

# A Comparison of Large Deflection Analysis of Bending Plates by Dynamic Relaxation

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## Abstract

In this paper, various dynamic relaxation methods are investigated for geometric nonlinear analysis of bending plates. Sixteen wellknown algorithms are employed. Dynamic relaxation fictitious parameters are the mass matrix, the damping matrix and the time step. The difference between the mentioned tactics is how to implement these parameters. To compare the efficiency of these strategies, several bending plates' problems with large deflections are solved. Based on the number of iterations and analysis time, the scores of the different schemes are calculated. These scores determine the ranking of each technique. The numerical results indicate the appropriate efficiency of Underwood and Rezaiee-Pajand & Alamatian processes for the nonlinear analysis of bending plates.

## Keywords

Dynamic relaxation · Mass · Damping · Time step · Large deformations · Bending plate

## 1 Introduction

Bending plate structures are important in mechanical and civil engineering. Therefore, a great deal of research has been done to describe their behavior [1]. Von Kármán formulated differential equations for large deformations of plates in 1910. Levy solved Von Kármán's equations using trigonometric functions and Fourier series. He obtained the equations that govern the behavior of quadrilateral thin plate bending with large displacements [2]. Bergan and Clough established finite element models for thin plates and shells based on Rayleigh-Ritz method [3]. Yang and Bhatti formulated a nonlinear element for solving the static and dynamic cases of plate bending by using the updated Lagrangian approach [4].

The analysis of elastic plates with large deformations is very difficult and there are few approaches to find an exact solution. Numerical and approximate solution procedures were developed for large displacements with the increasing processing power of modern computers. One of these tactics is called dynamic relaxation. Rushton is the first one to apply this scheme to find the solution of nonlinear problems [5]. Moreover, this investigator employed dynamic relaxation technique to analyze stress and post-buckling behavior of plates [6]. Taking into account the geometric nonlinearity, the following equations describing the state of plate movements were used in the mentioned reference:

$$u = u_0 - z \frac{\partial w}{\partial x}, \quad v = v_0 - z \frac{\partial w}{\partial y}, \quad w = w_0 \quad (1)$$

In these relations,  $u_0$ ,  $v_0$  and  $w_0$  are the mid-plane displacements in the  $x$ ,  $y$  and  $z$ -direction, respectively. The nonlinear stiffness formulations for large deflection analysis of plates are utilized for the rectangular plate finite element. The elements have five degrees of freedom at each nodal point. Two of these are in-plane displacements,  $u$  and  $v$  in the  $x$  and  $y$  directions, correspondingly. One transverse deflection  $w$ ; two rotations  $w_x$ , and  $w_y$ , about the  $y$  and  $x$  axes, respectively, are the other three degrees of freedom. In this structure, the normal strains in the  $x$ ,  $y$  directions are  $\varepsilon_x$  and  $\varepsilon_y$ , correspondingly. Moreover,  $\gamma_{xy}$  shows the shear strain. The plate strains can be written in terms of the

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middle surface deflections, in the subsequent form:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \quad (2)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 - z \frac{\partial^2 w}{\partial y^2} \quad (3)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + 2z \frac{\partial^2 w}{\partial x \partial y} \quad (4)$$

What come in the rest of paper has been carried out on bending plate based on the dynamic relaxation scheme. Basu and Dawson used dynamic relaxation method to study the small deflection behavior of rectangular orthotropic and isotropic sandwich plates with uniform or varying cross section [7]. Rushton evaluated the buckling behavior of initially curved plates subjected to lateral loading [8]. Turvey and Wittrick analyzed the post-buckling behavior of laminated plates with large deformations [9]. Alwar and Rao proposed the nonlinear solution of orthotropic clamped plates with constant thickness and subjected to uniform lateral loading [10, 11]. Rushton and Hook obtained the response of plates and beams with large displacements and nonlinear stress and strain behavior [12]. They also carried out buckling analysis of beams and plates onto an intermediate support [13]. Frieze carried out the analysis of plates, including both the nonlinear effects of material and geometry [14]. Turvey investigated the dynamic relaxation solution in the geometric nonlinear behavior of tapered annular plates [15]. Pica carried out transient and pseudo-transient analysis of the Mindlin plates [16].

Al-Shawi and Mardirosoian used a combination of finite element and dynamic relaxation method along with weighted coefficients for mass and damping to find the response of bending plates [17]. Zhang and Yu proposed an improved adaptive dynamic relaxation algorithm and used it to solve the elastoplastic bending of circular plates [18]. Turvey and Osman utilized finite difference dynamic relaxation strategy to geometrically nonlinear analysis of isotropic rectangular Mindlin plates [19]. Turvey and Salehi analyzed the large deformation of sector plates subject to uniform loading by using the dynamic relaxation and finite difference methods [20]. They made comparison studies, as well [21]. Kadkhodayan and Zhang carried out buckling and post-buckling analysis of plates using dynamic stability criteria and the dynamic relaxation process [22]. Turvey and Salehi studied linear and nonlinear analysis of the composite plates [23]. Salehi and Aghaei investigated circular viscoelastic plates. They applied the effect of higher-order shear deformations in their formulation [24]. Falahatgar and Salehi carried out a post-buckling analysis of the annular sector plate. They also evaluated geometrically nonlinear analysis of polymeric plates. They analyzed higher-order shear deformations by utilizing a finite difference form of the dynamic relaxation technique [25]. Moreover, they studied geometrically nonlinear viscoelastic analysis of annular sector composite plates [26]. Gol-

makani and Kadkhodayan investigated the nonlinear bending of FGM annular sector plates [27]. Falahatgar and Salehi found the solution to the bending response of unidirectional polymeric laminated composite plates [28].

At this stage, a brief review of the fictitious parameters' estimation for the dynamic relaxation procedure is presented. Brew and Brotton suggested that the mass of each degree of freedom be proportional to its diagonal element in the structural stiffness matrix [29]. Bunce presented a method for calculating critical damping in the dynamic relaxation tactic using Rayleigh principle [30]. Cassel and Hobbs estimated the artificial mass by using Gershgorin's circle theorem to check the convergence and numerical stability of nonlinear analysis with the dynamic relaxation scheme [31]. Papadrakakis suggested an automatic technique for finding the fictitious parameters. The difficulty of his solution is in the first assumption of two factors [32]. Underwood presented one of the most famous formulations for dynamic relaxation iterations [33]. Qiang obtained the artificial damping and time using Rayleigh principle [34]. Zhang et al. proposed the nodal damping model [35]. Munjiza et al. supposed that damping is proportional to the power of the mass and stiffness matrices. They showed that when the damping matrix is  $2M(M^{-1}S)^{0.5}$ , all modes will be critically damped [36, 37].

Rezaiee-Pajand and Taghavian Hakkak calculated displacement by utilizing the first three terms of Taylor series [38]. Kadkhodayan et al. obtained a relationship for the time step by minimizing the residual forces [39]. Rezaiee-Pajand and Sarafrazi presented the optimal time step ratio and the critical damping for nonlinear structural analysis [40]. Moreover, Rezaiee-Pajand and Alamatian proposed a new approach for estimating the fictitious damping and mass [41]. In addition, Rezaiee-Pajand et al. presented a new algorithm for the calculation of the damping matrix [42]. This was reached by minimizing the error between two successive steps. Sarafrazi and Rezaiee-Pajand suggested an equation for finding the time step ratio. The damping is zero in their process [43]. Rezaiee-Pajand et al. obtained another tactic for artificial time step based on the unbalanced energy [44]. Alamatian developed a new equation to evaluate the fictitious mass in kinetic dynamic relaxation [45]. Rezaiee-Pajand et al. investigated the efficiency of twelve methods of dynamic relaxation for the finite element analysis of frame and truss structures [46]. They introduced the top five procedures by comparing their performance. Rezaiee-Pajand and Rezaiee proposed a new time step for kinetic dynamic relaxation in the latest study [47].

The available research papers indicate that many studies have been carried out about the dynamic relaxation analysis of bending plate. According to what was stated so far, there are different solution approaches for finding the large deflection in these problems. However, the effectiveness of the various dynamic relaxation schemes in the geometrically nonlinear analyses of bending plate has not been studied yet. In this paper, sixteen different known dynamic relaxation processes are studied. It

should be noted that the difference between these schemes is in the estimation of the fictitious parameters. Dunkerley method is used to find the fictitious damping matrix. This approach has not been implemented so far. In order to improve the performance of the mentioned techniques, the authors also suggest some of the artificial factors for different tactics. A score will be assigned to each procedure based on the number of iterations or the total duration analysis. The final ranking of each algorithm will be obtained after solving all the problems.

