lemma 2.2, we have $\mathrm{D}\,\Phi(m) = -2\Psi(m) \otimes m + \alpha\,\mathrm{id}$ and therefore

$$\begin{aligned}
\nabla \cdot [\Phi(m)] &= \mathrm{tr}(\mathrm{D}\,\Phi(m) \nabla m) \\
&= -2\Psi(m) \cdot (\nabla m)^{\mathrm{T}} m + \alpha \nabla \cdot m \\
&= -\Psi(m) \cdot \nabla |m|^2 \\
&= \Psi(m) \cdot \nabla(1 - |m|^2).
\end{aligned}$$

□

Lemma 2.4. *There is a one-to-one correspondence between entropies $\Phi \in C_0^\infty(\mathbb{R}^2)^2$ and functions $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi(0) = 0$ via*

$$\Phi(z) = \varphi(z) z + (\nabla\varphi(z) \cdot \mathrm{R}\, z)\, \mathrm{R}\, z. \tag{2.6}$$

Proof of lemma 2.4. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi(0) = 0$ be given and Φ defined via (2.6). Obviously, $\Phi(0) = 0$. We have

$$\mathrm{D}\,\Phi(z) = z \otimes \nabla\varphi(z) + \varphi(z)\,\mathrm{id} + \mathrm{R}\, z \otimes (\mathrm{D}^2\, \varphi(z)\, \mathrm{R}\, z - \mathrm{R}\, \nabla\varphi(z)) + (\nabla\varphi(z) \cdot \mathrm{R}\, z)\, \mathrm{R}$$

and therefore $\mathrm{D}\,\Phi(0) = 0$ and

$$z \cdot \mathrm{D}\,\Phi(z)\, \mathrm{R}\, z = |z|^2 \nabla\varphi(z) \cdot \mathrm{R}\, z + (\nabla\varphi(z) \cdot \mathrm{R}\, z)(z \cdot \mathrm{R}\,\mathrm{R}\, z) = 0.$$

On the other hand, let $\Phi \in C_0^\infty(\mathbb{R}^2)^2$ be an entropy. Since $\Phi(0) = 0$ and $\mathrm{D}\,\Phi(0) = 0$,

$$|z|^2 \varphi(z) = \Phi(z) \cdot z \tag{2.7}$$

defines a $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\varphi(0) = 0$. Differentiating the identity (2.7) in the direction $\mathrm{R}\, z$ yields

$$\begin{aligned} |z|^2 \nabla\varphi(z)\, \mathrm{R}\, z &= z \cdot \mathrm{D}\, \Phi(z) \cdot \mathrm{R}\, z + \Phi(z) \cdot \mathrm{R}\, z \\ &= \Phi(z) \cdot \mathrm{R}\, z \quad \text{by (2.1).} \end{aligned} \tag{2.8}$$

Hence

$$\begin{aligned} |z|^2 \Phi(z) &= (\Phi(z) \cdot z) z + (\Phi(z) \cdot \mathrm{R}\, z)\, \mathrm{R}\, z \\ &= |z|^2 \varphi(z) z + |z|^2 (\nabla\varphi(z) \cdot \mathrm{R}\, z)\, \mathrm{R}\, z \quad \text{by (2.7), (2.8)} \\ &= |z|^2 (\varphi(z) z + (\nabla\varphi(z) \cdot \mathrm{R}\, z)\, \mathrm{R}\, z). \end{aligned}$$

By continuity, this implies (2.6). □

LEMMA 2.5. *Fix an $e \in S^1$, the set of unit vectors in $\mathbb{R}^2$. Then*

$$\Phi(z) = \begin{cases} |z|^2 e & \text{for } z \cdot e > 0, \\ 0 & \text{for } z \cdot e \leqslant 0 \end{cases} \tag{2.9}$$

is a generalized entropy in the sense that there exists a sequence $\{\Phi_\nu\}_{\nu\uparrow\infty}$ of entropies in $C_0^\infty(\mathbb{R}^2)^2$ such that

$$\{\Phi_\nu(z)\}_{\nu\uparrow\infty} \quad \text{is bounded uniformly for bounded } z, \tag{2.10}$$

$$\Phi_\nu(z) \xrightarrow{\nu\uparrow\infty} \Phi(z) \quad \text{for all } z. \tag{2.11}$$

