A Comparison of Joint/Independent State Particle Filters for Tracking Closely Spaced Targets in Clutter

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Abstract*—*In this paper, we present an overview of the literature for particle filtering under measurement origin uncertainty with an emphasize on single scan data association algorithms. We compare some of the existing and newly proposed joint state particle filtering algorithms for the closely spaced target tracking problem. Both maximum a posteriori (MAP) and minimum mean square error (MMSE) estimation outputs of four different algorithms are compared. We also include MMSE outputs of a non-joint (independent) state particle filter and Kalman filter in the comparison as a baseline.

Keywords-Particle filters; data association; measurement origin uncertainty; JPDAFC; JPDAF; NNJPDAF.

I. INTRODUCTION

The theory of single-sensor, single-target tracking is rather well understood in the literature over the last fifty years since the Kalman filter [1] was proposed. When multiple targets are present, however, the situation becomes much more complex and it is even more complicated when the sensor operates in the presence of false alarms and missed detections. Probably, the most important problem is the *measurement origin uncertainty* [2] where it is not known which measurement belongs to which target and more generally whether a measurement belongs to a target or not. How to use such measurements with uncertain origin to update a tracking filter constitutes one of the most challenging problems in tracking[1]. The literature devoted to solve this problem is abundant. However, the solution approaches can be roughly considered to be based on two main methodologies: Tracking with or without target identity.

In the first methodology, the aim is to know about where each individual target is. The identity information is preserved by carrying out a *data association* between measurements and targets using the rules of combinatorics. The algorithms based on this methodology can be classified into two categories as single hypothesis and multiple hypotheses algorithms.

Single hypothesis algorithms utilize only the current scan (frame) of measurements for the final association decision, i.e., they have *sequential decision logic* [4]. They are computationally cheap and especially suitable for real-time applications where quick decision is desired in accordance with the computational power at hand. The most widely used of such algorithms are those, the so-called *hard-assignment*[2] algorithms, such as, the Global Nearest Neighbor (GNN) [2], Nearest Neighbor Joint Probabilistic Data Association (NNJPDA), [5] and those, the so-called *soft-assignment*[3] algorithms, such as Joint Probabilistic Data Association (JPDA) [2], [6] and its variants [7].

Multiple hypotheses algorithms, on the other hand, utilize multiple scans (frames) of measurements for the final association decision. They propagate all possible hypotheses and defer the decision to a later time step (*deferred decision logic*) [4]. They are computationally expensive but produce better results than single hypothesis algorithms in dense clutter. The most common ones include the Multiple Hypotheses Tracking (MHT) [8] algorithm together with its efficient and practical implementations [9]–[12] and other multiple frame assignment (MFA) algorithms, such as, [13]–[16].

The second methodology, which has recently become more popular, is based on Random Finite Set (RFS) theory [17]. Both the set of targets and the set of measurements are considered as a RFS. The aim is to know about where there are targets without requiring any identification. Hence, the problem of data association is in a sense *circumvented* [18]. Practically used candidates of such algorithms include the probability hypothesis density (PHD) [19] and Cardinalized PHD (CPHD) [20] filters and their Gaussian Mixture approximations [21]. The references [17] and [22] provide a good literature overview and solid mathematical foundation of this methodology.

In this paper, we consider single scan algorithms belonging to the first methodology in which the targets are tracked with their identities. In conventional formulation, such algorithms are integrated into Kalman type of filters *analytically* as in the case of JPDAF [6]. The first work in which these algorithms are integrated into *particle filters* [23], [24] via a *simulation–based framework* was given in [25] where only the sketch of

[1] The problem was described, in the words of Li and Bar-Shalom, as the *crux* of tracking [3].

[2] Where each track is updated with a single measurement.

[3] Where each track is updated with a probabilistically weighted combination of all the measurements.

the algorithm was mentioned without a clear mathematical formulation. The formulation is explained more clearly in [26]. The same idea is also used later in [27] under the name of SIR/MCJPDAF which was claimed to be the particle filtering implementation of the so-called joint state (i.e., *coupled*) JPDAF [2]. The first attempt to integrate the decoupled (i.e., the so-called *independent* state) JPDA into particle filtering appeared in robotics [28]. Some of the equations in [28] were corrected later in [29]. Other particle filter implementations of decoupled/coupled JPDA were presented in [30]–[33].

Although in the aforementioned papers above, the JPDA (coupled or decoupled) is tried to be integrated into particle filtering, to the best of our knowledge, the decoupled JPDA is firstly integrated into particle filtering in a general and mathematically correct way in [34]. The formulation presented in [34] is more general than [29] in the sense that it takes multiple sensors and arbitrary proposal densities into account.

The other ideas on implementing single scan data association algorithms with particle filtering were proposed later in [35] and recently in [36]. In [35], the coupled JPDA, the NNJPDA and its more general track-coalescence avoiding version proposed in [37] were implemented in particle filtering framework. In [36], a particle filtering algorithm that considers the measurement-to-target association maximizing the *predictive likelihood*, called MPFF, was proposed. As explained in the following sections, this algorithm can be seen as a special case of the particle filtering implementation of the NNJPDA approach proposed in the current paper.

