

# A Comparison of Explicit Semi-Analytical Numerical Integration Methods for Solving Stiff ODE Systems

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**Abstract:** In this study, a comparison among three semi-analytical numerical integration algorithms for solving stiff ODE systems is presented. The algorithms are based on Differential Transform Method (DTM) which are Multiple-Step DTM (MsDTM), Enhanced MsDTM (E-MsDTM) and MsDTM with Padé approximants (MsDTM-P). These methods can be classified as explicit one step semi-analytical numerical integration methods. The error and stability analysis of each method is presented. New important relationships among the methods are introduced. To demonstrate our results, a comparison of the accuracy, stability and computational efficiency of the methods is presented through solving some linear and nonlinear problems arising in applied science and engineering.

**Keywords:** Multiple-Step Differential Transform Methods (MsDTM), Enhanced MsDTM, Padé Approximants, Stiff Problems, Accuracy, Stability

## Introduction

Many mathematical problems arising from the real world cannot be solved completely by analytical means. One of the most important mathematical problems arising in applied science and engineering is stiff ODE systems. Stiff ODE systems, which have significantly different timescales of change, occur in many fields of engineering particularly in the studies of electrical circuits, chemical and biochemical reactions, optimal control, vibrations and fluid mechanics; see for example (Schiesser, 1993; El-Zahar, 2015). The problem of stiffness has been known for some time and has attracted the attention of many numerical analysts leading to surveys of methods for stiff problems. It is well known that explicit numerical methods are intrinsically faster than commonly used implicit methods while implicit methods are commonly used for solving stiff problems because of their stability (Lambert, 1991; Hairer and Wanner, 1996; Butcher, 2008). In fact, the explicit numerical methods which recently are presented in (Wu, 1998; Wu and Xia, 2001; 2007; Novati, 2003; Ahmad *et al.*, 2004; Ahmad and Yaacob, 2005; Ebady *et al.*, 2012; Egbako and Adeboye, 2012; El-Zahar, 2013a; El-Zahar *et al.*, 2014a) can also satisfy the stiff problems.

Recently, semi-analytical numerical methods such as Homotopy Perturbation Method (HPM), Variational

Iteration Method (VIM), Differential Transform Method (DTM) and Adomian Decomposition Method (ADM), are applied for finding approximate analytical solutions for a wide classes of nonlinear problems. Solving stiff problems is one of the most recent applications of these methods; see for example (Mahmood *et al.*, 2005; Guzel and Bayram, 2005; Darvishi *et al.*, 2007; Hassan, 2008; Zhao and Xiao, 2010; Aminikhah and Hemmatnezhad, 2011; Aminikhah, 2012; Zou *et al.*, 2012; Atay and Kilic, 2013; Zhao *et al.*, 2014; El-Zahar *et al.*, 2014b). However, for some important classes of problems such as stiff ODE problems, singularly perturbed problems, chaotic and non-chaotic problems and nonlinear oscillators and for the sake of large convergence region, accuracy and efficiency, it is necessary to treat each of the above mentioned semi-analytical numerical methods as an algorithm in a sequence of time intervals. Therefore, different multiple-step HPM, DTM, ADM and VIM are presented for solving many important classes of problems; see for example (Adomian *et al.*, 1988; Mahmood *et al.*, 2005; Ghosh *et al.*, 2007; Goh *et al.*, 2008; Chowdhury *et al.*, 2009; 2012; 2014; Gonzalez-Parra *et al.*, 2009; Alomari *et al.*, 2010; Wang *et al.*, 2010; Odibat *et al.*, 2010; Molli *et al.*, 2013; Keimanesh *et al.*, 2011; Gokdogan *et al.*, 2012a; 2012b; Yildirim *et al.*, 2012; Erturk *et al.*, 2012; El-Zahar 2012a; 2012b; Do and

Jang, 2012; Alquran and Al-Khaled, 2012; Lee *et al.*, 2013; Patra and Ray, 2013; Smarda *et al.*, 2013; Heydari *et al.*, 2014; Momani *et al.*, 2014; Olvera *et al.*, 2014; El-Zahar, 2015).

Although the treatment of HPM, ADM and VIM as multiple-step methods speeds up the convergence rate and improves the accuracy, these methods have some limitations which hamper their applications.

An analytical integration is required for each step, each solution series term in HPM and ADM and each iteration in VIM and so some difficulties may arise as follows:

- The nonlinear and non-homogeneous parts of the ODE may be ill-conditioned such that the integration became very complicated
- By increasing the number of iterations or the number of solution series terms, the number of terms of approximate solution may increase, so rapidly that the integration becomes both complicated and time consuming
- "In VIM for some differential equations, we may have more than one Lagrange multiplier. Let  $L_1$  and  $L_2$  be different Lagrange multipliers for an ODE. We assume that  $L_1$  is more accurate than  $L_2$  and then by using  $L_1$  the more fast the approximation to the exact solution can be obtained, but the computation will be complicated. The mentioned notation for  $L_2$  is totally opposite" (Heydari *et al.*, 2013; 2014)
- In ADM, for nonlinear problems, we need to evaluate the Adomian polynomials that mostly require tedious algebraic calculations
- In HPM, we should suitably choose an initial guess, or infinite iterations are required

Due to the above mentioned difficulties, the Multiple-step DTM (MsDTM), which does not require analytical integration, evaluation of the Lagrangian multiplier, or rather difficult computation for finding the Adomian polynomials, is one of the most effective multiple-step semi-analytical numerical methods.

EL-Zahar, (2015) presented an adaptive step size MsDTM for solving singularly perturbed ODE systems, also known as stiff ODE systems. The results showed that the proposed method is an accurate and efficient method compared to classical MsDTM and RK4 method. However, for large values of the admissible local error or when a fixed step size is used, the problem of stability arises. A studying of this obstacle and how to overcome it is introduced in this study through presenting a comparison of the stability and accuracy of three semi-analytical numerical differential transform based methods. These methods are MsDTM, Enhanced MsDTM (E-MsDTM) and MsDTM with Padé approximants (MsDTM-P). These methods can be

classified as explicit one step semi-analytical numerical methods. The stability and error analysis of each method is presented. New important relationships among these methods are introduced. To demonstrate our results, a comparison of the accuracy, stability and computational efficiency of the methods is presented through solving some linear and nonlinear problems of practical importance in applied science and engineering.

## Multiple-Step DTM

The basic definition and the fundamental theorems of the MsDTM are given in (Odibat *et al.*, 2010; Keimanesh *et al.*, 2011; Gokdogan *et al.*, 2012b; Yildirim *et al.*, 2012; Erturk *et al.*, 2012; El-Zahar, 2012b; Khader and Megahed, 2014). For convenience of the reader, we present a review of the method as follows.

Consider the following non-linear initial-value problem:

$$\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = C_0 \quad (1)$$

Let  $[t_0, T]$  be the interval over which we want to find the solution of the initial value problem (1). In actual applications of the DTM, the  $N$ th-order approximate solution of the initial value problem (1) can be expressed by the finite series (Zhou, 1986; Nik and Soleymani, 2013; Rashidi *et al.*, 2013; 2014; Al-Amr, 2014; Benhammouda *et al.*, 2014):

$$y(t) \approx \sum_{k=0}^N Y(k)(t-t_0)^k = S(t), \quad t \in [t_0, T] \quad (2)$$

where:

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k y(t)}{dt^k} \right]_{t=t_0} \quad (3)$$

Equation (2) and (3) imply that the concept of differential transformation is derived from the Taylor series expansion. The following theorems can be deduced from (2) and (3)

### Theorem 1

$$\text{If } y(t) = \beta(u(t) \pm v(t)), \text{ then } Y(k) = \beta U(k) \pm \beta V(k)$$

### Theorem 2

$$\text{If } y(t) = u(t) v(t), \text{ then } Y(k) = \sum_{\ell=0}^k U(\ell) V(k-\ell).$$

### Theorem 3

$$\text{If } y(t) = \frac{d^m u(t)}{dt^m}, \text{ then } Y(k) = \frac{(k+m)!}{k!} U(k+m).$$

**Theorem 4**

If  $y(t) = (\beta + t)^m$ , then  $Y(k) = \frac{H_{m,k} m!}{k!(m-k)!} (\beta + t_0)^{m-k}$ ,

where  $H_{m,k} = \begin{cases} 1, & \text{if } m - k \geq 0 \\ 0, & \text{if } m - k < 0 \end{cases}$ .

