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# A comparative study of heat transfer analysis of fractional Maxwell fluid by using Caputo and Caputo-Fabrizio derivatives



Nauman Raza*, Muhammad Asad Ullah

*Department of Mathematics, University of the Punjab, Lahore-54590, Pakistan.*

**Abstract**

A comparative analysis is carried out to study the unsteady flow of a Maxwell fluid in the presence of Newtonian heating near a vertical flat plate. The fractional derivatives presented by Caputo and Caputo-Fabrizio are applied to make a physical model for a Maxwell fluid. Exact solutions of the non-dimensional temperature and velocity fields for Caputo and Caputo-Fabrizio time-fractional derivatives are determined via the Laplace transform technique. Numerical solutions of partial differential equations are obtained by employing Tzou's and Stehfest's algorithms in order to compare the results of both models. Exact solutions with integer order derivative (fractional parameter $\alpha = 1$) are also obtained for both temperature and velocity distributions as a special case. A graphical illustration is made to discuss the effect of Prandtl number $Pr$ and time $t$ on the temperature field. Similarly, the effect of Maxwell fluid parameter $\lambda$ and other flow parameters on the velocity field is graphically discussed, along with tabular forms.

**Keywords:** Maxwell fluid; Temperature field; Velocity field; Caputo and Caputo-Fabrizio fractional derivatives; Laplace transformation.

**Nomenclature**

$T$ - Fluid temperature $[K]$

$k$- Thermal conductivity of the fluid $[Wm^{-1}K^{-1}]$

$u$ - Velocity of the fluid $[ms^{-1}]$

$Pr$ - Prandtl number $[\frac{\mu \mathbb{C}_p}{k}]$
$C_p$ - Specific heat at a constant pressure $[jkg^{-1}K^{-1}]$
$Gr$ - Thermal Grashof number $[\beta T_w]$
$T_\infty$ - Temperature of fluid far away from the plate [K]
$h$ - Coefficient of heat transfer $[Wm^{-2}K^{-1}]$
$g$ - Gravitational acceleration $[ms^{-2}]$
$q$ - Laplace transforms parameter [-]
$Nu$ - Nusselt number [-]
$y$ - Cartesian coordinate [m]
**Greek symbols**
$\rho$ - Fluid density $[kgm^{-3}]$
$\lambda_1$, $\lambda$ - Maxwell fluid parameter [s]
$\mu$ - Dynamic viscosity $[kgm^{-1}s^{-1}]$
$\nu$ - Fluid kinematics viscosity $[m^2s^{-1}]$
$\beta$ - Volumetric thermal expansion coefficient $[K^{-1}]$
$\theta$ - Dimensionless temperature [-]
**Subscripts**
$\infty$ - Condition far away from the surface [-]



# 1 Introduction

The Maxwell fluid model is one of the simplest viscoelastic rate-type fluid model as compare to other non-Newtonian models [1]-[7]. At first, the Maxwell fluid model was developed to depict the viscous and elastic reaction of air. Presently, it is extensively studied to explain the reactions of some polymeric liquids. However, it has some restrictions. Initially, the classical definition of derivative was used to model the Maxwell fluid problems without analyzing the heat transfer. Therefore, numerous analysts are interested in generalizing classical flow problems to fractional dynamics [8]-[11]. The application of fractional calculus demonstrate functional electrical simulation and bio potential recording. Fractional calculus can be used to generalize the circuit models representing the non-integer order differential equation. Therefore, fractional calculus have a lot of application in economics [12], probability and statistics [13], physics [14] and fluid mechanics [15]. Olsson and Ystrm [16] discussed viscoelastic fluid flow and studied Maxwell fluid properties. Vieru and Rauf

[17] investigated the Stokes flows of a Maxwell fluid by using the wall slip condition. Fetecau and Corina Fetecau [18]-[19] discussed the Rayleigh-Stokes problem and obtained new solutions for a Maxwell fluid. Khan et al. [20] studied the unsteady magnetohydrodynamic oscillatory flow and found exact solutions for Maxwell fluid in a permeable medium. Zhenga et al. [21] applied the finite Hankel transform, Laplace transform and combined sequential fractional derivatives to study the unsteady rotating flows of a generalized Maxwell fluid and obtained exact solutions in the presence of oscillating pressure gradient between coaxial cylinders. Abro et al. [22] obtained a solution by an analytical method for a magnetohydrodynamic generalized Burger fluid over permeable oscillating flat plate.

Newtonian heating conditions are considered by a lot of researchers in their problems due to its many applications. Anwer et al. [23] used Newtonian heating to describe the unsteady natural convection flow of viscous fluid near an oscillating plate and find exact solution via Laplace transform technique. Vieru et al. [24] applied Newtonian heating on an incompressible viscous fluid with chemical reaction of first order and discussed the free convection flow and mass diffusion close a vertical plate. Hussanan et al. [25] explained the exact analysis of mass and heat transfer with Newtonian heating past a vertical plate.

Many real world problems can be modeled through derivatives. Especially, fractional derivatives are more applicable for certain situations than ordinary derivatives. Therefore, fractional derivatives are used in many fields, especially in electrochemistry, bio-engineering, finance, tracer in fluent currents and in fluid dynamics. Mathur and Khandelwal [26] studied Oldroyd-B fluid flow between coaxial cylinders and used fractional derivatives to find exact solutions. Abro et al. [27] examined the shape effects of molybdenum disulfide nanofluids in the mixed convection flow with the permeable medium and magnetic field by employing Atangana-Baleanu fractional derivatives. Due to some complications, Caputo and Fabrizio (CF) [28] defined new fractional derivatives with an exponential kernel without singularities. The CF derivative has a lot of physical applications in many fields, including in signal processing, the edge detection of images [29], multi-scale filtering of vascular images [30], identifying an arrhythmia in ECG signals and image processing applications including contrast improvement and color [31]. Imran et al. [32] considered the slip condition and Newtonian heating effects to examine the unsteady natural convection flow of Maxwell fluid over an exponentially accelerated infinite vertical plate via CF derivative and obtained semi analytical solutions by Stehfest's and Tzou's algorithms. Imran et al. [33] applied the Caputo and CF derivatives on second-grade fluid with Newtonian heating to find the exact solution by Laplace transform and also discussed the comparison between them. Shah and Khan [34] applied CF derivative on a second grade fluid to discuss the heat transfer analysis over an oscillating vertical plate and applied the Laplace transform to obtain an exact solution. Ali et al. [35] took the generalized Walters-B fluid and used the CF derivative to describe the MHD free convection flow in it. Khan et al. [36] analyzed the heat transfer analysis over an oscillating vertical plate in Maxwell fluid with CF derivative. Different problems have been explored through the CF derivative, a few of them are [37]-[40].

In the present paper, we describe the free convection flow of a fractional Maxwell fluid in the presence of Newtonian heating near a vertical plate. Solutions are determined by two different methods using the Caputo and CF fractional derivatives. The Laplace transform is used to obtain the exact solutions for non-dimensional temperature and velocity fields. Solutions for fractional order (fractional parameter $\alpha \to 1$) are also obtained. We comparatively analyze the solutions for the ordinary case and for fractional derivatives. Finally, we graphically observe the effect of Maxwell fluid parameter and fractional parameter $\alpha$ as well as the contributions of some flow parameters on temperature and velocity fields. We also discuss the effect of flow parameters in tabular form.



## 2 Mathematical formulation of the problem

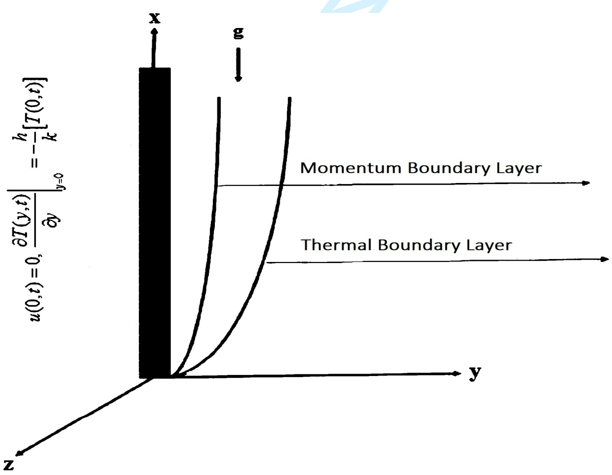


Figure 1: Physical geometry and schematic diagram of the problem

Consider the unsteady flow of an incompressible Maxwell fluid over an infinite vertical plate in the presence of Newtonian heating at the boundary. The $x$-axis is taken along the plate and $y$-axis is taken normal to the plate. Initially, the fluid and plate are at rest with constant temperature $T_\infty$. At the beginning $t = 0^+$, the local surface temperature $T$ is proportional to the heat transfer from the plate to the fluid. We consider that all physical variables are functions of $y$ and $t$ only. The incompressibility constraint is identically satisfied for such flow as shown in Fig. 1. The governing equations of motion under the Boussinesq approximation are written in usual notation as [8]:

$$\rho\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t}=\mu\frac{\partial^2 u(y,t)}{\partial y^2}+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\rho g\beta(T(y,t)-T_\infty), \tag{1}$$

$$\frac{\partial T(y,t)}{\partial t}=\frac{k}{\rho C_p}\frac{\partial^2 T(y,t)}{\partial y^2}. \tag{2}$$