In this paper, we present an overview of the literature and compare some of the existing and newly proposed particle filtering algorithms for tracking multiple targets in clutter. In section II, we give the problem formulation and the conceptual solution by emphasizing its main difficulties. Then, in section III and IV, we discuss how to overcome these difficulties in particle filtering and present some of the existing and newly proposed solutions. The paper ends with a simulation section where all the algorithms are compared through a set of artificial scenarios relevant for closely spaced target tracking.

II. Problem Formulation and Conceptual Solution

We consider tracking N_T targets in clutter using a single sensor. We formulate the problem based on the following assumptions.

A. Assumptions

A1. The number of targets is assumed fixed and known.

A2. Each target state at time step k, denoted by x_k^t $t=1,2,\ldots,N_T$, *is assumed independent of each other and makes its transition to the next time step, according to a known nonlinear transition model*:

$$x_{k+1}^t = f\left(x_k^t\right) + \xi_{k+1}^t \tag{1}$$

where the process noise $\left\{\xi_k^t\right\}$ *is a white sequence of known distribution* $\xi_k^t \sim p_\xi^t(\cdot)$.

Following *A2*, each target state becomes a Markov process [38, p.66]:

$$p\left(x_k^t \mid x_{0:k-1}^t\right) = p\left(x_k^t \mid x_{k-1}^t\right),\ t=1,2,\ldots,N_T \tag{2}$$

where $x_{0:k}^t \triangleq \left\{x_0^t, x_1^t, \ldots, x_k^t\right\}$ denotes the state sequence up to and including time step *k*. The state transition density $p\left(x_k^t \mid x_{k-1}^t\right)$ for target *t*, can be determined from the known state transition model in (1) as

$$p\left(x_k^t \mid x_{k-1}^t\right) = p_\xi^t\left(x_k^t - f\left(x_{k-1}^t\right)\right) \tag{3}$$

A3. An initial density for each target is also assumed available: $x_0^t \sim p_0^t(\cdot)$, $t=1,2,\ldots,N_T$ *(Bayesian framework).*

A4. The target-originated measurements are given by the known nonlinear transition model:

$$z_k^t = h\left(x_k^t\right) + \eta_k,\ t=1,2,\ldots,N_T \tag{4}$$

where $h(\cdot)$ *is a nonlinear transformation and* $\{\eta_k\}$ *is the measurement error noise, assumed a white sequence of known distribution* $\eta_k \sim p_\eta(\cdot)$.

A5. We assume the process noise sequences, $\left\{\xi_k^t\right\}$, *the initial states,* x_0^t *for* $t=1,2,\ldots,N_T$ *and the measurement noise sequence* $\{\eta_k\}$ *are mutually independent for all k.*

The joint-state of all targets is formed by stacking each individual target state into a single vector, i.e., $X_k \triangleq \left[x_k^1; x_k^2; \ldots; x_k^{N_T}\right]$ where we use a semicolon ";" to denote stacking operation. Due to *A2*, the state transition density for the joint state can be factorized over individual target state transition densities as

$$p\left(X_k \mid X_{k-1}\right) = \prod_{t=1}^{N_T} p\left(x_k^t \mid x_{k-1}^t\right) \tag{5}$$

Similarly, a priori information for the joint state becomes:

$$X_0 \sim p_0(\cdot) \triangleq \prod_{t=1}^{N_T} p_0^t(\cdot) \tag{6}$$

At each time step *k*, the sensor produces a set of m_k measurements: $Z_k \triangleq \left\{z_k^1, z_k^2, \ldots, z_k^{m_k}\right\}$ which consists of the measurements originated from true targets, if detected, and false alarms due to clutter. As noted before, it is not known with certainty which measurement belongs to which target and more generally whether a measurement belongs to a target or not (*measurement origin uncertainty* [2]).

A6. The target-originated measurements given by (4) are assumed to be resolved and they are available with a known detection probability, $P_D^t(k) \le 1$ *which is possibly time & target-varying.*

A7. Each measurement, whether it is target-originated or a false alarm, is assumed independent of each other.

A8. Clutter-originated measurements are assumed identically and uniformly distributed over the sensor's surveillance region of volume V. *That is, the probability density function (pdf) for the spatial position of false alarms is*

$$p_C(z) = V^{-1} \tag{7}$$

A9. Let m_k^C *be the number of clutter-originated measurements (false alarms) in a volume of interest* V *at time* *step k. Then, m_k^C is assumed a Poisson-distributed random variable with mean $\lambda_C V$ where $\lambda_C \triangleq P_{FA}/V_{RC}$ is the spatial density[4] of false alarms with P_{FA} being the probability of false alarm per scan per resolution cell and V_{RC} being the resolution cell volume, assumed constant and the same for all the cells. Then, the probability mass function (pmf) of m_k^C is:*

$$m_k^C \sim \mu_C(m) = \mathcal{P}(m;\lambda_C V) \triangleq \frac{(\lambda_C V)^m \exp(-\lambda_C V)}{m!} \tag{8}$$

where $\mathcal{P}(m;\bar{m})$ denotes a Poisson pmf with mean $\bar{m}$ for a dummy variable m.