**Theorem 5**

If  $y(t) = e^{\lambda t}$ , then  $Y(k) = \frac{\lambda^k}{k!} e^{\lambda t_0}$ .

**Theorem 6**

If  $y(t) = \sin(\omega t + \beta)$ , then  $Y(k) = \frac{\omega^k}{k!} \sin\left(\omega t_0 + \beta + \frac{k\pi}{2}\right)$ .

**Theorem 7**

If  $y(t) = \cos(\omega t + \beta)$ , then  $Y(k) = \frac{\omega^k}{k!} \cos\left(\omega t_0 + \beta + \frac{k\pi}{2}\right)$ .

Using some fundamental operations of DTM, we have the following recurrence relation:

$$Y(k+1) = F(k, Y(k)), Y(0) = C_0 \tag{4}$$

where,  $F(k, Y(k))$  is the differential transform of the function  $f(t, y(t))$ . The differential transform  $Y(k)$  of the unknown function  $y(k)$  can be obtained by solving the iterating algebraic system (4). In order to speed up the convergence rate and to improve the accuracy of resulting solutions, the entire domain  $[t_0, T]$  is usually split into sub-intervals and the algorithm of MsDTM is applied as follows:

Assume that the interval  $[t_0, T]$  is divided into  $M$  subintervals  $[t_m, t_{m+1}]$ ,  $m = 0, 1, \dots, M-1$  of equal step size  $h = (T-t_0)/M$ . The main ideas of the MsDTM are as follows. First, we apply the DTM to Equation (1) over the interval  $[t_0, t_1]$  to obtain the following approximate solution:

$$y_0(t) \approx \sum_{k=0}^N Y_0(k)(t-t_0)^k = S_0(t), \quad t \in [t_0, t_1] \tag{5}$$

Using the initial conditions  $y_0(t_0) = C_0$ . For  $m \geq 1$  and at each subinterval  $[t_m, t_{m+1}]$  we will use the initial conditions  $y_m(t_m) = S_{m-1}(t_m)$  and apply the DTM to Equation (1) over the interval  $[t_m, t_{m+1}]$ , where  $t_0$  in Equation (5) is replaced by  $t_m$ . The process is repeated and generates a sequence of approximate solutions  $y_m(t)$ ,  $m = 0, 1, \dots, M-1$ , for the solution  $y(t)$ :

$$y_m(t) \approx \sum_{k=0}^N Y_m(k)(t-t_m)^k = S_m(t). \tag{6}$$

Finally, the MsDTM assumes the following solution:

$$y(t) = \begin{cases} y_0(t) \approx S_0(t), & t \in [t_0, t_1], \\ y_1(t) \approx S_1(t), & t \in [t_1, t_2], \\ \vdots \\ y_{M-1}(t) \approx S_{M-1}(t), & t \in [t_{M-1}, t_M] \end{cases} \tag{7}$$

**Error Analysis of MsDTM**

The local truncation error of the approximate solution  $S_m(t)$  is estimated by the inequality (Do and Jang, 2012; Bervillier, 2012; El-Zahar, 2013b):

$$\|y_m(t) - S_m(t)\| \leq C \frac{(t-t_m)^{N+1}}{(N+1)!}, C = \max_{\xi \in [t_m, t_{m+1}]} |y_m^{(N+1)}(\xi)|, \tag{8}$$

$$t_m \leq t \leq t_{m+1}, m = 0, 1, \dots, M-1$$

It is well known that stiff ODE systems are associated with the existence of eigenvalues with very large magnitudes which consequentially results in solutions usually having exponential behavior.

Therefore, if we let  $C = K^{N+1}$ , in Equation (8), where  $K = \max_{\xi \in [t_m, t_{m+1}]} |y_m^{(1)}(\xi)|$ , then Equation (8) results in:

$$\|y_m(t) - S_m(t)\| \leq \frac{[K(t-t_m)]^{N+1}}{(N+1)!}, K = \max_{(t,y) \in R} |f(t,y)|, \tag{9}$$

$$t_m \leq t \leq t_{m+1}, m = 0, 1, \dots, M-1$$

From the local truncation error (8), it is clear that, for a fixed order  $N$ , as the step size  $h = (t-t_m)$  tends to zero the approximate solution  $S_m(t)$  tends to the exact solution  $y_m(t)$  and thus the method is consistent, while numerically, from (9), for a fixed step size, the convergence of the MsDTM only begins when  $N$  is of order  $O(Kh)$ .

**Stability Analysis of MsDTM**

In order to examine the MsDTM for the stability, let us consider the differential equation:

$$y'(t) = \lambda y(t) \tag{10}$$

where,  $\lambda$  is a complex constant and  $Re(\lambda) < 0$ . For this equation, the MsDTM solution, Equation (6), yields:

$$S_m(t_{m+1}) = \sum_{k=0}^N Y_m(k) h^k \quad \text{where } h = t_{m+1} - t_m \tag{11}$$

Form Equation (10) we have  $k! Y_m(k) = \lambda^k Y_m(0)$  and  $Y_m(0) = S_m(t_m)$ , then we get:

$$S_m(t_{m+1}) = \left[ \sum_{k=0}^N \frac{(h\lambda)^k}{k!} \right] S_m(t_m) \tag{12}$$

Setting  $z = \lambda h$  in Equation (12) results in the amplification factor:

$$R_N(z) = \sum_{k=0}^N \frac{z^k}{k!} \quad (13)$$

Which is the  $N^{\text{th}}$  order Taylor expansion of  $e^z$  and thus the MsDTM is not A-stable method.

### Enhanced MsDTM

Since the approximate solution obtained by MsDTM may yield an increasing error due to the nonlinearity (Do and Jang, 2012), a modification of MsDTM, which is called Enhanced MsDTM (Do and Jang, 2012) is presented as a multi-stage integration method as follows:

Suppose that we have solved the problem (1) up to a point  $t_{m-1}$ ,  $m = 1, 2, \dots, M-1$  and have obtained a value  $S_{m-1}(t_m)$  as an approximation of  $y_{m-1}(t_m)$ . Assuming the localization hypothesis (Lambert, 1991),  $y_m(t_m) = S_{m-1}(t_m)$ , we are interested in obtaining an approximate value for the true one  $y_m(t_{m+1})$ . For that purpose, the following method is developed.

The solution  $y_m(t)$  verifies the following IVP:

$$\frac{dy_m}{dt} = f(t, y_m(t)), \quad y_m(t_m) = S_{m-1}(t_m) \quad (14)$$

Equation (14) can be written as:

$$y_m(t) = S_{m-1}(t_m) + \int_{t_m}^t f(\tau, y_m(\tau)) d\tau \quad (15)$$

We can replace  $y_m(\tau)$  in Equation (15) by the approximate solution  $S_m(\tau)$  from Equation (6) to obtain an improved approximation  $S_m^1(t)$  to  $y_m(t)$ :

$$S_m^1(t) = S_{m-1}(t_m) + \int_{t_m}^t f(\tau, S_m^0(\tau)) d\tau \quad (16)$$

where,  $S_m^0(\tau) = S_m(\tau)$  and thus we can update the value  $S_m(t_{m+1})$  by:

$$S_m^1(t_{m+1}) = S_{m-1}(t_m) + \int_{t_m}^{t_{m+1}} f(\tau, S_m^0(\tau)) d\tau \quad (17)$$

Then,  $Y_{m+1}(0) = S_m^1(t_{m+1})$  from Equation (17) instead of  $Y_{m+1}(0) = S_m(t_{m+1})$  from Equation (6). For each subinterval, repeat this process, Equation (17), to update the initial condition of the followed subinterval,  $Y_{m+1}(0)$  and consequently to obtain enhanced approximation to  $y_{m+1}(t)$ .