Initial and boundary conditions are:

$$u(y,0)=0,\quad \left.\frac{\partial u(y,t)}{\partial t}\right|_{t=0}=0,\quad T(y,0)=T_\infty,\qquad \forall\ \ y\geq 0, \tag{3}$$

$$u(0,t)=0,\quad \left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0}=\frac{-h}{k}T(0,t),\qquad t>0, \tag{4}$$

$$u(\infty,t)=0,\quad T(\infty,t)=T_\infty,\qquad t>0. \tag{5}$$

Introducing the following dimensionless quantities,

$$y^\star=\frac{y}{\frac{k}{h}},\qquad \theta=\frac{T-T_\infty}{T_\infty},\qquad u^\star=\frac{u}{\frac{g}{\nu}(\frac{k}{h})^2},$$
$$Pr=\frac{\mu\mathbb{C}_p}{k},\qquad t^\star=\frac{t}{\frac{1}{\nu}(\frac{k}{h})^2},\qquad \lambda^\star=\frac{\lambda_1}{\frac{1}{\nu}.\left(\frac{k}{h}\right)^2},\qquad Gr=\beta T_\infty, \tag{6}$$

into Eqs.(1)-(5) and after dropping star notation we get the following dimensionless equations

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t}=\frac{\partial^2 u(y,t)}{\partial y^2}+\left(1+\lambda\frac{\partial}{\partial t}\right)\theta(y,t)Gr, \tag{7}$$

$$Pr\,\frac{\partial\theta(y,t)}{\partial t}=\frac{\partial^2\theta(y,t)}{\partial y^2}, \tag{8}$$

Here $\lambda$, $Gr$ and $Pr$ are the Maxwell fluid parameter, Grashof number and Prandtl number, respectively. In order to build up a time-fractional derivatives model, we simply supplant one order time derivative with $\alpha$ order time-fractional derivative and we obtain the problem as follows:

$$(1+\lambda D_t^\alpha)\frac{\partial u(y,t)}{\partial t}=\frac{\partial^2 u(y,t)}{\partial y^2}+(1+\lambda D_t^\alpha)\,\theta(y,t)Gr,\quad y,t\ >0, \tag{9}$$

$$PrD_t^{\alpha}\theta(y,t) = \frac{\partial^2\theta(y,t)}{\partial y^2}, \quad y,t > 0. \tag{10}$$

The initial and boundary conditions become:

$$u(y,0) = 0, \qquad \frac{\partial u(y,t)}{\partial t}\bigg|_{t=0} = 0, \qquad \theta(y,0) = 0, \qquad y \geq 0, \tag{11}$$

$$u(0,t) = 0, \quad \frac{\partial \theta(0,t)}{\partial y} = -[\theta(0,t) + 1] \quad t > 0, \tag{12}$$

$$u(\infty,t) = 0, \qquad \theta(\infty,t) = 0. \tag{13}$$



# 3 Preliminaries on fractional derivatives

## 3.1 Caputo fractional operator

The Caputo derivative of fractional order is define as

$${}^{C}D_t^{\alpha}f(y,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-p)^{-\alpha} f'(y,p)dp, & 0 \leq \alpha < 1, \\ \frac{\partial f(y,t)}{\partial t} & \alpha = 1. \end{cases}$$

The Laplace transform of the Caputo derivative is

$$L\{{}^{C}D_t^{\alpha}f(y,t)\} = q^{\alpha}L\{f(y,t)\} - q^{\alpha-1}f(y,0). \tag{14}$$

## 3.2 Caputo-Fabrizio fractional operator

The Caputo Fabrizio fractional derivative is defined

$${}^{CF}D_t^{\alpha}f(y,t) = \begin{cases} \frac{1}{1-\alpha}\int_0^t \; exp\left(\frac{-\alpha(t-p)}{1-\alpha}\right) f'(y,p)dp, & 0 \leq \alpha < 1, \\ \frac{\partial f(y,t)}{\partial t} & \alpha = 1. \end{cases}$$

The Laplace transform of the CF derivative is

$$L\{{}^{CF}D_t^{\alpha}f(y,t)\} = \frac{qL\{f(y,t)\} - f(y,0)}{(1-\alpha)q + \alpha} \tag{15}$$

**Remark**: Indeed, Caputo and Fabrizio have defined the time-fractional derivative with a normalization function $M(\alpha)$, with $M(0) = M(1) = 1$. In this paper, for simplicity, we consider the normalization function $M(\alpha) = 1$.

# 4 Extraction of temperature profile

## 4.1 For Caputo fractional derivative

Imran et al. [33] obtained the solution for the temperature field via Caputo fractional derivative for the same initial and boundary conditions. The Laplace transform is

$$\overline{\theta}(y,q) = \frac{1}{\sqrt{Pr}\ q^{\alpha} - 1}\ \frac{1}{q}\ \exp\left(-y\ \sqrt{Pr}\ q^{\frac{\alpha}{2}}\right). \tag{16}$$

We obtain the solution of Eq. (16) by using (A.2) and (A.4) from the appendix

$$\theta(y,t) = \frac{1}{\sqrt{Pr}}\ \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\ \mathbf{E}_{\frac{\alpha}{2},\frac{\alpha}{2}}\left(\frac{(t-\tau)^{\frac{\alpha}{2}}}{\sqrt{Pr}}\right)\ \phi\left(1, \frac{-\alpha}{2};\ -y\ \sqrt{Pr}\ \tau^{-\alpha/2}\right) d\tau, \tag{17}$$

where $\mathbf{E_{a,b}(z)}$ is the Mittag-Leffler function.

The local coefficient of heat transfer corresponding to Nusselt number from the plate to the fluid is

$$Nu = -\lim_{y\to 0} L^{-1}\left\{\frac{\partial\overline{\theta}(y,q)}{\partial y}\right\} = -L^{-1}\left\{\lim_{y\to 0}\frac{\partial\overline{\theta}(y,q)}{\partial y}\right\} = \mathbf{E}_{\alpha/2,1}\left(\frac{1}{\sqrt{Pr}} t^{\alpha/2}\right). \tag{18}$$

## 4.2 For Caputo-Fabrizio fractional derivative

Imran et al. [33] applied the same initial and boundary conditions to find the solution for the temperature field. So, we present its solution directly

$$\overline{\theta}(y,q) = \frac{1}{\sqrt{\frac{a_o Prq}{q+\alpha a_o}} - 1}.\frac{1}{q}\ \exp\left(-y\ \sqrt{\frac{a_o Prq}{q+\alpha a_o}}\right). \tag{19}$$

The inverse Laplace of Eq. (19), is determined by using convolution theorem as well as (A.1) and (A.5)-(A.7) from the appendix is given as

$$\theta(y,t) = v_1(t) * v_2(y,t) = \int_0^t v_1(t-\tau)v_2(y,\tau)d\tau, \tag{20}$$

where

$$v_1(t) = \delta(t)\int_0^\infty e^{-ua_0 Pr}\left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]du$$
$$+\int_0^\infty \left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]\sqrt{\frac{u\alpha a_0^2 Pr}{t}} I_1\left(2\sqrt{u\alpha a_0^2 Prt}\right)e^{a_0 uPr - \alpha a_0 t}du, \tag{21}$$

where $\delta(t)$ is distribution function.

$$v_2(y,t) = 1 - \frac{2a_o Pr}{\pi}\int_0^\infty \frac{sin\ yu}{u(u^2 + a_o Pr)}\ exp\left(-\frac{\alpha a_o t u^2}{u^2 + a_o Pr}\right)du. \tag{22}$$

The following relations are used to get the local coefficient of heat transfer regarding the Nusselt number:

$$Nu = -\lim_{y\to 0} L^{-1}\left\{\frac{\partial\overline{\theta}(y,q)}{\partial y}\right\} = -L^{-1}\left\{\lim_{y\to 0}\frac{\partial\overline{\theta}(y,q)}{\partial y}\right\} = \int_0^t N_1(t-\tau)N_2(u,\tau)d\tau, \tag{23}$$

where

$$N_1(t) = \sqrt{a_0 Pr} I_o\left(\frac{\alpha a_0}{2}t\right) exp\left(\frac{-\alpha a_0}{2}t\right), \tag{24}$$

and

$$N_2(u,t) = \delta(t)\int_0^\infty e^{-ua_0 Pr}\left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]du$$
$$+\int_0^\infty \left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]\sqrt{\frac{u\alpha a_0^2 Pr}{t}} I_1\left(2\sqrt{u\alpha a_0^2 Prt}\right)e^{a_0 uPr - \alpha a_0 t}du. \tag{25}$$

## 4.3 For $\alpha = 1$

For $\alpha = 1$, the temperature field given in Eq. (16) and Eq. (19) obtained by the Caputo and CF time-fractional derivatives is given as

$$\theta(y,t) = exp\left(-y + \frac{t}{Pr}\right) erfc\left(\frac{y\sqrt{Pr}}{2\sqrt{t}} - \sqrt{\frac{t}{Pr}}\right) - erfc\left(\frac{y\sqrt{Pr}}{2\sqrt{t}}\right), \tag{26}$$

and the Nusselt number is

$$Nu = exp\left(\frac{t}{Pr}\right)\left(2 - erfc\sqrt{\frac{t}{Pr}}\right). \tag{27}$$

# 5 Extraction of velocity profile

## 5.1 For Caputo fractional derivative

By taking the Laplace transform of Eq. (9) and keeping in mind the Eq. (16), as well as the associated initial and boundary conditions, we have

$$\frac{\partial^2 \bar{u}}{\partial y^2} - q\left(1+\lambda q^{\alpha}\right)\bar{u}(y,q) = -\frac{Gr\left(1+\lambda q^{\alpha}\right)}{q\left(\sqrt{Prq^{\alpha}}-1\right)}\, exp\left(-y\sqrt{Prq^{\alpha}}\right). \tag{28}$$

$$\bar{u}(0,q)=0, \quad \bar{u}(\infty,q)=0. \tag{29}$$

Eq. (28) using Eq. (29) gives:

$$\bar{u}(y,q) = -\frac{Gr}{\left(\sqrt{Prq^{\alpha}}-1\right)}\cdot\frac{q\left(1+\lambda q^{\alpha}\right)}{Prq^{\alpha}-q\left(1+\lambda q^{\alpha}\right)}\left[\frac{1}{q^2}exp\left(-y\sqrt{Prq^{\alpha}}\right)-\frac{1}{q^2}exp\left(-y\sqrt{q\left(1+\lambda q^{\alpha}\right)}\right)\right]. \tag{30}$$