B. Problem Definition

The problem of multi-target tracking is to estimate the joint-state X_k, given all the available information, which is the union of the sets of measurements up to and including time step *k*: $\mathcal{Z}_k \triangleq Z_{1:k} \triangleq \{Z_1, Z_2, \ldots, Z_k\}$.

C. Conceptual Solution

Solving the multitarget tracking problem defined above is achieved if one could estimate the a posteriori probability density function (pdf) $p(X_k \mid \mathcal{Z}_k)$ of the joint state X_k. Because, in that case, the solution (an estimate of X_k) can be extracted by either taking mode or mean of $p(X_k \mid \mathcal{Z}_k)$ which corresponds to two well-known point estimation schemes, *maximum a posteriori* (MAP) or *minimum mean square error* (MMSE) estimation, respectively:

$$\hat{X}_{k|k}^{MAP} \triangleq \mathrm{MODE}\left[p(X_k \mid \mathcal{Z}_k)\right] \triangleq \arg\max_{X_k} p(X_k \mid \mathcal{Z}_k) \tag{9}$$

$$\hat{X}_{k|k}^{MMSE} \triangleq \mathrm{MEAN}\left[p(X_k \mid \mathcal{Z}_k)\right] \triangleq \int X_k p(X_k \mid \mathcal{Z}_k)\, dX_k \tag{10}$$

Under the Markov assumptions:

$$p(X_k \mid X_{k-1}, \mathcal{Z}_{k-1}) = p(X_k \mid X_{k-1}) \tag{11}$$

$$p(Z_k \mid X_k, \mathcal{Z}_{k-1}) = p(Z_k \mid X_k), \tag{12}$$

The posterior can be conceptually calculated via Bayesian formalism as:

$$p(X_k \mid \mathcal{Z}_k) = \frac{1}{p(Z_k \mid \mathcal{Z}_{k-1})} p(Z_k \mid X_k)\, p(X_k \mid \mathcal{Z}_{k-1}) \tag{13}$$

where

$$\begin{aligned} p(Z_k \mid \mathcal{Z}_{k-1}) &\triangleq \int p(Z_k, X_k \mid \mathcal{Z}_{k-1}) dX_k \\ &= \int p(Z_k \mid X_k) p(X_k \mid \mathcal{Z}_{k-1})\, dX_k \end{aligned} \tag{14}$$

is a normalizing density, called the *predicted measurement density* or the *predictive likelihood*, and,

$$\begin{aligned} p(X_k \mid \mathcal{Z}_{k-1}) &\triangleq \int p(X_k, X_{k-1} \mid \mathcal{Z}_{k-1})\, dX_{k-1} \\ &= \int p(X_k \mid X_{k-1})\, p(X_{k-1} \mid \mathcal{Z}_{k-1})\, dX_{k-1} \end{aligned} \tag{15}$$

is the *one-step ahead prediction* density. Note that, the state transition density $p(X_k \mid X_{k-1})$ is already defined in (5) and to be complete, one needs to define the *likelihood* $p(Z_k \mid X_k)$. Indeed, as explained in the following sections, how to define this likelihood constitutes one of the main difficulties in this conceptual solution framework.

D. Main Difficulties in Applying the Conceptual Solution

Although the conceptual solution to the multitarget tracking problem is simply given by either (9) or (10), there are two main difficulties in applying such a conceptual framework.

The first main difficulty, as mentioned above, is to define the likelihood $p(Z_k \mid X_k)$ which is not trivial under measurement origin uncertainty. Indeed, solving this problem is equivalent to deciding how to update the tracking filter with measurements of uncertain origin, and as mentioned before, still challenges the tracking community.

Even one could get rid of the measurement origin uncertainty problem by some means, the second main difficulty of the conceptual solution framework lies in evaluating the multidimensional integrals given in (14) and (15). These integrals are intractable for most of the practical scenarios. They can be analytically evaluated in closed-form only in very few special cases. The most common is the one that the system and measurement models given in (1) and (4) obey the so-called *Linear-Gaussian (LG)* assumption [38, pp. 201]. In that case, the Kalman filter [1] provides the analytic solution to the mean and the covariance which fully characterize the Gaussian posterior.

In the following sections, we will discuss how to overcome the two difficulties mentioned above in a single elegant framework, namely, *particle filtering under measurement origin uncertainty*. First, we will present the "plain" particle filtering idea, which overcomes the second difficulty above, in section III. Then, in section IV, we will present some examples of how this idea can be implemented under measurement origin uncertainty to overcome the first difficulty as well.

III. A TRACTABLE EVALUATION OF BAYESIAN RECURSION: PARTICLE FILTERING

The particle filters [23], [24] provide a practical means for calculating the intractable Bayesian integrals, given in (14) and (15), by applying well-known numerical Monte Carlo integration methods *sequentially* in time, hence the name *Sequential Monte Carlo* methods [40]. The simple idea is to represent probability density functions (pdfs) with random samples (called *particles*) with associated *importance weights*. Let's assume that the posterior at time step $k-1$ is represented by a set of N particles and associated importance weights: $\left\{X_{k-1}^{(n)}, w_{k-1}^{(n)}\right\}_{n=1}^{N}$. Then, particle filtering basically consists of generating N new samples from a suitably designed *proposal distribution*, which may depend on the old state and the new measurements:

$$X_k^{(n)} \sim q\left(X_k \mid X_{k-1}^{(n)}, Z_k\right),\ n = 1, 2, \ldots, N \tag{16}$$

and calculating the corresponding new importance weights:

