A Comparative Study of Formal Concept Analysis and Rough Set Theory in Data Analysis

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Abstract. The theory of rough sets and formal concept analysis are compared in a common framework based on formal contexts. Different concept lattices can be constructed. Formal concept analysis focuses on concepts that are definable by conjuctions of properties, rough set theory focuses on concepts that are definable by disjunctions of properties. They produce different types of rules summarizing knowledge embedded in data.

1 Introduction

Rough set theory and formal concept analysis offer related and complementary approaches for data analysis. Many efforts have been made to compare and combine the two theories [1, 4–8, 11, 13]. The results have improved our understanding of their similarities and differences. However, there is still a need for systematic and comparative studies of relationships and interconnections of the two theories. This paper presents new results and interpretations on the topic.

The theory of rough sets is traditionally formulated based on an equivalence relation on a set of objects called the universe [9, 10]. A pair of unary set-theoretic operators, called approximation operators, are defined [15]. A concept, represented by a subset of objects, is called a definable concept if its lower and upper approximations are the same as the set itself. An arbitrary concept is approximated from below and above by two definable concepts. The notion of approximation operators can be defined based on two universes linked by a binary relation [14, 18].

Formal concept analysis is formulated based on the notion of a formal context, which is a binary relation between a set of objects and a set of properties or attributes [3, 12]. The binary relation induces set-theoretic operators from sets of objects to sets of properties, and from sets of properties to sets of objects, respectively. A formal concept is defined as a pair of a set of objects and a set of properties connected by the two set-theoretic operators.

The notion of formal contexts provides a common framework for the study of rough set theory and formal concept analysis, if rough set theory is formulated based on two universes. Düntsch and Gediga pointed out that the set-theoretic operators used in the two theories have been considered in modal logics, and therefore referred to them as modal-style operators [1, 4, 5]. They have demonstrated that modal-style operators are useful in data analysis.

In this paper, we present a comparative study of rough set theory and formal concept analysis. The two theories aim at different goals and summarize different types of knowledge. Rough set theory is used for the goal of prediction, and formal concept analysis is used for the goal of description. Two new concept lattices are introduced in rough set theory. Rough set theory involves concepts described by disjunctions of properties, formal concept analysis deals with concepts described by conjunctions of properties.

2 Concept Lattices Induced by Formal Contexts

The notion of formal contexts is used to define two pairs of modal-style operators, one for formal concept analysis and the other for rough set theory [1, 4].

2.1 Binary relations as formal contexts

Let U and V be two finite and nonempty sets. Elements of U are called objects, and elements of V are called properties or attributes. The relationships between objects and properties are described by a binary relation R between U and V, which is a subset of the Cartesian product $U \times V$. For a pair of elements $x \in U$ and $y \in V$, if $(x, y) \in R$, also written as xRy, we say that x has the property y, or the property y is possessed by object x.

An object $x \in U$ has the set of properties:

$$xR = \{y \in V \mid xRy\} \subseteq V. \tag{1}$$

A property y is possessed by the set of objects:

$$Ry = \{x \in U \mid xRy\} \subseteq U. \tag{2}$$

The complement of a binary relation is defined by:

$$R^c = U \times V - R = \{(x, y) \mid \neg(xRy)\}, \tag{3}$$

where ^{c} denotes the set complement. That is, xR^cy if and only if $\neg(xRy)$. An object $x \in U$ does not have the set of properties, $xR^c = \{y \in V \mid xR^cy\} = (xR)^c \subseteq V$. A property y is not possessed by the set of objects, $R^cy = \{x \in V \mid xR^cy\} = (Ry)^c \subseteq U$.

The triplet (U, V, R) is called a binary formal context. For simplicity, we only consider the binary formal context in the subsequent discussion.

2.2 Formal concept analysis

For a formal context (U, V, R), we define a set-theoretic operator $^* : 2^U \longrightarrow 2^V$:

$$\begin{aligned} X^* &= \{y \in V \mid \forall x \in U(x \in X \Longrightarrow xRy)\} \\ &= \{y \in V \mid X \subseteq Ry\} \\ &= \bigcap_{x \in X} xR. \end{aligned} \tag{4}$$

It associates a subset of properties X^* to the subset of objects X. Similarly, for any subset of properties $Y \subseteq V$, we can associate a subset of objects $Y^* \subseteq U$:

$$\begin{aligned} Y^* &= \{x \in U \mid \forall y \in V(y \in Y \Longrightarrow xRy)\} \\ &= \{x \in U \mid Y \subseteq xR\} \\ &= \bigcap_{y \in Y} Ry. \end{aligned} \tag{5}$$