We write Eq. (30) in following suitable form to determine its inverse Laplace transform

$$\bar{u}(y,q) = \bar{x}_1(q)[\bar{x}_2(y,q)-\bar{x}_3(y,q)], \tag{31}$$

where

$$\bar{x}_1(q) = -\frac{Gr}{\left(\sqrt{Prq^{\alpha}}-1\right)}\cdot\frac{q\left(1+\lambda q^{\alpha}\right)}{Prq^{\alpha}-q\left(1+\lambda q^{\alpha}\right)}, \tag{32}$$

$$\bar{x}_2(y,q) = \frac{1}{q^2}exp\left(-y\sqrt{Prq^{\alpha}}\right), \tag{33}$$

$$\bar{x}_3(y,q) = \frac{1}{q^2}exp\left(-y\sqrt{q\left(1+\lambda q^{\alpha}\right)}\right). \tag{34}$$

The inverse Laplace of Eqs. (31)-(33) is obtained by using convolution theorem as well as (A.2)-(A.4) from the appendix is given as

$$u(y,t) = x_1(t) * [x_2(y,t)-x_3(y,t)], \tag{35}$$

where convolution product is represented by $*$.

$$x_1(t) = \sum_{k=0}^{\infty} \frac{Gr(Pr)^{k+1/2}}{\lambda^{k+1}} \int_0^t \mathbf{G}_{\alpha,(\alpha-1)(k+1),(k+1)} \left(t-\tau, -\lambda^{-1}\right) \left[\tau^{\alpha/2-1} \mathbf{E}_{\alpha/2,\alpha/2} \left(\frac{1}{\sqrt{Pr}} \tau^{\alpha/2}\right)\right] d\tau$$
$$-t^{\alpha/2-1} \mathbf{E}_{\alpha/2,\alpha/2} \left(\frac{1}{\sqrt{Pr}} t^{\alpha/2}\right), \quad (36)$$

where $\mathbf{G_{a,b,c}(d,t)}$ is G-function.

$$x_2(y,t) = t.\phi\left(2, -\alpha/2, -y\sqrt{Pr}t^{-\alpha/2}\right). \quad (37)$$

We are unable to find the exact solution of $\overline{x}_3(y,q)$, given by Eq. (34). By employing Stehfest's and Tzou's algorithms, we have obtained its solution numerically.

## 5.2 For Caputo-Fabrizio fractional derivative

Keeping in mind Eq. (19) and taking the Laplace transform of Eq. (9) under the given initial and boundary conditions, we have

$$\left(1 + \frac{\lambda q}{(1-\alpha)q+\alpha}\right) q\bar{u}(y,q) = \frac{\partial^2 \bar{u}(y,q)}{\partial y^2} + \left(1 + \frac{\lambda q}{(1-\alpha)q+\alpha}\right) Gr\overline{T}(y,q). \quad (38)$$

$$\bar{u}(0,q) = 0, \qquad \bar{u}(\infty,q) = 0. \quad (39)$$

By using condition (39), the general solution of Eq. (38) in simplest form is

$$\bar{u}(y,q) = -Gr\frac{q+a_2}{q(q+b_1)}.\frac{1}{\sqrt{\frac{aq}{q+b}}-1}\left[\frac{1}{q}exp\left(-y\sqrt{\frac{a_1 q(q+a_2)}{q+b}}\right) - \frac{1}{q}exp\left(-y\sqrt{\frac{aq}{q+b}}\right)\right], \quad (40)$$

where

$$a_0 = \frac{1}{1-\alpha}, \quad a = a_0 Pr, \quad a_1 = 1+\lambda a_0, \quad a_2 = \frac{\alpha a_0}{1+\lambda a_0},$$
$$b = \alpha a_0, \qquad b_1 = \frac{a_0(\alpha - Pr)}{1+\lambda a_0}. \quad (41)$$

Eq. (40) can be written as

$$\overline{u}(y,q) = \overline{x}_1(q)[\overline{x}_2(y,q) - \overline{x}_3(y,q)], \quad (42)$$

where

$$\overline{x}_1(q) = -Gr\frac{q+a_2}{q(q+b_1)}.\frac{1}{\sqrt{\frac{aq}{q+b}}-1}, \tag{43}$$

$$\overline{x}_2(y,q) = \frac{1}{q}exp\left(-y\sqrt{\frac{a_1q(q+a_2)}{q+b}}\right), \tag{44}$$

$$\overline{x}_3(y,q) = \frac{1}{q}exp\left(-y\sqrt{\frac{aq}{q+b}}\right). \tag{45}$$

The inverse Laplace of Eq. (42) is

$$u(y,t) = x_1(t) * [x_2(y,t) - x_3(y,t)], \tag{46}$$

where $*$ denotes the convolution product.

$$x_1(t) = -Gr\int_0^t w_1(t-\tau)w_2(\tau)d\tau, \tag{47}$$

where

$$w_1(t) = \frac{a_2}{b_1} - \frac{b_1-a_2}{b_1}exp(-b_1t), \tag{48}$$

$$\begin{aligned} w_2(t) = \delta(t)\int_0^\infty e^{-ua_0Pr}\left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]du \\ + \int_0^\infty \left[\frac{1}{\sqrt{\pi t}} + e^u\left(2 - erfc(\sqrt{u})\right)\right]\sqrt{\frac{u\alpha a_0^2 Pr}{t}}I_1\left(2\sqrt{u\alpha a_0^2 Prt}\right)e^{a_0uPr-\alpha a_0t}du. \end{aligned} \tag{49}$$

and

$$\overline{x}_2(y,q) = \frac{a_1(q+a_2)}{q+b}.\frac{exp\left(-y\sqrt{\frac{a_1q(q+a_2)}{q+b}}\right)}{\frac{a_1q(q+a_2)}{q+b}}, \tag{50}$$

$$\overline{x_2}(y,q) = a_1\overline{w}_3(q).\overline{w}_4(q), \tag{51}$$

where

$$\overline{w}_3(q) = \frac{(q+a_2)}{q+b}, \tag{52}$$

and

$$\overline{w}_4(q) = \frac{exp\left(-y\sqrt{\frac{a_1 q(q+a_2)}{q+b}}\right)}{\frac{a_1 q(q+a_2)}{q+b}}. \tag{53}$$

The inverse Laplace of Eq. (52) is

$$w_3(t) = \delta(t) + (a_2 - b)exp(-bt), \tag{54}$$

and

$$w_4(t) = e^{(b-a_2)t} erfc\left(\frac{y\sqrt{a_1}}{2\sqrt{t}}\right) - \sqrt{a_1 b(b-a_2)}e^{-bt}\int_0^\infty \sqrt{\frac{u}{t-a_1 u}} e^{(2a_1 b - a_2)u}$$
$$erfc\left(\frac{y\sqrt{a_1}}{2\sqrt{u}}\right) J_1\left(2\sqrt{a_1 b(b-a_2)u(t-u)}\right) du, \tag{55}$$

so, finally

$$x_2(y,t) = \int_0^t w_4(t) + (a_2 - b)e^{-b(t-\tau)} w_4(\tau) d\tau. \tag{56}$$

The solution of Eq. (45) by using (A.7) is given as

$$x_3(y,t) = 1 - \frac{2b}{\pi}\int_0^\infty \frac{sin\left(y\sqrt{\frac{a}{b}}x\right)}{x(b+x^2)} exp\left(\frac{-bx^2 t}{b+x^2}\right) dx. \tag{57}$$

## 5.3 For $(\alpha = 1)$

For $\alpha = 1$, the Maxwell fluid velocity given in Eq. (30) and Eq. (40) obtained by the Caputo and CF time-fractional derivatives is given as

$$\bar{u}(y,q) = \frac{-Gr}{\sqrt{Pr}} \frac{q+A}{q(q+B)} . \frac{1}{\sqrt{q}-C} . \left[\frac{1}{q}.exp\left(-y\sqrt{q(1+\lambda q)}\right) - \frac{1}{q}.exp\left(-y\sqrt{Prq}\right)\right], \tag{58}$$

where

$$A = \frac{1}{\lambda}, \qquad B = \frac{1-Pr}{\lambda}, \qquad C = \frac{1}{\sqrt{Pr}}. \tag{59}$$

Eq. (58) can be written as

$$u(y,t) = u_{11}(t) * [u_{22}(y,t) - u_{33}(y,t)], \tag{60}$$

where $*$ represents the convolution product.

$$u_{11}(t) = \frac{-Gr}{\sqrt{Pr}} \int_0^t \left[ \frac{1}{\sqrt{t-\tau}} \mathbf{E}_{1/2,1/2} \left( C(t-\tau)^{1/2} \right) \right] . \left( \frac{A}{B} - \frac{B-A}{B} exp(-B\tau) d\tau \right). \quad (61)$$

We can write $\overline{u}_{22}(y,q)$ in simplest form as

$$\overline{u}_{22}(y,q) = exp \left[ -y\sqrt{\lambda} \sqrt{ \left( q + \frac{1}{2\lambda} \right)^2 - \left( \frac{1}{2\lambda} \right)^2 } \right] \quad (62)$$