[4] i.e., the "intensity rate" parameter of the underlying Poisson process.

$$w_k^{(n)} \propto p\left(Z_k \mid X_k^{(n)}\right) \frac{p\left(X_k^{(n)} \mid X_{k-1}^{(n)}\right)}{q\left(X_k^{(n)} \mid X_{k-1}^{(n)}, Z_k\right)} w_{k-1}^{(n)} \text{ with } \sum_{n=1}^{N} w_k^{(n)} = 1 . \qquad (17)$$

The new particle set $\left\{X_k^{(n)}, w_k^{(n)}\right\}_{n=1}^{N}$ is then approximately distributed according to the posterior at time step *k*. The algorithm recursion is initiated by sampling from the initial distribution given in (6), i.e., $\left\{X_0^{(n)}\right\}_{n=1}^{N} \sim p(X_0)$ and setting the weights to $w_0^{(n)} = 1/N$ for $n = 1,2,\ldots,N$. From time to time, a *resampling* stage is necessary to avoid *degeneracy* of the importance weights [24]. At any time step k, the MAP and MMSE outputs can be approximated using particles as

$$\hat{X}_{k|k}^{MAP} \approx \underset{X_k^{(n)}}{\arg\max}\, p\left(Z_k \mid X_k^{(n)}\right) \sum_{p=1}^{N} p\left(X_k^{(n)} \mid X_{k-1}^{(p)}\right) w_{k-1}^{(p)} \qquad (18)$$

$$\hat{X}_{k|k}^{MMSE} \approx \sum_{n=1}^{N} X_k^{(n)} w_k^{(n)} \qquad (19)$$

It is noted in [41] that the MAP point estimation given in (18) is a better approximation than the straightforward "maximum weight approximation" where the particle with the maximum weight is returned as the MAP output.

IV. Particle Filtering Under Measurement Origin Uncertainty

As sketched above, after initialization, particle filtering is nothing but the consecutive application of (16) and (17) and then applying resampling when necessary. At any time step, MAP or MMSE estimation outputs can be extracted from the particle representation as explained above. The only requirements to be able to run such a generic particle filtering algorithm are

1. To be able to sample from $p_0(\cdot)$,
2. To be able to sample from and evaluate $q(X_k \mid X_{k-1}, Z_k)$ for a given values of X_{k-1} and Z_k,
3. To be able to evaluate $p(Z_k \mid X_k)$ for a given value of X_k.

The first requirement can easily be satisfied in practice. The second one is also satisfied usually by selecting the proposal density as $q(X_k \mid X_{k-1}, Z_k) = p(X_k \mid X_{k-1})$.[5] Such a proposal selection reduces the second requirement to only being able to sample from $p_\xi(\cdot)$, since the density evaluation requirement is straightforwardly achieved through use of (5) and (3).

The most important difficulty in applying particle filtering under measurement origin uncertainty is to satisfy the third requirement, because, as explained before, the corresponding likelihood can only be evaluated if the *true assignment* between measurements and the sources (targets or clutter) is known. But in practice such knowledge is missing due to the problem of measurement origin uncertainty.

[5] Although there are better alternatives depending on the problem at hand [39] [40], we consider the same selection here. Better proposals and how they can improve the effciency of particle representation is out of scope of this paper.

The first particle filtering under measurement origin uncertainty was proposed in [25] where only a sketch of the algorithm was given without much mathematical rigor. The algorithm is explained rigorously later in [26]. The idea was very reasonable: The required likelihood is proposed as a weighted combination of all possible likelihoods, each conditioned on a specific assignment between measurements and sources, with weights being the corresponding *prior* probabilities of the assignments:

$$p(Z_k \mid X_k) = \sum_{\theta_k \in \Theta_k} p(Z_k \mid X_k, \theta_k) \Pr\{\theta_k \mid m_k\} \qquad (20)$$

where θ_k represents any of such specific assignments , called a *joint association event (or assignment hypothesis)* [2] and Θ_k is the set of all such events. A more clear explanation of the same idea was presented later in [27] under the name of SIR/MCJPDAF where more explicit formulas were given for the terms in (20). Although in [27] the authors claimed that they present a particle filtering version of the so-called *coupled* JPDA (i.e., a joint state JPDA) approach [2], this was not the case. Because, the weights in (20) were defined as the "prior" probabilities of the assignments (not the "posterior" probabilities). In that respect, their approach can be regarded as a particle filtering implementation of some sort of "degenerate" coupled JPDA approach. We call it as **JS-D-JPDA-PF** where JS denotes "joint state" and D denotes the "degenerate". Other various ad-hoc implementations of the same idea were also presented in [30] and [32]. In [33], the approach was further assisted with a clustering routine similar to [42] to solve the so-called *mixed labeling* problem [42], [43] which is prominent in joint state particle filters when the targets are closely spaced.

