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A COMPARATIVE EVALUATION OF HEURISTIC LINE BALANCING TECHNIQUES

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Abstract

In this paper, we report on a computational experiment designed to assess the efficacy of twenty-six heuristic decision rules which group work tasks into work stations along an assembly line such that the number of work stations required is minimized. The heuristic decision rules investigated vary from simple list processing procedures that consider a single attribute of each work task for assignment, to procedures which are optimal seeking, but which have had their search terminated through the imposition of a limit on the amount of computation time that can be devoted to each search. Also included are heuristic decision rules which backtrack in an attempt to locate an improved solution, and decision rules which probabilistically search for improved solutions. Our investigation differs from those reported previously, in that the objective in balancing each line is to determine the minimum number of work stations for a given limit on the time available for assembly at each work station (the cycle time). Previous approaches have investigated the problem of determining the minimum cycle time for a given line length. We also compare the results obtained with the optimal solution for a subset of the problems investigated. Both problems which have appeared in the open literature and problems which have been solved for the first time are included. Since some of the results reported in this paper differ from those reported previously, we suggest why these differences have occurred. Guidelines are also given to those balancing industrial assembly lines on the choice of the heuristic decision rule to use whether one is attempting to obtain a minimum station balance given a limit on the time available for assembly at each work station, or whether one is attempting to minimize the time devoted to assembly at a work station given a limit on the number work stations available.

1. INTRODUCTION

One of the problems inherent in organizing for mass production is how to group work tasks to be performed into "work packages" or work stations so as to achieve a desired level of performance. When the number of work stations or production employees is fixed, the objective is to allocate the work to be performed to the work stations in such a way that the maximum time required for assembly at any given station (i.e., the cycle time) is minimal across all feasible station balances. The optimal solution to this problem, called the Type II problem (Mastor, 1970), maximizes the output rate or the number of units that are produced for the given line length. A Type I problem, on the other hand, is characterized by a forecast of demand or a production plan dictating the output or the production rate. The objective in this problem is to detemine the minimum number of work stations required to meet the specified production requirements. A line with fewer stations translates into lower labor costs and reduced space requirements, and hence, a more cost effective production plan. In this paper, we examine in detail the latter of these two problem types, and offer guidelines for applying our results to either one.

There have been two previous investigations of the performance of various heuristic decision rules for solving the Type II assembly line balancing (ALB) problem for the case in which heuristic performance is assessed relative to the best heuristic result obtained. Mastor (1966) evaluated the performance of ten heuristic decision rules on twenty and forty task assembly problems for three different order strengths (ratio of the number of ordering relations among the tasks to be grouped into work stations to the total number of possible orderings), and for line lengths (number of work stations) ranging

from four to fifteen for the forty task problems and from three to twelve for the twenty task problems. Since the majority of the line balancing heuristics investigated by Mastor were designed to solve the Type I line balancing problem, Mastor obtained minimal cycle time balances by iteratively employing each of the evaluated techniques, increasing the cycle time in one percent increments above the lower bound cycle time until a balance was achieved for the specified number of work stations. Mastor's general conclusion is that best results are obtained by the Held et al. (1963) dynamic programming technique, although the range of experimental conditions over which this procedure was evaluated was less than for most of the other procedures. (This was due to excessive computation times being required on certain categories of problems using the dynamic programming approach. The difficulty reported by Mastor in solving certain problem types using dynamic programming is consistent with our experience with this approach, as described later. Also, Magazine and Wee (1981b) observed this same result when attempting to obtain optimal solutions to the line balancing problem using a dynamic programming procedure.) The results obtained with the Held et al. dynamic programming technique were followed closely by Arcus's COMSOAL (1963) in Mastor's investigation. Further, the range of experimental conditions over which this procedure was evaluated was not curtailed as with the Held et al. dynamic programming procedure.

Dar-El (Mansoor) (1975) investigated twelve heuristic decision rules, also for Type II problems. Dar-El used fifty, one-hundred and five, and one-hundred and forty task assembly problems, varied across three task precedence configurations, and four ratios of work elements per station (WEST ratios) in his evaluation. Dar-El's general conclusion is that MALB, a technique he developed (1973), which is based upon a backtracking extension to the Ranked

Positional Weight heuristic of Helgeson and Birnie (1961), gives consistently superior results to the Arcus or to the other techniques investigated. (A procedure we have found effective, a backtracking approach due to Hoffmann (1963), was not included in Dar-El's investigation.)

In this paper, we report on a computational experiment designed to assess the efficacy of heuristic procedures for solving the Type I assembly line balancing problem. Heuristic decision rules that have been found effective elsewhere are included in this evaluation, as are some that are reported on for the first time. Four data sets are used to assess heuristic performance. The first data set consists of 120 computer generated problems each of which is solved for 11 cycle times, yielding 1,320 balances. The second data set consists of 12 problems which have appeared in the open literature. These 12 problems are solved for a variety of cycle times, yielding 64 balances in total. The third and fourth data sets each consist of 110 "difficult" balances. These latter problems have been generated in order to assess the performance of several of the heuristic decision rules evaluated under a "worst case" scenario.

We first compare the various approaches examined to the minimum heuristic result obtained for each problem. This measure provides us with a relative measure of heuristic performance, and thus is similar to previous investigations. Then, because many of the heuristic procedures evaluated are able to determine and verify optimal solutions within the specified time limit for a subset of the problems examined, we also compare heuristic performance to the known optimal solution for this subset of problems. In so doing, we obtain an estimate of absolute performance for each of the heuristic decision rules that previous investigations have not provided.

In Section 2, each of the heuristic decision rules investigated is described. Section 3 then describes the test problems used to assess heuristic performance. In Section 4 we present the computational results. Because the results reported in Section 4 are somewhat at variance with other reported results, in Section 5 we suggest why the variability in results has occurred. We also offer guidelines for applying the heuristic decision rules to either type of line balancing problem. Concluding remarks are given in Section 6.

2. HEURISTIC LINE BALANCING TECHNIQUES INVESTIGATED

In this section, each of the heuristic decision rules investigated is described. These decision rules vary from simple list processing prioritizing schemes to optimal-seeking procedures which have had the amount of time to devote to each search limited. To aid in our description of these procedures, we have placed each decision rule into one of four categories. These categories allow us to capture salient features of each approach, and to emphasize the similarities and differences among them. The first category consists of those decision rules which implement a list processing prioritizing scheme for task assignment based upon a single attribute of each assembly task. We term this category "Single-Pass Decision Rules" (Dar-El (1975)). The second category consists of decision rules which are a composite of the Single-Pass Decision Rules, or which otherwise produce multiple single-pass solutions for a given problem, selecting that solution for implementation which results in the fewest number of work stations. We include in this category the Arcus COMSOAL (1966) procedure found effective by Mastor, as well as a composite rule which selects the best of the solutions determined by our application of the single-pass decision rules included in Category I. This second category is

termed "Composite Decision Rules." Two of the procedures evaluated consist of backtracking approaches which attempt to improve upon a solution or station assignment previously obtained. These procedures are contained in a third category labeled "Backtracking Decision Rules." Procedures in this category include the Precedence Matrix, Enumeration method of Hoffmann (1963), and MALB (1973). Finally, six procedures are included which are optimizing approaches, but which have been programmed with a time limit set to restrict the time permitted for each solution. The procedures included in this category are two versions of Branch and Bound (Magazine and Wee, 1981b), two variations of Integer Programming (Talbot and Patterson, 1984), Dynamic Programming (Schrage and Baker, 1978), and MUST (Dar-El and Rubinovitch, 1979). This last category is titled, "Optimal Seeking Decision Rules."

2.1 Single-Pass Decision Rules

In this category, thirteen single-pass, single attribute, priority dispatch scheduling rules such as Maximum Ranked Positional Weight (Helgeson and Birnie, 1961), Maximum Number of Immediate Followers (Tonge, 1961), Maximum Task Time First (Moodie and Young, 1965), etc., are included. The heuristic decision rules in this set include those which have been found effective elsewhere, plus two "benchmark" heuristic decision rules. Several techniques are additionally included which have been recently developed to compete against the set of established rules.

Each of the decision rules included in this first category consists of a simple, computationally efficient, list-processing procedure that assigns tasks to work stations according to a task's computed priority. Operationally, in the implementation of each of these procedures, a task is first assigned a numerical priority specified by the logic of the heuristic decision rule.

Then, tasks that are both precedence and cycle-time feasible (i.e., all predecessors have been assigned to a work station and the task time is not larger than the remaining time available at the work station), are placed on an available list. The task on the available list with the highest priority is assigned first. The available list is then updated to reflect the possible addition of tasks that are now precedence feasible, and the amount of time available to be assigned to tasks in the work station is reduced by the task time of the assigned task. This process continues for a station until no more tasks can be assigned to it. The assignment process then continues to the next station, and so on, until all tasks have been assigned to some work station. When the final task has been assigned, a complete balance has been obtained.

The thirteen decision rules included in our investigation are delineated in Table 1. Also included in Table 1 is the basis for determining task priority. A random rule is included in this set in order to provide a comparison with the more logically appealing decision rules. Rules which are being investigated for the first time include: MINLB, MINUB, MINSLK, MAXAVGRPW, MIN(UB/TFOL), MAX(DUR/UB), and MAX(TFOL/SLK). This set thus includes the majority of those single-pass, heuristic decision rules which have been found effective elsewhere, as well as includes several new approaches for solving this problem.

2.2 Composite Decision Rules

2.2.1 The Best of Thirteen Rule (COMPOSITE-13)

The first composite decision rule evaluated selects the best solution available from Category I decision rules. Because a given decision rule does not always discriminate among tasks on an available list (for example, two or more tasks available for assignment to a work station could tie on their task

TABLE 1

SINGLE-PASS DECISION RULES

Rule	Notation	Basis for Determining Task Priority
1. Maximum Ranked Positional Weight	MAXRPW	$RPW_i = t_i + \sum_{j \in S_i} t_j$
2. Maximum Total Number of Follower Tasks	MAXTFOL	NS_i
3. Maximum Task Time	MAXDUR	t_i
4. Maximum Number of Immediate Follower Tasks	MAXIFOL	NIS_i
5. Minimum Slack	MINTSLK	$UB_i - LB_i$
6. Random Task Assignment	RANDOM	random (uniform)
7. Minimum Lower Bound	MINLB	$LB_i = [(t_i + \sum_{j \in P_i} t_j)/C]^+$
8. Minimum Upper Bound	MINUB	$UB_i = N+1-[(t_i + \sum_{j \in S_i} t_j)/C]^+$
9. Minimum Task Number	MINTSKNO	task number, i
10. Maximum Average Ranked Positional Weight	MAXAVGRPW	$RPW_i/(NS_i+1)$
11. Minimum Upper Bound Divided by the Total Number of Followers	MIN(UB/TFOL)	$UB_i/(NS_i+1)$
12. Maximum Task Time Divided by Task Upper Bound	MAX(DUR/UB)	t_i/UB_i
13. Maximum Total Task Followers Divided by Task Slack	MAX(TFOL/SLK)	$NS_i/Slack_i$

C — Station cycle time.