The inverse Laplace of Eq. (62) and $\overline{u}_{33}(y,q)$ is

$$\begin{aligned} u_{22}(y,t) = \frac{1}{2\lambda} exp \left[ -\frac{1}{2\lambda} t \right] \int_0^t \frac{y\sqrt{\lambda}}{2q\sqrt{\pi q}} exp \left[ -\frac{y^2\lambda}{4q} \right] \frac{q}{\sqrt{t^2-q^2}} I_1 \left[ \frac{1}{2\lambda} \sqrt{t^2-q^2} \right] dq \\ + \frac{y\sqrt{\lambda}}{2t\sqrt{\pi t}} exp \left[ -\frac{y^2\lambda}{4t} - \frac{1}{2\lambda} t \right] \end{aligned} \quad (63)$$

and

$$u_{33}(y,t) = erfc \left( \frac{y\sqrt{Pr}}{2\sqrt{t}} \right). \quad (64)$$

# 6 Numerical discussion and results

In this paper, we have discussed the free convection flow of Maxwell fluid with Newtonian heating near a vertical plate for time greater than zero. Two different non-integer order definitions are used, namely, Caputo and CF definition. The differential model is summed up to the fractional model. To observe the physical appearance of the problem, graphical illustration and numerical outcome are plotted. An important fact of the model is how the temperature and velocity fields are effected by time and fractional parameter. These hypothetical results can be valuable for some practical issues. We also interested in comparing the results of Caputo and CF time-fractional models. In Fig. 2, we show the influence of time by taking different values of non-integer order $\alpha$. We can see from Fig. 2 that when the value of time $t$ is greater, the Caputo fractional model has less temperature as compared to CF fractional model.

Furthermore, the thermal boundary layer thickness becomes greater for greater time. Due to this, the temperature also increases. The influence of $Pr$ is observed in Fig. 3. It is noted that by increasing $Pr$, the temperature decreases. Physically, for greater $Pr$ input, fluid have greater

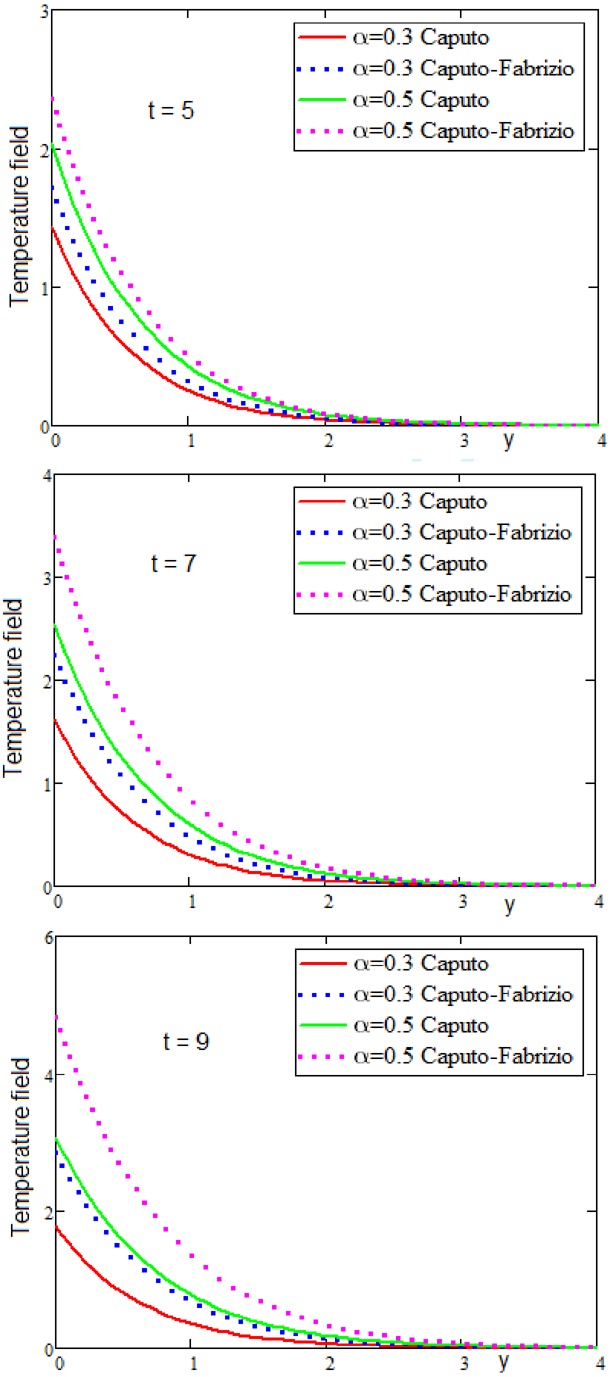


Fig: 2 Temperature profiles for $Pr = 5$ and multiple values of time.

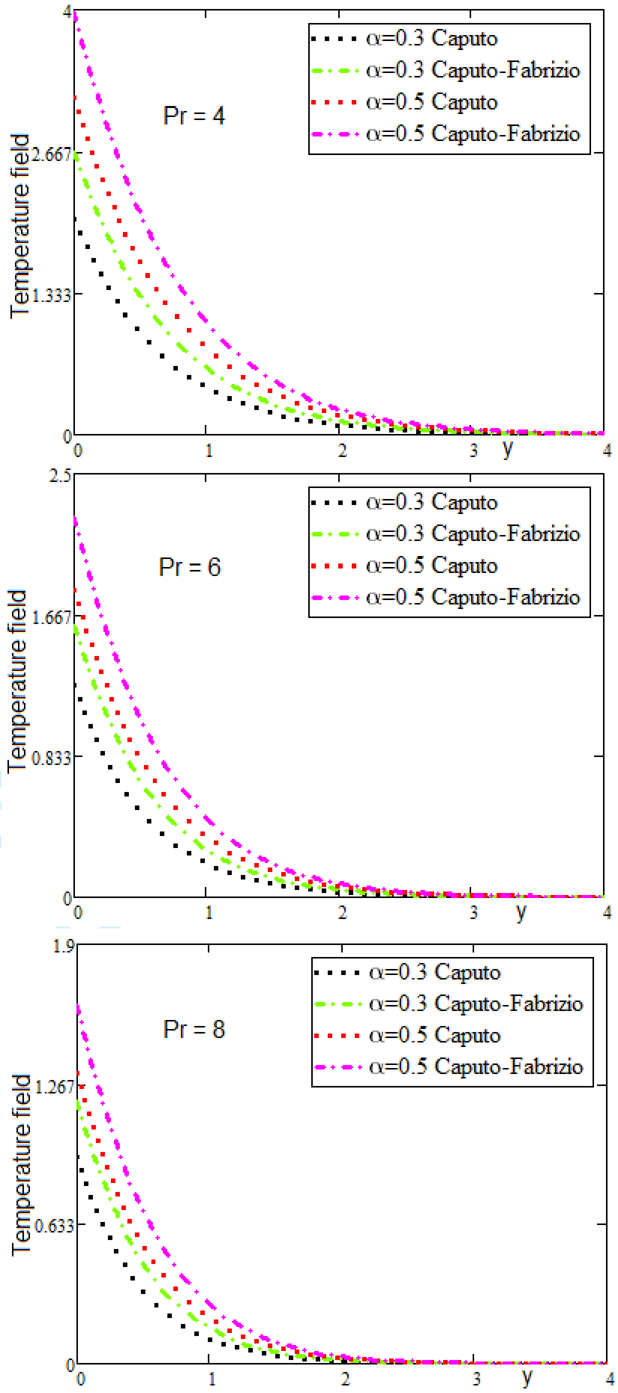


Fig: 3 Temperature profiles for $t = 6$ and multiple values of $Pr$.

viscosity but lower thermal conductivity. Due to this reason, boundary layer thickness decreases. The heat transfer from the plate to the fluid corresponding to Nusselt number with multiple values of $Pr$ is shown in Fig. 4. It is discovered that heat transfer rate of CF model is higher as compared to Caputo model, but value of the Nusselt number decreases when $Pr$ increases. A comparison between temperature field corresponding to Caputo derivative, CF derivative and ordinary case is shown in Fig. 5. We observed that temperature of the ordinary fluid is less than Caputo and CF derivative for time $t = 1$. But when we increase the time $t = 2$, then the temperature of ordinary fluid also increases.

In Fig. 6, we presented the influence of time on velocity field for various values of $\alpha$, and one can see that for smaller values of time, the velocity of CF model has greater value than Caputo model. We noted that CF fluid velocity increases and become greatest near the plate for greater time domain. In Fig. 7, we displayed the effect of the Maxwell fluid parameter on the fluid velocity. It clearly appears that the size of velocity for both fractional models is an increasing function of the Maxwell fluid parameter. The impact of $Gr$ and $Pr$ on the velocity field are depicted in Figs. 8 and 9 respectively. It is noted that for large values of $Gr$, the fluid velocity for both models increases, but the fluid velocity for both models decreases for large values of $Pr$. Fig. 10 is plotted to show the comparison between ordinary, Caputo and CF velocity profiles. We found that ordinary fluid has higher velocity than Caputo and CF for greater value of time.

In order to highlight the differences between both models, we must analyze the evolution of the Caputo kernel $k_1(\alpha, t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, and Caputo-Fabrizio kernel $k_2(\alpha, t) = \frac{1}{1-\alpha} exp\left(\frac{-\alpha t}{1-\alpha}\right), \alpha \in [0, 1]$. Fig. 11 shows the curves $k_1(\alpha, 5), k_2(\alpha, 5)$ and $k_1(\alpha, 7), k_2(\alpha, 7)$. It is observed from these graphs that values of Caputo kernel are bigger than values of Caputo-Fabrizio kernel, therefore the values of weighted function of the thermal flux are bigger for the Caputo derivative than Caputo-Fabrizio derivative. Physically, in the case of Caputo derivative, the heat transfer is stronger attenuated than for Caputo-Fabrizio derivatives. From the results, the values of fluid temperature are lower for the Caputo derivative case. This aspect can clearly be observed from Figs 2 and 3. Similar behavior can also be seen for the velocity field as well.