The first of our proposals in the present paper removes the letter "D" in the name of the above algorithm by modifying the likelihood in (20) as:

$$p(Z_k \mid X_k) = \sum_{\theta_k \in \Theta_k} p(Z_k \mid X_k, \theta_k) \Pr\{\theta_k \mid \mathcal{Z}_k\} \qquad (21)$$

where, different from (20), not the "prior" but the "posterior" assignment probabilities are used in weighting each conditional likelihood. The likelihood in (21) exactly corresponds to the likelihood in coupled JPDA [2], hence, constitutes a *proper* implementation of the coupled JPDA in particle filtering framework. We call the resulting algorithm as **JS-JPDA-PF**.

Another likelihood proposal of the present paper is a generalized version of likelihood in MPPF [36] algorithm whose likelihood was defined as:

$$\begin{aligned} p(Z_k \mid X_k) &= p\left(Z_k \mid X_k, \theta_k^*\right) \\ \theta_k^* &\triangleq \underset{\theta_k \in \Theta_k}{\arg\max}\, p\left(Z_k \mid \mathcal{Z}_{k-1}, \theta_k\right) \end{aligned} \qquad (22)$$

where

$$p(Z_k \mid \mathcal{Z}_{k-1}, \theta_k) \triangleq \int p(Z_k \mid X_k, \theta_k) p(X_k \mid \mathcal{Z}_{k-1})\, dX_k \qquad (23)$$

is the conditional version of the predictive likelihood given in (14) which is calculated for a specific measurement assignment hypothesis θ_k. We call this approach as **JS-MAXPLA-PF**. In this paper, we propose to use the following likelihood:

$$p(Z_k \mid X_k) = p(Z_k \mid X_k, \theta_k^*)$$
$$\theta_k^* \triangleq \arg\max_{\theta_k \in \Theta_k} \Pr\{\theta_k \mid \mathcal{Z}_k\} \tag{24}$$

where $\Pr\{\theta_k \mid \mathcal{Z}_k\}$ is the posterior assignment probability, which is defined by [2]:

$$\Pr\{\theta_k \mid \mathcal{Z}_k\} = \frac{1}{c_k} p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k) \Pr\{\theta_k \mid m_k\} \tag{25}$$

where the term $p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k)$ is the same with (23), c_k is a (normalizing) θ_k-independent term and $\Pr\{\theta_k \mid m_k\}$ is the prior assignment probability. Note that (22) can be viewed as a special case of (24) where the term $\Pr\{\theta_k \mid m_k\}$ is ignored in maximization. Using such likelihood in (24) corresponds to the NNJPDA approach [5]. Hence, we call the resulting algorithm as **JS-NNJPDA-PF**. It is well-known that the NNJPDA improves the performance for *track-coalescence*[6] problem [44] which exists especially when the targets are closely spaced. The likelihoods of each algorithm are summarized in Table I.

TABLE I. THE LIKELIHOODS DEFINED IN EACH ALGORITHM

Algorithm	Likelihood	Ref.
JS-D-JPDA-PF	$p(Z_k \mid X_k) = \sum_{\theta_k \in \Theta_k} p(Z_k \mid X_k, \theta_k) \Pr\{\theta_k \mid m_k\}$	[25]–[27]
JS-JPDA-PF	$p(Z_k \mid X_k) = \sum_{\theta_k \in \Theta_k} p(Z_k \mid X_k, \theta_k) \Pr\{\theta_k \mid \mathcal{Z}_k\}$	This paper
JS-MAXPLA-PF	$p(Z_k \mid X_k) = p(Z_k \mid X_k, \theta_k^*)$ where $\theta_k^* \triangleq \arg\max_{\theta_k \in \Theta_k} p(Z_k \mid \mathcal{Z}_{k-1}, \theta_k)$	[36]
JS-NNJPDA-PF	$p(Z_k \mid X_k) = p(Z_k \mid X_k, \theta_k^*)$ where $\theta_k^* \triangleq \arg\max_{\theta_k \in \Theta_k} \Pr\{\theta_k \mid \mathcal{Z}_k\}$	This paper

For the sake of completeness, the terms $\Pr\{\theta_k \mid m_k\}$, $p(Z_k \mid X_k, \theta_k)$, $p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k)$ and $\Pr\{\theta_k \mid \mathcal{Z}_k\}$ appeared in the above likelihood expressions will be given next.

A. The Prior Probability of a Joint Association Event

Lemma-1: Let $T_D(\theta_k)$ be the set of indices of *detected targets* and $m_k^C(\theta_k)$ be the number of false alarms hypothesized under the event θ_k. Then, assuming that $T_D(\theta_k)$ and $m_k^C(\theta_k)$ are independent, the a priori probability of a joint association event conditioned on m_k number of measurements is given by

$$\begin{aligned}\Pr\{\theta_k \mid m_k\} &= \gamma_k m_k^C(\theta_k)! \mu_C\left(m_k^C(\theta_k)\right) \\ &\times \prod_{t \in T_D(\theta_k)} P_D^t(k) \prod_{t \in T_\varnothing(\theta_k)} \left(1 - P_D^t(k)\right)\end{aligned} \tag{26}$$

where $\gamma_k \triangleq \left(m_k! \Pr\{m_k\}\right)^{-1}$ is a θ_k-independent term, $\mu_C\left(m_k^C(\theta_k)\right)$ is the probability mass function for the number of false alarms and $T_\varnothing(\theta_k)$ is the set of indices of *missed (undetected) targets* under θ_k.