N — The number of tasks to be balanced into work stations.

$NS_i(NP_i)$ — The total number of tasks which succeed (precede) task i (i.e., the number of elements of $S_i(P_i)$).

$NIS_i(NIP_i)$ — The number of tasks which must immediately succeed (precede) task i.

$S_i(P_i)$ — The set of tasks which must succeed (precede) task i.

t_i — Assembly time required to complete task i.

$[X]^+$ — The smallest integer greater than or equal to X.

times when the prioritizing scheme is based upon task duration), it is also possible to use one of the remaining twelve decision rules to break ties. Including the tie-breaker decision rules yields 156 (13 x 12) rule/tie-breaker rule combinations for solving each problem. Further, the possibility also exists to balance a given line beginning with the last task in the network and working forward, achieving what previous researchers of the line balancing problem have termed a "reverse" balance. Considering decision rule/tie-breaker rule combinations as well as balancing a line in reverse yields 312 possible decision rules for solving a given problem (13 x 12 x 2) using the thirteen heuristic decision rules described in Category I. The Best of Thirteen Rule selects the best of these 312 balances, implemented as described below.

Whenever a balance is determined by any decision rule that is equal to the smallest integer greater than or equal to the sum of the task times divided by the cycle time, the optimal solution (that is, the solution requiring the minimum number of work stations) has been determined. This is because it is not possible to obtain a balance requiring fewer work stations than given by this quantity, which we refer to as the LOWER BOUND (M) on the value of a solution.[1] In the implementation of the Best of Thirteen Decision Rule (as well as with all of the decision rules which follow), the procedure terminates the search for the minimum solution whenever a balance is obtained that is equal to this lower bound. The effect of this process is to eliminate unnecessary computations and, hence, to reduce the computation time required to solve many of the problems investigated.

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[1]The quantity

$$M = [\Sigma_i \, t_i/C]^+$$

has been termed the "theoretical minimum" number of work stations by previous researchers investigating the line balancing problem. This is an unfortunate choice of words, as a given problem may have an optimal solution in which the number of work stations exceeds the "theoretical minimum."

2.2.2 Arcus' Biased Sampling Procedure (ARCUS)

The Arcus (1966) procedure uses a biased sampling approach to generate feasible sequences of tasks for assignment to a work station. A "fit list," consisting of those tasks which can be assigned to a work station is constructed, and weights governing the probability the task will be selected for assignment to the work station are assigned to each task. Tasks so assigned are removed from the fit list, and a new fit list, consisting of the tasks which can currently be assigned to a work station is constructed. The process continues until all tasks have been assigned to some work station. A given line is then balanced several times, resulting in different assignments based upon the probabilistic selection of tasks from the fit list. To be consistent with the work of Arcus (1963), Mastor (1970), and Dar-El (1975), Arcus' Rule IX (which is the product of five separate weights) is used in our evaluation. The procedure we use is programmed to determine eighty balances for each problem, and to then select the balance requiring the fewest number of work stations. However, if a balance is obtained during execution of this procedure that is equal to the lower bound M, then the procedure terminates, yielding M as the number of work stations required.

2.3 Backtracking Decision Rules

2.3.1 Hoffmann's Enumeration Procedure (HOFFMANN-0.0)

Hoffmann (1963) developed a heuristic method that assigns to each work station, in numerical order, a combination of tasks which minimizes station idle time. Starting with station one, a precedent feasible list of tasks is maintained from which the combination of tasks which will minimize station idle time is found via complete enumeration. These tasks are assigned to station one, and the process continues with station number two using an updated precedent feasible list. This procedure is repeated for each station

in numerical order, until all tasks have been assigned. Hoffmann uses a special zero-one precedence matrix and index vector to implement the enumeration process, which results in a very simple computer code, a description and FORTRAN listing of which is in his original article (1963). The Hoffmann methodology can be applied to tasks in decreasing order as well as in numerical order, which results in a second solution to a problem. In this investigation, all balances are solved in both the forward and reverse direction (where a forward balance did not determine the minimum required number of work stations, M), with the best solution being reported.

2.3.2 Hoffmann's Modified Enumeration Procedure (HOFFMANN-0.5
HOFFMANN-1.0
HOFFMANN-2.0)

Because the original Hoffmann approach considers stations in numerical order, it has a tendency to concentrate idle time in the later stations. In addition, a significant amount of computation time can be spent at those stations where the task precedence feasible list is long, and where a zero idle time solution doesn't exist or is difficult to determine. In the latter case, all feasible combinations of task assignments may have to be enumerated. In order to overcome these difficulties, Gehrlein and Patterson (1975) proposed a very slight modification to the original Hoffmann procedure: instead of determining the minimum idle time solution at each work station, determine one that is 'acceptably' close to minimum.

To define acceptably close, Gehrlein and Patterson use a measure of average idle time per station, which is derived in the following manner. The quantity (M x C) gives the minimum amount of time that will be devoted to producing one unit of product. This quantity bears the following relationship to the total work content ($\sum_i t_i$) for a given line:

$$(M \times C) \geq \sum_i t_i .$$

The difference between these two quantities $((M \times C) - \sum_i t_i)$ represents the minimum total line idle time that will necessarily be present in any balance. This quantity divided by the minimum number of work stations required, M, yields an estimate of the average amount of idle time present in a work station.

The Hoffmann procedure is thus modified by curtailing the search for the set of tasks resulting in the minimum amount of idle time for a given station whenever a subset of tasks σ is found which satisfies the following relation:

$$0 \leq C - \sum_\sigma t_i \leq \Theta \; ([(C \times M) - \sum_i t_i]/M)$$

where the parameter Θ is varied in an attempt to obtain a balance requiring the fewest number of work stations. Gehrlein and Patterson (1975 and 1978) have shown that this slight modification to Hoffmann's original procedure can have a significant impact on reducing the amount of time devoted to each search, while simultaneously smoothing the idle time present among work stations for certain classes of line balancing problems. Further, this improvement in computation time is often achieved with no increase in the number of work stations required.

Based upon previous experience in solving line balancing problems with this approach, we allow Θ to assume the values 0.5, 1.0, and 2.0, and denote these modified Hoffmann procedures by HOFFMANN-0.5, HOFFMANN-1.0, and HOFFMANN-2.0, respectively.

2.3.3 Dar-El's Line Balancing Heuristic (MALB)

Dar-El developed MALB (1973) as a heuristic variant of his earlier optimal-seeking iterative procedure (1964). His optimal seeking procedure is based upon the Rank Positional Weight heuristic method of Helgeson and Birnie

(1961), enhanced with a backtracking algorithm that generates all feasible sequences of task assignments. Dar-El (1973) found that the computation time of his optimal seeking method restricted its applicability, so he sought to develop a heuristic approach that would retain most of the power of the optimal approach, without incurring its computational expense. MALB is the result of these efforts. To limit the number of sequences generated, and hence, to reduce computation time, MALB incorporates four types of heuristics that control the amount of backtracking permitted. In a comparative study (1975), Dar-El found that MALB dominated both ARCUS and the best of the ten single-pass rules he evaluated. On the basis of these results, MALB was included in the present investigation.

MALB, however, was originally written to solve the Type II problem. Hence, it was necessary to modify it slightly here. Like most of the methods developed to solve Type II problems, MALB, in effect, solves a series of Type I problems. In the Type I problem the cycle time is fixed, and the minimum number of stations is sought. The Type II problem fixes the number of stations, and seeks to determine the minimum cycle time. Operationally however, MALB, and most of the other procedures, solves the Type II problem by fixing the cycle time at the theoretical minimum time (the largest task time), and determines the minimum number of stations less than a specified 'goal' number of stations, γ. If no solution is found, then the cycle time is increased one time unit, and the process is repeated. The first feasible solution found with the number of work stations less than or equal to γ yields the desired balance. (In general, if the procedure is optimal-seeking, then the first feasible solution is optimal. Otherwise, as with MALB, it is heuristic.)

For the present investigation, instead of permitting the cycle time to increase when no feasible solution is determined, the goal γ is increased by

one work station, and another attempt is made to find a feasible solution. The first feasible solution found is the minimum (heuristic) number of stations. Specifically, then, the only modification made to MALB (other than changing input and output formats) was to defeat the logic in the procedure that permits the cycle time to increase. The cycle time is fixed at the desired experimental level. Initially, γ is set equal to M. If no feasible solution is determined, the value of γ is increased by one station, and the process is repeated until a feasible solution is determined.

2.4 Optimal-Seeking Decision Rules

The methods described in this section are included in our evaluation in order to gain insight into the benefit obtained by using optimal-seeking algorithms given a computational time constraint roughly comparable to the time required to solve a problem using the more sophisticated heuristic procedures. Clearly, given enough computation time, any optimal-seeking procedure will dominate a heuristic one. Our purpose is not to verify this fact, but to give an indication of the quality of solutions obtained for a given amount of computation time expended. All of the solution routines listed here have thus been run with an internal CPU time trap of 3.0 seconds per problem (balance). The reported solution is the minimum one obtained within the 3.0 second limit.

2.4.1 Branch and Bound Methods (MAG-1 and MAG-2)

Magazine and Wee (1981b) report excellent results for the Type I line balancing problem with their branch and bound procedure. With their method, each node in the solution tree corresponds to a feasible set of task assignments to a particular work station, where all nodes at the same depth in the tree refer to the same station number. Starting with node zero (the null set), descendent nodes from a node of depth (d) are generated, which are

maximal feasible assignments of tasks to station (d+1). The branching direction, fathoming criteria, and growth rate of the tree are controlled via a number of heuristics and dominance tests incorporated into their procedure.

Two decision rules of special significance used in their procedures are IUFFD (Immediate Update First Fit Decreasing), which we refer to as MAXDUR, and IUBRPW (Immediate Update Backward Recursive Positional Weight), a variation of MAXRPW. Although their branch and bound procedure incorporates a breadth-first (versus depth-first) branching strategy, an upper bound on the solution is found at each node using one of these two heuristics for the unassigned tasks. This permits their procedure to terminate prior to verification of optimality with a given heuristic solution to a problem. To facilitate the use of this feature, they allow the user to select one of these heuristics and to specify a time limit for implementation of their approach. This is the time limit capability used in our experiments. The references to MAG-1 and MAG-2 correspond, respectively, to the use of the heuristics IUFFD and IUBRPW.

2.4.2 Integer Programming (ALBCUT and ALBHOFF)

Two variations of Talbot and Patterson's (1984) integer programming procedure are included in our evaluation. The basic algorithm is a depth-first, implicit enumeration, backtracking procedure to which various search, fathoming, and backtracking decision rules are applied. The first variation included (ALBCUT) contains network cuts (1984), where search and backtracking are controlled with the heuristic decision rule MINUB (with ties broken using MAXDUR). The second variation (ALBHOFF) also uses MINUB and MAXDUR for search and backtracking, but does not use any cut associated fathoming rules. Also, prior to enumeration, the problem is first solved with HOFFMANN-0.5, which

provides an initial heuristic starting solution and computational upper bounds.