Proof of lemma 2.5. Consider the function φ,

$$\varphi(z) = \begin{cases} z \cdot e & \text{for } z \cdot e > 0, \\ 0 & \text{for } z \cdot e \leqslant 0 \end{cases}$$

and the map ξ given by

$$\xi(z) = \begin{cases} e & \text{for } z \cdot e > 0, \\ 0 & \text{for } z \cdot e \leqslant 0. \end{cases}$$

Observe that ξ is the gradient of φ wherever the latter is differentiable. Obviously, there exists a sequence $\{\varphi_\nu\}_{\nu\uparrow\infty}$ in $C_0^\infty(\mathbb{R}^2)$ with $\varphi_\nu(0) = 0$ such that

$$\{(\varphi_\nu(z), \nabla\varphi_\nu(z))\}_{\nu\uparrow\infty} \quad \text{is bounded uniformly for bounded } z, \tag{2.12}$$

$$(\varphi_\nu(z), \nabla\varphi_\nu(z)) \xrightarrow{\nu\uparrow\infty} (\varphi(z), \xi(z)) \quad \text{for all } z. \tag{2.13}$$

According to lemma 2.4,

$$\Phi_\nu(z) = \varphi_\nu(z) z + (\nabla\varphi_\nu(z) \cdot \mathrm{R}\, z)\, \mathrm{R}\, z$$

is an entropy. Equation (2.12) implies (2.10) and, according to (2.13),

$$\begin{aligned}\Phi_\nu(z) \xrightarrow{\nu\uparrow\infty} & \varphi(z)z + (\xi(z)\cdot \mathrm{R}\,z)\,\mathrm{R}\,z \\ &= \begin{cases} (z\cdot e)z + (e\cdot \mathrm{R}\,z)\,\mathrm{R}\,z & \text{for } z\cdot e > 0, \\ 0 & \text{for } z\cdot e \leqslant 0, \end{cases} \\ &= \begin{cases} |z|^2 e & \text{for } z\cdot e > 0, \\ 0 & \text{for } z\cdot e \leqslant 0, \end{cases}\end{aligned}$$

which turns into (2.11). □

Lemma 2.6. *Let μ be a probability measure on $\mathbb{R}^2$ supported on S^1. Suppose it has the property*

$$\int \Phi\cdot \mathrm{R}\,\tilde{\Phi}\,\mathrm{d}\mu = \int \Phi\,\mathrm{d}\mu \cdot \int \mathrm{R}\,\tilde{\Phi}\,\mathrm{d}\mu \quad \textit{for all entropies } \Phi,\ \tilde{\Phi}.$$

Then μ is a Dirac measure.

Proof of lemma 2.6. According to lemma 2.5, we are allowed to use the generalized entropies of the form (2.9). As μ is supported on S^1, this yields

$$e\cdot \mathrm{R}\,\tilde{e}\mu(\{z\cdot e > 0\}\cap\{z\cdot\tilde{e} > 0\}) = e\cdot \mathrm{R}\,\tilde{e}\mu(\{z\cdot e > 0\})\mu(\{z\cdot\tilde{e} > 0\}) \quad \text{for all } e,\tilde{e}\in S^1$$

or

$$\mu(\{z\cdot e > 0\}\cap\{z\cdot\tilde{e} > 0\}) = \mu(\{z\cdot e > 0\})\mu(\{z\cdot\tilde{e} > 0\}) \\ \text{for all } \tilde{e}\in S^1 - \{e,-e\} \text{ and all } e\in S^1.$$