Proof: See JPDAF derivations in [2].

Remark-1: Under *A9* (i.e., substituting $\mu_C\left(m_k^C(\theta_k)\right)$ in (8)), (26) can be written as,

$$\Pr\{\theta_k \mid m_k\} = \kappa_k (\lambda_C V)^{m_k^C(\theta_k)} \prod_{t \in T_D(\theta_k)} P_D^t(k) \prod_{t \in T_\varnothing(\theta_k)} \left(1 - P_D^t(k)\right) \tag{27}$$

where $\kappa_k \triangleq \exp(-\lambda_C V)\gamma_k$ is a θ_k-independent term.

B. The Likelihood / Predictive Likelihood Conditioned on a Specific Joint Association Event

Under *A7*, the likelihood conditioned on a specific joint association event can be expressed as:

$$p(Z_k \mid X_k, \theta_k) = \prod_{j \in J_C(\theta_k)} p_C(z_k^j) \prod_{j \in J_T(\theta_k)} p_T\left(z_k^j \mid x_k^{\theta_k(j)}\right) \tag{28}$$

where $p_C(z)$ is the pdf of the spatial position of false alarms and $p_T(z \mid x) = p_\eta(z - h(x))$ is the likelihood of a target-originated measurement with $p_\eta(\cdot)$ being the pdf of the measurement noise. $J_C(\theta_k)$ and $J_T(\theta_k)$ denote the sets of indices of *clutter-originated measurements (false alarms)* and *target-originated measurements* under θ_k, respectively.[7] Based on *A7*, the predictive likelihood conditioned on a specific joint association event can be written as

$$p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k) = \prod_{j \in J_C(\theta_k)} p_C(z_k^j) \prod_{j \in J_T(\theta_k)} p_{\theta_k(j)}\left(z_k^j \mid \mathcal{Z}_{k-1}\right) \tag{29}$$

where

$$p_t\left(z_k^j \mid \mathcal{Z}_{k-1}\right) \triangleq \int p_T\left(z_k^j \mid x_k^t\right) p\left(x_k^t \mid \mathcal{Z}_{k-1}\right) dx_k^t \tag{30}$$

is the *predicted measurement* density, or the *predictive likelihood* for the target *t*.

Remark-2: Under *A8*, (i.e., substituting $p_C(z_k^j)$ in (7)), (28) and (29) can be respectively written as,

$$p(Z_k \mid X_k, \theta_k) = V^{-m_k^C(\theta_k)} \prod_{j \in J_T(\theta_k)} p_T\left(z_k^j \mid x_k^{\theta_k(j)}\right) \tag{31}$$

and

$$p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k) = V^{-m_k^C(\theta_k)} \prod_{j \in J_T(\theta_k)} p_{\theta_k(j)}\left(z_k^j \mid \mathcal{Z}_{k-1}\right). \tag{32}$$

Remark-3: Note that, as opposed to the JPDAF derivations in [2], we keep the general non-Gaussian nature for the predicted measurement density (a.k.a *innovation density* in linear case) in (30). This integral can be approximated with particles as

$$p_t\left(z_k^j \mid \mathcal{Z}_{k-1}\right) \approx \sum_{n=1}^{N} p_T\left(z_k^j \mid x_k^{t,(n)}\right) w_{k-1}^{(n)} \tag{33}$$

where we use the fact that $q(X_k \mid X_{k-1}, Z_k) = p(X_k \mid X_{k-1})$.

[6] This is closely related to the mixed labelling problem in joint state particle filters.

[7] Mathematically, a joint association event, θ_k can be defined as a mapping $\theta_k(\cdot): J \to S$ where, $J = \{1, 2, \ldots, m_k\}$ and $S = \{0, 1, \ldots, N_T\}$ are the sets of indices of measurements and sources (targets or clutter), respectively.

C. The Posterior Probability of a Joint Association Event

The *posterior* probability of a joint association event is given as [2]

$$\Pr\{\theta_k \mid \mathcal{Z}_k\} = \frac{1}{c_k} p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k) \Pr\{\theta_k \mid m_k\} \tag{34}$$

where $\Pr\{\theta_k \mid m_k\}$ is the *prior* probability defined in (26) and $p(Z_k \mid \mathcal{Z}_{k-1}, m_k, \theta_k)$ is the conditional predictive likelihood defined in (29).