2.4.3 Dynamic Programming (DYNAMIC)

Schrage and Baker (1978) have proposed an efficient method for implementing the dynamic programming approach of Held et. al. (1963) through improved procedures for generating feasible subsets, and for labeling. Magazine and Wee (1981b) programmed and tested the Schrage and Baker approach for solving the Type I line balancing problem. Magazine and Wee concluded that their branch and bound solution procedure is preferred to dynamic programming for solving these types of line balancing problems, both with regard to computation time and computer storage required. For completeness, however, we have included the dynamic programming approach in our evaluation.

2.4.4 Multiple Solutions Technique (MUST)

MUST, by Dar-El and Rubinovitch (1979), employs exhaustive enumeration to generate all solutions, or some subset of them, for solving the Type II line balancing problem. As with MALB, the Type II problem is solved as a sequence of Type I problems. The cycle time is fixed, and a "hurdle" number of work stations, γ, is specified. If a feasible number of work stations cannot be found given γ, the cycle time is increased one time unit, and the process is repeated. Starting with station number one, MUST generates all feasible task assignments. Redundancies are eliminated, and the remaining feasible task assignments (subsets) for this station are saved. At station two, and subsequent stations taken in numerical order, these subsets are extended by adding all feasible task assignments at a work station. Redundancies are then eliminated, and the extended subsets are saved. This process is repeated until all tasks have been assigned (all subsets are fully extended).

This exhaustive enumeration procedure results in an exponential growth in the number of subsets saved. In order to make this process manageable, MUST saves sets as a sequence of bits, and also uses circular storage buffers. Even with these storage-saving features however, MUST, unlike the other methods evaluated, is programmed to use external storage. To control the growth in the number of subsets saved, MUST contains three user-specified parameters. These can be set to permit the generation of either multiple optimal solutions, or multiple heuristic solutions.

As a result of experiments by Dar-El and Rubinovitch (1979) which demonstrated that MUST dominates MALB, MUST was included in our investigation. It was modified to solve the Type I problem in the same manner as was MALB, by permitting the hurdle number of work stations γ to increase by one station if a feasible solution could not be found, rather than (as with the Type II approach) increasing the cycle time by one time unit in each iteration.

3. DESCRIPTION OF PROBLEMS SOLVED

In order to generalize the results of our investigation, both randomly generated problems and problems taken from the open literature are solved. The characteristics used for problem generation are based upon an examination of actual assembly line balancing problems, and are experimentally varied over a wide range of representative values. There are four data sets in total: three comprising randomly generated problems, and a fourth data set consisting of problems available in the open literature.

3.1 Main Experimental Data Set

This data set contains 120 unique assembly networks, each of which is solved for 11 cycle times. Details of problem characteristics and generation schemes for these problems are described below.

3.1.1 Problem Size

Sixty 50-task and sixty 100-task assembly networks are randomly generated. These line sizes are sufficiently large to permit generalization of our results to actual line balancing problems, and are comparable in size to those used by Dar-El (1975), and are significantly larger than those examined by Mastor (1970).

3.1.2 Distribution of Task Times

An investigation of actual line balancing problems appearing in the open literature suggests parameters for generating task times. These times are derived from the binomial distribution (n=30, and π=0.25), with zero duration tasks having their task times increased to one time unit.

3.1.3 Distribution of Cycle Times

When the magnitude of the cycle time approaches the maximum task time in an assembly line balancing problem, the alternatives available for scheduling tasks into work stations decrease, as do the opportunities for the heuristic scheduling rule to exercise its intended logic in assigning tasks to a work station. Alternatively, as the cycle time increases relative to the maximum task time, the number of alternatives for assigning tasks to work stations increases, as does the discretionary "power" of the investigated decision rule. In order to investigate the efficacy of the heuristic decision rules evaluated over a wide range of problem types, each problem generated is solved for the cycle time equal to the maximum task time. The cycle time is then increased in 10 percent increments of the maximum task time to the closest integer equal to 10 percent, 20 percent, 30 percent, etc., above the minimum possible cycle time until eleven balances are obtained for each problem.

3.1.4 Strength of the Precedence-Ordering Relations Among Tasks

The strength of the precedence-ordering relations among tasks affects the number of alternative production lines that may be established. This in turn

influences the discretionary power of the heuristic decision rule and, hence, its effectiveness. Network density, a characteristic which measures the strength of this relation, has been found to be an important factor in influencing heuristic performance in previous investigations of the line balancing problem.

To define density, let W be an N x N 0-1 matrix that represents a precedence-ordering relation P. For a given element ω_{ij} of W, let ω_{ij} = 1 if $X_i \in P_j$ (i.e., if task i precedes task j) and $\omega_{ij} = 0$ if task i does not precede task j, or $X_i \notin P_j$. Let d be the total number of relations contained in P, or

$$d = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{ij}$$

The maximum number of relations that can be included in P is N(N-1)/2. The density of the assembly network is defined to be the ratio:

$$D = 2d/[N(N-1)],$$

or the ratio of the number of relations that exist to the number which could exist. The measure 1-D is the F-ratio used by Dar-El (1975), and the measure D is referred to as order strength by Mastor (1970).

Values of D close to 1 indicate a highly interconnected network, and fewer alternatives available for assigning tasks to a work station. Values of D close to 0 indicate relatively fewer precedence relationships, and more opportunities for assigning tasks to a work station. In our evaluation, values of D equal to 0.2, 0.3, 0.5, and 0.8 are used for problem generation.

3.1.5 Partial-Order Generating Methods

It is possible to test not only for the effect of network density on heuristic performance, but also for the effect of the method used to generate precedence-ordering relations. That is, several methods for generating a

precedence-ordering relation on a set of tasks can be devised, each resulting in the identical problem density. However, various factors of the ordering relation structure can be different for different generating methods. Three methods of generating precedence-ordering relations are investigated, and their effects on problem solution are assessed.

All three methods of generating precedence-ordering relations work in the same general way. We begin to describe how each method works by recognizing that precedence-ordering relations are partial ordering relations. A binary relation P on a set X is a partial order if it is irreflexive ($x_i P x_i$ for no x_i in X), asymmetric ($x_i P x_j$ requires not $x_j P x_i$ for all x_i and x_j in X), and transitive (if $x_i P x_j$ and $x_i P x_k$ then $x_i P x_k$ for all x_i, x_j, and x_k in X). Given a linear ordering relation $L = x_1 x_2 \ldots x_N$, we wish to generate a random partial order P that is contained in L, or $P \subseteq L$. That is, if $x_i P x_j$, then $i < j$. We do not lose any generality by requiring this, because any partial order can be transformed so that if $x_i P x_j$ then $i < j$ simply by interchanging the subscripts attached to the x's.

The first step in each generating process is to generate a suborder P_0. Suborders are binary relations that are irreflexive and asymmetric, but not necessarily transitive. P_0 is then changed to a partial order by taking the transitive closure of P_0, denoted by P_0^t. This indicates that all necessary P relations are added so that P_0^t is transitive. For example, if the suborder P_0 has $x_i P_0 x_j$ and $x_j P_0 x_k$, it is not necessarily true that $x_i P_0 x_k$, since suborders are not always transitive. In forming P_0^t, all relations, such as $x_i P_0 x_k$, are appended so that P_0^t is transitive. Since P_0^t is a transitive suborder, it is a partial order, and it is used as the generated partial order in network synthesis (Fishburn & Gehrlein (1974)).

We now describe the three partial order generating methods used in our investigation:

Method 1: Let σ be a randomly selected permutation on $\{1, 2, \ldots, N\}$. For each j from 2 through N select a y_i at random from $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(j-1)}\}$. Let $P_0 = L \cap \{[y_2, x_{\sigma(2)}], [x_{\sigma(2)}, y_2], \ldots, [y_N, x_{\sigma(N)}], [x_{\sigma(N)}, y_N]\}$ so that P_0 contains N-1 order pairs, and form the transitive closure P_0^t. Then P_0^t is made more dense by adding q additional pairs from $L-P_0^t$. The transitive closure is taken after each pair is added and P results after the final transitive closure is taken.

Method 2: Form P_0 by taking each (i,j) pair with $i < j$ and enter $x_i P_0 x_j$ with probability p. The same p is used when each (i,j) pair is considered for entry into P_0. Then $P = P_0^t$.

Method 3: Form P_0 by taking each (i,j) pair with $i < j$ and enter $x_i P_0 x_j$ with probability $\left(\frac{(j-i)}{(N-1)}\right)k$ for some positive k. The same k is used when each (i,j) pair is considered for entry into P_0. Then $P = P_0^t$.

By examinating three distinct methods for generating the partial orders necessary to construct each network, we can assess the sensitivity of our results to the network generation scheme used. Many examples of simulation analysis can be given where the results reported are dependent on the procedure used to generate random partial orderings. (For example, see Fishburn & Gehrlein (1974) or Patterson (1976).) Hence, an examination of their effect is prerequisite to making general statements concerning the efficacy of the heuristic procedures investigated. (For a more comprehensive discussion of how the desired density measures are determined by using partial order generating methods, see Gehrlein (1980).)

3.1.6 Main Experimental Problems Solved

For each line size (50 and 100 tasks), each density (0.2, 0.3, 0.5, and 0.8), and each partial-order generating method (Method 1, Method 2, and Method 3), five line balancing problems are constructed. The density of the partial orders generated by Methods 1, 2, and 3 is controlled by the values of q, p, and k used, respectively, in each method. Each of the generated problems is then solved for cycle times equal to the largest task time, the largest task time plus 10 percent of the largest task time (rounded to the nearest integer), the largest task time plus 20 percent of the largest time (rounded to the nearest integer), etc., until the cycle time equals twice the largest task time. This yields 1,320 balances in total [(2)(4)(3)(5)(11)] in the Main Experimental Data Set.

3.2 Literature Problems

A number of problems, ranging in size from 7 assembly tasks to 111 assembly tasks, were taken from the open literature to be used in this evaluation: Arcus (1963), Bowman (1960), Dar-El (1964), Heskia (1968), Jackson (1956), Jaeschke (1964), Kilbridge and Wester (1961), Merten (1967), Sawyer (1970), and Tonge (1961 and 1969). These problems also serve as the basis for determining the network generator parameters used for the main experimental data set. The cycle times used to assess heuristic performance in evaluating the procedures described are the same as those used by Talbot and Patterson (1984) in their evaluation of an integer programming algorithm for obtaining optimal solutions to this line balancing problem, with the exception of the Bowman and the Dar-El problems, which they did not include. For these latter two problems, the same cycle times as reported by the original authors were used (C = 20 for the Bowman problem, and C = 48, 62, and 94 for the Dar-El problem). For a more complete description of this data, see Talbot et. al.

(1985). Sixty-four balances were obtained using this literature data set over the range of cycle times considered.

3.3 "Difficult" Problem Set One

Five 50-task and five 100-task assembly line networks with task times uniformly distributed between 1 and 10,000, and with network density equal to 0.2 were randomly generated and solved for 11 trial cycle times equal to 10,000, 11,000, ..., 20,000. These balances were included in the experiment because with few precedence relations in the network and a very wide range of task times, the enumeration procedures evaluated can be expected to exhibit large increases in computation times. There are two arguments to support this notion: With an enumeration procedure, there will be fewer opportunities to terminate the search procedure at a given station before full enumeration is complete, because there are fewer sets of sums of task times that equal, or nearly equal, the cycle time. Also, with fewer precedence relationships, many more assignments are possible, resulting in an anticipated increase in the computation time required to solve a given problem.