## 2 The dynamic relaxation method

Dynamic relaxation is one of the explicit methods to solve a system of simultaneous equations. In this process, a fictitious mass and damping are added to the static structural equations system to obtain a fictitious dynamic system. In the dynamic relaxation approach, the velocity variations are assumed to be linear and the acceleration is supposed to be constant for each time step  $t$ . Thus, the following equalities can be obtained for the iterative relations of this tactic by using central finite differences:

$$\dot{X}_i^{n+\frac{1}{2}} = \frac{2m_{ii}^n - C_{ii}^n t^n}{2m_{ii}^n + C_{ii}^n t^n} \dot{X}_i^{n-\frac{1}{2}} + \frac{2t^n}{2m_{ii}^n + C_{ii}^n t^n} (p_i^n - f_i^n), \quad (5)$$

$$i = 1, 2, \dots, ndof$$

$$X_i^{n+1} = X_i^n + t^{n+1} \dot{X}_i^{n+\frac{1}{2}}, \quad i = 1, 2, \dots, ndof \quad (6)$$

The terms  $m_{ii}^n$ ,  $C_{ii}^n$ ,  $t^n$  and  $f_i^n$ , are the  $i$ -th diagonal entry of the fictitious mass and damping matrices, the virtual time step, the  $i$ -th entry of the internal force vector in the  $n$ -th iteration of the dynamic relaxation procedure, respectively. The external load of the static structure is displayed with  $p_i^n$ . Moreover,  $ndof$  denotes the number of degrees of freedom for the system. Furthermore, the vectors  $X$  and  $\dot{X}$  indicate the displacement and velocity, correspondingly. Equalities (5) and (6) are repeated until convergence to a stable response is achieved. It is assumed that the mass and damping matrices are diagonal. This assumption leads to the explicit solutions for the relations of the dynamic relaxation scheme, and they are solved by using vector operators alone. It should be noted that the vector of residual force  $\mathbf{R}$  that causes fictitious oscillations of the structure has following relation:

$$R = P - F \quad (7)$$

There are various techniques to estimate the artificial parameters. It is worth noting that the stability of dynamic relaxation solution greatly depends on the mass and the damping. Hence, extensive researches have been carried out to find these matrices. Moreover, some schemes have also been proposed for the time step. The most well-known strategies for the fictitious parameters of dynamic relaxation approach are presented in the rest of the paper.

### 2.1 Papadrakakis method

Papadrakakis proposed an automated algorithm for finding the factors needed in the dynamic relaxation scheme [32]. He assumed that the mass and damping matrices are as follows:

$$M = \rho D, \quad C = cD \quad (8)$$

Where  $\rho$  and  $c$  are the mass and damping factors, respectively. Moreover,  $\mathbf{D}$  is a diagonal matrix whose entries are the main diagonal entries of the stiffness matrix. Papadrakakis used the following equality for estimating the optimum factors:

$$\left(\frac{t^2}{\rho}\right)_{opt} = \frac{4}{\lambda_{B\max} + \lambda_{B\min}} \quad (9)$$

$$\left(\frac{ct}{\rho}\right)_{opt} = \frac{4\sqrt{\lambda_{B\max} \cdot \lambda_{B\min}}}{\lambda_{B\max} + \lambda_{B\min}} \quad (10)$$

In the present relation,  $\lambda_{B\min}$  and  $\lambda_{B\max}$  are the minimum and maximum eigenvalues for the matrix  $B = D^{-1}S$ , correspondingly. The stiffness matrix is shown by  $\mathbf{S}$ . Lower and upper bounds can be obtained from the Eqs. (11) and (12):

$$\lambda_{B\min} = -\frac{\lambda_{DR}^2 - \beta\lambda_{DR} + \alpha}{\lambda_{DR}\gamma} \quad (11)$$

$$|\lambda_{B\max}| < \max_i \sum_{j=1}^{ndof} |b_{ij}| \quad (12)$$

The rate of error reduction between two successive iterations is shown by  $\lambda_{DR}$  and can be calculated from Eq. (13) as follows:

$$\lambda_{DR} = \frac{\|X^{n+1} - X^n\|}{\|X^n - X^{n-1}\|} \quad (13)$$

Also, the factors  $\alpha$ ,  $\beta$  and  $\gamma$  can be found from Eq. (14) to (16):

$$\alpha = \frac{2 - ct/\rho}{2 + ct/\rho} \quad (14)$$

$$\beta = \alpha + 1 \quad (15)$$

$$\gamma = \frac{2t^2/\rho}{2 + ct/\rho} \quad (16)$$

First, the values  $\lambda_{B\min}$  and  $\lambda_{B\max}$  are assigned, and the dynamic relaxation process starts. The term  $\lambda_{B\min}$  is obtained from Eq. (11) when  $\lambda_{DR}$  converges to a constant value. In Papadrakakis strategy, the time step is assumed to be a constant. In this paper, the time step is set to be equal 1.

## 2.2 Underwood procedure

In this method, the mass matrix is calculated using Geršgorin circle theory. Eq. (17) shows that. Here, the symbol  $S$  indicates the stiffness matrix. It is worth noting that Underwood assigned time step equal to 1.1 to ensure the stability of solution technique.

$$m_{ii} = \frac{t^2}{4} \sum_{j=1}^{ndof} |S_{ij}| \quad (17)$$

Moreover, the fictitious damping matrix is calculated from  $C = 2\omega_0 M$ . In this equality,  $\omega_0$  is the minimum frequency of the fictitious dynamic system. This factor is estimated by using Rayleigh principle as follows.

$$\omega_0 = \sqrt{\frac{X^T S^L X}{X^T M X}} \quad (18)$$

Here,  $S^L$  is the local stiffness matrix, and its elements are obtained by the following equation [33].

$$S_{ii}^{L,n} = \frac{f_i(X^n) - f_i(X^{n-1})}{t \dot{X}_i^{n-\frac{1}{2}}} \quad (19)$$

In the Eq. (18), if the value of the second root is not positive, then it is assumed that the damping is zero. The maximum value of angular frequency is two [33]. Therefore, if  $\omega_0$  is greater than 2, a value less than 2 is used for it (for example, 1.9).

## 2.3 Qiang tactic

In this approach, the mass matrix is computed from the sum of the absolute values of the elements of rows of the stiffness matrix. Qiang suggested Eqs. (20) and (21) for the optimal values of damping and time step [34].

$$C_{ii} = 2 \sqrt{\frac{\omega_0}{1 + \omega_0}} m_{ii} \quad (20)$$

$$t = \frac{2}{\sqrt{1 + \omega_0}} \quad (21)$$

Furthermore, the minimum system frequency when it is in free oscillation can be calculated as follows by using Rayleigh principle [34].

$$\omega_0 = \sqrt{\frac{X^T S X}{X^T M X}} \quad (22)$$

## 2.4 Zhang approach

Zhang and Yu achieved the damping using Rayleigh principle as follows [18]:

$$\omega_0 = \sqrt{\frac{X^T F}{X^T M X}} \quad (23)$$

$$C = 2\omega_0 M \quad (24)$$

The fictitious mass matrix in this tactic, is like that of Underwood solution. The time step is also equal to one. It is worth

emphasizing; Zhang et al. have suggested a formula for the initial displacement. In other words, these researchers used a value other than zero for the initial displacement. It should be noted that the dynamic relaxation scheme will converge to a stable response with any arbitrary initial displacement [48]. Rezaiee-Pajand et al. showed that this strategy has not a good performance for the initial displacement [46]. Hence, in this paper, zero vector is used to begin the dynamic relaxation process.

## 2.5 The nodal damping algorithm

In the previous procedures, the damping factor is the same for all degrees of freedom of the structure. Noted that this factor is calculated again in each iteration. Dynamic relaxation technique can be improved by using various damping factors. On this basis, Kadkhodayan et al. proposed the following equality to calculate the damping [35].

$$C_k = \zeta_k m_{kk} k = 1, 2, \dots, N \quad (25)$$

$$\zeta_k^n = 2 \left[ \frac{(X_k^n)^T f_k^n}{(X_k^n)^T m_{kk}^n (X_k^n)} \right]^{\frac{1}{2}} \quad (26)$$

Here, the number of nodes of the structure is shown by  $N$ . The Eq. (26) is summed up over all the degrees of freedom at each node. The above equality shows that in this approach, the damping factor is the same for degrees of freedom of every node. The fictitious mass matrix is obtained from Eq. (17). Moreover, the time step is equal to 1.

## 2.6 Rezaiee-Pajand and Taghavian Hakkak technique

In this tactic, the diagonal elements of the mass matrix are considered to be proportional to their corresponding values in the stiffness matrix. Eq. (27) shows the mathematical formula [38]. Rezaiee-Pajand and Taghavian Hakkak proposed that  $\alpha = 0.6$ .

$$m_{ii} = \alpha S_{ii} \quad (27)$$

Moreover, the damping is obtained from Qiang solution. The time step is equal to one. These researchers suggested the following equation for the displacement by using the first three polynomials of the Taylor series.

$$X^{n+1} = X^n + t \dot{X}^n + \frac{t^2}{2} \ddot{X}^n \quad (28)$$

Acceleration  $\ddot{X}^n$  and velocity  $\dot{X}^n$  can be obtained from equations (29) and (30), correspondingly [38].

$$R^n = M^n \ddot{X}^n + C^n \dot{X}^n \quad (29)$$

$$\dot{X}^n = \frac{X^n - X^{n-1}}{t} \quad (30)$$

## 2.7 Kinetic damping process

Dynamic relaxation is two types: viscous and kinetic. Damping is negligible in the kinetic dynamic relaxation. In other words, matrix  $C$  in Eq. (5) is set to be equal to zero. When a reduction occurs in the amount of kinetic energy of the structure, it shows that a maximum point in the structural kinetic energy graph is passed. In this time, the velocity of all the degrees of freedom reset to zero. If the response at this time is used to start the next step, then convergence will not be achieved. Topping assumed that the peak of the kinetic energy occurs in the middle of the time step [49]. Thus, the displacement calculates in the middle of the step which is obtained from the following equality.

$$X_i^{n-\frac{1}{2}} = X_i^{n+1} - \frac{3}{2}t\dot{X}_i^{n+\frac{1}{2}} + \frac{t^2}{2m_{ii}}r_i^n \quad (31)$$