Remark-4: Following *Remark-1* and *Remark-2*, a reduced expression for $\Pr\{\theta_k \mid \mathcal{Z}_k\}$ is given as,

$$\begin{aligned}\Pr\{\theta_k \mid \mathcal{Z}_k\} = \rho_k (\lambda_C)^{m_k^C(\theta_k)} \prod_{t \in T_D(\theta_k)} P_D^t(k) \prod_{t \in T_\varnothing(\theta_k)} \left(1 - P_D^t(k)\right) \\ \times \prod_{j \in J_T(\theta_k)} p_{\theta_k(j)}\left(z_k^j \mid \mathcal{Z}_{k-1}\right)\end{aligned} \tag{35}$$

Remark-5: Defining the expression related to the detection parameters as:

$$\mathcal{D}(\theta_k) \triangleq (\lambda_C)^{m_k^C(\theta_k)} \prod_{t \in T_D(\theta_k)} P_D^t(k) \prod_{t \in T_\varnothing(\theta_k)} \left(1 - P_D^t(k)\right) \tag{36}$$

and following the simplified expressions given in *Remark-1*, *Remark-2* and *Remark-4*, the likelihoods for **JS-D-JPDA-PF** and **JS-JPDA-PF** given in Table I can be written as

$$p(Z_k \mid X_k) = \kappa_k \sum_{\theta_k \in \Theta_k} \mathcal{D}(\theta_k) \prod_{j \in J_T(\theta_k)} p_T\left(z_k^j \mid x_k^{\theta_k(j)}\right) \tag{37}$$

and

$$\begin{aligned}p(Z_k \mid X_k) = \rho_k \sum_{\theta_k \in \Theta_k} \Big\{ V^{-m_k^C(\theta_k)} \mathcal{D}(\theta_k) \\ \times \prod_{j \in J_T(\theta_k)} p_T\left(z_k^j \mid x_k^{\theta_k(j)}\right) p_{\theta_k(j)}\left(z_k^j \mid \mathcal{Z}_{k-1}\right) \Big\}\end{aligned} \tag{38}$$

respectively. Here, $\rho_k \triangleq \kappa_k / c_k$ is a θ_k-independent term.

Remark-6: For the time & target-invariant detection probabilities case, i.e., $P_D^t(k) = P_D$, $\forall t$, (36) reduces to

$$\mathcal{D}(\theta_k) \triangleq (\lambda_C)^{m_k^C(\theta_k)} (P_D)^{m_k^T(\theta_k)} (1 - P_D)^{N_T - m_k^T(\theta_k)} \tag{39}$$

Remark-7: The likelihood defined in [27] is exactly based on (37) and (39) except the term $\lambda_C^{m_k^C(\theta_k)} \triangleq (P_{FA}/V_{RC})^{m_k^C(\theta_k)}$ is replaced with $\lambda_C^{m_k^C(\theta_k)} \approx P_{FA}^{\,m_k^C(\theta_k)}$ in [27] and the variable Z_n in (6) of [27] which was wrongly defined as "the number of false alarms in hypothesis n" should be corrected as "the number of missed (undetected) targets in hypothesis n". A similar correction is also needed in [45] and [32].

V. Simulation Experiments

To be able to characterize the performance of different particle filtering algorithms for the data association problem, we follow a "divide and conquer" approach and take a simplified scenario where the data association problem is to be made more pronounced and the problems, such as nonlinearity, non-Gaussianity, model-mismatch (maneuver) [38] and curse-of dimensionality of particle filters in high dimensional state spaces [46], are all suppressed. We are particularly interested in situations where the targets are *closely spaced*. Along this line, we consider tracking (or more properly *localization* of) two closely spaced *stationary* targets in one dimensional (1D) geometry. The state of the targets consists of only position and assumed to obey the simple random walk model given by

$$x_{k+1}^t = x_k^t + \xi_k^t \ , \ t = 1, 2 \tag{40}$$

where ξ_k^t, called the process noise, is assumed to be zero mean white Gaussian sequence with variance σ_p^2, i.e., $\xi_k^t \sim \mathcal{N}(0, \sigma_p^2)$. While we generate ground truths without a process noise which means that the targets are perfectly stationary, we assume a very small value for this parameter during filtering ($\sigma_p^2 = 0.01$). Targets are located at initial positions $x_0^1 = -d/2$ and $x_0^2 = +d/2$. The distance between targets (d) is parameterized by the sensor's measurement accuracy as, $d = a \times \sigma_m$ where $\sigma_m = 10$ [m] is the measurement noise standard deviation and a being a scenario-dependent scalar. We consider a scenario with closely spaced targets by selecting $a = 2$.

The sensor is assumed to produce position only measurements of each target for every $\Delta_k = 1$ second as

$$z_k^t = x_k^t + \eta_k \ , \ t = 1, 2 \tag{41}$$

where the measurement noise $\eta_k \sim \mathcal{N}(0, \sigma_m^2)$ is assumed to be zero mean white Gaussian sequence with variance σ_m^2. The surveillance region of the sensor is taken as the interval $[-10\sigma_m, +10\sigma_m]$, hence its volume, the length of the interval, is $V = 20\sigma_m$. Both targets are assumed to have a unity probability of detection ($P_D^1 = P_D^2 = P_D = 1$) which is constant and time-invariant. We assume that sensor may produce false alarms which are uniformly distributed over the surveillance region. At each time step *k*, the number of such false alarms is assumed to be Poisson distributed with mean $\lambda_C V$ where λ_C is the spatial false alarm density in [m^{-1}]. In the sequel, we perform experiments for $\lambda_C V \in \{0, 5\}$ which correspond to "no clutter" and "dense clutter" scenarios. We run each scenario until $K = 50$ time steps and repeated each scenario for $N_{MC} = 100$ Monte Carlo runs.