3.4 "Difficult" Problem Set Two

Preliminary experimental results using "Difficult" Problem Set One, led to the development of "Difficult" Problem Set Two. This set contains the same 110 networks as in Set One, except that all the odd task times are increased by one time unit to make them <u>even numbers</u>. The eleven cycle times include the largest task time and added increments of 10% of this value (to the nearest integer) up to twice the largest task time, where all even cycle times are increased by one time unit to make them <u>odd numbers</u>. With all problems having even task times and odd cycle times, <u>no</u> zero idle time station assignments of tasks or zero balance delay solutions are possible, thus

forcing enumeration methods such as HOFFMANN-0.0 to completely enumerate all feasible task assignments at every work station.

We emphasize that these latter two data sets were not generated to represent actual problem types such as those found in examining industrial line balancing problems. Rather, they were generated to assess "worst case" performance for several of the techniques investigated, especially those employing enumerative and backtracking approaches.

4. COMPARATIVE RESULTS

4.1 Experimental Conditions

An attempt was made to solve each of the problems in all four data sets using the 26 solution procedures described in Section 3. A time limit of 3.0 CPU seconds per balance for each solution procedure was implemented as described. This is an internal time trap that excludes input time, but includes minimal computer output time per balance. All methods (with the exception of MUST) are programmed in FORTRAN and were run under the same operating system and compiler (IBM FORTRAN H OPT=2) and on the same computer, an AMDAHL 470/V8. MUST is programmed in PL1, and was run on the same AMDAHL 470/V8 using the same operating system after being compiled by the IBM PL1 optimization compiler (OPT=2). All methods, including MUST, were programmed to read from the same data set in an identical manner, and to write summary results to a common output data base. A concerted effort was made to run each program under similar operating conditions.

With the exception of MALB, HOFFMANN-0.0, MUST and DYNAMIC, the methods were able to solve all problems within the time limit specified per problem. Unfortunately, neither MUST nor DYNAMIC were able to solve more than just a few of the smaller problems within this time limit. It became very clear that

their performance would be even worse on the larger problems, and it would be a misuse of a constrained computer budget to have them attempt the larger problems. Hence, they were dropped from further experimentation, leaving 24 solution methods to be evaluated.

Before proceeding with the main results, an indication of the results obtained using MUST and DYNAMIC will be given to justify excluding them from further consideration. (The following are representative of our results--many more tests were run than are described here.)

As mentioned in Section 2.4.4, MUST contains three user-specified parameters that control the growth of the solution space, which in turn affect computation time and solution quality. We first input the recommended parameter values as given in Table 2 of Dar-El and Rubinovitch (1979). Using these parameter values, MUST was able to solve only 27 of the 64 literature balances (the largest solved consisting of 21 tasks) within the 3.0 second time limit specified. Even with the limit doubled to 6.0 seconds per balance, only 29 of the 64 problems were solved, the largest consisting of 28-tasks. Within the 6.0 second limit, MUST was not able to solve any of the problems in either of the "Difficult" Data Sets One or Two, or any of the first 11 50-task problems in the Main Experimental Data Set.

In an effort to obtain more solutions using MUST, even at the possible expense of lower quality solutions, the parameter values were all set equal to one, which would have the anticipated effect of reducing computation time and storage requirements for the algorithm. With these parameter values, MUST was able to solve 40 of the literature balances (the largest being a 30-task problem), two of the 110 balances (both 50-task) in the "Difficult" Set One, and none of the problems in the "Difficult" Set Two, within the 3.0 second time limit. A number of other tests were conducted with similar results. It

should be noted that these timing results are consistent with those of Dar-El and Rubinovitch, who report an average of 125.4 seconds per balance on an IBM 370/168 for a set of 50 to 140-task problems, for which they also report MALB required an average of 22.4 seconds on the much slower IBM 7044.

The dynamic programming procedure was able to solve only the smallest 27 of the 64 literature balances within the 3.0 second time limit per balance, the largest problem being Mitchell's 21-task problem (C=39). Attempts to solve larger problems (e.g., 28-task and 45-task) also failed with a time limit of 6.0 seconds per balance. These findings are completely consistent with Magazine and Wee's results (1981b), and reinforced our decision to exclude this procedure from further evaluation.

4.2 Main Experimental Data Set Results

Table 2 contains summary results for all heuristic decision rules retained, and for each data set. The first column under each data set is a relative measure of heuristic performance, the average percent above the best solution found by any method. The second column under each data set is the average percent above optimum, an absolute measure of heuristic performance. Whereas Table 2 indicates average heuristic performance, Table 3 shows the worst single result using each method. Specifically, the first column under each data set lists for each method the maximum percent above best heuristic solution determined for a single balance, and the second column lists the maximum percent above optimum. (All methods determined each balance within the 3.0 second time limit specified, except as noted in Table 2.)

In order to more clearly illustrate the relationships depicted in Table 2 for the Main Experimental Data Set, Figures 1A and 1B were prepared. Figure 1A is a histogram of first column data with the heuristics grouped by category, and ranked within category. Figure 1B is a similar histogram of second

TABLE 2

Average Percent Results For All Data Sets

DATA SET:	MAIN EXPERIMENTAL DATA SET		LITERATURE		DIFFICULT SET ONE		DIFFICULT SET TWO	
CRITERIA:	Above Best (1) Heuristic	Above (2) Optimum	Above Best (3) Heuristic	Above (4) Optimum	Above Best (5) Heuristic	Above (6) Optimum	Above Best (7) Heuristic	Above (8) Optimum
SOLUTION METHOD								
1 MAXRPW	5.16	5.77	4.62	4.85	4.76	4.35	4.43	4.33
2 MAXTFOL	4.89	5.55	4.27	4.31	4.70	4.46	4.22	4.02
3 MAXDUR	4.71	5.20	4.81	4.70	4.99	5.22	4.72	4.87
4 MAXIFOL	6.00	6.24	3.96	3.94	4.79	4.37	4.59	4.62
5 MINTSLK	6.50	6.81	5.16	5.34	5.43	5.20	5.09	4.75
6 RANDOM	9.33	9.48	10.26	10.71	9.11	8.32	8.96	8.22
7 MINLB	6.03	6.45	5.81	6.03	5.15	5.19	4.94	4.73
8 MINUB	4.51	5.11	3.43	3.34	3.91	3.75	3.40	3.43
9 MINTSKNO	6.63	7.21	8.96	9.02	7.08	6.95	7.04	6.75
10 MAXAVGRPW	6.35	6.76	5.66	5.84	5.15	4.93	5.09	5.09
11 MIN(UB/TFOL)	5.00	5.61	4.27	4.31	4.82	4.51	4.30	4.07
12 MAX(DUR/UB)	4.50	5.14	4.40	4.52	4.87	4.88	4.59	4.75
13 MAX(TFOL/SLK)	5.05	5.65	4.71	4.83	4.60	4.22	4.26	4.03
14 COMPOSITE-13	2.34	2.97	0.46	0.23	1.90	2.47	1.65	1.72
15 ARCUS	2.73	3.35	2.78	2.94	2.06	2.00	1.75	1.80
16 MALB	2.15	1.78	0.31	0.36	2.80	2.40	2.78	2.52
17 HOFFMANN-0.0	0.46	0.04	1.34	1.48	1.75	0.65	1.42	0.44
18 HOFFMANN-0.5	0.47	0.04	3.29	3.44	1.58	0.15	1.61	0.33
19 HOFFMANN-1.0	0.50	0.09	3.29	3.44	1.61	0.39	1.56	0.20
20 HOFFMANN-2.0	0.65	0.16	3.41	3.44	2.41	1.68	1.96	1.34
21 MAG-1	1.05	0.92	0.00	0.00	0.89	0.26	1.09	0.31
22 MAG-2	2.39	2.21	0.52	0.61	2.16	1.34	2.27	1.18
23 ALBCUT	3.68	4.13	0.60	0.26	3.52	2.92	3.22	3.23
24 ALBHOFF	0.42	0.02	0.73	0.45	1.13	0.01	1.10	0.13
Number of Balances	1,320[a]	635[a]	64	55	110[b]	50[b]	110[c]	45[c]

a. The results for MALB are based upon 872 balances (out of 1,320) and 431 (out of 635) that it solved.
b. The results for MALB are based upon 79 (out of 110) and 30 (out of 50) balances.
c. The results for MALB are based upon 80 (out of 110) and 24 (out of 45) balances. The results for HOFFMANN-0.0 are based upon 69 (out of 110) and 25 (out of 45) balances, respectively.

TABLE 3

Maximum Percent Results For All Data Sets

DATA SET:	MAIN EXPERIMENTAL DATA SET		LITERATURE		DIFFICULT SET ONE		DIFFICULT SET TWO	
CRITERIA:	Above Best (1) Heuristic	Above (2) Optimum	Above Best (3) Heuristic	Above (4) Optimum	Above Best (5) Heuristic	Above (6) Optimum	Above Best (7) Heuristic	Above (8) Optimum
SOLUTION METHOD								
1 MAXRPW	17.4	15.0	50.0	50.0	18.2	15.8	21.1	21.1
2 MAXTFOL	17.4	15.8	33.3	33.3	21.1	21.1	21.1	21.1
3 MAXDUR	13.6	13.6	33.3	33.3	15.0	15.0	13.3	13.3
4 MAXIFOL	17.4	17.4	33.3	33.3	21.1	21.1	21.1	21.1
5 MINTSLK	17.6	16.7	33.3	33.3	18.2	15.0	14.3	11.1
6 RANDOM	25.0	25.0	50.0	50.0	22.7	20.0	17.6	17.6
7 MINLB	20.0	20.0	33.3	33.3	20.0	20.0	15.8	15.8
8 MINUB	14.3	14.3	33.3	33.3	15.0	15.0	15.8	15.8
9 MINTSKNO	17.6	17.6	50.0	50.0	18.2	15.8	16.7	15.8
10 MAXAVGRPW	17.9	15.8	33.3	33.3	18.2	13.3	15.8	11.8
11 MIN(UB/TFOL)	17.4	15.8	33.3	33.3	21.1	21.1	21.1	21.1
12 MAX(DUR/UB)	15.0	15.0	33.3	33.3	15.8	15.0	15.8	13.3
13 MAX(TFOL/SLK)	17.4	15.0	33.3	33.3	19.0	15.8	19.0	15.8
14 COMPOSITE-13	10.0	7.7	16.7	12.5	7.7	7.7	7.7	7.7
15 ARCUS	10.5	10.5	33.3	33.3	9.1	7.1	8.7	7.1
16 MALB	13.6	10.5	20.0	20.0	15.8	15.8	15.8	15.8
17 HOFFMANN-0.0	10.0	7.1	25.0	25.0	9.5	5.9	7.1	5.9
18 HOFFMANN-0.5	10.0	7.1	33.3	33.3	10.5	4.8	7.7	5.6
19 HOFFMANN-1.0	10.0	7.1	33.3	33.3	10.5	5.0	10.0	5.9
20 HOFFMANN-2.0	10.0	7.1	33.3	33.3	10.5	7.7	9.1	6.7
21 MAG-1	11.6	11.6	0.0	0.0	14.3	6.9	14.3	10.5
22 MAG-2	13.5	11.4	33.3	33.3	14.3	11.8	14.3	10.5
23 ALBCUT	12.5	12.5	16.7	14.3	15.0	15.0	15.8	15.8
24 ALBHOFF	8.3	7.1	25.0	25.0	9.5	2.9	6.7	3.0
Number of Balances	1,320	635	64	55	110	50	110	45

NOTE: The footnotes to Table 2 also apply to Table 3.