The above process, Equation (17), is called Enhanced MsDTM with one iteration ( $E_1$ MsDTM). To apply Enhanced MsDTM with two iterations, we solve the integral equation, Equation (16) and replace  $S_m^0(\tau)$  by

$S_m^1(\tau)$  to get  $S_m^2(t)$ . Repeating this successive approximations  $I$ -times by replacing  $S_m^{I-1}(\tau)$  by  $S_m^I(\tau)$ ,  $I = 1, 2, 3, \dots$  results in enhanced MsDTM with  $I$  iteration ( $E_I$ MsDTM) as follows:

$$S_m^I(t) = S_{m-1}(t_m) + \int_{t_m}^t f(\tau, S_m^{I-1}(\tau)) d\tau, \quad (18)$$

$$I = 1, 2, 3, \dots, m = 0, 1, \dots, M-1$$

and hence we get an enhanced initial condition  $Y_{m+1}(0) = S_m^I(t_{m+1})$ .

In fact the above process, Equation (18), is a multiple-step successive approximations method and it is equivalent to multiple-step Picard iteration method (Djang, 1948; Lal and Moffatt, 1982; Wazwaz, 2011) with DTM solution as the zeroth approximation of  $y_m(t)$ .

### Error Analysis of E-MsDTM

It is well known that (Innocentini, 1999; Youssef and El-Arabawy, 2007) when the function  $f(t, y(t))$  satisfies Lipschitz condition with respect to  $y(t)$  in a rectangle

$$R, R = \{ |t - t_m| \leq a, |y(t) - S_{m-1}(t_m)| \leq b \},$$

$|f(t, y(t)) - f(t, \bar{y}(t))| \leq L|y(t) - \bar{y}(t)|$ ,  $L = \text{constant}$ , then, irrespective of the choice of the initial approximation, the successive approximations  $S_m^I(t)$  converge on the

interval  $\left[ t_m, t_m \pm \min\left(a, \frac{b}{H}\right) \right]$ ,  $H = \max_{\{t, y\} \in R} |f(t, y)|$  to the

solution of the problem (1). Also, the error of the approximate solution  $S_m^I(t)$  can be estimated as follows:

$$\begin{aligned} \|y_m(t) - S_m^I(t)\| &\leq L \int_{t_m}^t \|y_m(\mu_1) - S_m^{I-1}(\mu_1)\| d\mu_1 \\ &\leq L^2 \int_{t_m}^t \int_{t_m}^{\mu_1} \|y_m(\mu_2) - S_m^{I-2}(\mu_2)\| d\mu_2 d\mu_1 \\ &\leq L^I \int_{t_m}^t \int_{t_m}^{\mu_1} \dots \int_{t_m}^{\mu_{I-1}} \|y_m(\mu_I) - S_m^0(\mu_I)\| d\mu_I \dots d\mu_2 d\mu_1 \quad (19) \\ &\leq C L^I \frac{(t - t_m)^{N+I+1}}{(N+I+1)!} \leq C L^I \frac{(t - t_m)^{N+I+1}}{(N+I+1)!} \\ &\leq \frac{C}{L^{N+1}} \frac{[L(t - t_m)]^{N+I+1}}{(N+I+1)!}, C = \max_{\xi \in [t_m, t_{m+1}]} |y_m^{(N+1)}(\xi)|, \\ t_m \leq t \leq t_{m+1}, m = 0, 1, \dots, M-1 \end{aligned}$$

It is clear that the choice of the initial approximation,  $S_m^0(t) = S_{m-1}(t_m)$ , i.e.,  $N = 0$ , results in the following bounded error:

$$\|y_m(t) - S_m^I(t)\| \leq \frac{C}{L} \frac{[L(t-t_m)]^{I+1}}{(I+1)!}, C = \max_{\xi \in [t_m, t_{m+1}]} |y_m^{(I)}(\xi)|, \leq \frac{H}{L} \frac{[L(t-t_m)]^{I+1}}{(I+1)!}, \quad (20)$$

$$H = \max_{(t,y) \in K} |f(t,y)|, t_m \leq t \leq t_{m+1}, m = 0, 1, \dots, M-1$$

which is the classical Picard iteration method error (Youssef and El-Arabawy, 2007).

From the local truncation error (19), it is clear that, for a fixed number of iterations  $I$ , as the step size  $h = (t - t_m)$  tends to zero the approximate solution  $S_m^I(t) \rightarrow y_m(t)$  and thus the method is consistent, while numerically, for a fixed step size, the convergence of  $E_I$ -MsDTM only begins when  $I$  is of order  $O(Lh)$ .

### Stability Analysis of E-MsDTM

In order to examine the stability of E-MsDTM with  $N^{th}$  order and  $I$  iterations, we consider again Equation (10).

Using  $E_I$ -MsDTM, the approximate solution of Equation (10) is given by:

$$S_m^1(t) = S_m^0(t_m) + \int_{t_m}^t \lambda S_m^0(\tau) d\tau \quad (21)$$

where,  $S_m^0(t)$  is the zeroth approximation of  $y_m(t)$  and given by:

$$S_m^0(t) = \sum_{k=0}^N Y_m(k)(t-t_m)^k \quad (22)$$

From Equation (21) and (22) we have:

$$S_m^1(t) = S_m^0(t_m) + \sum_{k=0}^N \lambda Y_m(k) \frac{(t-t_m)^{k+1}}{k+1} \quad (23)$$

Since  $(1+k)Y_m(k+1) = \lambda Y_m(k)$  for Equation (10), then Equation (23) becomes:

$$S_m^1(t) = S_m^0(t_m) + \sum_{k=0}^N Y_m(k+1)(t-t_m)^{k+1} \quad (24)$$

and can be written as:

$$S_m^1(t) = \sum_{k=0}^{N+1} Y_m(k)(t-t_m)^k \quad (25)$$

which is the  $(N+1)^{th}$  order approximate solution of (10) obtained by DTM. Replacing the solution in Equation (22) by that obtained in Equation (25) and repeating the above algorithm  $I$  times results in:

$$S_m^I = \sum_{k=0}^{N+I} Y_m(k)(t-t_m)^k \quad (26)$$

which is the  $(N+I)^{th}$  order approximate solution of Equation (10) obtained by DTM and thus the amplification factor of  $E_I$ -MsDTM with  $N^{th}$  order and  $I$  iterations is given by:

$$R_{N,I}(z) = \sum_{k=0}^{N+I} \frac{z^k}{k!} \quad (27)$$

which is the  $(N+I)^{th}$  order Taylor expansion of  $e^z$  and thus the  $E_I$ -MsDTM is not A-stable method.

From the stability analysis of MsDTM and E-MsDTM, we get the following theorem.

### Theorem 8

*Enhanced-MsDTM with order  $N$  and iterations  $I$  and the MsDTM with order  $(N+I)$  have the same stability region.*

Also, we observe that applying the above procedure from Equation (21) to (26) on Equation (10) in a vector form results in the following theorem.

### Theorem 9

*Enhanced-MsDTM with order  $N$  and iterations  $I$  is equivalent to the MsDTM with order  $(N+I)$  for the homogenous autonomous ODE system:*

$$\frac{dy}{dt} = A y(t), \quad y(t_0) = y_0$$

where,  $A$  is an  $(n \times n)$  constant matrix and  $y$  is an  $(n \times 1)$  solution vector.

The stability regions of MsDTM and  $E_I$ -MsDTM at different orders ( $N$ ) and number of iterations ( $I$ ), respectively, are shown in Fig. 1.