Thus, at the beginning of the dynamic relaxation process, displacement should be equal to (31). On the other hand, under these conditions, Eq. (5) cannot be used to calculate the speed. Hence the following equality is suggested [49].

$$\dot{X}_i^{n+\frac{1}{2}} = \frac{t}{2m_{ii}}r_i^n \quad (32)$$

Where the vector of the residual force  $r_i^n$  is evaluated in the position  $X_i^{n-\frac{1}{2}}$  calculated from (31). Then, the iterations of dynamic relaxation restart with these displacement and velocity vectors in order to maximize the kinetic energy again. This cycle continues until a suitable convergence is achieved. It should be noted that the fictitious time step is equal to one in this algorithm. Moreover, the mass matrix is available by Eq. (33) [50].

$$m_{ii} = \frac{t^2}{2} \sum_{j=1}^{ndof} |S_{ij}| \quad (33)$$

## 2.8 Minimum residual force method

The time step is effective in the numerical stability and convergence rate of the dynamic relaxation procedure. The dynamic behavior of structures is a time-dependent process. Hence, it needs to use a fictitious value for the time step for the transfer of the static problem to the dynamic space. This value should be obtained in such a way as to not only maintain the numerical stability of the scheme, but also reduce the number of iterations required for convergence. Kadkhodayan et al. proposed an optimum time step by minimizing the unbalanced forces [39]. Eq. (34) gives this value:

$$t^{n+1} = \frac{\sum_{i=1}^{ndof} r_i^n f_i^{n+\frac{1}{2}}}{\sum_{i=1}^{ndof} (f_i^{n+\frac{1}{2}})^2} \quad (34)$$

In this relation,  $f_i^{n+\frac{1}{2}}$  is the internal force increment. The following equation shows this quantity [44].

$$f_i^{n+\frac{1}{2}} = \sum_{j=1}^{ndof} S_{ij,T}^n \dot{X}_i^{n+\frac{1}{2}} \quad (35)$$

Here,  $S_{ij,T}^n$  is a tangential stiffness matrix in the middle of the step. It is worth emphasizing; it is difficult to calculate the stiffness matrix at the middle of the step. Therefore, the values of the previous step are employed in the Eq. (35) [44]. It should be noted that the fictitious mass and damping matrices are obtained from Eqs. (17) and (24), respectively.

## 2.9 Rezaiee-Pajand and Alamatian procedure

In 2002, Rezaiee-Pajand and Alamatian presented another equality for the fictitious mass by minimizing the displacement error between two successive iterations and the linearity assumption [41].

$$m_{ii} = \text{Max} \left( \frac{(t^n)^2}{2} S_{ii}^n, \frac{(t^n)^2}{4} \sum_{j=1}^{ndof} |S_{ij}^n| \right) \quad (36)$$

Moreover, They suggested Eq. (37) to calculate the damping [41]. The fictitious minimum frequency  $\omega_0$  is obtained by Eq. (23). The time step is also set equal to 1.

$$C_{ii} = \sqrt{\omega_0^2(4 - t^2\omega_0^2)}m_{ii} \quad (37)$$

## 2.10 Minimum unbalanced energy tactic

Rezaiee-Pajand et al. wrote the out-of-balance energy function of the artificial dynamic system as follows [44]:

$$UBE = \sum_{i=1}^{ndof} \left[ t^{n+1} \dot{X}_i^{n+\frac{1}{2}} \left( r_i^n - t^{n+1} f_i^{n+\frac{1}{2}} \right) \right]^2 \quad (38)$$

They suggested another time step based on the minimal amount of Eq. (38). This proposal leads to two answers for this factor. One of these responses minimizes Eq. (38). If the characteristic equation of Eq. (38) does not have a real answer, the time step of Kadkhodayan method (Eq. (34)) will be used [44]. The mass and damping matrices are achieved from Eqs. (36) and (37), correspondingly.

## 2.11 Rezaiee-Pajand and Sarafrazi approach

In most techniques of calculating the fictitious damping, Rayleigh principle is used to obtain a minimum of eigenvalue. This principle provides an upper bound to the minimum eigenvalue [30]. Rezaiee-Pajand and Sarafrazi used the power iterative process to determine the minimum eigenvalue [40]. In each iteration of the dynamic relaxation, they used one step of the iterative power procedure. On this basis, the damping matrix is obtained from Eq. (39):

$$C_{ii}^n = \sqrt{\lambda_1^n(4 - \lambda_1^n)}m_{ii}^n \quad (39)$$

Here,  $\lambda_1^n$  is the transferred eigenvalue. Its value is estimated from the relation  $\lambda_1^n = \lambda^n + 4$ . The factor  $\lambda^n$  is the eigenvalue that is obtained from the iterative power algorithm. This transfer is carried out to calculate the minimum of the eigenvalue since this method yields the largest eigenvalue [46]. Moreover, this factor is compared with the minimum eigenvalue that are achieved by

using Rayleigh principle, and the lower value is chosen. In this tactic, the time step is constant, and it is equal to one. Moreover, the mass is estimated by Eq. (36).

### 2.12 Zero damping technique

Rezaiee-Pajand and Sarafrazi suggested the relation between the critical damping ratio and the time step [43]. Then, they assumed the damping parameter to be zero and obtained the following formula for the time step ratio  $\gamma$ :

$$\gamma = \frac{t^{n+1}}{t^n} = \frac{1}{(1 + \sqrt{\lambda_1})^2} \quad (40)$$

Here, the power iteration process is used for finding the minimal eigenvalue  $\lambda_1$ . In this strategy, the mass matrix is provided from the Eq. (36). Furthermore, the following equation is utilized instead of Eqs. (5) and (6) in order to calculate the velocity and displacement vectors [43]:

$$\dot{X}^{n+1} = \gamma^n (M^{-1}R + \dot{X}^n) \quad (41)$$

$$X^{n+1} = X^n + \dot{X}^{n+1} \quad (42)$$

### 2.13 Dunkerley algorithm

Dunkerley method obtains a lower limit for the main frequency of oscillation [51]. In this method, the minimum frequency is obtained by Eq. (43) as follows:

$$\frac{1}{\omega_0^2} = \sum_{i=1}^{ndof} a_{ii}m_{ii} \quad (43)$$

In this relationship,  $a_{ii}m_{ii}$  are the contributions of each degree of freedom when the other degrees-of freedom are absent. Hence:

$$a_{ii}m_{ii} = \frac{1}{\omega_{ii}^2} \quad (44)$$

The symbol  $\omega_{ii}$  is the system frequency with a single degree of freedom with mass  $m_{ii}$ , for the  $i$ -th degree of freedom. On this basis, Dunkerley relationship is as follows:

$$\frac{1}{\omega_0^2} = \sum_{i=1}^{ndof} \frac{1}{\omega_{ii}^2} \quad (45)$$

The calculation of the mass and damping matrices is performed by Eqs. (36) and (37). Moreover, the applied time step is equal to 1.

## 3 Numerical examples

All the methods presented in the previous section were programmed in the FORTRAN language. The geometrically non-linear analysis of various plates was carried out using this program. The loads were entered into ten increments, and then the

answers were obtained. The number of iterations and analysis time for each structure are listed in the tables. The load-displacement curves are plotted. The accuracy is the same for these approaches. However, each tactic requires a different number of iterations to achieve the desired accuracy. The steps in the dynamic relaxation process for the analysis of structures are as follows:

*Step 1* - Choose the initial velocity (zero) and initial displacement (zero or results of previous increment).

*Step 2* - Form the internal force vector and the stiffness matrix of each element.

*Step 3* - Assembling the internal force vector.

*Step 4* - Estimate the residual force vector using Eq. (7).

*Step 5* - Go to step 12, if  $\left\| \frac{R^T R}{P^T P} \right\| < e_R$ . Otherwise, continue.

*Step 6* - Form the artificial mass matrix.

*Step 7* - Form the fictitious damping matrix.

*Step 8* - Update the velocities using Eq. (5).

*Step 9* - Update the time step.

*Step 10* - Update the nodal displacements using Eq. (6).

*Step 11* - Go to step 2.

*Step 12* - Print the displacements for this increment.

*Step 13* - If  $N > 10$ , the analysis is finished; otherwise continue.

*Step 14* -  $N = N + 1$  and go to step 2.