FIGURE 1A

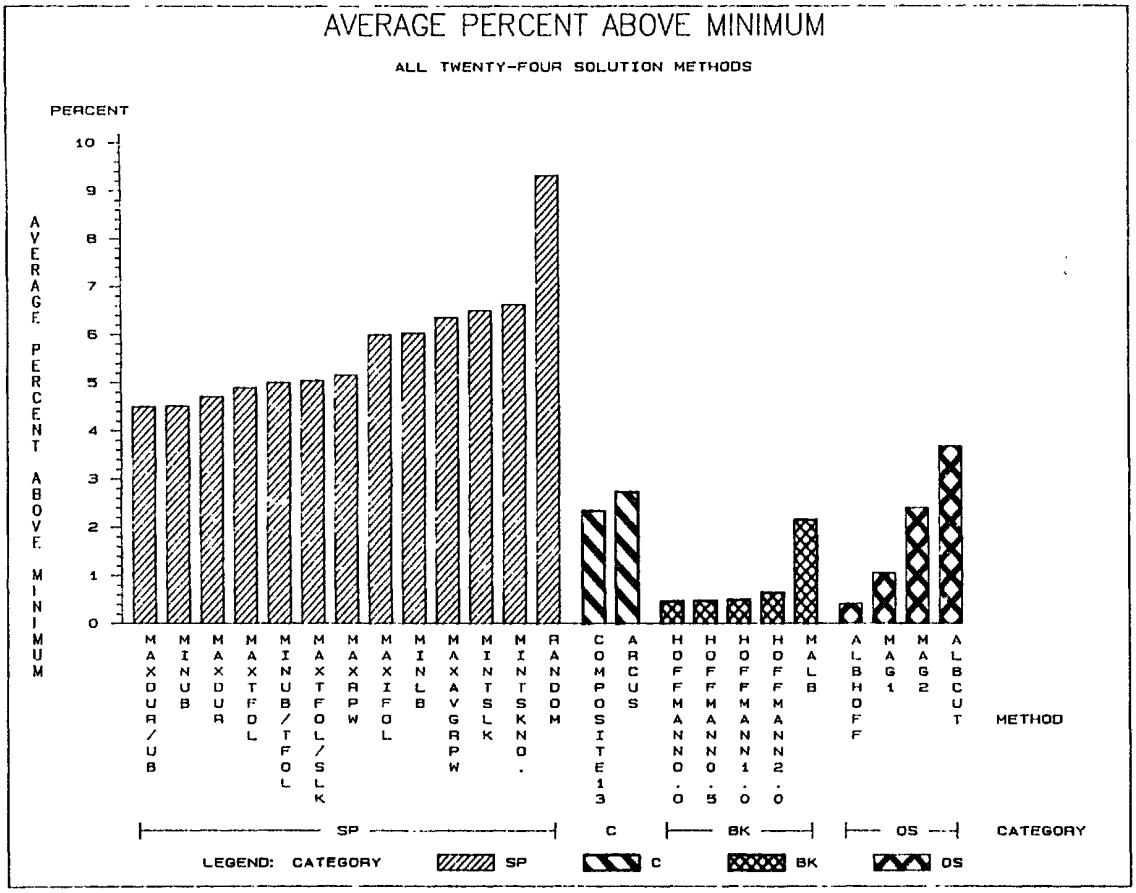
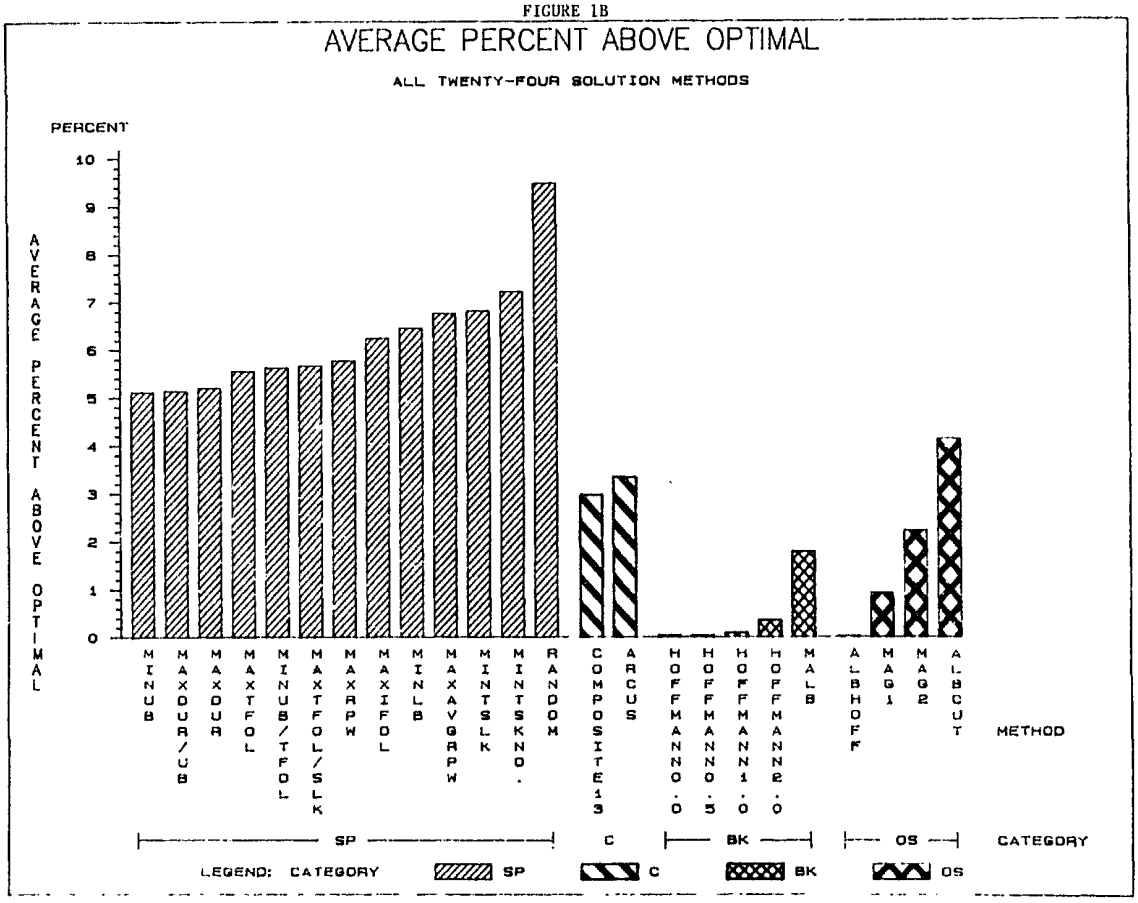


FIGURE 1B



column data. As indicated in Figure 1 (A or B), the enumeration procedure of Hoffmann, or a variant of it, performs best on these problems. The original Hoffmann procedure, HOFFMANN-0.0, yields an average increase of 0.46 percent above the best heuristic solution result obtained, and an average increase of 0.04 percent above the optimal number of work stations. The Gehrlein and Patterson modification to improve computation time and idle time distribution characteristics yields only slightly larger results, as seen with HOFFMANN-0.5, -1.0, and -2.0. ALBHOFF, which begins with the HOFFMANN-0.5 solution and systematically attempts to improve upon it, has the best overall results of 0.42 percent above the best heuristic result obtained, and 0.02 percent above optimal solution on this group of test problems. ALBHOFF also found and verified the optimal solution more frequently than did any of the other optimizing methods (632 out of 635 optimal balances examined) and has the best "worst case" performance for a single balance (8.3 percent above best heuristic, and 7.1 percent above optimal, respectively).

The next best results obtained are 1.06 percent above best heuristic and 0.92 percent above optimum for MAG-1. The performance of MAG-2, at more than twice the average percentage levels of MAG-1, and the much larger differences between ALBHOFF and ALBCUT, indicate the importance of the incorporated heuristics in implementing each of the optimization approaches. The primary difference between MAG-1 and MAG-2, for example, is in the use of IUFFD versus IUBRPW influencing the search strategy. Similarly, the overriding reason for the difference in performance between ALBHOFF and ALBCUT lies in the use of different initializing heuristics with each approach (HOFFMANN-0.5 versus MINUB). Although each of these methods will eventually determine the optimal solution given sufficient time and computer storage, these results suggest that the computational effort required in using these procedures as

heuristics is significantly influenced by the choice of the internal heuristic procedure employed.

For the 872 balances it obtained, MALB has an average percent increase above the best heuristic result of 2.15, and an average percent increase above optimum of 1.78. These percentages are not directly comparable to the percentages for all of the other methods which are based upon the larger set of 1,320 balances. In an effort to correct for this difference, all methods were reranked on just the 872 balances that MALB obtained. This resulted in no changes in the rankings for the percent above optimum criterion. For percent above best heuristic, the ranking of MALB did not change. However, there were two pairwise exchanges: MINUB and MAX(DUR/UB) exchanged rankings, and COMPOSITE-13 and MAG-2 exchanged rankings.

A more detailed examination of MALB's results indicate that both problem size and cycle time affect computational performance. Of the 448 balances not obtained, 308 are from 100-task and 140 are from 50-task problems. More striking, however, is the impact the cycle time increment has on heuristic performance. The most difficult problems for MALB, in terms of computation time, are those with cycle times in close proximity to the largest task time. A plot of balances not solved against cycle time increment looks surprisingly exponential, with 112 not solved when the cycle time equals the largest task time, 96 not solved for the first increment above the lowest cycle time, etc., decreasing to 14 not solved at twice the largest task time. In terms of solution quality, however, our results with MALB are generally consistent with Dar-El's (1975) findings that MALB yields better average results than ARCUS or a composite of the single-pass heuristics he tested.

Predictably, the composite methods perform better than the single-pass heuristics in terms of average and worst case performance. Somewhat

surprising is that over all methods considered, COMPOSITE-13 has the second best worst case results, even though a number of other methods have better average results.

Among the single-pass methods examined, two new heuristics, MINUB and MAXDUR/UB, slightly outperformed MAXDUR as the best of the group. The poorest average and worst case results come from the benchmark rule RANDOM, as one designing heuristics would hope. (This is not always the case, however. Davis and Patterson (1975) found for the resource-constrained project scheduling problem, that a random rule was often better than several apparently reasonable heuristics.) Although no attempt was made to number the tasks in a structured fashion during problem generation, solving the problems by assigning tasks in numerical order using MINTSKNO, our second benchmark rule, did produce slightly better solutions than when an explicit effort was made to randomly assign tasks with the RANDOM rule. This indicates the need to explicitly apply a random rule in this type of experiment, rather than rely on an 'arbitrary' surrogate.