## MsDTM-Padé Approximants

The goal of Padé approximants is to make the maximum error of MsDTM, Equation (8), as small as possible by maintaining the accuracy far outside the radius of convergence of the series solution (Ehle, 1973; Rashidi and Mohimani Pour, 2010a; 2010b; Lu, 2012; Kanth and Aruna, 2013; Domairry and Hatami, 2014) and to improve the stability property of the solution. Let the Padé approximation of the MsDTM series solution  $S_m(t)$  on  $[t_m, t_{m+1}]$  is the quotient of two polynomials  $p_m(t)$  and  $q_m(t)$  of order  $p$  and  $q$  respectively and  $p+q = N$  (Ehle, 1973). We use the notation  $S_{m,pa}(t)$  to denote the quotient:

$$S_{m,pa}(t) = \frac{p_m(t)}{q_m(t)}, \text{ for } t \in [t_m, t_{m+1}], m = 0, 1, \dots, M-1 \quad (28)$$

Let the polynomials used in (28) are:

$$p_m(t) = \sum_{k=0}^p P(k)(t-t_m)^k, \quad (29)$$

$$q_m(t) = \sum_{k=0}^q Q(k)(t-t_m)^k, \text{ with } Q(0) = 1$$

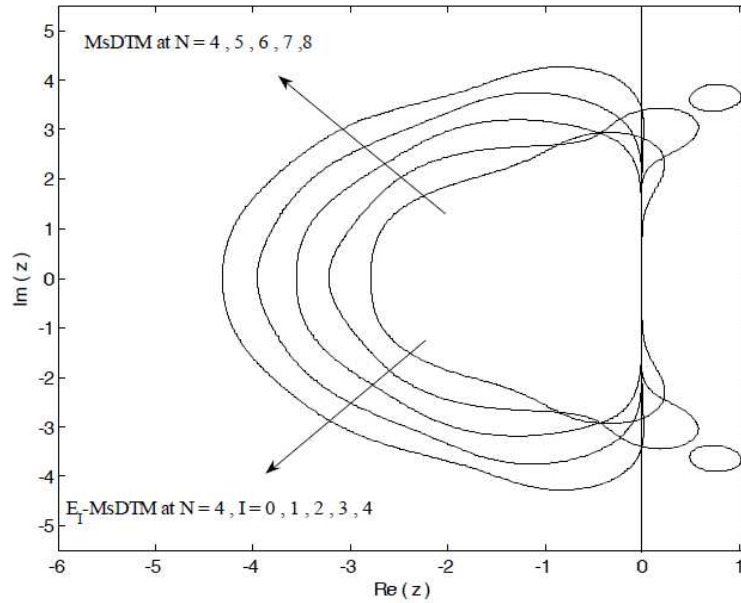


Fig. 1. Stability regions of MsDTM and E<sub>1</sub>-MsDTM at different orders (N) and number of iterations (I), respectively

The polynomials in (28) are constructed so that  $S_m(t)$  and  $S_{m,pa}(t)$  agree at  $t = t_m$  and their derivatives up to  $N$ . Since  $S_m(t)$  has a Taylor expansion defined in Equation (6), the formal power series:

$$S_m(t) - S_{m,pa}(t) = S_m(t) - \frac{p_m(t)}{q_m(t)} = O((t - t_m)^{N+1}) \quad (30)$$

Determines the coefficients of  $p_m(t)$  and  $q_m(t)$ . Multiplying (30) by  $q_m(t)$  gives:

$$\left( \sum_{k=0}^N Y(k)(t - t_m)^k \right) \left( \sum_{k=0}^q Q(k)(t - t_m)^k \right) - \left( \sum_{k=0}^p P(k)(t - t_m)^k \right) = \sum_{k=N+1}^{qN} C(k)(t - t_m)^k \quad (31)$$

When the left side of (31) multiplied out and the coefficients of the powers of  $(t - t_m)^k$  are set equal to zero for  $k = 0, 1, \dots, N$ , the result is a system of  $N+1$  linear equations in the  $N+1$  unknown coefficients of  $p_m(t)$  and  $q_m(t)$ . Solving this linear system we get  $S_{m,pa}(t)$  rational approximation of  $S_m(t)$ . From the local truncation error (30), it is clear that, for a fixed order  $N$ , as  $(t - t_m) \rightarrow 0$ ,  $S_{m,pa}(t) \rightarrow S_m(t)$  and thus the method is consistent.

#### Stability Analysis of MsDTM-P

For the stability of MsDTM with Padé approximants, it is well known that the Padé approximants of order  $(p, q)$  are A-stable if and only if  $2 \geq p - q \geq 0$  (Ehle, 1973; Hairer and Wanner, 1996,

Theorem 4.12), that is, the main diagonal and two sub diagonals in Padé approximants are A-stable. The absolute stability of MsDTM with diagonal and two sub diagonals Padé approximants is shown in Fig. 2.

#### Numerical Experiments

To demonstrate our results, a comparison of the accuracy, stability regions and computational efficiency of the MsDTM, E-MsDTM and MsDTM-P is presented through solving some linear and nonlinear practical problems with known exact solutions arising in electrical and chemical engineering. All calculations are carried out by MAPLE 14 software in a PC with a Pentium 2 GHz and 512 MB of RAM.

#### Example 1

Figure 3 is a schematic diagram of a separately excited D.C motor where  $R$  the armature resistance;  $L$  the armature inductance;  $I$  the armature current;  $v$  the terminal voltage of the motor;  $e$  the back electromotive force;  $T$  the load torque;  $J$  the torque of inertia and  $\dot{\theta} = \omega$  is the motor speed. Taking  $\omega_r = \omega / \Omega$ ,  $I_r = iR / K_b \Omega$ ,  $u = v / K_b \Omega$ ,

$B = bR / K_b^2$ ,  $T_L = TR / \Omega K_b^2$ ,  $\varepsilon = T_e / T_m$  as dimensionless variables, where  $K_b$  is the constant of the torque,  $\Omega$  is a nominal operating point,  $T_e = L / R$  is the armature circuit time constant,  $T_m = JR / K_b^2$  the mechanical time constant and  $\tau = t / T_m$  as the time variable, then the transient behavior of the D.C motor shown in Fig. 3 can be described by following stiff ODE system.

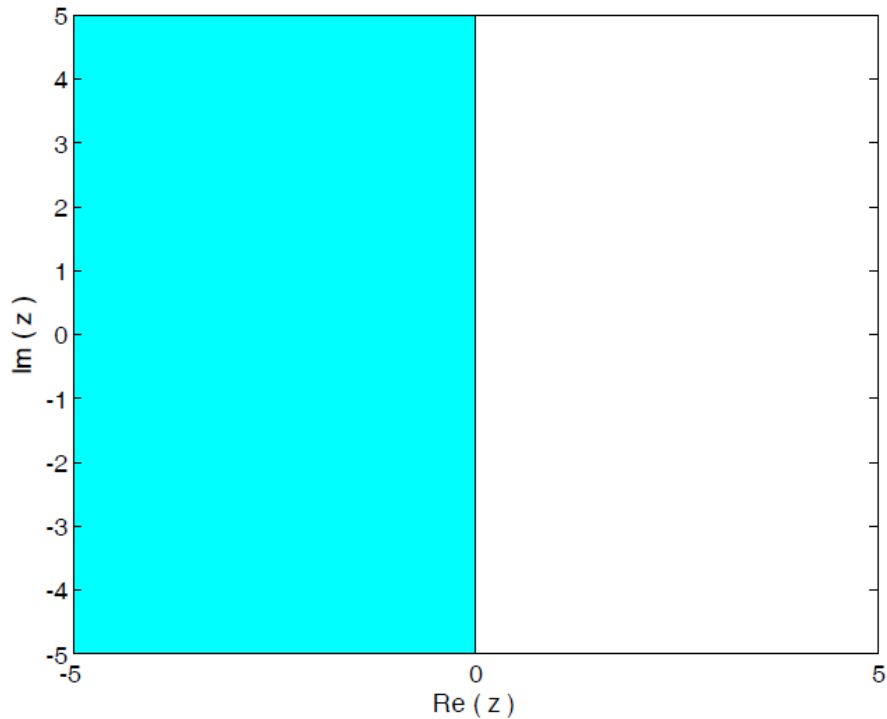


Fig. 2. Absolute stability of MsDTM with main and two sub diagonals Padé approximants