Here, The number of increments is shown with  $N$ . The acceptable error of residual force ( $e_R$ ) is the same for all schemes, and its value equals  $10^{-4}$ . It is dimensionless. The thickness of the plates  $h$ , the elasticity modulus  $E$  and Poisson's ratio  $\nu$  are assumed to be 1 cm, and 200 GPa and 0.3, respectively in all the samples. The dimensionless load parameter is  $\frac{12qb^2(1-\nu^2)}{Eh^4}$ . In this relation, the uniform load and the width or diameter of the plates are shown with  $q$  and  $b$ , correspondingly. Moreover, the flexural rigidity of plate is obtained from  $D = \frac{Eh^3}{12(1-\nu^2)}$ . The node that has the maximum deflection is shown with the symbol  $\mathbf{M}$  in the figures. Furthermore, the simple boundaries are shown with dashed lines, and the clamped supports are shown in shaded areas. The horizontal axis in the load-displacement curves is the deflection to the thickness ratio. The merit of the various techniques is estimated using the following equation, and it is based on the number of iterations ( $E_I$ ) and the analysis durations ( $E_T$ ).

$$E_I = 100 \times \left( \frac{I_{\max} - I}{I_{\max} - I_{\min}} \right) \quad (46)$$

$$E_T = 100 \times \left( \frac{T_{\max} - T}{T_{\max} - T_{\min}} \right) \quad (47)$$

The number of iterations and the analysis time are shown with  $I$  &  $T$ , respectively. Zero is the lowest score, and it is associated with a solution that has required the largest number of iterations or the longest time for the analysis. On the other hand, a procedure with the minimum number of iterations or time taken for the analysis is given a score of 100. Then, the results are compared and ranked. Moreover, the authors have suggested other

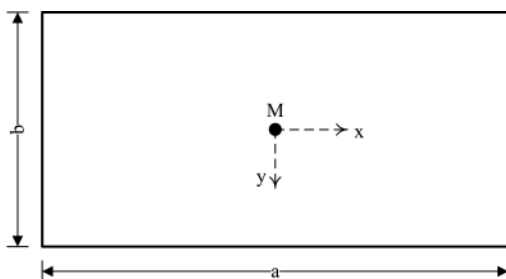
**Tab. 1.** The used dynamic relaxation methods and their indexes

Number	Method	Index
1	Papadrakakis	Papadrakakis
2	Underwood	Underwood
3	Qiang	Qiang
4	Zhang 1	Zhang1
5	Zhang 2	Zhang2
6	Nodal Damping	Nodal Damping
7	Rezaiee-Pajand & Taghavian Hakkak 1	RPTH1
8	Kinetic Damping Dynamic Relaxation	kdDR
9	Minimizing the residual force	MFT
10	Rezaiee-Pajand & Alamatian 1	mdDR1
11	Rezaiee-Pajand & Alamatian 2	mdDR2
12	Minimizing the residual Energy	MRE
13	Rezaiee-Pajand & Sarafrazi	RPS
14	Zero Damping	zdDR
15	Dunkerley	Dunkerley
16	Rezaiee-Pajand & Taghavian Hakkak 2	RPTH2

fictitious parameters for some strategies. Table 1 shows the various methods used in this study. In Zhang1 and Zhang2, the time steps to find the mass matrices are assumed to be 1 and 1.1, correspondingly. Damping is calculated by Zhang approach in RPTH2 tactic. Furthermore, the minimum residual force procedure is used in mdDR2 scheme to obtain the time step.

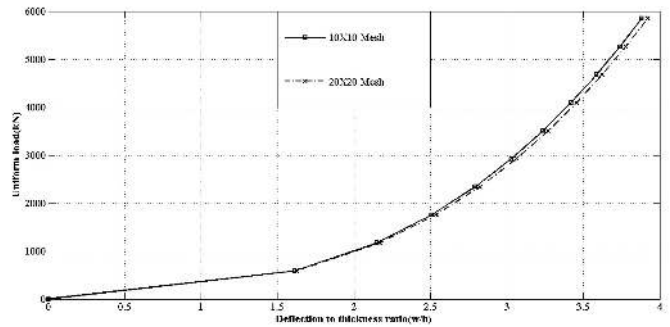
### 3.1 The quadrilateral plate with various supports

The first, analysis of the quadrilateral plate shown in Fig. 1 is performed in three modes. In two cases, a square plate is considered. One case is clamped while the other structure has simple boundaries. The third plate is a clamped rectangular plate with a length to the width ratio equal to 2. The width of plate  $b$  is equal to 1 meter. To study the effect of the mesh, both 10 x 10 and 20 x 20 configurations are used. Due to symmetry, a quarter of plates are modeled. Figs. 2 to 4 show the maximal load-deflection curves. It is worth emphasizing; the maximum displacement is in the middle of the plate. The result obtained from the computer program of the authors is the same as the results reported in Ref. [5].

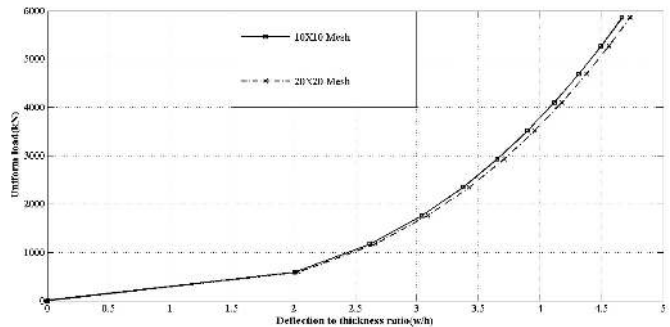


**Fig. 1.** The quadrilateral plate

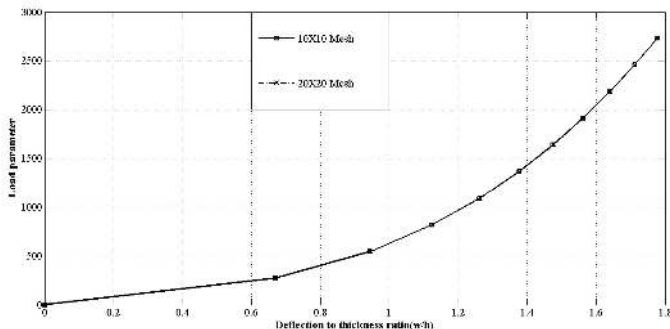
The number of iterations and the analysis time of the methods are presented in Tables 2 to 7. Based on Table 2, the Rezaiee-Pajand and Taghavian Hakkak approach in the analysis of clamped square plate with 10 x 10 mesh converged to in-



**Fig. 2.** The load- maximum deflection curves for the clamped square plate



**Fig. 3.** The load- maximum deflection curves for the clamped rectangular plate



**Fig. 4.** The load- maximum deflection curves for the simple square plate

**Tab. 2.** The ranking of methods for the clamped square plate (mesh 10 X 10)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	1399	1275	1350	1380	1441	1435	1477	1199	501	499	11956	0	13	49.11	0	14
2	272	238	214	201	200	179	217	189	172	165	2047	100	1	8.172	100	1
3	290	257	240	229	222	216	211	208	205	202	2280	97.649	3	9.25	97.367	2
4	300	280	266	258	248	239	234	228	220	219	2492	95.509	6	9.843	95.918	4
5	314	294	279	270	261	251	245	239	231	229	2613	94.288	10	10.312	94.773	8
6	332	278	251	257	247	239	232	227	222	218	2503	95.398	7	9.938	95.686	6
7	172	134	Error	Error	Error	Error	Error	Error	Error	Error			0			0
8	525	443	415	379	371	354	365	373	366	362	3953	80.765	11	19	73.55	12
9	320	295	278	269	259	248	241	238	229	226	2603	94.389	9	10.797	93.588	11
10	300	280	265	257	248	239	234	228	220	219	2490	95.529	5	9.922	95.725	5
11	343	282	273	256	246	237	229	225	218	217	2526	95.166	8	10.625	94.008	9
12	340	282	273	256	246	237	229	225	218	220	2526	95.166	8	10.64	93.971	10
13	290	257	240	229	221	215	211	207	204	201	2275	97.699	2	9.516	96.717	3
14	305	270	254	243	236	230	226	222	219	217	2422	96.216	4	10.125	95.229	7
15	439	464	478	496	503	508	518	522	516	527	4971	70.491	12	19.469	72.405	13
16	183	160	Error	Error	Error	Error	Error	Error	Error	Error			0			0

**Tab. 3.** The ranking of methods for the clamped square plate (mesh 20 X 20)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	3771	3680	2867	2412	2013	2561	3706	2718	1879	1857	27464	0	13	827.141	0	13
2	701	572	509	475	419	429	413	408	365	362	4653	100	1	103.891	100	1
3	791	648	589	551	523	502	485	471	460	449	5469	96.423	3	134.687	95.742	4
4	828	714	642	624	597	571	552	537	520	508	6093	93.687	6	134.844	95.72	5
5	868	750	673	655	627	601	578	563	546	533	6394	92.368	9	141	94.869	6
6	939	705	628	580	546	522	548	532	518	505	6023	93.994	5	133.156	95.954	2
7	671	510	452	Error	Error	Error	Error	Error	Error	Error			0			0
8	1083	938	822	779	706	726	730	626	692	635	7737	86.48	11	187.484	88.442	10
9	1587	749	675	650	625	599	577	562	546	532	7102	89.264	10	184.422	88.865	9
10	827	714	643	623	597	574	551	537	521	508	6095	93.678	7	134.485	95.77	3
11	863	714	643	623	597	572	551	537	521	508	6129	93.529	8	160.437	92.182	7
12	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
13	791	648	589	551	523	502	485	471	459	449	5468	96.427	2	182.562	89.123	8
14	805	661	634	591	562	538	520	485	473	463	5732	95.27	4	191.546	87.88	11
15	951	983	1003	1039	1044	1053	1067	1084	1063	1082	10369	74.942	12	227.563	82.901	12
16	692	559	490	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0



correct answers after the third increment. In other words, this tactic achieves acceptable error, but the answers are incorrect. On the other hand, Table 3 shows that the minimum residual energy procedure (MRE) is not able to obtain an acceptable error. Hence, it is divergent. RPTH2 solution is similar to that after the third increment. Moreover, RPTH1 process after third increment gives incorrect answers.