Sending $\tilde{e}$ to e yields

$$\mu(\{z\cdot e > 0\}) \leqslant \mu(\{z\cdot e > 0\})\mu(\{z\cdot e \geqslant 0\}) \quad \text{for all } e\in S^1$$

or

$$\mu(\{z\cdot e > 0\}) = 0 \quad \text{or} \quad \mu(\{z\cdot e \geqslant 0\}) \geqslant 1 \quad \text{for all } e\in S^1.$$

As μ is a probability measure, this implies

$$\operatorname{supp}\mu \subset \{z\cdot e\leqslant 0\} \quad \text{or} \quad \operatorname{supp}\mu \subset \{z\cdot e\geqslant 0\} \quad \text{for all } e\in S^1.$$

As the measure μ is concentrated on S^1, this forces it to be concentrated on a single point on S^1. □

3. Compensated compactness and Young measures

Proof of the propositions. We may focus on proposition 1.2, since, as explained in § 1, it implies proposition 1.1.

The first step is to show that, for any entropy $\Phi\in C_0^\infty(\mathbb{R}^2)^2$,

$$\{\nabla\cdot[\Phi(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{is compact in } H^{-1}(\Omega). \tag{3.1}$$

According to (1.4) and lemmas 2.2, 2.3, there exists a $\Psi \in C_0^\infty(\mathbb{R}^2)^2$ such that

$$\nabla \cdot [\Phi(m_\nu)] = \Psi(m_\nu) \cdot \nabla(1 - |m_\nu|^2) \quad \text{a.e. in } \Omega. \tag{3.2}$$

Since Ψ is bounded and according to (1.5), $\{(1 - |m_\nu|^2)\Psi(m_\nu)\}_{\nu\uparrow\infty}$ converges to zero in $L^2(\Omega)$. As a consequence, $\{\nabla \cdot [(1 - |m_\nu|^2)\Psi(m_\nu)]\}_{\nu\uparrow\infty}$ converges to zero in $H^{-1}(\Omega)$. Therefore, equation (3.1) will follow from the assertion that

$$\{\nabla \cdot [\Phi(m_\nu) - (1 - |m_\nu|^2)\Psi(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{is compact in } H^{-1}(\Omega), \tag{3.3}$$

which we show now. Thanks to (3.2), we have

$$\nabla \cdot [\Phi(m_\nu) - (1 - |m_\nu|^2)\Psi(m_\nu)] = -\nabla \cdot [\Psi(m_\nu)](1 - |m_\nu|^2) \quad \text{a.e. in } \Omega. \tag{3.4}$$

We observe that, since Φ and Ψ are bounded and according to (1.5),

$$\{|\Phi(m_\nu) - (1 - |m_\nu|^2)\Psi(m_\nu)|^2\}_{\nu\uparrow\infty} \quad \text{is uniformly integrable.} \tag{3.5}$$

Since $\mathrm{D}\Psi$ is bounded, and according to (1.6),

$$\{\nabla \cdot [\Psi(m_\nu)](1 - |m_\nu|^2)\}_{\nu\uparrow\infty} \quad \text{is bounded in } L^1(\Omega). \tag{3.6}$$

A variation of a lemma by Murat [16] (see also [18, lemma 28]) now shows that in the presence of (3.5) and (3.6), the identity (3.4) implies (3.3). (Recall that this in turn implies (3.1).) For the convenience of the reader, we formulate and prove the lemma.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let the sequence $\{f_\nu\}_{\nu\uparrow\infty} \subset L^2(\Omega)^N$ satisfy*

$$\{\nabla \cdot f_\nu\}_{\nu\uparrow\infty} \quad \textit{is bounded in } L^1(\Omega), \tag{3.7}$$

$$\{|f_\nu|^2\}_{\nu\uparrow\infty} \quad \textit{is uniformly integrable on } \Omega. \tag{3.8}$$

Then

$$\{\nabla \cdot f_\nu\}_{\nu\uparrow\infty} \quad \textit{is compact in } H^{-1}(\Omega).$$