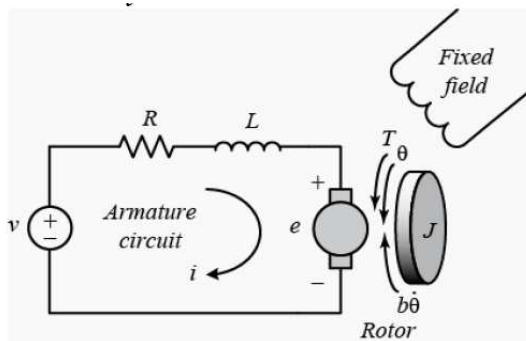


Fig. 3. A schematic diagram of a separately excited D.C motor

$$\left. \begin{aligned} \frac{d\omega_r}{d\tau} &= -B\omega_r + I_r - T_L \\ \epsilon \frac{dI_r}{d\tau} &= -\omega_r - I_r + u \end{aligned} \right\} \quad (32)$$

System (32) with parameters listed in Table 1 has been integrated using fourth order MsDTM (4MsDTM), enhanced fourth order MsDTM with one iteration (E<sub>1</sub>-4MsDTM) and fourth order MsDTM with Padé approximant (4MsDTM-P) on the interval [0, 4] and the results are presented in figures and tables for different values of the step size *h*.

The errors have been defined as the maximum of the absolute pointwise errors for each component of the solution as:

$$E_{\omega_r} = \|\omega_r(\tau_j) - (\omega_r)_j\|, \\ E_{I_r} = \|I_r(\tau_j) - (I_r)_j\|, E_{\omega_d} = \max\{E_{\omega_r}, E_{I_r}\}$$

For system (32), at  $\epsilon = 0.001$ , the eigenvalues are  $\lambda = -9.998e + 003, -1.2446e + 00$ , the stiffness ratio is 802.625 and the drawback of MsDTM and E-MsDTM are more severe, where for (32), the 4MsDTM is stable only if  $h \leq h_{s4}$ , ( $h_{s4} = 2.7881e-003$ ) and the E<sub>1</sub>-4MsDTM is stable only if  $h \leq h_{s5}$ , ( $h_{s5} = 3.2202e-003$ ).

Figure 4-6 show the stable and unstable solutions of (32) obtained by using 4MsDTM, E<sub>1</sub>-4MsDTM and 4MsDTM-P at different values of time-step size *h*. In each figure, we choose the time step and the plotting range so that the differences between the obtained solutions are clear.

Figure 4 shows that the 4MsDTM solutions obtained at  $h = 0.0025, 0.0027$  ( $h < h_{s4}$ ), converge to the true solutions, while the solution obtained at  $h = 0.0028$  ( $h > h_{s4}$ ), is unstable and diverges far from the true solution. To overcome the stability restriction on using 4MsDTM with  $h = 0.0028$  for (32), we should use smaller time-step size or higher order MsDTM, where the 5MsDTM has a larger stability region ( $h_{s5} = 3.2202e-003$ ) than 4MsDTM.

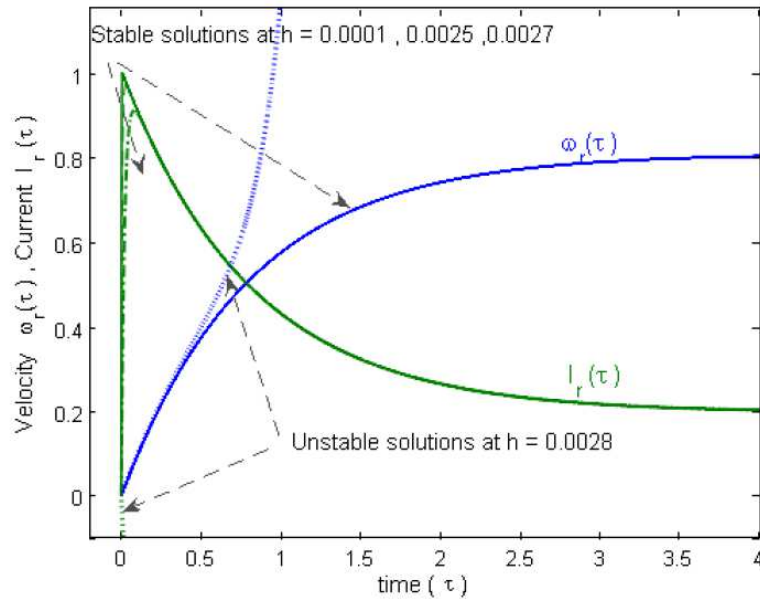


Fig. 4. 4MsDTM solutions for system (32) at different values of time-step size  $h$

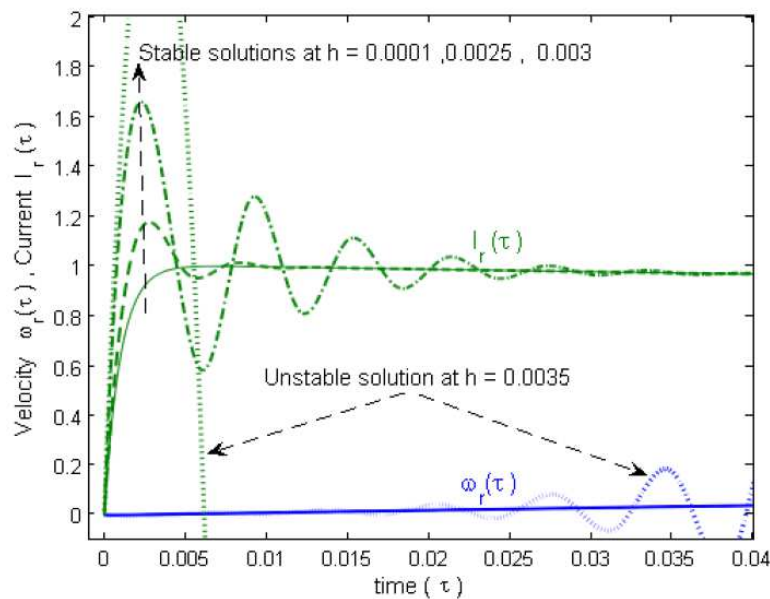


Fig. 5.  $E_1$ -4MsDTM solutions for system (32) at different values of time-step size  $h$

As shown in Fig. 5, although the  $E_1$ -4MsDTM solutions, at  $h = 0.0025, 0.003$  ( $h < h_{s5}$ ), exhibiting spurious oscillations at the layer region (Hairer and Wanner, 1996; Shyy *et al.*, 1992; Yost and Rao, 2000), they remain bounded and decrease with time and thus the solutions are stable. At  $h = 0.0035$  ( $h > h_{s5}$ ), the  $E_1$ -4MsDTM solutions oscillate wildly and quickly exit the range of the graph and therefore, there is a need to increase the order of the method or the number of iterations  $I$ , where  $E_1$ -5MsDTM or  $E_2$ -4MsDTM has a larger stability region ( $h_{s6} = 3.557e-003$ ) than  $E_1$ -4MsDTM.

Figure 6 shows that the 4MsDTM-P solutions are stable and converge to the true solutions even for larger time-step size.

Results in Table 2, show us that the 4MsDTM and  $E_1$ -4MsDTM solutions at  $h \geq 0.003$  and  $h \geq 0.004$  respectively, do not approximate the true solutions of (32) correctly. We insert dashes (-) to indicate this phenomenon. Obviously, these step sizes are not enough to meet the stability restrictions where 4MsDTM and  $E_1$ -4MsDTM are not A-stable methods.