As shown in Tables 4 and 5, Rezaiee-Pajand and Taghavian Hakkak technique is not useful for the rectangular plate. Furthermore, in the 20 x 20 configuration for this structure, the MFT strategy yields an inappropriate response from the beginning. The results obtained for the clamped supports show that the mdDR2 and MRE methods require the same number of iterations. In these algorithms, a similar relationship is used for estimating the mass and damping. It is worth noting that if Eq. (38) has no real answer, the calculated time step by the MRF approach is used in the MRE tactic. Thus, it can conclude that for these plates, the minimum unbalanced energy scheme is unsuitable for calculating the time step. In other words, the minimum residual force is used to find this parameter. This has happened after the fifth increment for a rectangular plate with a 10 x 10 mesh.

The MdDR1 and Zhang1 algorithms more or less have the same performance. This behavior can be for the reason that in both techniques, the fictitious frequency is estimated by using Rayleigh principle. The MFT and Zhang2 techniques approximately require the same number of iterations. However, the number of iterations is different in each loading increment. It is worth emphasizing; these two procedures also use Rayleigh principle to estimate the smallest frequency. Table 3 shows that the number of iterations of the first increment of these two strategies is significantly different. Their numbers of iterations are different at each incremental loading since the time step for these two methods is different. It should be noted that the mass and damping matrices have a greater effect on the dynamic relaxation process. Hence, the total numbers of iterations of other increments of these two tactics get close to each other.

Moreover, Qiang and RPS approaches have the same behavior. As mentioned in the RPS technique, the minimum calculated eigenvalue by Rayleigh principle, and the iterative power procedure is utilized to find the damping. Qiang scheme also uses Rayleigh principle. Hence, it is concluded that in analysis of these structures, the minimum frequency by using Rayleigh principle is always less when compared with the iterative power algorithm.

The results of analysis of simple square plates are inserted in Tables 6 and 7. These tables show that all the methods converge to an acceptable answer. What was mentioned above about the similar behavior of some of the tactics more or less holds true here. The results obtained indicate that Underwood approach is the most efficient one and Papadrakakis process is the worst procedure to solve the quadrilateral plate problem. The mesh used does not have a significant effect on the response as shown

in Fig. 2 to 4. Hence, a similar configuration is used for the other samples.

### 3.2 The circular plate

In this section, the circular plate, showed in Fig. 5, is analyzed. Due to the symmetry, one-quarter of the structure is modeled. The number of used bending elements is 67. The results of the analysis are shown in Fig. 6 and inserted in Table 8. According to the Table 8, Rezaiee-Pajand and Taghavian Hakkak technique is not able to solve this plate. In other words, the residual force in these methods is more than the acceptable value. For this reason, this approach diverges. Hence, the ranking of this strategy is zero. It should be noted that the similar behavior observed for some of the schemes in the previous problem is also seen in this sample. Based on the obtained results, Underwood and Qiang procedures are the best and Papadrakakis algorithm is the worst tactic to solve this problem.

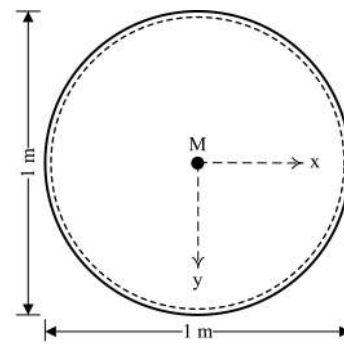


Fig. 5. The circular plate

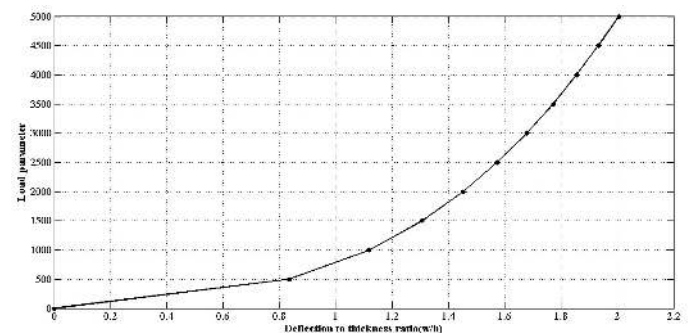


Fig. 6. The load- maximum deflection curve for the circular plate

### 3.3 The rectangular plate with opening

Now, the quadrilateral plate with opening shown in Fig. 7 is investigated. Rezaiee-Pajand and Alamatian have analyzed this structure before [41]. In the current paper, this plate will be solved with 20 X 20 mesh. Maximum displacement occurs in the middle of the upper boundary. The load-maximum deflection curve is shown in Fig. 8. The achieved responses are the same as those reported by Ref [41]. The ranking of the methods is inserted in Table 9. Based on Table 9, the responses of RPTH2 and RPTH1 processes are divergent. The MRE and mdDR2 approaches are the most efficient tactics to analyze this plate. The

**Tab. 4.** The ranking of methods for the clamped rectangular plate (mesh 10 X 10)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	1340	1151	1350	2215	755	790	822	927	911	1023	11284	0	14	46.297	0	14
2	284	243	200	203	214	218	221	247	221	201	2252	100	1	8.859	100	1
3	309	286	281	284	292	301	311	320	328	336	3048	91.187	10	12.296	90.819	9
4	316	286	265	251	242	234	227	221	217	214	2473	97.553	3	9.704	97.743	3
5	331	300	278	261	254	246	239	232	228	223	2592	96.236	7	10.188	96.45	6
6	344	293	274	263	256	246	242	237	230	225	2610	96.036	8	10.062	96.787	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	525	459	411	409	412	385	411	409	425	433	4279	77.558	12	20.344	69.323	12
9	276	299	274	265	260	251	245	240	232	229	2571	96.468	6	10.359	95.993	7
10	315	286	269	252	242	234	228	224	217	213	2480	97.476	4	9.562	98.122	2
11	268	284	259	252	246	238	233	225	221	216	2442	97.896	2	9.797	97.495	4
12	333	292	275	268	246	238	233	225	228	216	2554	96.656	5	10.469	95.7	8
13	309	285	280	283	291	300	310	319	327	335	3039	91.287	9	12.344	90.691	10
14	326	302	298	302	311	322	333	344	353	362	3253	88.917	11	13.359	87.98	11
15	577	589	599	600	595	603	598	605	602	606	5974	58.791	13	22.828	62.688	13
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 5.** The ranking of methods for the clamped rectangular plate (mesh 20 X 20)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	3246	3312	3266	3060	3470	4576	4793	5186	3503	4048	38460	0	11	1157.454	0	13
2	689	548	488	482	432	460	439	433	408	357	4736	100	1	105.969	100	1
3	788	663	611	578	555	536	522	510	499	491	5753	96.984	3	141.969	96.576	6
4	826	716	649	606	579	555	537	521	508	497	5994	96.27	5	134.078	97.327	2
5	866	750	680	636	607	582	563	546	532	522	6284	95.41	7	140.844	96.683	4
6	936	715	646	605	609	587	570	555	541	530	6294	95.38	8	141.031	96.665	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1126	907	810	799	750	738	756	692	685	670	7933	90.52	9	193.141	91.71	9
9	Error	Error	Error	Error	Error	Error	Error	Error	Error	Error			0			0
10	826	716	649	606	579	555	537	521	508	497	5994	96.27	5	134.594	97.278	3
11	828	716	649	606	579	555	537	521	508	497	5996	96.264	6	158.625	94.992	7
12	828	716	649	606	579	555	537	521	508	497	5996	96.264	6	159.328	94.925	8
13	788	662	611	578	554	536	521	509	499	490	5748	96.999	2	195.297	91.505	10
14	815	677	627	594	570	552	538	526	516	507	5922	96.483	4	200.922	90.97	11
15	1159	1175	1232	1254	1263	1271	1281	1293	1308	1291	12527	76.898	10	279.625	83.485	12
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 6.** The ranking of methods for the simple square plate (mesh 10 X 10)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	1652	801	777	815	855	566	518	572	723	706	7985	0	16	33.891	0	16
2	205	176	137	128	171	137	108	115	85	91	1353	100	1	5.484	100	1
3	259	197	174	161	151	144	139	135	132	129	1621	95.959	8	6.75	95.543	9
4	228	192	164	148	158	151	143	137	132	130	1583	96.532	4	6.329	97.025	4
5	239	201	173	177	144	157	150	144	139	136	1660	95.371	11	6.641	95.927	7
6	305	191	163	147	134	150	143	137	132	130	1632	95.793	10	6.375	96.863	6
7	250	191	167	152	142	134	128	124	119	116	1523	97.437	2	6.281	97.194	3
8	324	183	250	243	201	212	204	191	202	184	2194	87.319	15	10.625	81.902	15
9	277	198	175	164	155	148	141	141	136	132	1667	95.265	12	7.031	94.554	12
10	228	192	165	148	159	151	144	138	132	130	1587	96.472	5	6.343	96.976	5
11	277	198	166	156	148	135	135	134	130	124	1603	96.23	6	6.75	95.543	9
12	285	198	178	156	151	135	135	134	130	125	1627	95.869	9	6.907	94.991	11
13	259	197	174	160	151	144	139	135	131	129	1619	95.989	7	6.906	94.994	10
14	268	206	183	169	160	154	148	144	141	139	1712	94.587	13	7.313	93.561	13
15	222	192	191	196	180	195	186	200	192	190	1944	91.089	14	7.765	91.97	14
16	221	189	159	166	156	147	140	135	129	125	1567	96.773	3	6.219	97.413	2