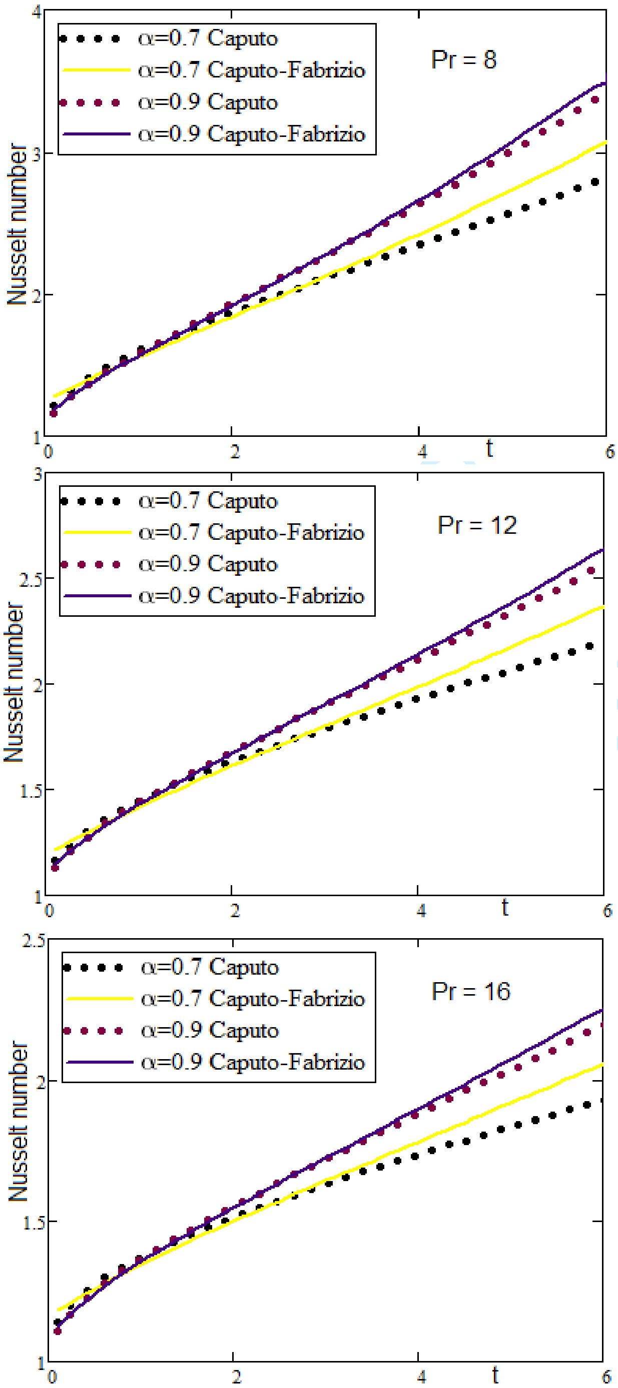


Fig: 4 Nusselt number profiles given by Eqs. (18) and (23) for multiple values of $Pr$.

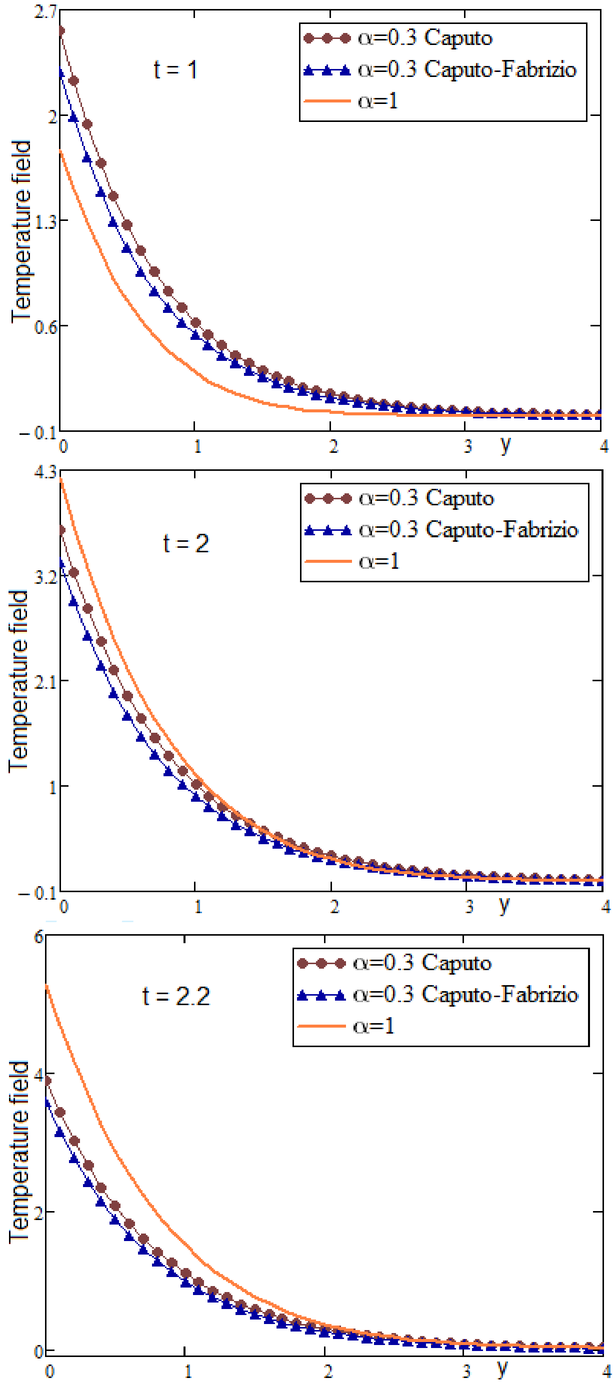


Fig: 5 Comparison between ordinary, Caputo and CF temperature profiles for $Pr = 2$ and multiple values of time.

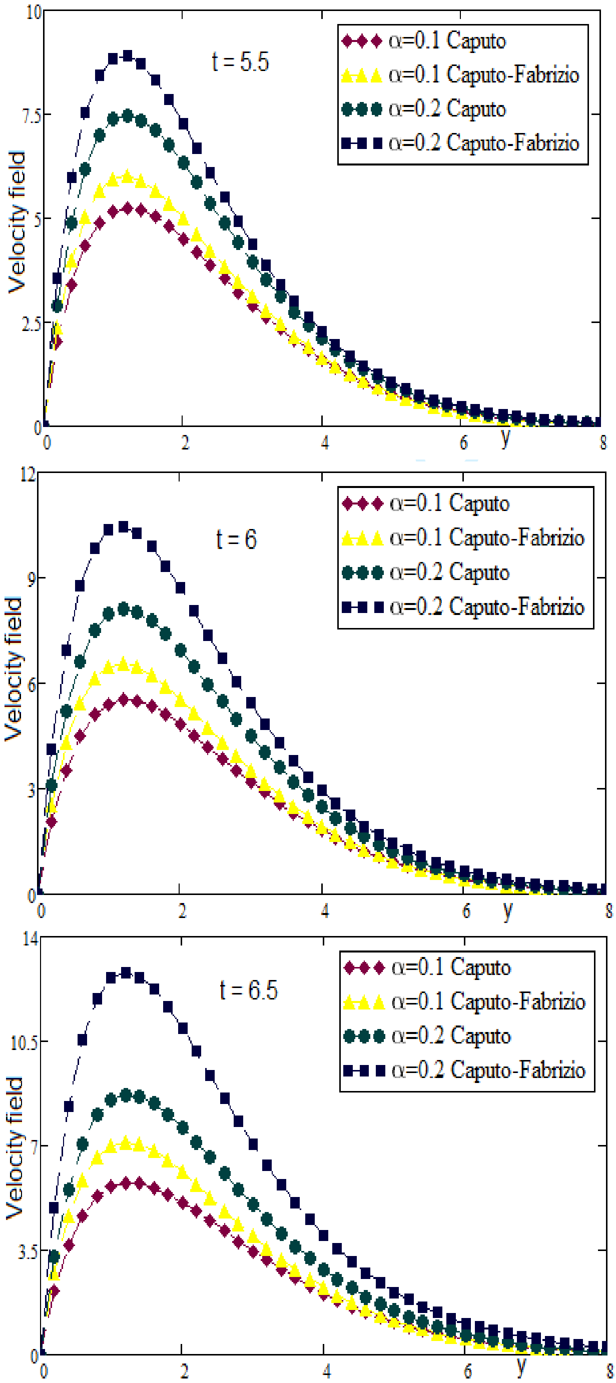


Fig: 6 Velocity profiles for $Gr = 4$, $Pr = 2$, $\lambda = 0.5$ and multiple values of time.