Proof of lemma 3.1. We have to show that, for any sequence $\{\varphi_\nu\}_{\nu\uparrow\infty} \subset H_0^1(\Omega)$ with

$$\varphi_\nu \overset{w}{\rightharpoonup} 0 \quad \text{in } H^1(\Omega), \tag{3.9}$$

we have

$$\int_\Omega \varphi_\nu \nabla \cdot f_\nu \to 0. \tag{3.10}$$

According to Rellich, equation (3.9) implies strong convergence of $\{\varphi_\nu\}_{\nu\uparrow\infty}$ in $L^2(\Omega)$ to zero, which entails convergence in measure, that is,

$$\mathcal{L}^N(\{|\varphi_\nu| \geqslant \delta\}) \to 0 \tag{3.11}$$

for any fixed $\delta > 0$. We split φ_ν into

$$\varphi_\nu = \varphi_\nu^{(1)} + \varphi_\nu^{(2)},$$

where

$$\varphi_\nu^{(1)} = \begin{cases} -\delta & \text{on } \{\varphi_\nu < -\delta\}, \\ \varphi_\nu & \text{on } \{|\varphi_\nu| \leqslant \delta\}, \\ \delta & \text{on } \{\varphi_\nu > \delta\}. \end{cases}$$

Our construction is such that $\varphi_\nu^{(1)}, \varphi_\nu^{(2)} \in H_0^1(\Omega)$ and

$$\left.\begin{array}{ll} |\varphi_\nu^{(1)}| \leqslant \delta & \text{on } \Omega, \\ \nabla\varphi_\nu^{(2)} = 0 & \text{a.e. on } \{|\varphi_\nu| \geqslant \delta\}, \\ \nabla\varphi_\nu^{(2)} = \nabla\varphi_\nu & \text{a.e. on } \{|\varphi_\nu| < \delta\}. \end{array}\right\} \tag{3.12}$$

Now,

$$\int_\Omega \varphi_\nu \nabla \cdot f_\nu = \int_\Omega \varphi_\nu^{(1)} \nabla \cdot f_\nu - \int_\Omega f_\nu \cdot \nabla\varphi_\nu^{(2)},$$

so that by (3.12),

$$\left|\int_\Omega \varphi_\nu \nabla \cdot f_\nu\right| \leqslant \delta \int_\Omega |\nabla \cdot f_\nu| + \left(\int_{\{|\varphi_\nu| \geqslant \delta\}} |f_\nu|^2 \int_\Omega |\nabla\varphi_\nu|^2\right)^{1/2}.$$

We observe that (3.9) in particular implies the boundedness of

$$\left\{\int_\Omega |\nabla\varphi_\nu|^2\right\}_{\nu\uparrow\infty}.$$

Hence (3.8) and (3.11) yield

$$\limsup_{\nu\uparrow\infty}\left|\int_\Omega \varphi_\nu \nabla \cdot f_\nu\right| \leqslant \delta \limsup_{\nu\uparrow\infty} \int_\Omega |\nabla \cdot f_\nu|.$$

Since $\delta > 0$ was arbitrary, we obtain (3.10) as desired from (3.7). □

In the second step, we apply the tools of Young measures and compensated compactness in the spirit of Tartar [18]. According to Young's theory of generalized functions (also called Young measures), there exists a non-negative Borel measure μ_x such that, for a subsequence,

$$\int_\Omega \int \zeta(z, x)\, \mathrm{d}\mu_x(z)\, \mathrm{d}x = \lim_{\nu\uparrow\infty} \int_\Omega \zeta(m_\nu(x), x)\, \mathrm{d}x \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^2 \times \Omega), \tag{3.13}$$

with the understanding that the function

$$\Omega \ni x \mapsto \int \zeta(z, x)\, \mathrm{d}\mu_x(z)$$

is integrable for any $\zeta \in C_0^\infty(\mathbb{R}^2 \times \Omega)$ (see [4, 5, 18]). The family $\{\mu_x\}_{x\in\Omega}$ is called the Young measure associated to the subsequence $\{m_\nu\}_{\nu\uparrow\infty}$. According to (1.5), $\{|m_\nu|^2\}_{\nu\uparrow\infty}$ is uniformly integrable. Therefore, equation (3.13) can be improved to