**Tab. 7.** The ranking of methods for the simple square plate (mesh 20 X 20)

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	6947	5590	4744	5384	4274	3975	3422	2780	3866	3605	44587	0	16	1425.875	0	16
2	766	573	549	464	461	373	376	391	350	358	4661	100	1	107	100	1
3	923	694	605	551	513	485	463	446	431	418	5529	97.826	7	138.047	97.646	7
4	840	698	598	533	492	457	502	480	461	443	5504	97.889	3	125.078	98.629	3
5	881	732	621	560	510	479	526	504	483	465	5761	97.245	12	130.469	98.221	5
6	1178	701	587	522	482	528	502	481	462	373	5816	97.107	13	132.407	98.074	6
7	949	713	620	563	524	494	471	452	436	422	5644	97.538	10	141.422	97.39	9
8	775	610	580	666	562	586	472	530	522	525	5828	97.077	14	142.344	97.32	10
9	916	720	620	559	553	496	461	470	458	451	5704	97.388	11	153.406	96.481	13
10	840	698	593	533	486	456	435	480	398	382	5301	98.397	2	120.609	98.968	2
11	898	687	596	540	524	495	471	445	433	419	5508	97.879	4	148.625	96.844	11
12	898	687	596	540	524	496	471	445	433	419	5509	97.876	5	149.016	96.814	12
13	923	694	605	551	513	485	463	445	430	418	5527	97.831	6	195.422	93.296	14
14	931	702	613	559	522	494	472	454	439	427	5613	97.616	8	200.063	92.944	15
15	1372	598	531	539	494	532	505	485	530	516	6102	96.391	15	140.172	97.485	8
16	861	718	615	553	497	538	514	416	474	456	5642	97.543	9	128.125	98.398	4

**Tab. 8.** The ranking of methods for the circular plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	1543	983	981	978	1008	1054	1224	1409	1588	1823	12591	0	16	632.484	0	16
2	392	355	342	267	291	266	258	274	253	269	2967	89.501	3	103.906	93.003	3
3	444	390	363	344	331	320	312	305	299	294	3402	85.455	12	145.094	85.756	11
4	327	334	327	315	308	298	295	287	281	277	3049	88.738	4	106.593	92.53	4
5	343	351	344	330	320	313	309	301	294	290	3195	87.38	10	111.828	91.609	7
6	355	344	314	326	311	298	286	278	296	291	3099	88.273	8	108.609	92.176	6
7	318	247	215	196	183	173	166	160	155	206	2019	98.317	2	85.719	96.203	2
8	715	637	582	554	505	475	493	466	457	456	5340	67.432	14	208.375	74.622	12
9	314	356	343	329	319	313	307	301	294	288	3164	87.669	9	144.281	85.899	10
10	326	334	328	315	308	299	295	287	281	277	3050	88.729	5	106.688	92.514	5
11	347	334	328	315	305	299	295	287	281	277	3068	88.561	6	139.937	86.664	8
12	366	334	328	315	305	299	295	287	281	277	3087	88.385	7	141.562	86.378	9
13	443	389	362	344	331	320	312	305	299	294	3399	85.483	11	217.844	72.956	13
14	460	403	376	358	345	335	326	319	314	309	3545	84.125	13	227.297	71.293	14
15	830	901	939	964	990	1015	1027	1042	1046	1072	9826	25.714	15	343.641	50.822	15
16	235	216	201	188	182	174	168	161	158	155	1838	100	1	64.14	100	1

**Tab. 9.** The ranking of methods for the rectangular plate with opening

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	2440	1869	2549	2060	2255	2458	2165	1553	1419	1134	19902	0	13	1227.25	0	14
2	578	511	509	489	410	439	426	356	377	381	4476	99.478	4	183.469	99.553	3
3	906	826	773	742	721	705	694	685	678	673	7403	80.602	10	398.922	79.004	10
4	579	503	465	444	425	417	401	394	388	379	4395	100	1	180.313	99.854	2
5	607	528	488	465	446	438	420	413	407	398	4610	98.614	5	188.343	99.088	4
6	664	527	484	494	476	461	454	440	421	415	4836	97.156	7	198.359	98.133	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	968	800	741	718	670	678	640	606	623	621	7065	82.782	8	314.281	87.076	9
9	623	528	488	464	446	429	420	413	405	396	4612	98.601	6	268.922	91.403	8
10	579	504	465	445	425	418	401	394	388	380	4399	99.974	2	178.781	100	1
11	597	504	465	445	425	418	401	394	388	380	4417	99.858	3	259.016	92.347	6
12	597	504	465	445	425	418	401	394	388	380	4417	99.858	3	260.016	92.252	7
13	906	825	773	741	720	705	693	685	678	672	7398	80.635	9	658.938	54.204	12
14	931	851	800	769	748	733	723	714	708	703	7680	78.816	11	690.594	51.185	13
15	1126	1172	1217	1247	1274	1281	1310	1317	1323	1332	12599	47.095	12	519.437	67.509	11
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

numbers of iterations required in these two methods are the same in all increments. The reason for this behavior was reported earlier. Moreover, Papadrakakis algorithm consumes the longest time. The similarity in the behavior of the techniques reported earlier also holds true here.

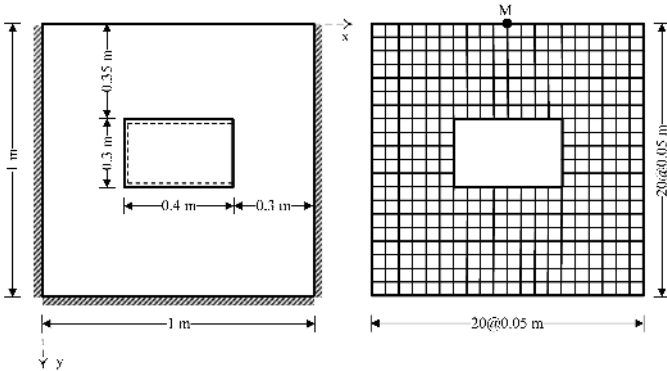


Fig. 7. The rectangular plate with opening

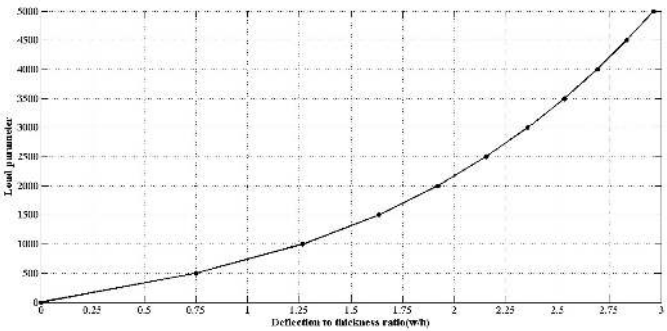


Fig. 8. The load- maximum deflection curves for the rectangular plate with opening

### 3.4 The L-shaped plate

Here, an L-shaped plate, which is shown in Fig. 9, is analyzed [39]. This structure is modeled with 150 bending elements. Some symbols are used for boundary conditions. Free boundaries, clamped and simple supports are displayed by F, C and S, respectively. For example, a plate SFSSSF has a free edge on the borders 2 and 6, and the other edges are simple support. The load-maximum deflection curves are shown in Fig. 10. The maximum displacement of CSCSSS and CCCCC plates occurs at a node with coordinates (0.5,0.5) m. On the other hand; this node is in the middle of the edge 2 of the SFSSSF structure. Tables 10 to 12 include the rating and ranking of the methods. The number of iterations and the required time to converge in Rezaiee-Pajand and Taghavian Hakkak approach are less than the other techniques for the CSCSSS and CCCCC plates. Moreover, Zhang1 and mdDR1 procedures are the best solutions for the analysis of the SFSSSF structure. It should be noted that these two schemes have the same behavior. Table 11 shows that Rezaiee-Pajand and Taghavian Hakkak tactic is not able to analyze the SFSSSF plate. Papadrakakis process requires the largest number of iterations and the longest time to converge among the strategies that converged.

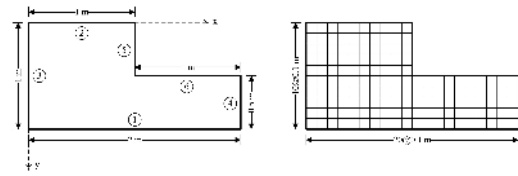


Fig. 9. The L-shaped plate

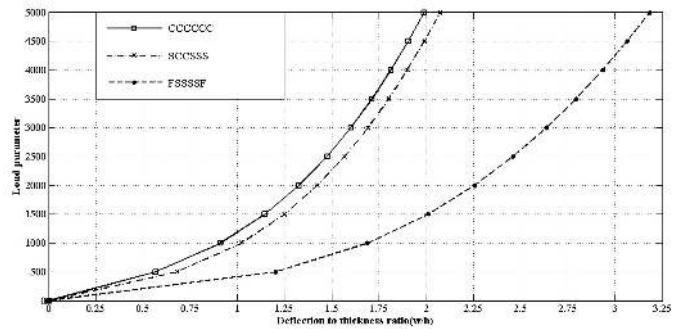


Fig. 10. The load- maximum deflection curves for the L-shaped plate

### 3.5 The triangular plate

Now, the triangular plate shown in Fig. 11 will be investigated. Sixty bending elements are used for the structural analysis. Fig. 12 shows the load-displacement curve. The maximum deflection occurs in the middle of the triangle. The results are inserted in Table 13. Underwood method requires the lowest number of iterations, and the shortest time required to analyze this plate. On the other hand, the rank of RPTH1 and RPTH2 is zero. In other words, these techniques are divergent from the first step. The similar behaviors of some tactics that in first sample have been discussed are also true here.