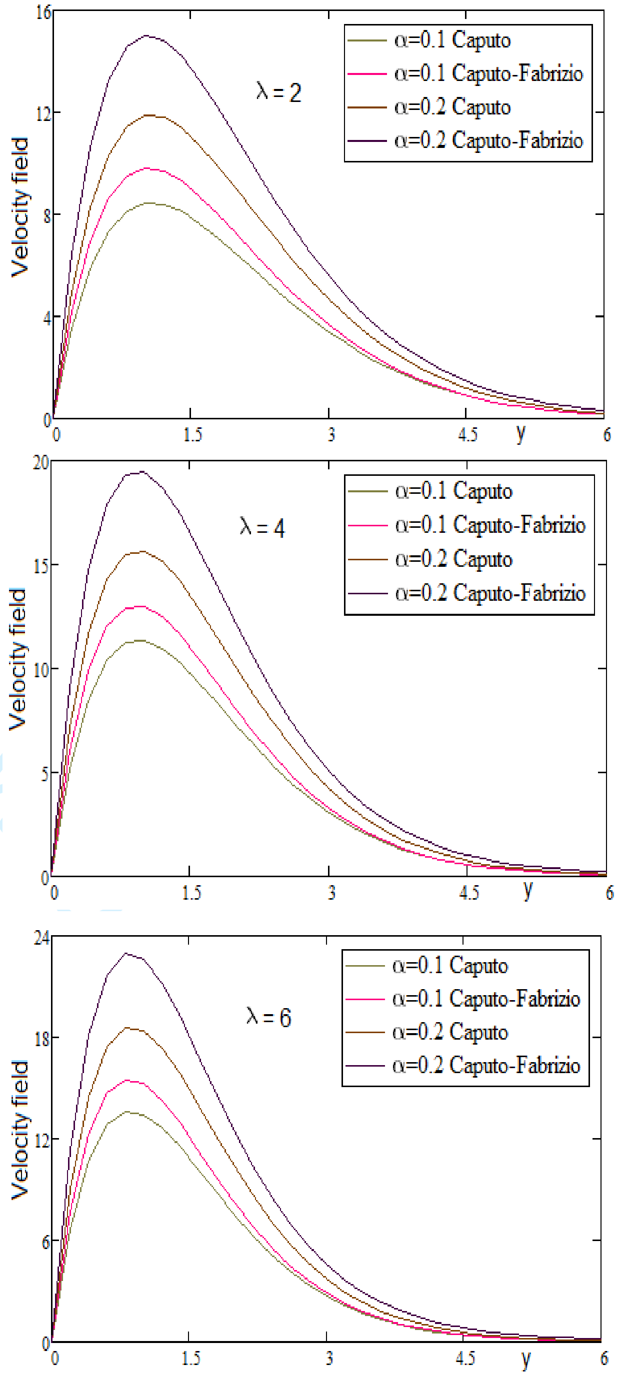


Fig: 7 Velocity profiles for $Gr = 4$, $Pr = 2$, $t = 6$ and multiple values of $\lambda$.

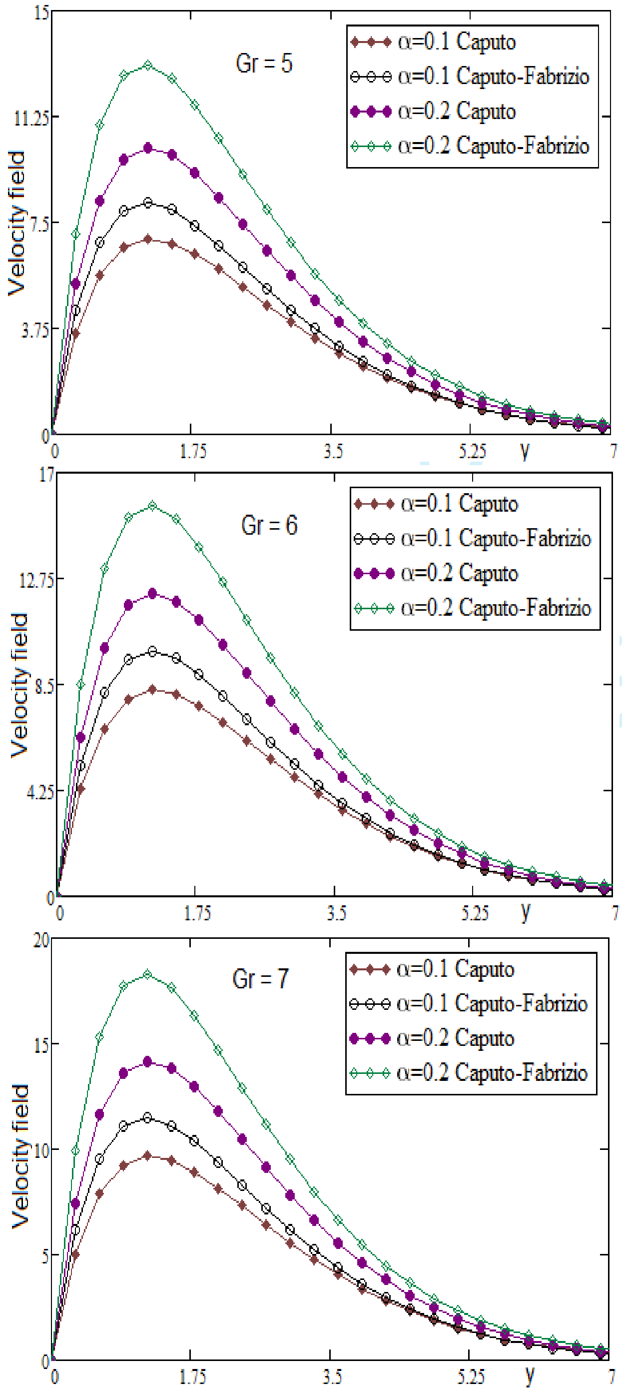


Fig: 8 Velocity profiles for $t = 6$, $Pr = 2$, $\lambda = 0.5$ and multiple values of $Gr$.

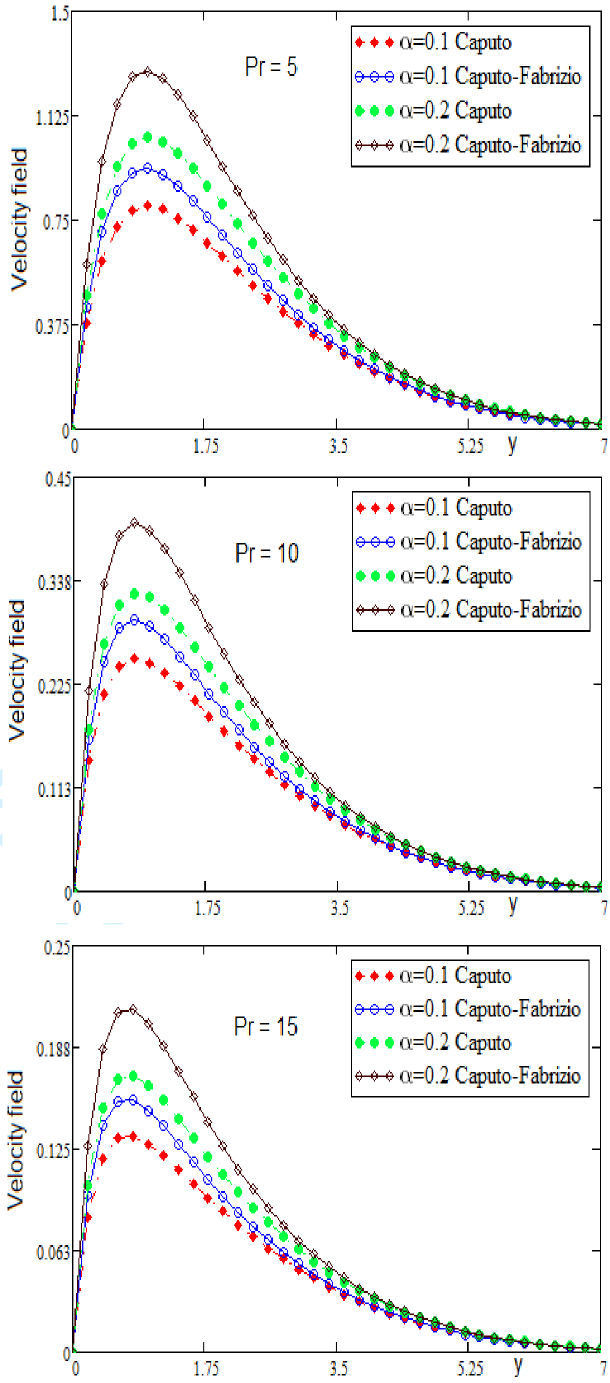


Fig: 9 Velocity profiles for $Gr = 4$, $\lambda = 0.5$, $t = 6$ and multiple values of $Pr$.

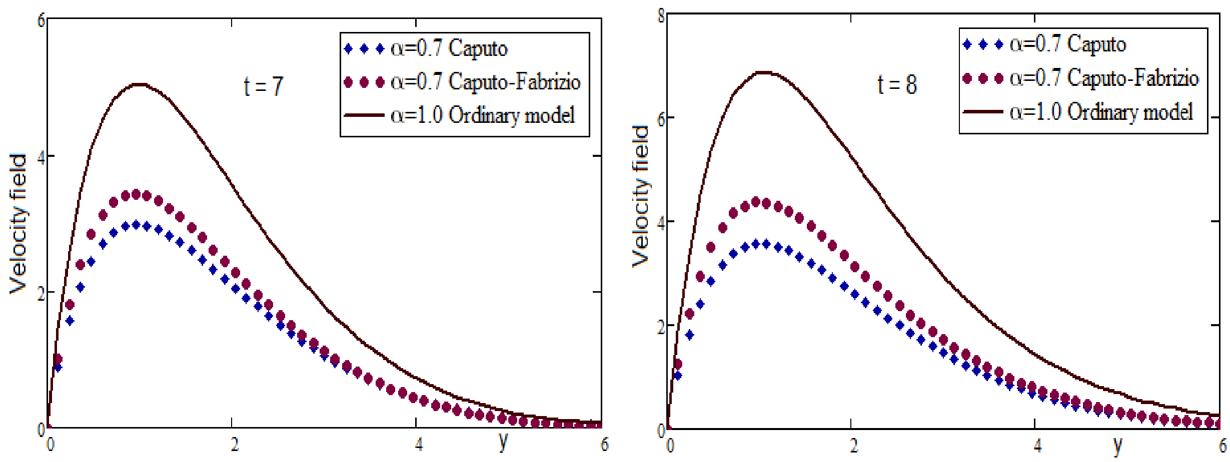


Fig: 10 Comparison between ordinary, Caputo and Caputo-Fabrizio velocity profiles for $\lambda = 1.5$, $Gr = 6$, $Pr = 8$ and different values of time.