They have the properties: for $X, X_1, X_2 \subseteq U$ and $Y, Y_1, Y_2 \subseteq V$,

$$\begin{array}{lll} (1) & X_1 \subseteq X_2 \Longrightarrow X_1^* \supseteq X_2^*, & Y_1 \subseteq Y_2 \Longrightarrow Y_1^* \supseteq Y_2^*, \\ (2) & X \subseteq X^{**}, \quad Y \subseteq Y^{**}, & \\ (3) & X^{***} = X^*, \quad Y^{***} = Y^*, & \\ (4) & (X_1 \cup X_2)^* = X_1^* \cap X_2^*, \quad (Y_1 \cup Y_2)^* = Y_1^* \cap Y_2^*. & \end{array}$$

A pair of mappings is called a Galois connection if it satisfies (1) and (2), and hence (3).

Consider now the dual operator of * defined by [1]:

$$\begin{aligned} X^\# &= X^{c*c} \\ &= \{y \in V \mid \exists x \in U(x \in X^c \wedge \neg(xRy))\} \\ &= \{y \in V \mid \neg(X^c \subseteq Ry)\} \\ &= \{y \in V \mid X^c \cap (Ry)^c \neq \emptyset\}. \end{aligned} \tag{6}$$

For a subset of properties $Y \subseteq V$, $Y^\#$ can be similarly defined. Properties of $^\#$ can be obtained from the properties of *. For example, we have $(X_1 \cap X_2)^\# = X_1^\# \cup X_2^\#$.

By definition, $\{x\}^* = xR$ is the set of properties possessed by x, and $\{y\}^* = Ry$ is the set of objects having property y. For a set of objects X, X^* is the *maximal* set of properties shared by *all* objects in X. Similarly, for a set of properties Y, Y^* is the *maximal* set of objects that have *all* properties in Y. For a subset $X \subseteq U$, a property in $X^\#$ is not possessed by at least one object not in X.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a formal concept if $X = Y^*$ and $Y = X^*$. The set of objects X is referred to as the extension of the concept, and the set of properties is referred to as the intension of the concept. Objects in X share all properties Y, and only properties Y are possessed by all objects in X. The set of all formal concepts forms a complete lattice called a concept lattice [3]. The meet and join of the lattice is given by:

$$\begin{aligned} (X_1, Y_1) \wedge (X_2, Y_2) &= (X_1 \cap X_2, (Y_1 \cup Y_2)^{**}), \\ (X_1, Y_1) \vee (X_2, Y_2) &= ((X_1 \cup X_2)^{**}, Y_1 \cap Y_2). \end{aligned} \tag{7}$$

By property (3), for any subset X of U, we have a formal concept (X^{**}, X^*), and for any subset Y of V, we have a formal concept (Y^*, Y^{**}).

2.3 Rough sets

We consider a slightly different formulation of rough set theory based on a binary relation between two universes [4, 14, 18].

Given a formal context, we define a pair of dual approximation operators ${}^{\square}, {}^{\diamond} : 2^U \longrightarrow 2^V$,

$$\begin{aligned} X^{\square} &= \{y \in V \mid \forall x \in U (xRy \Longrightarrow x \in X)\} \\ &= \{y \in V \mid Ry \subseteq X\}, \qquad (8) \\ X^{\diamond} &= \{y \in V \mid \exists x \in U (xRy \wedge x \in X)\} \\ &= \{y \in V \mid Ry \cap X \neq \emptyset\} \\ &= \bigcup_{x \in X} xR. \qquad (9) \end{aligned}$$

Similarly, we define another pair of approximation operators ${}^{\square}, {}^{\diamond} : 2^V \longrightarrow 2^U$,

$$\begin{aligned} Y^{\square} &= \{x \in U \mid \forall y \in V (xRy \Longrightarrow y \in Y)\} \\ &= \{x \in U \mid xR \subseteq Y\}, \qquad (10) \\ Y^{\diamond} &= \{x \in U \mid \exists y \in V (xRy \wedge y \in Y)\} \\ &= \{x \in U \mid xR \cap Y \neq \emptyset\} \\ &= \bigcup_{y \in Y} Ry. \qquad (11) \end{aligned}$$

They have the properties: for $X, X_1, X_2 \subseteq U$ and $Y, Y_1, Y_2 \subseteq V$,

(i) $X_1 \subseteq X_2 \Longrightarrow [X_1^{\square} \subseteq X_2^{\square}, X_1^{\diamond} \subseteq X_2^{\diamond}]$,
$Y_1 \subseteq Y_2 \Longrightarrow [Y_1^{\square} \subseteq Y_2^{\square}, Y_1^{\diamond} \subseteq Y_2^{\diamond}]$,