Not indicated in the tables and figures is that while certain rules produced superior results on average, or in terms of worst case results, no one procedure consistently dominated any other. All procedures, including RANDOM, are able to produce minimum, and often unique, solutions to certain problems, albeit is rare for some of them.

In order to more systematically analyze the results for the Main Experimental Data Set, a randomized complete block, full factorial analysis of variance was performed.[1] Assessed are the effects of the number of tasks (T),

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[1]MALB was omitted from this analysis because of the overall size of the experiment and the consequent effect unequal cell sizes had on the analysis. The SAS statistical software system (PROC ANOVA) on an Amdahl 470/V8 computer was used to analyze these results.

network density (D), method of generating partial orders (G), number of increments above the lower bound cycle time [i.e., $MAX_i(t_i)$] (I), and heuristic solution technique (H) on line balancing performance. The response variable measured is the Percent Increase Over the Minimum Heuristic Result obtained. Those variables found to be significant in this analysis include (in order of significance): the heuristic solution technique (H) examined ($r^2 = 0.47$), and the number of increments (I) above the lower bound cycle time ($r^2 = 0.30$). Variables found relatively insignificant in contributing to heuristic performance include network density ($r^2 = 0.015$), number of tasks in the line balancing problem ($r^2 = 0.0006$), and the method used to generate random partial orders ($r^2 = 0.002$). The only significant interaction effect identified is the interaction between the heuristic solution technique employed and the number of increments above the lower bound cycle time (H*I).[1] Thus, it appears as though heuristic performance is not contingent in a statistical sense upon these latter measures (T, D, G) of problem structure.

In order to illustrate the (H*I) interaction, we include Figure 2, which is a three-dimensional histogram of the Percent Increase Above the Best Heuristic Solution for seven of the methods examined and for all eleven cycle times (C = maximum task time at Increment 1. At Increment 2, C = 110% of maximum task time, etc.). What is clear from this chart is that the average performance of each rule varies as a function of the cycle time, and in some cases by a considerable amount (by a factor of six for COMPOSITE-13). For each method, the distribution of Average Percent Above the Best Heuristic

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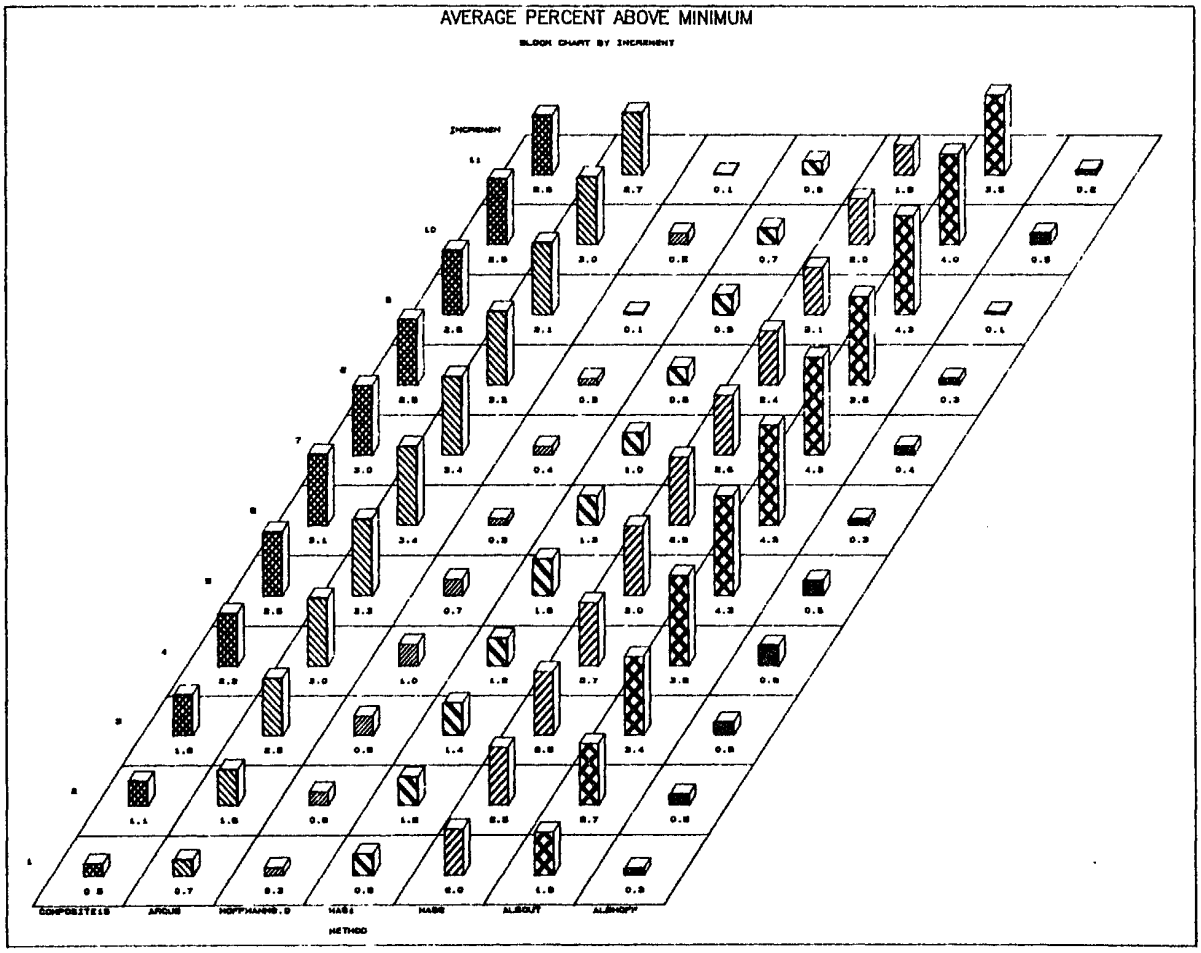
[1]MAG-1 did generally (1) decline in performance when the number of tasks in an assembly network increased from fifty to one-hundred, and did generally (2) improve in performance over the other techniques examined when the density of a network increased (especially to 0.8). These results, however, were not found to be statistically significant.

FIGURE 2



Result Obtained is low at Increment 1, then gradually rises, and then decreases (eventually to zero for all methods when $C = \Sigma t_i$, although this latter situation is not specifically plotted in Figure 2). In an effort to better understand this relationship, the distributions of the average number of stations (instead of average percent) above best heuristic solution were also plotted to control for the fact that as C gets larger, the number of stations in a solution generally gets smaller. These distributions are very similar in appearance to those shown in Figure 2, revealing the same general types of information. In general, with both plots, one would expect there to be much less variability among the methods when the cycle time is equal to or is close to the maximum task time. This is because there are fewer opportunities to effect alternative task assignments. As the cycle time increases

beyond the maximum task time, however, there become more opportunities to effect different assignments as there are more combinations of tasks that can be feasibly assigned into each work station. This phenomenon continues until the cycle time approaches the work content of the product, in which case all methods will assign each of the tasks to the one work station required, giving zero variability in heuristic performance.

One has to exercise a great deal of caution in interpreting Figure 2. It is tempting to conclude, for example, that the best performance overall is achieved by the heuristic decision rules when the cycle time is equal to the maximum task time. What is minimal when the cycle time is equal to the maximum task time is the variance in the performance of the procedures. There are simply fewer opportunities using these procedures to effect alternate balances at this cycle time. And, in fact, it is at the lower cycle times that the optimization approaches used in this evaluation (as heuristic procedures) have the greatest amount of difficulty in determining and verifying the optimality of the solutions obtained. Specifically, for only one balance out of 120 at the first increment was an optimal solution found and verified using the four approaches described, whereas 112 out of 120 optimal balances were found and verified when the cycle time was 200% (increment 11) of the maximum task time. The entire range of values for the eleven increments are (1, 6, 18, 25, 43, 68, 69, 87, 103, 103, 112). Hence, without additional verified optimal solutions at the lower cycle times, it is simply not possible to claim that all of the methods studied perform better in an absolute sense at the lower cycle time increments.

Tukey's studentized range test (HSD) was used to compare the results obtained for the performance measure Percent Increase Above Minimum Heuristic Result Obtained. An alpha value of 0.05 was used for reporting significant

differences. It is encouraging that using this procedure, the RANDOM rule which produced the highest mean value overall was placed into a group by itself, and the five variants of the HOFFMANN procedure which produced the lowest overall sample mean values were also grouped together. These results were identical when the performance measure became Percent Increase Over Optimal. Employment of Duncan's multiple range test and Scheffe's comparison procedure produced similar results on this group of test problems.

4.2 Literature Data Set

As indicated in Tables 2 and 3, MAG-1 performed relatively best on the Literature Data Set by determining minimum balance solutions to all problems, and verified optimum solutions for 55 balances. The second best results were determined by MALB and COMPOSITE-13, for the measure Percent Increase Above Best Heuristic and Above Optimum, respectively. COMPOSITE-13, MALB, MAG-2, ALBCUT, and ALBHOFF each found 54 of the 55 verified optimum solutions.

Although it is instructive to compare methods with a common set of test problems appearing in the open literature, some caution should be exercised in extrapolating percentage results from this data set. First, it is a very small sample. More important however, is the use of Percentage Increase Above Minimum Heuristic Results criterion. Over half of the balances obtained using this data have solutions consisting of less than ten stations (ranging from two stations up to 29), with only two balances containing more than 20 stations. With such small numbers in the denominator using this criterion, even a one station difference between methods often yields a large percentage increase. In contrast, this does not seem to be such a problem with the Main Experimental Data Set where solutions range from 13 to 90 stations, with over 90% distributed approximately uniformly in the 15 to 60 station range.

4.3 "Difficult" Data Set One

This data set was generated with low network density and high variability in task times to especially challenge the backtracking and optimal-seeking methods which have the best performance on the other two data sets. As can be seen from Tables 2 and 3, however, MAG-1 and the Hoffmann-based codes yield the best results on this data set, overall.

4.4 "Difficult" Data Set Two

This 'pathological' data set with all even task times and odd cycle times was also created to tax backtracking and optimal-seeking methods, but specifically HOFFMANN-0.0, which was expected to be unable to solve problems in this data set within the time limit specified. This expectation is based upon the notion that HOFFMANN-0.0 uses exhaustive enumeration of task assignments unless a zero idle time solution is first found. As is shown in Tables 2 and 3, HOFFMANN-0.0 was able to solve 69 of the 110 balances. However, along with MAG-1, the other Hoffmann-based procedures yield the five best average results reported.

4.5 Computer Time and Computer Storage Requirements

Table 4 indicates the actual computer time and storage used by the 24 methods. The times quoted are the average seconds of CPU time required by each method to solve, or attempt to solve, all balances in each data set. For each balance, the minimum of the actual CPU time and 3.0 seconds is used. These values are summed over all balances in the data set and divided by the number of balances in the data set to obtain the times appearing in Table 4. The storage values are given in kilobytes of primary storage required.