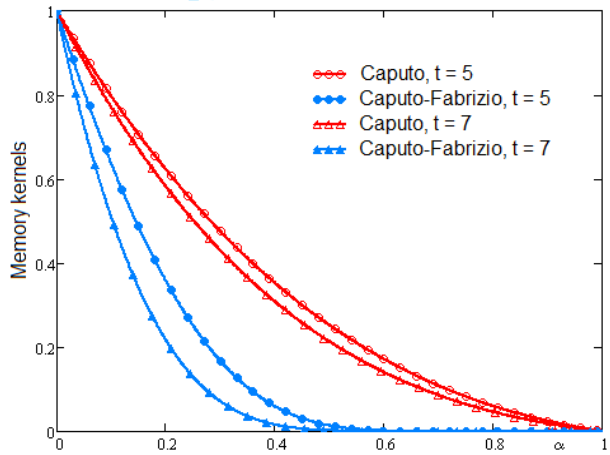


Fig: 11 Variation with alpha of the Caputo kernel $k_1(\alpha, t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ and Caputo-Fabrizio kernel $k_2(\alpha, t) = \frac{1}{1-\alpha} exp\left(\frac{-\alpha t}{1-\alpha}\right)$.

The effect of $Pr$ and time with respect to $\alpha$ on Nusselt number is brought into light in Table 1. It is noted that the heat transfer rate from plate to the fluid is greater for Caputo model when $t = 1.3$. We observe that for higher values of time, flow of heat transfer for CF model increases to attain maximum value. In Table 2, some numerical calculations for the Nusselt number and temperature field are carried out by Stehfest's and Tzou's algorithms. In Tables 3 and 4, we consider the fixed values of $\alpha$ to discuss the impact of Maxwell fluid parameter $\lambda$ and time $t$. It is noticed that for $\alpha = 0.2$, the CF fractional model has less velocity than Caputo model. To investigate the influence of the fractional parameter, we put $\alpha = 0.6$ and time $t = 5$ and found that velocity of CF model

Table 1: Values of the Nusselt number for variation of $Pr$ and time

| $Pr$ | $t = 1.3,\ \alpha = 0.3$ | | $t = 3.5,\ \alpha = 0.3$ | | $t = 5.3,\ \alpha = 0.3$ | |
|---|---|---|---|---|---|---|
| | $Nu.$ (18) | $Nu.$ (23) | $Nu.$ (18) | $Nu.$ (23) | $Nu.$ (18) | $Nu.$ (23) |
| 2 | 3.909 | 3.583 | 6.296 | 6.450 | 8.382 | 15.945 |
| 3 | 2.631 | 2.471 | 3.444 | 3.570 | 3.991 | 4.324 |
| 4 | 2.181 | 2.075 | 2.649 | 2.747 | 2.935 | 3.382 |
| 5 | 1.948 | 1.868 | 2.277 | 2.356 | 2.467 | 2.794 |
| 6 | 1.804 | 1.739 | 2.058 | 2.12565 | 2.201 | 2.462 |
| 7 | 1.705 | 1.651 | 1.914 | 1.973 | 2.028 | 2.246 |
| 8 | 1.633 | 1.585 | 1.811 | 1.863 | 1.906 | 2.095 |
| 9 | 1.577 | 1.534 | 1.733 | 1.780 | 1.815 | 1.983 |
| 10 | 1.533 | 1.494 | 1.672 | 1.715 | 1.745 | 1.896 |
| 11 | 1.496 | 1.461 | 1.622 | 1.662 | 1.688 | 1.825 |
| 12 | 1.466 | 1.433 | 1.581 | 1.619 | 1.641 | 1.7685 |
| 13 | 1.441 | 1.409 | 1.547 | 1.582 | 1.602 | 1.719 |
| 14 | 1.417 | 1.389 | 1.517011 | 1.55033 | 1.568 | 1.678 |
| 15 | 1.398 | 1.371 | 1.491 | 1.524 | 1.539 | 1.643 |
| 16 | 1.381 | 1.355 | 1.481 | 1.451 | 1.514 | 1.612 |



Table 2: Numerical values of temperature and Nusselt number by Stehfest's and Tzou's

| | Caputo | | Caputo Fabrizio | | Caputo | | Caputo Fabriziio | |
|---|---|---|---|---|---|---|---|---|
| $Pr$ | $T(y,t)$ [Stehfest's] | $T(y,t)$ [Tzou's] | $T(y,t)$ [Stehfest's] | $T(y,t)$ [Tzou's] | $Nu$ [Stehfest's] | $Nu$ [Tzou's] | $Nu$ [Stehfest's] | $Nu$ [Tzou's] |
| 2 | 0.971733 | 0.966405 | 1.202298 | 1.202888 | 4.518931 | 4.502033 | 5.189606 | 5.19253 |
| 4 | 0.203403 | 0.202029 | 0.248277 | 0.248384 | 2.243233 | 2.237295 | 2.408415 | 2.40989 |
| 6 | 0.09037 | 0.089749 | 0.110854 | 0.110901 | 1.828623 | 1.825026 | 1.924897 | 1.926132 |
| 8 | 0.050048 | 0.049709 | 0.061813 | 0.061838 | 1.646575 | 1.644002 | 1.716404 | 1.717533 |
| 10 | 0.030981 | 0.030777 | 0.038518 | 0.038534 | 1.54167 | 1.539679 | 1.59747 | 1.598537 |

Table 3: Numerical values of velocity for multiple values of $\lambda$ and $t$

| $\lambda$ | $t = 1,\ \alpha = 0.2$ | | $t = 3,\ \alpha = 0.2$ | | $t = 5,\ \alpha = 0.2$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) |
| 1 | 0.497 | 0.475 | 1.206 | 1.175601 | 1.643 | 1.847 |
| 2 | 0.547 | 0.527 | 1.515 | 1.474 | 2.116 | 2.355 |
| 3 | 0.559 | 0.544 | 1.750 | 1.697 | 2.514 | 2.781 |
| 4 | 0.554 | 0.544 | 1.929 | 1.864 | 2.851 | 3.14 |
| 5 | 0.542 | 0.537 | 2.067 | 1.989 | 3.141 | 3.445 |
| 6 | 0.528 | 0.528 | 2.173 | 2.084 | 3.389 | 3.704 |
| 7 | 0.515 | 0.517 | 2.254 | 2.155 | 3.604 | 3.924 |
| 8 | 0.502 | 0.507 | 2.315 | 2.207 | 3.789 | 4.113 |
| 9 | 0.490 | 0.498 | 2.360 | 2.244 | 3.950 | 4.274 |
| 10 | 0.4808 | 0.488 | 2.393 | 2.271 | 4.089 | 4.411 |
| 11 | 0.472 | 0.480 | 2.415 | 2.288 | 4.209 | 4.529 |
| 12 | 0.464 | 0.472 | 2.430 | 2.298 | 4.314 | 4.629 |
| 13 | 0.458 | 0.466 | 2.438 | 2.303 | 4.404 | 4.714 |
| 14 | 0.453 | 0.460 | 2.441 | 2.304 | 4.482 | 4.786 |
| 15 | 0.447 | 0.455 | 2.440 | 2.301 | 4.549 | 4.847 |

has become greater. We determined the influence of time by considering various values of $\lambda$ for $\alpha = 0.2, 0.6$ on Caputo and CF velocities in Tables 5 and 6 respectively. When the value of time is small, the value of CF velocity is also small as compare to Caputo velocity. In Tables 7 and 8, we discussed the impact of $y$ and time on velocities. It is noted that the velocities reduce with increasing $y$. We use Tzou's and Stehfest's algorithms to obtain the numerical values for fluid velocities in Table 9.

Table 4: Numerical values of velocity for multiple values of $\lambda$ and $t$

| $\lambda$ | $t = 1,\ \alpha = 0.6$ | | $t = 3,\ \alpha = 0.6$ | | $t = 5,\ \alpha = 0.6$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) |
| 1 | 0.351 | 0.341 | 1.730 | 1.652 | 3.266 | 3.506 |
| 2 | 0.347 | 0.353 | 2.142 | 2.049 | 4.014 | 4.313 |
| 3 | 0.311 | 0.340 | 2.463 | 2.343 | 4.672 | 5.022 |
| 4 | 0.278 | 0.322 | 2.710 | 2.557 | 5.250 | 5.643 |
| 5 | 0.254 | 0.305 | 2.896 | 2.708 | 5.761 | 6.185 |
| 6 | 0.240 | 0.291 | 3.031 | 2.811 | 6.210 | 6.655 |
| 7 | 0.230 | 0.279 | 3.127 | 2.878 | 6.607 | 7.063 |
| 8 | 0.224 | 0.270 | 3.190 | 2.917 | 6.957 | 7.415 |
| 9 | 0.219 | 0.263 | 3.227 | 2.936 | 7.265 | 7.718 |
| 10 | 0.215 | 0.257 | 3.244 | 2.940 | 7.535 | 7.977 |
| 11 | 0.212 | 0.252 | 3.245 | 2.932 | 7.772 | 8.197 |
| 12 | 0.209 | 0.248 | 3.234 | 2.918 | 7.979 | 8.384 |
| 13 | 0.207 | 0.244 | 3.214 | 2.897 | 8.159 | 8.541 |
| 14 | 0.204 | 0.241 | 3.188 | 2.873 | 8.315 | 8.671 |
| 15 | 0.203 | 0.239 | 3.158 | 2.847 | 8.449 | 8.778 |



Numerical results are obtained by using 2.0 GHz AMD A6 machine with 4.0 GB RAM. Tzou's and Stehfest's algorithms have been implemented through MATHCAD software. For convergence, 200 terms of the series of algorithms are considered to obtain the inverse Laplace transform. Time taken by CPU is approximately 0.6 sec. The CPU time also depends on the complexity of problems and number of terms considered in algorithms used to obtain inverse Laplace transform.