(ii) $X^{\square\diamond} \subseteq X \subseteq X^{\diamond\square}, \quad Y^{\square\diamond} \subseteq Y \subseteq Y^{\diamond\square}$,

(iii) $X^{\diamond\square\diamond} = X^{\diamond}, \quad Y^{\diamond\square\diamond} = Y^{\diamond}$,
$X^{\square\diamond\square} = X^{\square}, \quad Y^{\square\diamond\square} = Y^{\square}$,

(iv) $(X_1 \cap X_2)^{\square} = X_1^{\square} \cap X_2^{\square}, \quad (X_1 \cup X_2)^{\diamond} = X_1^{\diamond} \cup X_2^{\diamond}$,
$(Y_1 \cap Y_2)^{\square} = Y_1^{\square} \cap Y_2^{\square}, \quad (Y_1 \cup Y_2)^{\diamond} = Y_1^{\diamond} \cup Y_2^{\diamond}$.

Based on the notion of approximation operators, we introduce two new concept lattices in rough set theory.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called an object oriented formal concept if $X = Y^{\diamond}$ and $Y = X^{\square}$. If an object has a property in Y then the object belongs to X. Furthermore, only objects in X have properties in Y. The family of all object oriented formal concepts forms a lattice. Specifically, the meet $\wedge$ and join $\vee$ are defined by:

$$\begin{aligned} (X_1, Y_1) \wedge (X_2, Y_2) &= ((Y_1 \cap Y_2)^{\diamond}, Y_1 \cap Y_2), \\ (X_1, Y_1) \vee (X_2, Y_2) &= (X_1 \cup X_2, (X_1 \cup X_2)^{\square}). \qquad (12) \end{aligned}$$

For a set of objects $X \subseteq U$, we have a formal concept $(X^{\Box\Diamond}, X^{\Box})$. For a set of properties $Y \subseteq V$, we have $(Y^{\Diamond}, Y^{\Diamond\Box})$.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a property oriented formal concept if $X = Y^{\Box}$ and $Y = X^{\Diamond}$. If a property is possessed by an object in X then the property must be in Y. Furthermore, only properties Y are possessed by objects in X. The family of all property oriented formal concepts forms a lattice with meet $\wedge$ and join $\vee$ defined by:

$$\begin{aligned}(X_1, Y_1) \wedge (X_2, Y_2) &= (X_1 \cap X_2, (X_1 \cap X_2)^{\Diamond}),\\ (X_1, Y_1) \vee (X_2, Y_2) &= ((Y_1 \cup Y_2)^{\Box}, Y_1 \cup Y_2).\end{aligned} \tag{13}$$

For a set of objects $X \subseteq U$, we can construct a property oriented formal concept $(X^{\Diamond\Box}, X^{\Diamond})$. For a set of properties $Y \subseteq V$, there is a property oriented formal concept $(Y^{\Box}, Y^{\Box\Diamond})$. The property oriented concept lattice was introduced by Düntsch and Gediga [4].

2.4 Relationships between operators and other representations

Düntsch and Gediga referred to the four operators *, $^{\#}$, $^{\Box}$, and $^{\Diamond}$ as modal-style operators, called the sufficiency, dual sufficiency, necessity and possibility operators, respectively [1, 4].

The relationships between four modal-style operators can be stated as follows:

$$\begin{aligned}(X)^{\Box}_R &= \{y \in V \mid Ry \subseteq X\}\\ &= \{y \in V \mid X^c \subseteq (Ry)^c\}\\ &= \{y \in V \mid X^c \subseteq R^c y\}\\ &= (X^c)^*_{R^c};\end{aligned} \tag{14}$$

$$\begin{aligned}(X)^{\Diamond}_R &= \{y \in V \mid X \cap Ry \neq \emptyset\}\\ &= \{y \in V \mid X^{cc} \cap (Ry)^{cc} \neq \emptyset\}\\ &= (X^c)^{\#}_{R^c}.\end{aligned} \tag{15}$$

where the subscription R indicates that the operator $^{\Box}$ is defined with respect to the relation R. Conversely, we have $(X)^*_R = (X^c)^{\Box}_{R^c}$ and $(X)^{\#}_R = (X^c)^{\Diamond}_{R^c}$.