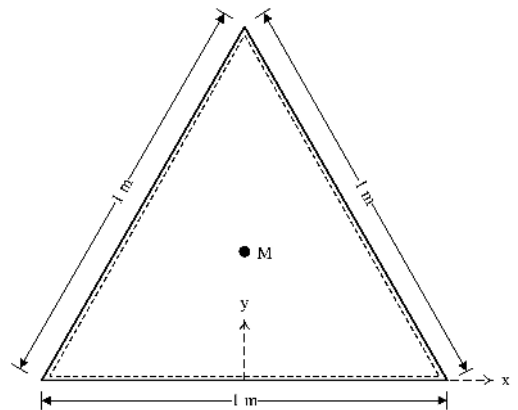


Fig. 11. The triangular plate

### 3.6 Parallelogram plate

Fig. 13 shows a parallelogram plate, and the mesh used in this article. The results are shown in Table 14 and Fig. 14. Based on Table 14, RPTH1 and RPTH2 approaches were not able to achieve acceptable residual errors. In other words, the responses were divergent. Hence, the ranking of these techniques is zero. On the other hand, Underwood, mdDR1 and Zhang1 methods have the first to the third rankings for the number of iterations, respectively. It is worth emphasizing; Zhang1 and mdDR1 so-

**Tab. 10.** The ranking of methods for CCCCCC plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	1966	1092	1481	7682	5149	816	829	832	864	872	21583	0	16	1097.203	0	16
2	424	373	300	300	323	303	286	267	269	284	3129	93.937	3	119.938	95.063	5
3	478	409	380	362	348	338	330	323	317	312	3597	91.555	11	169.188	90.272	10
4	357	360	341	328	317	304	296	288	284	280	3155	93.805	4	119.687	95.088	4
5	607	528	488	465	446	438	420	413	407	398	4610	86.399	13	188.343	88.409	11
6	425	375	346	324	332	322	313	304	298	292	3331	92.909	9	120.375	95.021	6
7	340	258	227	208	195	186	178	172	167	279	2210	98.615	2	99.563	97.045	2
8	695	594	614	534	486	449	490	491	463	462	5278	82.998	14	212.375	86.071	12
9	365	371	357	343	332	318	309	301	297	294	3287	93.133	8	159.156	91.248	9
10	356	361	342	328	317	305	296	288	284	280	3157	93.795	5	114.266	95.615	3
11	388	361	341	328	317	305	296	288	284	280	3188	93.637	6	154.453	91.706	7
12	412	361	341	328	317	305	296	288	284	280	3212	93.515	7	155.969	91.558	8
13	478	409	380	361	348	338	329	322	317	311	3593	91.575	10	249.468	82.463	13
14	495	424	395	377	364	354	345	339	333	328	3754	90.756	12	261.219	81.32	14
15	886	962	1003	1018	1051	1054	1079	1090	1099	1111	10353	57.165	15	373.656	70.383	15
16	259	233	212	198	187	178	174	170	165	162	1938	100	1	69.187	100	1

**Tab. 11.** The ranking of methods for SFSSSF plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	7666	13501	6969	12199	5390	5709	8489	5538	9861	11601	86923	0	12	23363.859	0	14
2	1396	1284	1074	1237	940	1112	827	911	1022	940	10743	99.487	4	789.735	99.89	3
3	2034	2037	1998	1941	1887	1839	1797	1761	1729	1701	18724	89.064	10	2802.485	90.984	11
4	1399	1136	1080	1051	1016	989	966	950	931	912	10430	99.896	3	775.594	99.953	2
5	1467	1188	1134	1103	1066	1032	1012	997	974	958	10931	99.241	7	800.672	99.842	4
6	1322	1171	1165	1097	1031	1079	1052	1025	1000	975	10917	99.26	6	802.344	99.834	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1949	1844	1519	1645	1538	1578	1565	1591	1369	1333	15931	92.712	8	1229.64	97.944	6
9	1402	1193	1144	1108	1070	1038	1015	997	977	958	10902	99.279	5	1750.906	95.637	10
10	1399	1132	1080	1051	1015	984	963	950	929	913	10416	99.914	2	764.937	100	1
11	1307	1137	1089	1051	1015	989	967	950	932	913	10350	100	1	1700.421	95.86	9
12	1307	1137	1089	1051	1015	989	967	950	932	913	10350	100	1	1694.469	95.887	8
13	2034	2037	1998	1941	1887	1839	1797	1761	1729	1701	18724	89.064	10	7920.516	68.337	13
14	2057	2060	2023	1968	1915	1867	1826	1790	1759	1731	18996	88.709	11	7559.203	69.935	12
15	1571	1649	1683	1702	1727	1746	1761	1774	1790	1791	17194	91.062	9	1272.313	97.755	7
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 12.** The ranking of methods for CSCSSS plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	13588	9215	12627	10460	11797	11320	12603	10115	8373	10220	110318	15.998	10	5005.109	19.328	10
2	2232	2013	1828	1696	1746	1628	1615	1777	1656	1410	17601	100	1	709.094	100	1
3	12730	12596	12629	12576	12489	12387	12281	12175	12071	11969	123903	3.6901	12	5425.625	11.432	11
4	7156	6850	6778	6696	6612	6531	6453	6380	6311	6245	66012	56.14	8	2664.187	63.287	7
5	7503	7183	7108	7022	6934	6849	6768	6691	6618	6549	69225	53.229	9	2816.516	60.426	9
6	7909	6814	6065	5676	5377	5073	4808	4633	4547	4415	55317	65.829	5	2253.157	71.005	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	3039	2235	1959	2209	1907	1842	1864	1852	2267	1752	20926	96.988	2	888.235	96.636	2
9	2803	3320	5970	2444	1813	2172	1879	2106	1514	1667	25688	92.673	3	1136.438	91.975	3
10	7155	6849	6777	6695	6611	6530	6452	6379	6309	6244	66001	56.149	7	2695.547	62.698	8
11	6579	6297	5995	5991	5802	5521	5336	4079	6276	6235	58111	63.298	6	2567.765	65.097	6
12	2416	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
13	12727	12591	12623	12570	12482	12380	12273	12167	12062	11960	123835	3.7518	11	5813.203	4.1538	12
14	13020	12931	13001	12974	12905	12819	12725	12629	12533	12439	127976	0	13	6034.406	0	13
15	4995	3884	3355	2998	2731	2521	2348	2201	2075	1964	29072	89.607	4	1193.359	90.906	4
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 13.** The ranking of methods for the triangular plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	8963	7892	6264	5930	7527	9460	7559	5706	4146	4674	68121	0	12	1988.375	0	14
2	1056	898	666	701	620	688	700	685	671	564	7249	100	1	202.984	100	1
3	1888	1551	1427	1351	1299	1260	1229	1204	1183	1164	13556	89.639	10	391.61	89.435	11
4	1270	1131	1025	950	905	864	832	805	785	766	9333	96.576	3	261.078	96.746	2
5	1332	1180	1073	995	951	906	864	840	823	805	9769	95.86	5	274.828	95.976	5
6	1767	1326	1174	1113	1053	1028	1025	979	980	953	11398	93.184	7	324.172	93.212	9
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1632	1224	1373	1234	1131	993	1073	1085	928	993	11666	92.744	8	351.547	91.679	10
9	1502	1180	1073	993	947	906	864	840	823	805	9933	95.591	6	294.719	94.862	8
10	1270	1132	1026	946	906	863	823	806	785	771	9328	96.585	2	262.296	96.678	3
11	1390	1131	1025	945	905	863	823	806	785	771	9444	96.394	4	278.875	95.749	6
12	1390	1131	1025	945	905	863	823	806	785	771	9444	96.394	4	280.859	95.638	7
13	1887	1551	1427	1351	1299	1259	1229	1203	1182	1164	13552	89.645	9	410.906	88.354	12
14	1914	1575	1452	1377	1325	1286	1256	1231	1210	1193	13819	89.207	11	423.985	87.622	13
15	1456	874	856	871	896	881	901	901	900	908	9444	96.394	4	265.359	96.506	4
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 14.** The ranking of methods for the parallelogram plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	15964	6676	6564	6118	5447	5832	5377	6245	6435	6487	71145	0	12	2741.328	0	14
2	1167	846	787	725	685	618	585	565	558	543	7079	100	1	261.469	100	1
3	1298	1030	924	858	811	776	747	724	704	687	8559	97.69	2	323.437	97.501	2
4	1169	1132	1006	926	874	825	796	773	744	717	8962	97.061	5	323.609	97.494	3
5	1226	1186	1053	973	915	866	835	801	781	758	9394	96.387	8	338.093	96.91	6
6	1750	1144	993	911	911	868	835	809	783	762	9766	95.806	9	353.266	96.298	9
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1578	1362	1279	1039	1018	1037	950	959	891	910	11023	93.844	10	427.75	93.295	12
9	3643	1186	1053	973	915	866	835	801	770	758	11800	92.631	11	455.563	92.173	13
10	1168	1133	1006	927	874	827	790	765	744	717	8951	97.078	4	323.734	97.489	4
11	1613	1132	1006	926	874	827	790	765	744	717	9394	96.387	8	363.297	95.894	11
12	1456	1132	1006	926	874	827	790	765	744	717	9237	96.632	6	358.281	96.096	10
13	1298	1030	924	858	811	776	747	724	704	687	8559	97.69	2	341.859	96.758	7
14	1313	1043	937	871	824	789	761	738	718	701	8695	97.478	3	347.359	96.536	8
15	1720	839	832	839	839	823	847	831	854	845	9269	96.582	7	335.485	97.015	5
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 15.** The ranking of methods for the heterogeneous quadrilateral plate I