Although the values in Table 4 are accurate measurements, one should interpret them cautiously in evaluating the relative efficacy of each solution

TABLE 4

Computation Time and Amount of Computer

Storage Required to Solve Line Balancing Problems

Solution Procedure	Average CPU Time to Solve Each Balance in Each Data Set[a]				Primary Storage Required[b]
	Main Experimental Data Set 1,320 Balances	Literature Data Set 64 Balances	Difficult Data Set One 110 Balances	Difficult Data Set Two 110 Balances	
Each Single-Pass Decision Rule[c]	0.004	0.002	0.005	0.005	8
COMPOSITE-13	1.37	0.57	1.52	1.52	91
ARCUS	0.49	0.05	2.14	2.19	63
HOFFMANN-0.0	0.17	0.53	1.48	8.36[e]	251
HOFFMANN-0.5	0.18	0.35	0.28	0.29	251
HOFFMANN-1.0	0.18	0.29	0.22	0.24	251
HOFFMANN-2.0	0.18	0.29	0.19	0.20	251
MALB	1.85	0.07	1.57	1.60	175
MAG-1	1.48	0.03	1.15	1.18	1,711
MAG-2	1.87	0.03	1.64	1.82	1,711
ALBCUT	2.76	0.52	2.40	2.52	248
ALBHOFF[d]	1.67	0.91	1.76	1.91	172

[a]AMDAHL 470/V8 CPU time, in seconds with a 3.0 second time limit per balance. FORTRAN-H Compiler used, with OPT=2.
[b]In kilobytes.
[c]One computer program was used to find individual single-pass results and COMPOSITE-13 results. The storage value is an estimate, however, of what would be required to code any one single-pass decision rule.
[d]Half-word integers are used in the HOFFMANN-0.5 subroutine in ALBHOFF. This has the effect of increasing the computation time and decreasing the storage over the standard HOFFMANN-0.5 procedure.
[e]The location of the internal timer in this code permitted some balances for which no solution was found to be attempted for more than 3.0 seconds in this data set.

procedure. First of all, through clever programming one could reduce both storage and time requirements for any of the methods examined. Second, the methods vary in the predictability of their use of time and storage, which affects the potential expense and ease of their use. For example, it is not possible to specify precisely, and in advance, the amount of computer storage that will be required to solve a given problem with the MAG-1 and MAG-2 branch and bound procedures. (This is a generic problem with branch and bound codes, and has been observed elsewhere (Patterson, 1984).) Thus, the array dimensions, and hence storage values, had to be determined by trial and error (repetitively increasing dimensions until overflows disappeared). The two codes dropped from the study, DYNAMIC and MUST also have this drawback, although MUST has been programmed to directly use external storage, which

reduces the need to a priori specify storage precisely. None of the other 22 procedures evaluated have this drawback relative to storage needs. However, all methods other than the single-pass and composite methods (and the optimum codes with time traps) have difficult-to-predict time requirements for an arbitrary balance.

Within a given data set, it is fairly safe to compare the relative computation times of the methods. Across sets it is more risky, although in terms of number of tasks in each balance, the Main Experimental Data Set, and the "Difficult" Sets One and Two, are comparable. In creating the latter two data sets, it was expected that some of the methods, specifically the backtracking methods, would perform worse in terms of computation time. This is certainly true of HOFFMANN-0.0 on "Difficult" Data Set Two. In "Difficult" Data Set One, HOFFMANN-0.0 is about nine times slower than it is in the Main Experimental Data Set. However, relative to all other methods on "Difficult" Data Set Two, its time performance was average. The only other method that exhibits substantially increased time per balance is ARCUS.

5.0 A Comparison of These Results with Previous Investigations

There have been two previous major investigations of heuristics for solving the Type II line balancing problem [Mastor (1970) and Dar-El (1975)], but none, until the current investigation, for the Type I problem. Hence, it is difficult to make direct comparisons among these three evaluations. However, as indicated earlier, most methods for solving the Type II problem are in reality methods for solving the Type I problem that are repetitively applied to the same network with an increasing cycle time. This is certainly true for all of the methods included in our evaluation. Hence, we feel that the procedures examined will give the same relative performance for the Type II problem as they have for the Type I problem. In fact, by solving each

network using 11 increments of cycle time, we have in one sense mimicked the process needed to solve a Type II problem.

The major methods included in both our evaluation and Mastor's (with Mastor's terminology given in parentheses) are MINTSKNO (Lexicographical), MAXRPW (Helgeson), MAXIFOL (Immediate Follower), RANDOM (Random), HOFFMANN-0.0 (Hoffmann), and ARCUS (Arcus). Mastor also included a version of the Held, et. al. (1963), dynamic programming procedure. In general, the poorest results obtained by Mastor were for the Lexicographical rule, followed by the single-pass decision rules. The best performance was obtained by Held et. al., followed by Arcus and Hoffmann, although the Held et. al. and Hoffmann methods were tested over a subset of his data because both methods required excessive computation times. Dar-El included MALB, ARCUS, and the best of ten single-pass heuristics (10-SP) in his comparative evaluation. The ten single-pass methods are combinations of four procedures included in our investigation, MAXRPW, MAXTFOL, MAXDUR, and MAXIFOL. Dar-El excluded Hoffmann and Held et. al. from his investigation on the basis of Mastor's observation that these methods exhibit high computation times. Overall, Dar-El found that MALB consistently gave superior results to ARCUS, and that generally, ARCUS outperformed 10-SP.

All three investigations (Mastor's, Dar-El's and ours) are consistent in finding that the more complicated procedures give better solutions than the single-pass methods. Our results comparing MALB and ARCUS are not quite as conclusive as are Dar-El's. However, they are generally in agreement. More noteworthy are the differences among the results, the most important of which is the difference in the performance of HOFFMANN-0.0 in our study and the Hoffmann in Mastor's investigation. In FORTRAN, many decimal fractions are imprecisely represented in the computer. When these numbers are arithmetically

manipulated (e.g., normalized) and compared, this imprecision can give logically erroneous results. Specifically, by treating task times and cycle times as decimal fractions, the test in Mastor's version of the Hoffmann procedure to determine if a station's idle time is equal to zero is very rarely met thus forcing complete enumeration at each work station using this approach. This is in fact the situation we have artificially created in "Difficult" Data Set Two with all even task times and all odd cycle times. If one feels that these types of problems are likely to occur in balancing an industrial assembly line, then he or she should probably use one of the modified versions of the Hoffmann procedure (we prefer HOFFMANN-0.5). It has been our experience, however, that these instances arise rarely in practice, and one has to go to the extreme of normalizing the task times or else constructing a data set such as given in our "Difficult" Data Set Two in order to observe the HOFFMANN-0.0 procedure experiencing difficulty in deriving a satisfactory solution to a line balancing problem.

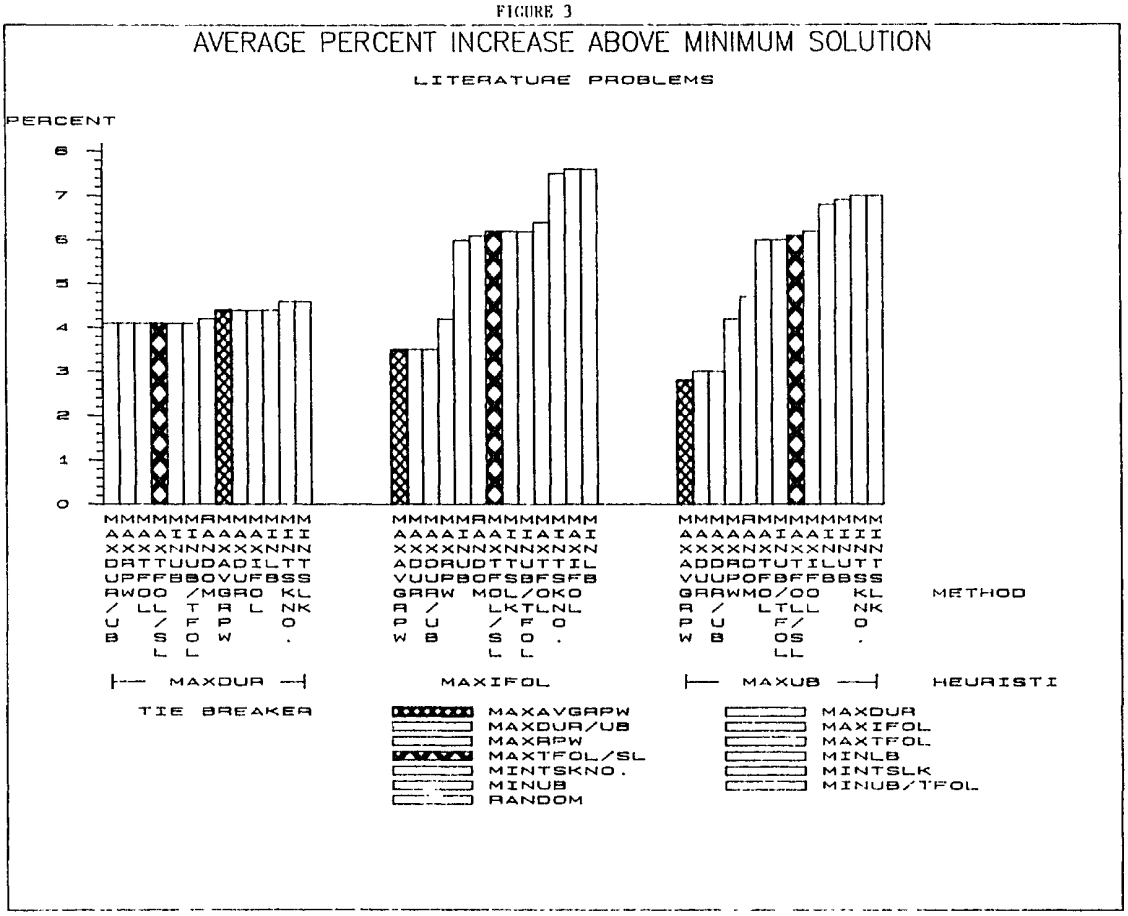
The less significant differences in the individual performance of single-pass decision rules among these three investigations may be explained by the lack of explicit (or at least stated) tie-breaking rules used by Mastor and Dar-El. We have observed that the effect of the tie-breaker rule can become quite significant, and that the tie-breaker's performance is highly dependent upon the main scheduling heuristic used. This is largely because of the heuristic discretion the tie-breaker is or is not able to exercise, a concept addressed previously for the project scheduling problem (Patterson, 1976). For example, when MINTSLK is the main scheduling rule employed, there are usually numerous opportunities for the tie-breaker to exercise its influence (discretion) because several tasks available for scheduling at a given station will possess equal amounts of slack. In this instance, the tie-breaker rule

becomes the dominating heuristic because of the inability of the main heuristic to discriminate among the competing tasks for assignment to a work station. The use of a heuristic such as RANDOM, on the other hand, produces relatively fewer opportunities to break ties since it is unlikely the random numbers assigned for task priority will be equal. To illustrate this point, in Figure 3 we give the Average Percent Increase Above the Minimum Heuristic Result Obtained using the single-pass scheduling heuristics of MAXDUR, MAXIFOL, and MAXUB with the other single-pass decision rules serving as tie-breakers for the literature problems data set. Apparent in this figure is that the performance of MAXDUR varies little regardless of the tie-breaker used because it leaves little discretion to the tie-breaker. However, the performance of MAXIFOL and MAXUB are strongly influenced by the tie-breaker employed because of the large number of ties these rules produce. Figure 3 also illustrates that a tie-breaker's effectiveness is a function of the main rule it is paired with. Two rules, MAXAVGRPW and MAXTFOL/SL, have been highlighted in Figure 3 to illustrate this point. If one were to evaluate these tie-breakers using a main scheduling heuristic such as MAXDUR, then one might conclude that MAXAVGRPW isn't a very effective tie-breaker. However, when paired with MAXIFOL and MAXUB, this tie-breaker actually delivers the best results. Differences in reported results previously may have been due simply to the manner in which ties are broken, or in the order in which tasks are labeled numerically as data input.