In filtering, we assume our prior knowledge on these initial states as Gaussian centered at each initial true state with variance $\sigma_{0|0}^2$. During experiments, we initialized each filter by using single point initialization from a single measurement prior to simulation, i.e., $\hat{x}_{0|0}^t = z_0^t$ and set the initial variance to the measurement error variance, $\sigma_{0|0}^2 = \sigma_m^2$. for preserving the filter's *consistency* [38, pp. 232]. We simulate four joint filters given in Table I, namely, **JS-D-JPDA-PF** [27], **JS-JPDA-PF**, **JS-MAXPLA-PF** [36] and **JS-NNJPDA-PF,** and two "baseline" filters. One of the baseline filters is chosen as the **MC-JPDAF** of [34] which is the first and exact particle filtering implementation of the decoupled (i.e., independent state) JPDA approach, and the other one is the Kalman filter

(**KF**) with *perfect* data association, which constitutes a kind of ultimate performance limit (actually a Cramer-Rao Lower Bound – CRLB) to the problem.[8] We implemented the joint state filters without any "plug-in"s, like clustering routine considered in [33] and [42]. In all the particle filters, the number of particles is chosen as $N = 250$ which seems enough for this low dimensional (1D) problem. No gating[9] is applied to the measurements. We consider MMSE outputs for all the filters. Moreover, for joint-state filters, MAP point estimates are also computed using (18).

The result of "no-clutter" scenario is given in Fig. 1. Note that we use deliberately soft colors and dashed lines for the soft assignment algorithms, **JS-D-JPDA-PF** [27] and **JS-JPDA-PF**, and more prominent colors and continuous lines for the hard assignment algorithms, **JS-MAXPLA-PF** [36] and **JS-NNJPDA-PF**. Moreover, we use a continuous line with "dot" markers for both of the baseline algorithms, **MC-JPDAF** [34] and **KF** with perfect data association.

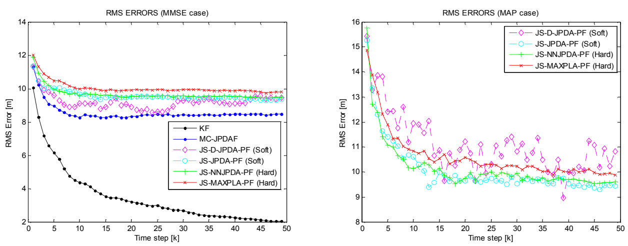

Figure 1. The filtering results smoothed over 100 Monte Carlo runs for "no-clutter" scenario. Both MMSE and MAP outputs are presented.

The conclusions drawn for the "no-clutter" scenario are as follows:

- The independent state filter **MC-JPDAF** [34] with MMSE output performs *better* than any "plain" joint-state filter (either with MMSE or MAP outputs).
- Among joint state filters, MMSE outputting performs better than their MAP counterparts. This is a contradictory result with the previously obtained results in the literature, such as, [41].
- Among joint state filters, the degenerate implementation of the coupled JPDAF, namely, **JS-D-JPDA-PF** [27] performs better than the proposed exact implementation **JS-JPDA-PF** in MMSE outputting case, but the situation is reversed in MAP outputting where the degenerate implementation behaves rather jumpy. This result clearly shows that how to decide on point estimate extraction from a given particle cloud is very important for the final filter performance.

The result of "dense-clutter" scenario is given in Fig. 2. The following conclusions are drawn:

- The independent state filter **MC-JPDAF** [34] with MMSE output performs *worse* than any "plain" joint-state filter (either with MMSE or MAP outputs).
- Among joint state filters, MMSE outputting performs better than their MAP counterparts.
- Among joint state filters, the soft assignment algorithms perform better than their hard assignment counterparts for MMSE outputting case. Indeed, the proposed exact implementation **JS-JPDA-PF** performs well in both MMSE and MAP output case.

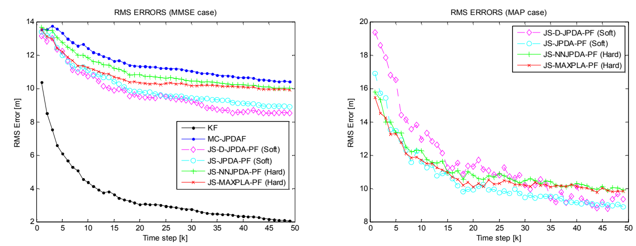

Figure 2. The filtering results smoothed over 100 Monte Carlo runs for "dense-clutter" scenario. Both MMSE and MAP outputs are presented.

VI. Conclusion

One should note that the conclusions drawn from simulation results are correct within the boundaries of the simulation setup. Especially the conclusion "MAP outputting performs better than MMSE" should be carefully investigated by accounting *track swap* and *mixed-labeling* phenomena. So a possible near future study might be performing extra simulations by taking these phenomena into account and considering other evaluation metrics such as MOSPA [49].

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[8] This is somewhat a looser CRLB under no measurement origin uncertainty case. For the actual problem, i.e., tracking under measurement origin uncertainty, a more tighter Posterior Cramer-Rao Lower Bound (PCRLB) can be found by utilizing a similar procedure with [47], see [2].

[9] The another interesting research path (see e.g., [48].) might be to investigate the effect of gating parameters on performance of *non-Gaussian nonlinear* multiple target tracking especailly when the targets are closely spaced.

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