The relationships between binary relations and operators are summarized by: for $x \in U$, $y \in V$,

$$\begin{aligned}&xR = \{x\}^* = \{x\}^{\Diamond}, \qquad Ry = \{y\}^* = \{y\}^{\Diamond},\\ &xRy \Longleftrightarrow x \in \{y\}^* \Longleftrightarrow y \in \{x\}^*,\\ &xRy \Longleftrightarrow x \in \{y\}^{\Diamond} \Longleftrightarrow y \in \{x\}^{\Diamond}.\end{aligned} \tag{16}$$

From a binary relation R, we can define an equivalence relation E_U on U:

$$xE_U x' \Longleftrightarrow xR = x'R. \tag{17}$$

Two objects are equivalent if they have exactly the same set of properties [11]. Similarly, we define an equivalence relation E_V on V:

$$yE_V y' \Longleftrightarrow Ry = Ry'. \tag{18}$$

Two properties are equivalent if they are possessed by exactly the same set of objects [11].

Now we define a mapping, $j : 2^U \longrightarrow 2^V$, called the basic set assignment as follows:

$$j(X) = \{y \in V \mid Ry = X\}. \tag{19}$$

A property y is assigned to the set of objects that have the property. The following set:

$$\{j(X) \neq \emptyset \mid X \subseteq U\}, \tag{20}$$

is in fact the partition induced by the equivalence relation E_V. Similarly, a basic set assignment $j : 2^V \longrightarrow 2^U$ is given by:

$$j(Y) = \{x \in U \mid xR = Y\}. \tag{21}$$

The set:

$$\{j(Y) \neq \emptyset \mid Y \subseteq V\}, \tag{22}$$

is the partition induced by the equivalence relation E_V.

In terms of the basic set assignment, we can re-express operators *, $^\#$, $^\Box$ and $^\Diamond$ as:

$$\begin{aligned} X^* &= \bigcup_{X \subseteq F} j(F), & X^\# &= \bigcup_{X \cup F \neq U} j(F), \\ X^\Box &= \bigcup_{F \subseteq X} j(F), & X^\Diamond &= \bigcup_{F \cap X \neq \emptyset} j(F). \end{aligned} \tag{23}$$

It follows that $X^* \cap X^\Box = j(X)$.

3 Data Analysis using Modal-style Operators

Modal-style operators provide useful tools for data analysis [1, 4]. Different operators lead to different types of rules summarizing the knowledge embedded in a formal context. By the duality of operators, we only consider * and $^\Box$.

3.1 Rough set theory: predicting the membership of an object based on its properties

For a set of objects $X \subseteq U$, we can construct a set of properties $X^\Box$. It can be used to derive rules that determine whether an object is in X. If an object has a property in $X^\Box$, the object must be in X. That is,

$$\forall x \in U[\exists y \in V(y \in X^\Box \wedge xRy) \Longrightarrow x \in X].$$

It can be re-expressed as a rule: for $x \in U$,

$$\bigvee_{y \in X^\Box} xRy \Longrightarrow x \in X. \tag{24}$$

In general, the reverse implication does not hold.

In order to derive a reverse implication, we construct another set of objects $X^{\Box\Diamond} \subseteq X$. For the set of objects, we have a rule: for $x \in U$,

$$x \in X^{\Box\Diamond} \Longrightarrow \bigvee_{y \in X^{\Box}} xRy. \tag{25}$$

This can be shown as follows:

$$\begin{aligned} x \in X^{\Box\Diamond} &\Longrightarrow xR \cap X^{\Box} \neq \emptyset \\ &\Longrightarrow \exists y \in V(xRy \wedge y \in X^{\Box}) \\ &\Longrightarrow \bigvee_{y \in X^{\Box}} xRy. \end{aligned} \tag{26}$$

In general, X is not the same as $X^{\Box\Diamond}$, which suggests that one can not establish a double implication rule for an arbitrary set.

For a set of objects $X \subseteq U$, the pair $(X^{\Box\Diamond}, X^{\Box})$ is an object oriented formal concept. From the property $X^{\Box\Diamond\Box} = X^{\Box}$ and the rule (24), it follows:

$$\bigvee_{y \in X^{\Box}} xRy \Longrightarrow x \in X^{\Box\Diamond}. \tag{27}$$

By combining it with rule (25), we have a double implication rule:

$$x \in X^{\Box\Diamond} \Longleftrightarrow \bigvee_{y \in X^{\Box}} xRy. \tag{28}$$

The results can be extended to any object oriented formal concept. For $(X = Y^{\Diamond}, Y = X^{\Box})$, we have a rule:

$$x \in X \Longleftrightarrow \bigvee_{y \in Y} xRy. \tag{29}$$

That is, the set of objects X and the set of properties Y in (X, Y) uniquely determine each other.