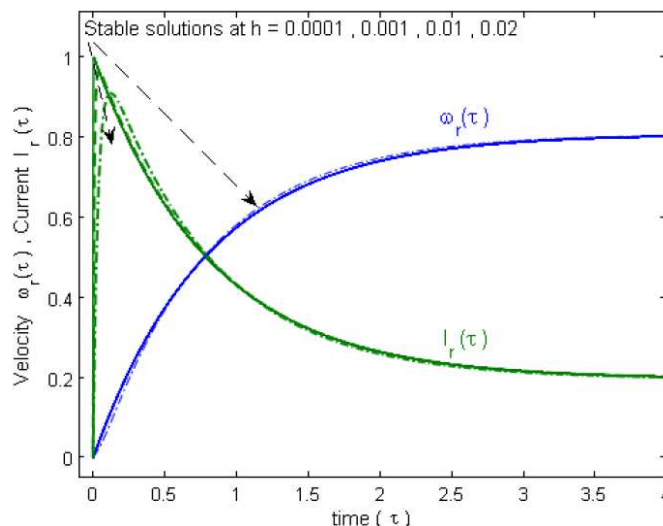


Fig. 6. 4MsDTM-P solutions for system (32) at different values of time-step size  $h$

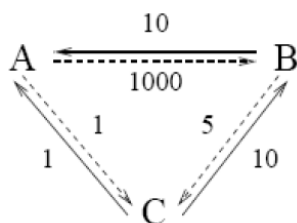


Fig. 7. A circular reaction

and therefore, there is a need to decrease the time-step to obtain accurate solutions, while the A-stable 4MsDTM-P performs better even for large time step.

Results in Table 2 also show us that the  $E_1$ -4MsDTM solutions are accurate better than the 4MsDTM solutions, while the 4MsDTM-P solutions are more accurate than  $E_1$ -4MsDTM solutions

Table 3 and 4 present a comparison of the maximum absolute error  $E_{\text{sol}}$  and the CPU used time, respectively, for solving (32) using the NMsDTM,  $E_1$ -4MsDTM and NMsDTM-P at different orders,  $N$ , number of iterations,  $I$  and time-step size,  $h$ .

As shown from Table 3, the results of (I+4) MsDTM and  $E_1$ -4MsDTM have the same accuracy and stability results, which confirms theoretical results in (Theorems 8, 9), while the results of (I+4) MsDTM-P are more accurate. In addition, for a fixed order in NMsDTM or a fixed number of iterations in  $E_1$ -4MsDTM, as the step size decreases the numerical solution converges to the exact one. While, for a fixed step size  $h$ , the convergence of (NMsDTM,  $E_1$ -4MsDTM) only begins when  $(N, I)$  are  $(O(Kh), O(Lh))$ , respectively, where for (32),  $K = L = O(\varepsilon^{-1})$  ([Vulanic, 1989, Theorem 1]; (El-Zahar and EL-Kabeir, 2013, Lemma 3.2); Zhao and Xiao, 2010; Zhao *et al.*, 2014).

Results in Table 4, show us that, for the same step size  $h$ , the (I+4) MsDTM has the lowest CPU usage and the  $E_1$ -4MsDTM has the highest CPU usage in solving (32). Moreover, increasing the order  $N$  of NMsDTM is computationally cheaper than increasing the order of NMsDTM-P, while increasing the number of iterations  $I$  in  $E_1$ -4MsDTM is computationally more expensive, where for  $I > 1$  successive analytical integrations are required ( $I-1$ )-times and therefore the MsDTM is the lowest computational cost while the E-MsDTM is the highest computational cost.

### Example 2

Figure 7 describes a circular reaction with 3 substances  $A, B$  and  $C$ , with initial values  $A(0) = 1, B(0) = 2$  and  $C(0) = 3$ . The system of ODEs describing the circular reaction in Fig. 7 is as follows:

$$\frac{d}{dt} \begin{bmatrix} A(t) \\ B(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} -1001 & 10 & 1 \\ 1000 & -15 & 10 \\ 1 & 5 & -11 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \\ C(t) \end{bmatrix} \quad (33)$$

For system (33), the eigenvalues are  $\lambda = -1.0110e03, -1.5964e01, 0.00$  and therefore the stiffness ratio is 63.33 and we have  $(h_{s4} = 2.7550e-003, h_{s5} = 3.1820e-003)$ . The system has been integrated on the interval  $[0, 1]$  and the numerical results are presented in Fig. 8-10 and Table 5 and 6 using MsDTM, E-MsDTM and MsDTM-P at different orders,  $N$ , number of iterations,  $I$  and time-step size  $h$ . The present numerical results for system (33) provide support for the earlier analysis of the results presented for system (32), which confirm our theoretical results.

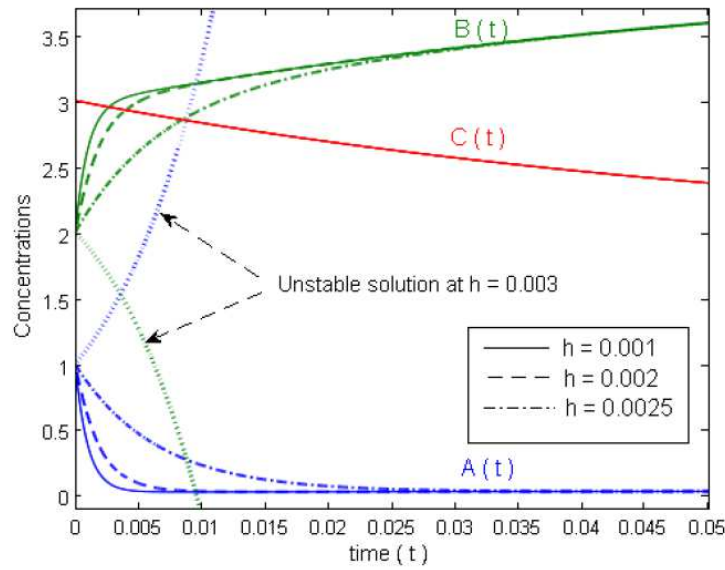


Fig. 8. 4MsDTM solutions for the system (33) at different values of time-step size  $h$

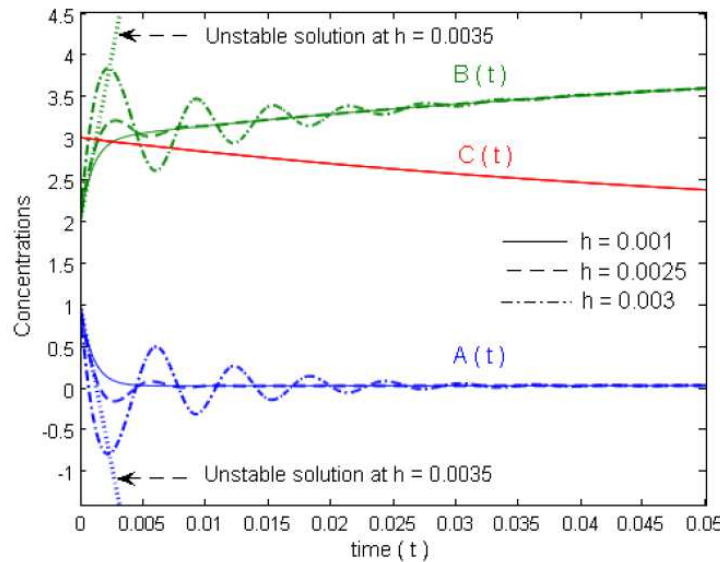


Fig. 9. E1-4MsDTM solutions for the system (33) at different values of time-step size  $h$

**Example 3**

Consider the nonlinear system of ODEs (Wu and Xia, 2001):

$$Y' = -BY + UW \tag{34}$$

where:

$$Y(0) = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, U = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, B = U \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{pmatrix} U,$$

$$W = \begin{pmatrix} z_1^2 \\ z_2^2 \\ z_3^2 \\ z_4^2 \end{pmatrix}, Z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = UY, D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 1000 \\ 800 \\ -10 \\ 0.001 \end{pmatrix}$$

The exact solution of (34) is:

$$Y = UZ$$

where:

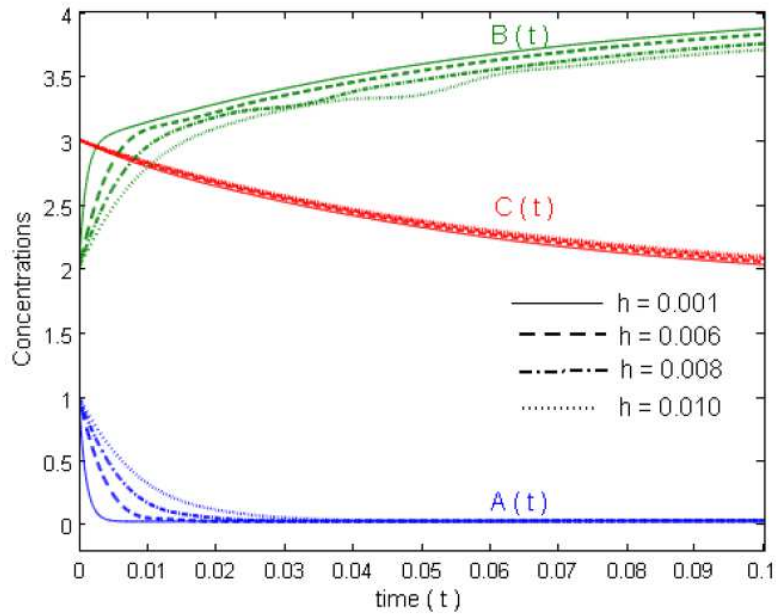


Fig. 10. 4MsDTM-P stable solutions for the system (33) at different values of time-step size h

Table 1. Parameters of the separately excited D.C motor

$J = 8 \times 10^{-3}$ kg.m <sup>2</sup>	$K_b = 0.043$ V/rad/s
$R = 1.5\Omega$	$\Omega = 279.07$ rad/s
$b = 3 \times 10^{-4}$ N.m/rad/s	$V_a = 12$ V
$L = 9.735$ mH	$T_L = 0$

Table 2. Maximum absolute errors of 4MsDTM, E<sub>1</sub>-4MsDTM and 4MsDTM-P at different time-step size h for system (32)

h	$E_{\omega_r}$			$E_{I_r}$		
	4MsDTM	E1-4MsDTM	4MsDTM-P	4MsDTM	E1-4MsDTM	4MsDTM-P
0.0200	-	-	2.7050e-2	-	-	5.3853e-1
0.0100	-	-	4.0418e-3	-	-	2.9919e-1
0.0050	-	-	4.6199e-4	-	-	9.7038e-2
0.0040	-	-	2.2318e-4	-	-	5.8205e-2
0.0030	-	6.9808e-4	8.6548e-5	-	6.9721e-1	2.6954e-2
0.0027	8.1048e-4	3.3816e-4	6.0986e-5	8.0947e-1	3.8269e-1	1.9728e-2
0.0025	5.6554e-4	2.4684e-4	4.7189e-5	5.6483e-1	2.4653e-1	1.5554e-2
0.0020	1.9770e-4	6.8496e-5	2.2180e-5	1.9746e-1	6.8410e-2	7.4707e-3
0.0010	1.1661e-6	1.2096e-6	1.9070e-6	1.1647e-3	1.2081e-3	5.3772e-4

Table 3. Maximum absolute errors  $E_{ol}$  of different methods at different orders and time-step size for system (32)

h	0.004	0.003	0.0025	0.002	0.001
5MsDTM	-	6.9721E-1	2.4653E-1	6.8410E-2	1.2081E-3
6MsDTM	-	3.1124E-1	9.1200E-2	2.0123E-2	1.7536E-4
7MsDTM	-	1.2052E-1	2.9297E-2	5.1465E-3	2.2167E-5
8MsDTM	5.0842e-1	4.1227E-2	8.3206E-3	1.1647E-3	2.4860E-6
5MsDTM-P [2/3]	5.7084E-2	2.6326E-2	1.5139E-2	7.2327E-3	5.0646E-4
6MsDTM-P [3/3]	5.3599E-3	1.5185E-3	6.3195E-3	2.0093E-4	3.8045E-6
7MsDTM-P [3/4]	2.6364E-4	4.2917E-5	1.2348E-5	4.0629E-6	5.4844E-8
8MsDTM-P [4/4]	2.9289E-4	4.9229E-5	1.4552E-5	3.0193E-6	1.4690E-8
E <sub>1</sub> -4MsDTM	-	6.9721E-1	2.4653E-1	6.8410E-2	1.2081E-3
E <sub>2</sub> -4MsDTM	-	3.1124E-1	9.1200E-2	2.0123E-2	1.7536E-4
E <sub>3</sub> -4MsDTM	-	1.2052E-1	2.9297E-2	5.1465E-3	2.2167E-5
E <sub>4</sub> -4MsDTM	5.0842e-1	4.1227E-2	8.3206E-3	1.1648E-3	2.4860E-6

Table 4. CPU time(s) in Sec used in solving system (32)

$h$	0.004	0.003	0.0025	0.002	0.001
5MsDTM	-	00.641	00.750	00.953	01.922
6MsDTM	-	00.703	00.813	01.000	01.953
7MsDTM	-	00.813	00.891	01.047	02.141
8MsDTM	00.766	00.906	01.031	01.140	02.281
5MsDTM-P [2/3]	02.625	03.344	04.016	05.015	10.125
6MsDTM-P [3/3]	02.750	03.703	04.500	05.593	11.078
7MsDTM-P [3/4]	03.437	04.641	05.500	06.875	14.000
8MsDTM-P [4/4]	03.860	06.313	06.094	07.687	15.375
E <sub>1</sub> -4MsDTM	-	03.469	04.188	05.188	10.375
E <sub>2</sub> -4MsDTM	-	06.968	08.407	10.531	20.938
E <sub>3</sub> -4MsDTM	-	10.125	12.109	16.047	33.875
E <sub>4</sub> -4MsDTM	10.512	12.365	15.674	20.021	41.108

Table 5. Maximum absolute errors of different methods at different orders and time-step size for system (33)

$h$	0.004	0.003	0.0025	0.002	0.001
5MsDTM	-	7.2348e-1	2.5594e-1	7.1061e-2	1.2565e-3
6MsDTM	-	3.2698e-1	9.5856e-2	2.1161e-2	1.8450e-4
7MsDTM	-	1.2818e-1	3.1173e-2	5.4785e-3	2.3621e-5
8MsDTM	5.4701e-1	4.4388e-2	8.9620e-3	1.2550e-3	2.6813e-6
5MsDTM-P [2/3]	1.3698e-1	3.7719e-2	1.6635e-2	6.0714e-3	2.3814e-4
6MsDTM-P [3/3]	5.3980e-3	1.5400e-3	6.4337e-4	2.0542e-4	3.9269e-6
7MsDTM-P [3/4]	2.2854e-3	5.5555e-4	2.0054e-4	5.4601e-5	8.0908e-7
8MsDTM-P [4/4]	3.4722e-4	5.8611e-5	1.7378e-5	3.6186e-6	3.2323e-7
E <sub>1</sub> -4MsDTM	-	7.2348e-1	2.5594e-1	7.1061e-2	1.2565e-3
E <sub>2</sub> -4MsDTM	-	3.2698e-1	9.5856e-2	2.1161e-2	1.8450e-4
E <sub>3</sub> -4MsDTM	-	1.2818e-1	3.1173e-2	5.4785e-3	2.3621e-5
E <sub>4</sub> -4MsDTM	5.4701e-1	4.4388e-2	8.9620e-3	1.2551e-3	2.6813e-6

Table 6. CPU time(s) in Sec used in solving system (33)