# 7 Conclusion

The free convection flow of a Maxwell fluid modelled with Caputo and CF differential operators with Newtonian heating is investigated. Exact and numerical calculations of the temperature and velocity fields have been computed using Caputo, CF and for fractional parameter $\alpha = 1$ as a special case. Numerical computations are made for Maxwell fluid parameter, $Pr$, $Gr$ and time-fractional parameter by varying time.

Table 5: Numerical values of velocity multiple values of $t$ and $\lambda$

| $t$ | $\lambda = 2,\ \alpha = 0.6$ | | $\lambda = 4,\ \alpha = 0.6$ | | $\lambda = 6,\ \alpha = 0.6$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) |
| 1 | 0.347 | 0.353 | 0.278 | 0.322 | 0.240 | 0.291 |
| 2 | 1.238 | 1.116 | 1.428 | 1.262 | 1.439 | 1.272 |
| 3 | 2.142 | 2.049 | 2.710 | 2.557 | 3.031 | 2.811 |
| 4 | 3.062 | 3.110 | 3.976 | 4.029 | 4.625 | 4.644 |
| 5 | 4.014 | 4.313 | 5.250 | 5.643 | 6.210 | 6.655 |
| 6 | 5.006 | 5.661 | 6.552 | 7.398 | 7.812 | 8.811 |
| 7 | 6.044 | 7.338 | 7.893 | 9.516 | 9.448 | 11.358 |
| 8 | 7.134 | 9.189 | 9.286 | 11.82 | 11.358 | 14.099 |
| 9 | 8.280 | 11.575 | 10.736 | 14.743 | 12.875 | 17.531 |
| 10 | 9.485 | 15.309 | 12.250 | 19.247 | 14.686 | 22.756 |



Table 6: Numerical values of velocity for multiple values of $t$ and $\lambda$

| $t$ | $\lambda = 2,\ \alpha = 0.8$ | | $\lambda = 4,\ \alpha = 0.8$ | | $\lambda = 6,\ \alpha = 0.8$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$(46) |
| 1 | 0.218 | 0.247 | 0.143 | 0.196 | 0.132 | 0.170 |
| 2 | 1.248 | 1.158 | 1.396 | 1.259 | 1.317 | 1.201 |
| 3 | 2.388 | 2.328 | 3.043 | 2.932 | 3.398 | 3.211 |
| 4 | 3.674 | 3.691 | 4.769 | 4.806 | 5.569 | 5.586 |
| 5 | 5.134 | 4.736 | 6.654 | 6.233 | 7.869 | 7.412 |
| 6 | 6.774 | 7.353 | 8.724 | 9.460 | 10.351 | 11.229 |
| 7 | 8.796 | 10.007 | 11.227 | 12.705 | 13.309 | 15.036 |
| 8 | 11.011 | 14.913 | 13.944 | 18.579 | 16.499 | 21.807 |
| 9 | 13.827 | 35.033 | 17.359 | 42.308 | 20.477 | 48.841 |

Table 7: Numerical values of velocity for fixed $\alpha = 0.6$

| $y$ | $t = 8,\ \alpha = 0.6$ | | $t = 10,\ \alpha = 0.6$ | | $t = 12,\ \alpha = 0.6$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 9.286 | 11.820 | 12.25 | 19.247 | 15.484 | 79.960 |
| 2 | 6.418 | 8.067 | 9.432 | 15.283 | 12.793 | 77.020 |
| 3 | 2.824 | 3.350 | 4.994 | 8.435 | 7.556 | 56.577 |
| 4 | 0.816 | 0.912 | 1.995 | 3.744 | 3.608 | 37.642 |
| 5 | 0.134 | 0.200 | 0.568 | 1.522 | 1.370 | 24.148 |
| 6 | $9.575\ .\ 10^{-3}$ | 0.077 | 0.099 | 0.726 | 0.384 | 15.493 |
| 7 | $2.701\ .\ 10^{-3}$ | 0.046 | $7.32\ .\ 10^{-3}$ | 0.446 | 0.066 | 10.09 |



Table 8: Variation of time for fixed $\alpha = 0.6$ on velocity

| $y$ | $\lambda = 2,\ \alpha = 0.6$ | | $\lambda = 4,\ \alpha = 0.6$ | | $\lambda = 6,\ \alpha = 0.6$ | |
|---|---|---|---|---|---|---|
| | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) | $u(y,t)$ (35) | $u(y,t)$ (46) |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 9.485 | 15.309 | 12.250 | 19.247 | 14.686 | 22.756 |
| 2 | 7.805 | 12.839 | 9.432 | 15.283 | 10.559 | 17.039 |
| 3 | 4.726 | 7.973 | 4.994 | 8.435 | 4.827 | 8.234 |
| 4 | 2.425 | 4.319 | 1.995 | 3.744 | 1.451 | 3.040 |
| 5 | 1.066 | 2.157 | 0.568 | 1.522 | 0.252 | 1.195 |
| 6 | 0.390 | 1.055 | 0.099 | 0.726 | 0.022 | 0.671 |
| 7 | 0.112 | 0.552 | $7.32\ .\ 10^{-3}$ | 0.446 | $5.936\ .\ 10{-}3$ | 0.422 |
| 8 | 0.022 | 0.330 | $2.002\ .\ 10^{-3}$ | 0.294 | $2.284\ .\ 10^{-3}$ | 0.254 |

Table 9: Numerical values of Caputo an Caputo-Fabrizio velocities by Stehfest's and Tzou's

| | Caputo | | Caputo Fabrizio | |
|---|---|---|---|---|
| y | [Stehfest's] | [Tzou's] | [Stehfest's] | [Tzou's] |
| 0.2 | 2.7036 | 2.70426 | 2.91354 | 2.91416 |
| 0.4 | 4.31227 | 4.31341 | 4.66352 | 4.66569 |
| 0.6 | 5.12707 | 5.12852 | 5.55074 | 5.55385 |
| 0.8 | 5.38045 | 5.38214 | 5.81438 | 5.81809 |
| 1 | 5.25044 | 5.25222 | 5.64345 | 5.64759 |
| 1.2 | 4.87213 | 4.87377 | 5.18695 | 5.19136 |
| 1.4 | 4.34674 | 4.34812 | 4.56216 | 4.56658 |
| 1.6 | 3.74912 | 3.75048 | 3.86073 | 3.86478 |
| 1.8 | 3.13399 | 3.13583 | 3.15246 | 3.1558 |
| 2 | 2.54085 | 2.54338 | 2.4876 | 2.49019 |

A few perceptions and final comments are given as follows:

- The CF model has less temperature than Caputo model for small time domain.
- For greater values of $Pr$, temperature of both types decreases.
- Fluid velocity becomes a decreasing function of $Pr$ and leads to slower fluid flow.
- The fluid velocity increases with respect to $Gr$, $\alpha$ and Maxwell fluid parameter $\lambda$.
- Velocity of Caputo model is higher than CF for small time domain, while velocity of CF model increases instead of Caputo model on increasing time domain.

**AppendixA.**

$$If\ L^{-1}[G(q)] = g(t)\ and\ h(u,t) = L^{-1}[e^{-uw(q)}]\ then$$
$$L^{-1}\{G[w(q)]\} = \int_0^\infty g(u)h(u,t)du, \qquad (A.1)$$

$$L^{-1}\left\{\frac{e^{-aq^b}}{q^c}\right\} = t^{c-1}\phi\left(c,-b,-at^{-b}\right); \phi(x,y,z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)\Gamma(x+ny)};\quad 0<b<1 \qquad (A.2)$$

$$\mathbf{G}_{a,b,c}(d,t) = L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = \sum_{n=0}^{\infty}\frac{d^n\Gamma(n+c)t^{(n+c)a-b-1}}{\Gamma(c)\Gamma(n+1)\Gamma[(n+c)a-b]}$$
$$if\ Re(ac-b)>0, Re(q)>0\ and\ \mid d\mid<\mid q^\alpha\mid \qquad (A.3)$$

$$L^{-1}\left\{\frac{q^{a-b}}{q^a+c}\right\} = t^{b-1}\mathbf{E}_{a,b}\left(-ct^a\right); \mathbf{E}_{a,b}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)};\quad a>0,\ b>0 \qquad (A.4)$$

$$L^{-1}\left\{\frac{1}{\sqrt{q}+m}\right\} = \frac{1}{\sqrt{\pi t}} - me^{m^2t}erfc(m\sqrt{t}) \qquad (A.5)$$

$$L^{-1}\left\{exp\left(\frac{a}{q+b}\right)-1\right\} = \sqrt{\frac{a}{t}}I_1(2\sqrt{at}).e^{-bt} \qquad (A.6)$$

$$L^{-1}\left\{\frac{1}{q}\ exp\left(-a\sqrt{\frac{\lambda q}{q+\lambda}}\right)\right\} = 1-\frac{2\lambda}{\pi}\int_0^\infty\frac{sin(ax)}{x(\lambda+x^2)}e^{\frac{-\lambda x^2 t}{\lambda+x^2}}dx \qquad (A.7)$$

Distribution function:

$$\delta(t) = \begin{cases} 0 & t\neq 0 \\ \infty & \text{t} = 0 \end{cases}$$

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