$$\int_\Omega \int \zeta(z, x)\, \mathrm{d}\mu_x(z)\, \mathrm{d}x = \lim_{\nu\uparrow\infty} \int_\Omega \zeta(m_\nu(x), x)\, \mathrm{d}x, \tag{3.14}$$

for all $\zeta \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ with $\sup |\zeta(z,x)|/(1+|z|^2) < \infty$. By choosing $\zeta = \zeta(x)$ in (3.14), we see that

$$\int \mathrm{d}\mu_x = 1 \quad \text{for a.e. } x \in \Omega. \tag{3.15}$$

Besides (3.13), the Young measure also satisfies

$$\int_\Omega \int \zeta(z,x)\,\mathrm{d}\mu(z)\,\mathrm{d}x \leqslant \limsup_{\nu\uparrow\infty} \int_\Omega \zeta(m_\nu(x),x)\,\mathrm{d}x$$
$$\text{for all non-negative } \zeta \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2). \tag{3.16}$$

By choosing $\zeta(z) = (1-|z|^2)^2$ in (3.16), we see that (1.5) implies

$$\operatorname{supp} \mu_x \subset S^1 \quad \text{for a.e. } x \in \Omega. \tag{3.17}$$

Let Φ, $\tilde{\Phi}$ be two entropies. According to our first step,

$$\{\nabla \cdot [\Phi(m_\nu)], \nabla \times [\mathrm{R}\,\tilde{\Phi}(m_\nu)] = \nabla \cdot [\tilde{\Phi}(m_\nu)]\}_{\nu\uparrow\infty} \quad \text{are compact in } H^{-1}(\Omega).$$

Therefore, by the div-curl lemma of Murat and Tartar [15,18], the weak* limit of the product of $\Phi(m_\nu)$ and $\mathrm{R}\,\tilde{\Phi}(m_\nu)$ in measures is the product of the weak limits in $L^2(\Omega)$ of $\Phi(m_\nu)$ and $\mathrm{R}\,\tilde{\Phi}(m_\nu)$, respectively. According to (3.13), these weak limits can be expressed in terms of the Young measure $\{\mu_x\}_{x\in\Omega}$; hence, on the level of the Young measure, we obtain the commutation relation

$$\int \Phi \cdot \mathrm{R}\,\tilde{\Phi}\,\mathrm{d}\mu_x = \left(\int \Phi\,\mathrm{d}\mu_x\right) \cdot \left(\int \mathrm{R}\,\tilde{\Phi}\,\mathrm{d}\mu_x\right) \quad \text{for a.e. } x \in \Omega.$$

Using this relation and (3.15), (3.17), we apply lemma 2.6 to conclude that

$$\mu_x \text{ is a Dirac measure for a.e. } x \in \Omega.$$

This entails

$$\int |z|^2\,\mathrm{d}\mu_x(z) = |m(x)|^2 \quad \text{where } m(x) = \int z\,\mathrm{d}\mu_x(z) \quad \text{for all } x \in \Omega, \tag{3.18}$$

where, according to (3.14), m is the weak* limit of $\{m_\nu\}_{\nu\uparrow\infty}$ in measures. As a consequence of (1.5), $\{|m_\nu|^2\}_{\nu\uparrow\infty}$ is uniformly integrable, so that m is the weak limit of $\{m_\nu\}_{\nu\uparrow\infty}$ in $L^2(\Omega)$. According to (3.18) and (3.14) for $\zeta(z,x) = |z|^2$, we have

$$\|m\|_{L^2(\Omega)} = \lim_{\nu\uparrow\infty} \|m_\nu\|_{L^2(\Omega)}.$$

As it is well known, convergence of the norm strengthens weak convergence to strong convergence in $L^2(\Omega)$, so that

$$\lim_{\nu\uparrow\infty} \|m_\nu - m\|_{L^2(\Omega)} = 0.$$

□

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