3.2 Formal concept analysis: summarizing the common properties of a set of objects

In formal concept analysis, we identify the properties shared by a set of objects, which provides a description of the objects. Through the operator *, one can infer the properties of an object based on its membership in a set X. More specifically, we have:

$$\forall y \in V \forall x \in U[(y \in X^* \wedge x \in X) \Longrightarrow xRy)].$$

This leads to a rule: for $x \in U$,

$$x \in X \Longrightarrow \bigwedge_{y \in X^*} xRy. \tag{30}$$

The rule suggests that an object in X must have all properties in X^*. The reverse implication does not hold in general.

For the construction of a reverse implication, we construct another set of objects $X^{**} \supseteq X$. In this case, we have:

$$\bigwedge_{y \in X^*} xRy \Longrightarrow x \in X^{**}. \tag{31}$$

An object having all properties in X^* must be in X^{**}. For an arbitrary set X, X may be only a subset of X^{**}. One therefore may not be able to establish a double implication rule for an arbitrary set of objects.

A set of objects X induces a formal concept (X^{**}, X^*). By property $X^{***} = X^*$ and rule (30), we have:

$$x \in X^{**} \Longrightarrow \bigwedge_{y \in X^*} xRy. \tag{32}$$

Combining it with rule (31) results in: for $x \in U$,

$$x \in X^{**} \Longleftrightarrow \bigwedge_{y \in X^*} xRy. \tag{33}$$

In general, for a formal concept $(X = Y^*, Y = X^*)$, we have:

$$x \in X \Longleftrightarrow \bigwedge_{y \in Y} xRy. \tag{34}$$

That is, the set of objects X and the set of properties Y determine each other.

3.3 Comparison

Rough set theory and formal concept analysis offer two different approaches for data analysis. A detailed comparison of the two methods may provide more insights into data analysis.

Fayyad *et al.* identified two high-level goals of data mining as prediction and description [2]. Prediction involves the use of some variables to predict the values of some other variables. Description focuses on patterns that describe the data. For a set of objects $X \subseteq U$, the operator $^\Box$ identifies a set of properties $X^\Box$ that can be used to predict the membership of an object x with respect to X. It attempts to achieve the goal of prediction. In contrast, the operator * identifies a set of properties X^* that are shared by all objects in X. In other words, it provides a method for description and summarization. In special cases, the tasks of prediction and description become the same one for certain sets of objects. In rough set theory, this happens for the family of object oriented formal concepts. In formal concept analysis, this happens for the family of formal concepts.

A property in $X^\Box$ is sufficient to decide that an object having the property is in X. The set $X^\Box$ consists of *sufficient* properties for an object to be in X. On the other hand, an object in X must have properties in X^*. The set X^* consists of *necessary* properties of an object in X. Therefore, rough set theory and formal concept analysis focus on two opposite directions of inference. The operator $^\Box$ enables us to infer the membership of an object based on its properties. On the other hand, through the operator *, one can infer the properties of an object based on its membership in X. By combining the two types of knowledge, we obtain a more complete picture of the data.

By comparing the rules derived by rough set theory and formal concept analysis, we can conclude that the two theories focus on different types of concepts. Rough set theory involves concepts described by disjunctions of properties, formal concept analysis deals with concepts described by conjunctions of properties. They represent two extreme cases. In general, one may consider other types of concepts.

By definition, * and $^\Diamond$ represent the two extremely cases in describing a set of objects based on their properties. Assume that $xR \neq \emptyset$ and $Ry \neq \emptyset$. Then we have the rules: for $x \in U$,

$$\begin{aligned} &x \in X \Longrightarrow \exists y \in V(y \in X^\Diamond \wedge xRy), \\ &x \in X \Longrightarrow \forall y \in V(y \in X^* \Longrightarrow xRy). \end{aligned} \tag{35}$$

That is, an object has all properties in X^* and at least one property in $X^\Diamond$. The pair $(X^*, X^\Diamond)$ with $X^* \subseteq X^\Diamond$ thus provides a characterization of X in terms of properties.

4 Conclusion

Both the theory of rough sets and formal concept analysis formalize in some meaningful way the notion of concepts. The two theories are compared in a common framework consisting of a formal context. Different types of concepts are considered in the two theories. They capture different aspects of concepts. Rough set theory involves concepts described by disjunctions of properties, formal concept analysis deals with concepts described by conjunctions of properties. One makes opposite directions of inferences using the two theories. The operator $^\Box$ enables us to infer the membership of an object based on its properties, and the operator * enables us to infer the properties of an object based on its membership in X.

The combination of the two theories leads to a better understanding of knowledge embedded in data. One may combine modal-style operators to obtain new modal-style operators and analyze data using the new operators [1, 4, 5]. Further studies on the relationships between the two theories would lead to new results [16, 17].

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