$h$	0.004	0.003	0.0025	0.002	0.001
5MsDTM	-	00.312	00.360	00.422	00.906
6MsDTM	-	00.359	00.359	00.500	01.031
7MsDTM	-	00.401	00.417	00.513	01.094
8MsDTM	00.312	00.406	00.469	00.578	01.156
5MsDTM-P [2/3]	01.015	01.266	01.500	01.922	03.879
6MsDTM-P [3/3]	01.094	01.407	01.656	02.156	04.188
7MsDTM-P [3/4]	01.469	01.766	02.156	02.672	05.234
8MsDTM-P [4/4]	01.484	02.031	02.250	02.859	05.781
E <sub>1</sub> -4MsDTM	-	01.375	01.594	01.938	03.875
E <sub>2</sub> -4MsDTM	-	02.625	03.047	03.828	07.734
E <sub>3</sub> -4MsDTM	-	12.031	09.906	12.282	18.140
E <sub>4</sub> -4MsDTM	19.325	24.841	29.898	35.128	48.125

Table 7. Maximum absolute errors of 4MsDTM, E<sub>1</sub>-4MsDTM and 4MsDTM-P for system (34)

$h$	4MsDTM	E <sub>1</sub> -4MsDTM	4MsDTM-P
0.0050	-	-	7.5894e-2
0.0040	-	-	4.7552e-2
0.0030	-	4.5933e-1	1.9585e-2
0.0027	5.6182e-1	2.5191e-1	1.4148e-2
0.0025	3.9178e-1	1.6220e-1	1.1054e-2
0.0020	1.3673e-1	4.4960e-2	5.1837e-3
0.0010	4.8999e-3	7.9266e-4	3.5344e-4

Table 8. Maximum absolute errors of different methods at different orders for system (34)

<i>h</i>	0.004	0.003	0.002	0.001
5MsDTM	-	4.7058e-1	4.6153e-2	8.1526e-4
6MsDTM	-	2.1115e-1	1.3697e-2	1.1990e-4
7MsDTM	8.9639e-1	8.6648e-2	3.7323e-3	1.6268e-5
8MsDTM	4.1510e-1	3.4088e-2	9.7866e-4	2.1338e-6
5MsDTM-P [2/3]	4.2018e-2	1.2326e-2	1.9659e-3	9.1152e-5
6MsDTM-P [3/3]	3.4206e-3	1.0154e-3	1.4181e-4	5.5071e-5
7MsDTM-P [3/4]	3.7350e-3	6.7155e-4	5.1114e-5	6.9606e-7
8MsDTM-P[4/4]	2.2286e-4	3.8427e-5	2.4243e-6	5.6783e-8
E <sub>1</sub> -4MsDTM	-	4.5933e-1	4.4960e-2	7.9266e-4
E <sub>2</sub> -4MsDTM	-	1.9605e-1	1.2665e-2	1.1029e-4
E <sub>3</sub> -4MsDTM	6.7896e-1	7.3224e-2	3.1266e-3	1.3482e-5
E <sub>4</sub> -4MsDTM	2.9982e-1	2.4318e-2	6.8748e-4	1.4706e-6

Table 9. CPU time(s) in Sec used in solving system (34)

<i>h</i>	0.004	0.003	0.002	0.001
5MsDTM	-	08.181	12.219	24.547
6MsDTM	-	09.859	14.703	29.579
7MsDTM	08.750	11.656	17.422	34.813
8MsDTM	10.125	13.516	20.156	40.406
5MsDTM-P [2/3]	07.109	09.532	14.235	28.594
6MsDTM-P [3/3]	08.516	11.359	16.984	34.047
7MsDTM-P [3/4]	10.140	13.547	20.344	40.719
8MsDTM-P[4/4]	11.641	15.672	23.359	46.875
E <sub>1</sub> -4MsDTM	-	10.141	15.156	30.219
E <sub>2</sub> -4MsDTM	-	14.215	21.894	48.716
E <sub>3</sub> -4MsDTM	12.014	35.097	43.810	65.125
E <sub>4</sub> -4MsDTM	28.451	44.384	55.127	78.124

$$Y = \begin{pmatrix} y_{1E}(t) \\ y_{2E}(t) \\ y_{3E}(t) \\ y_{4E}(t) \end{pmatrix}, Z = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}, z_i(t) = \frac{d_i}{1 - (1 + d_i)e^{d_i t}}, i=1,2,3,4$$

The problem has been integrated on the interval [0, 1] and the numerical results are presented in Table 7-9 using MsDTM, E-MsDTM and MsDTM-P at different orders, *N*, number of iterations, *I* and time-step size *h*.

Results in Table 7 show that, the E<sub>1</sub>-4MsDTM solutions are accurate better than the 4MsDTM solutions, while the 4MsDTM-P solutions are more accurate and have a good stability property.

Results in Table 8 show that, the E<sub>1</sub>-4MsDTM results are little better than the (I+4) MsDTM results, while the (I+4) MsDTM-P results are more accurate than the (I+4) MsDTM and E<sub>1</sub>-4MsDTM results.

Results in Table 4, 6, 9 show us that, the CPU usage in solving (34) is extremely higher than CPU usage in (32) and (33) and thus is due to the nonlinearity of (34). But we still have MsDTM with the lowest computational cost and E-MsDTM with the highest computational cost.

## Conclusion

In this study, we have presented an effective comparison among three explicit one step semi-analytical numerical methods for solving stiff ODE

systems. The methods are the MsDTM, E-MsDTM and MsDTM-P. The error and stability analysis of each method is presented. The error analysis shows that the methods are consistent and so they closely match the given IVP (1) when the step size *h* is sufficiently small. The stability analysis shows that the E<sub>1</sub>-*N*MsDTM have stability regions larger than those with *N*MsDTM and both of them are conditionally stable methods, while *N*MsDTM-P is A-stable method and larger time steps can be used hence suitable for the stiff ODE systems. In addition, the (*N*+1) MsDTM and E<sub>1</sub>-*N*MsDTM have the same stability regions (Theorem 8) and are equivalent for linear homogenous autonomous initial-value ODE system (Theorem 9). Also, the present analysis of the E<sub>1</sub>-*N*MsDTM shows that it is equivalent to the multiple-step Picard iteration method with *N*DTM solution as the zeroth approximation of *y<sub>m</sub>(t)*. We have applied the methods on three linear and nonlinear practical problems with known exact solutions and the numerical results are presented in figures and tables for comparison. The numerical results confirm our theoretical ones and additionally show that:

- As the step size decreases or the order increases or the number of iterations for E-MsDTM increases, the accuracy of the methods increases
- For the same order and step size, the E-MsDTM solutions are accurate better than the MsDTM

solutions, while the MsDTM-P solutions are more accurate than E-MsDTM solutions

- For the same step size, the  $E_I$ -NMsDTM and  $(N+I)$ MsDTM have the same results for linear homogenous autonomous ODE systems, while for nonlinear ODE systems the  $E_I$ -NMsDTM solutions are accurate little better than the  $(N+I)$ MsDTM solutions
- Numerically, the convergence of  $E_I$ -MsDTM begins when  $I$  is of order  $O(Lh)$ , while begins for MsDTM when  $N$  is of order  $O(Kh)$

From the view of computations, increasing the order of the MsDTM or MsDTM-P is computationally cheaper than increasing the number of iterations in E-MsDTM.

We conclude that the MsDTM-P is more suitable than classical MsDTM and E-MsDTM for solving stiff ODE systems because of its stability, higher accuracy and a relatively low computational cost.

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## Author's Contributions

**E.R. El-Zahar:** Participated in overall error and stability analysis, solving stiff systems, analysis and interpretation of the results, comparisons among algorithms, edited the manuscript.

**H.M. Habib:** Participated in error and stability analysis of MsDTM, solving stiff electrical system, analysis and interpretation of the results, critical revision, contributed to the writing of the manuscript.

**M.M. Rashidi:** Participated in error and stability analysis of E-MsDTM, solving stiff reaction system, analysis and interpretation of the results, critical revision, contributed to the writing of the manuscript.

**I.M. El-Desoky:** Participated in error and stability analysis of MsDTM-P, solving stiff nonlinear system, analysis and interpretation of the results, critical revision, contributed to the writing of the manuscript.

## Ethics

The authors confirm that this work is original and has not been published elsewhere.

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