6.0 Conclusions and Recommendations

Twenty-six heuristic decision rules are evaluated in this paper, which are designed to assign tasks into work stations such that given a cycle time, or production rate, the number of stations is minimized. The procedures vary from simple list processing prioritizing schemes, which

FIGURE 3



consider a single attribute of each work task, to optimal-seeking procedures which have had the amount of time permitted for search limited. These methods are tested on four data sets, one large set consisting of 1,320 balances, one set of problems found in the open literature, and two specially constructed sets of problems.

A randomized complete block, full factorial analysis of variance was performed to assess the effects of number of tasks, network density, method of generating precedence relationships, and cycle time on heuristic performance. The results indicate that heuristic performance is not significantly affected by network structure, but that performance does vary with the magnitude of the cycle time.

In general, our evaluation demonstrates that methods exist which will give optimal or near-optimal solutions to large line balancing problems with very little computational effort. Of the pure heuristic methods, variants of Hoffmann's Precedence Matrix approach (1963), followed by Dar-El's MALB (1973), appear to be among the best procedures. Of the optimal-seeking methods, Magazine and Wee's branch and bound procedure (1981b) using the heuristic IUFFD (MAXDUR), and Talbot and Patterson's implicit enumeration procedure (1984) initialized with the Hoffmann procedure yield the best overall results.

On the basis of our analysis and experience in using the computer implementations of each of these procedures, we make the following general recommendations. If one is constrained in terms of computer storage and computational speed (for example as with a personal computer), then solving the problem with HOFFMANN-0.5 first, and then improving upon it with an always-feasible optimal-seeking approach such as ALBHOFF (with or without the additional cutting based rules) and using a time trap, would seem to return the best results for the effort expended. If computer storage is not a limiting factor, then MAG-1 (used with a time trap) is likely to give among the best results. To reduce the risk of not determining a good solution, multiple methods should be attempted. This is especially true for balances with cycle times close to the largest task time, which appear to be the most difficult for which to find and verify an optimal solution for a problem.

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Appendix

Listing of Literature Data Set

Note: The format of the problems listed is as follows:

1. Problem title, number of tasks (N), cycle times.

2 - N+1. Task time, immediate successor tasks.

One blank record separates consecutive problems.

```
MERTEN'S 7-ELEMENT        7    6    7    8   10   15   18
     1  2  4
     5  3  5
     4
     3  7
     5  6
     6
     5

BOWMAN'S PROBLEM          9   20
    11  2
    17  3  4
     9  5  6
     5  6
     8  7
    12  8
    10  9
     3  9
     0

JAESCHKE'S 9-ELEMENT      9    6    7    8   10   18
     5  2  3
     3  4
     4  4
     5  5  6  7
     4  8
     5  9
     1  9
     4  9
     6

JACKSON'S 11-ELEMENT     11    7    9   10   13   14   21
     6  2  3  4  5
     2  6
     5  7
     7  7
     1  7
     2  8
     3  9
     6 10
     5 11
     5 11
     4

DAR-EL'S PROBLEM         11   48   62   94
     4  4
    38  4  5
    45 11
    12  6
    10  7
     8  8
    12  9
    10 10
     2 10
    10 11
    34

MITCHELL'S 21-ELEMEN     21   14   15   21   26   35   39
     4  2  3
     3 21
     9  4
     5  5 21
     9  6  7
     4  8
     8  8 14
     7  9
     5 10 11 12 13
     1 15
     3 15
     1 15
     5 17 18
     3 19
     5 16 18
     3 17
    13 20
     5 19
     2
     3
     7

HESKIAOFF'S 28-ELEM.     28  138  205  216  256  324  342
    70  3  4  5 21 22 23  8 19 24 26
    59  6 17
    33 28
     6 28
     1 28
    27  7
    17 18
    62  9
    31 10
    53 11 12
    21 15
    19 13
   108 14 16
    52 28
     5 28
     8 28
    97 28
     8 28
    47 20
    67 28
    17 28
     8 28
     3 28
    21 25
   107 28
     3 27
     2 28
    72

SAWYER'S 30-ELEMENT      30   25   27   30   36   41   54   75
     8  4  5
     7 11 12
    19 16 17
    10  7
     2  6
```

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117          6  7
118         14  8
119         10  9
120          1 26
121          4 24
122         14
123         15 13
124          5 14
125         12 15 20
126          9 22
127         10 20
128          2 18
129         10 19
130         18
131         16 21 24
132         21 22
133         14 23
134         16 27
135          7 25
136         17 26
137          9 27
138         25 28 29
139          7
140         14 30
141          2
142
143     KILBRIDGE & WESTER      45   57   79   92  110  138  184
144          9  3  7  0
145          9  4  8  0
146         10  5  0
147         10  6  0
148         17  9  0
149         17 10  0
150         13  9 14  0
151         13 10 14  0
152         20 41  0
153         20 41  0
154         10 13  0
155         11 13 37  0
156          6 14 15  0
157         22 29 30 31 32 25 17  0
158         11 16 18 23 24  0
159         19 19  0
160         12 26 27  0
161          3 19  0
162          7 20 33  0
163          4 21 00
164         55 22 00
165         14 28 00
166         27 33  0
167         29 33  0
168         26 26  0
169          6 38  0
170          5 28 33  0
171         24 38  0
172          4 41  0
173          5 41  0
174          7 41  0
175          4 41  0
176         15 34 35 36  0
177          3 38  0
178          7 40  0
179          9 38  0
180          4 43  0
181          7 40  0
182          5 41  0
183          4 41  0
184         21 42  0
185         12 44 45  0
186          6  0
187          5  0
188          5  0
189
190     TONGE'S 70-ELEMENT      70  176  364  410  468  527
191         17  2 41 69 70
192         66  3
193         54  4 68
194         52  6  7
195          6  6 24 30
196         88  8
197         21  8
198        128 12
199         68 10
200         70 11
201         85 12
202         21 13 14
203        134 23
204        135 23
205         94 16
206         90 17 18
207         50 19
208        143 19
209         19 20 22 57
210         54 21
211         50 23
212         40 23
213         73 25 31 33
214         12 25
215        152 26 27 28 29
216         42 35
217         45 35
218         74 35
219         26 35
220         11 31
221         31 32
222         50 35
223        102 34
224         46 35
225         35 36 51 48 44 53 56 60 62 61
226         40 37
227          2 38
228          1 39
229          3 40
230         13 42
231         16 42
232         25 43
```

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 21 50
 43 45
 30 46
 83 47
 89 50
 56 49
 59 50
 43
 11 52
 26 54
 44 54
121 55
 38
 68
 22 58
  7 59
 16 60
 32
 25 65
 27 63
156 64
 28 65 66 67
 15
 26
 18
 72
 23
 27

ARCUS' 83-ELEMENT      83 5048 5853 6842 7571 8412 889810816
1673  2
 985  3  4  5
1836  6
 973  6  7
1700  8
2881  9 10
2231 11
1040 77 78
1793 12
1250 13 14 25
 700 15
 464 16
 500 17 18 20
1133 19
 577 20 39
 483 77 78
 880 21 22 28
 667 23
 600 24
 233 26
 408 27
 847 27
 767 74
 850 28 29
 780 32
 912 74
 748 69
1863 32
 714 30
1004 31
 713 39
 642 33 34 35 36
 629 37
1234 77 78
1143 77 78
1266 38 39
 792 40
1251 41
1310 42 43 44 75
 663 77 78
 494 45
1288 77 78
 792 77 78
 578 46
 594 47
 578 48
 622 49
 578 50
 564 69
 578 51
 578 52
 578 53
 578 54
 578 55
 578 56
 578 57
 578 58
 578 59
 578 60
 578 61
 578 62
 578 63
 578 64
 578 65
 578 66
 578 67
 578 68
 578 74 75
 467 70 71
 887 72
 396 73
1296 73
1100 74 75
2543 76
 764 76
 357 77 78
 701 79
1164 79
 286 80 81
2100 82
 450 83
1300 83
3691

ARCUS' 111-ELEMENT     111 5755 884710027107431137817067
1960  2
```

```
349      1715  3
350       735  4
351      1715  5  6  7  8  9 10
352       490 39
353      1225 39
354       169 83
355      2252 71
356      1225 32
357      2319 11 12
358      1715 13 14 15 16 17 18 19 20 21
359       980 13 14 15 16 17 18 19 20 21
360       735 71
361      2281 22 23 24 25
362      2750 26
363        77 27
364        89 28
365        51 29
366       364 30
367       405 91
368      3060111
369       125 31 83
370      3429 32 33
371        43 69 70
372      3430 34
373      1960 82
374        29 35
375        27 36
376        15 37
377       121 38
378      1715 39
379      2127 41
380      1470111
381      4037 42
382        68 43
383        62 44 91
384        42 45 91
385       364 46 91
386      4998 40
387      1470111
388      2963 69 70
389      5689 47
390        68 48 49 91
391        18 50
392        10 51
393        81 52
394      5200 54 55 56 57 58 59 60
395        39 53
396        67 91
397        27111
398        15111
399       121111
400        58111
401      1715 69 70
402       125 61 62 63
403      4010 63 64
404      1470 65 91
405      1470 66 91
406      2303 67 91
407      1960 68 91
408      2205 69 70
409      4018 71
410      2744111
411      2999 72
412       735111
413       735111
414       735111
415       735111
416       545 77 78
417      3386 73
418      3234 91
419      2205 74 91
420      2206 75
421       490 76
422       825 77 78 91 79
423      3528 80 81 82
424      3568 83
425      1200 84
426       618 85
427      1470 86 91
428      1715 87 91
429       735111
430      1960 91
431      2889 88 89 91
432       618111
433       490111
434       735 90
435       490105
436       921105
437       326111
438      5390 92 93 94
439       243 95
440       371 95
441        58 95
442      5059 96 97 98 99100104
443      1225101
444       769102
445       768103
446      1670111
447      1670111
448       490105
449       202106107
450       203107108
451       202111
452      2744111
453       162109
454       324111
455       162110
456       121111
457       162111
458        91
459
460
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