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	10755	26064	11281	9888	10493	13845	8447	8176	7186	14534	120669	0	13	7610.969	0	14
2	2077	1473	1337	1237	1276	1236	1203	1175	1176	1170	13360	100	1	703.891	100	1
3	7748	6838	6380	6081	5865	5698	5564	5453	5358	5277	60262	56.293	11	3463.719	60.043	11
4	2368	2113	1942	1824	1747	1697	1644	1602	1565	1524	18026	95.652	2	937.984	96.611	2
5	2483	2217	2037	1925	1834	1779	1725	1681	1652	1599	18932	94.808	7	985.36	95.925	6
6	2926	2161	1959	1834	1773	1696	1645	1594	1574	1540	18702	95.022	6	973.234	96.1	4
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	3311	2260	2154	1864	2091	1887	1872	1811	1827	1516	20593	93.26	9	1122.781	93.935	9
9	2502	2216	2037	1925	1848	1780	1725	1682	1648	1599	18962	94.78	8	1142.688	93.647	10
10	2368	2113	1941	1825	1747	1697	1644	1602	1565	1525	18027	95.651	3	948.39	96.46	3
11	2411	2113	1941	1836	1747	1697	1644	1603	1575	1525	18092	95.59	4	1066.516	94.75	7
12	2411	2113	1941	1836	1747	1697	1644	1603	1575	1525	18092	95.59	4	1069.844	94.702	8
13	7748	6837	6379	6080	5864	5697	5562	5451	5357	5276	60251	56.303	10	4241.485	48.783	12
14	7848	6946	6494	6198	5985	5820	5687	5578	5485	5405	61446	55.189	12	4364.5	47.002	13
15	4861	2895	2146	1674	1232	1124	1183	1120	1179	1173	18587	95.129	5	975.297	96.071	5
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0



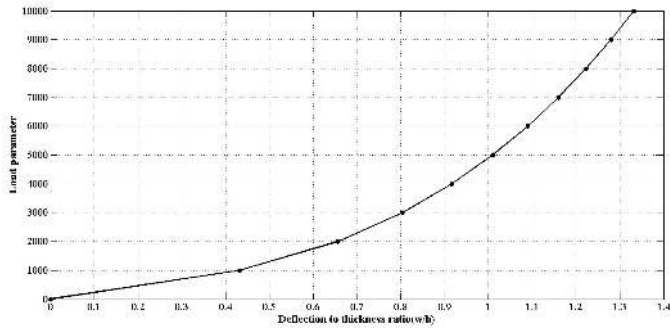


Fig. 12. The load- maximum deflection curves for the triangular plate

lutions have the same behavior.

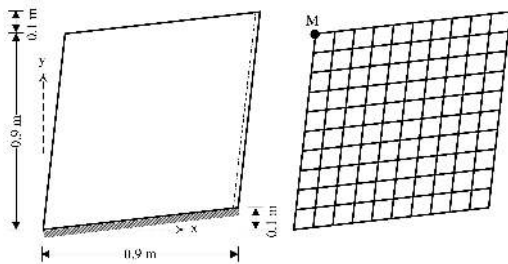


Fig. 13. The parallelogram plate

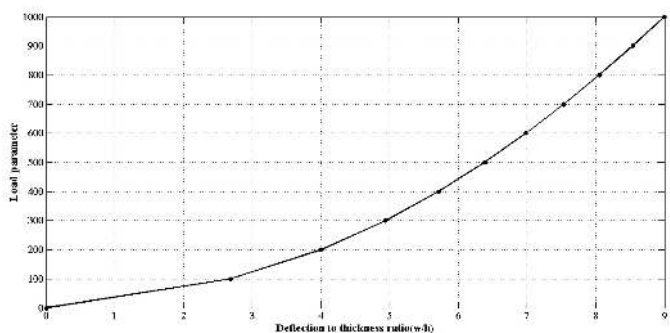


Fig. 14. The load- maximum deflection curves for the parallelogram plate

### 3.7 The heterogeneous quadrilateral plate I

The quadrilateral plate shown in Fig. 15 will be solved here. The number of elements is 233. Fig. 16 indicates load-deflection curve at node M. The coordinates of this node are (0.42, 0.24) m. Table 15 lists the ranking of all strategies. Based on this table, RPTH1 and RPTH2 methods are not capable of analyzing this structure. Underwood, Zhang1 and mdDR1 processes are the best ones for the analysis of this plate. Papadrakakis method is the worst one. It should be noted that some schemes have a similar behavior.

### 3.8 The heterogeneous quadrilateral plate II

A quadrilateral plate shown in Fig. 17 will be analyzed. Fig. 18 shows its load-maximum deflection curve. The plate was modeled with 108 elements. The maximum displacement occurs in the middle of the free edge. Moreover, the ranking of each tactic is inserted in Table 16. According to this table, the residual forces in both RPTH procedures are infinity. Hence,

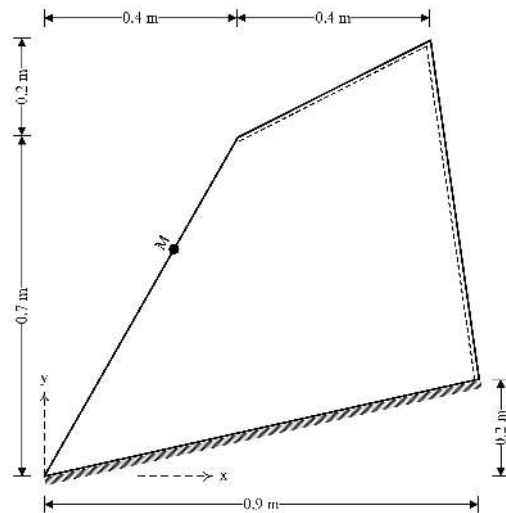


Fig. 15. The heterogeneous quadrilateral plate I

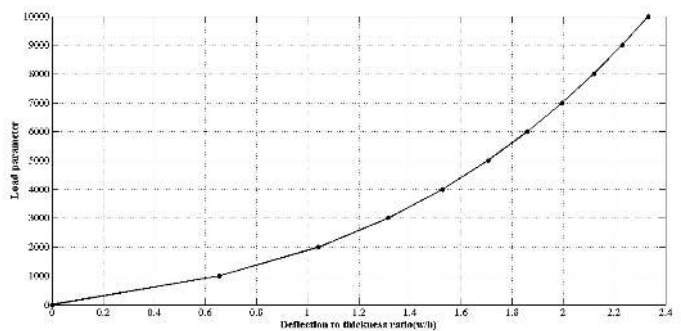


Fig. 16. The load- deflection curves for the heterogeneous quadrilateral plate I

these algorithms are not able to converge to the answer. Top ranking approaches for this sample are the same as the parallelogram plate.

### 3.9 The elliptical plate

Fig. 19 shows the elliptical plate. Due to symmetry, 64 bending plate elements are used to discretize a quarter of the structure. The maximum deflection occurs at the center. Results of the analysis are presented in Fig. 20 and Table 17. Underwood method is the best and Papadrakakis algorithm is the worst process among the converged techniques. Both RPTH schemes could not solve this structure.

### 3.10 The donut-shaped plate

Here, the plate in Fig. 21 is studied. Due to the symmetry, meshing is done in a quarter of the structure with 90 elements. Fig. 22 shows the load-maximum deflection curve. Table 18 demonstrates the rating of methods. Underwood procedure is the most efficient solution to analyze this structure. In other words, this approach requires the lowest number of iterations and the shortest time to yield a response. RPTH2 and RPTH1 strategies are not able to solve this plate. Also, Papadrakakis tactic takes the maximum number of iterations and the longest duration analysis.

**Tab. 16.** The ranking of methods for the heterogeneous quadrilateral plate II

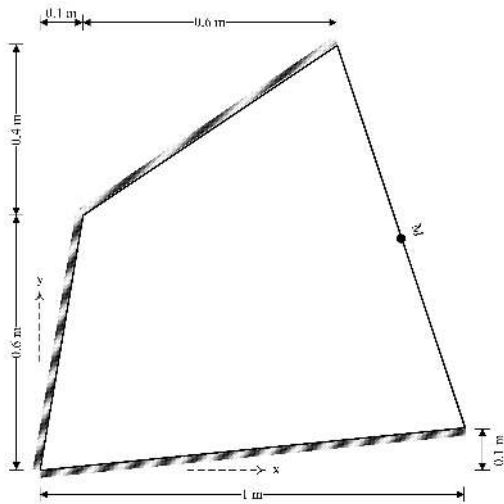
Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	3034	1767	2604	3224	1853	1595	2074	2267	1173	1337	20928	0	13	472.312	0	14
2	479	408	382	372	294	315	322	332	283	286	3473	100	1	76.688	100	1
3	559	486	451	428	412	400	390	382	375	369	4252	95.537	3	95.515	95.241	2
4	622	539	493	465	444	426	409	400	391	379	4568	93.727	6	100.203	94.056	4
5	652	564	518	486	466	445	430	419	409	398	4787	92.472	9	105.078	92.824	9
6	712	561	525	502	482	470	459	460	448	449	5068	90.862	10	111.265	91.26	11
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	813	651	625	572	573	570	564	541	560	488	5957	85.769	11	151.188	81.169	13
9	672	541	512	480	458	440	428	415	408	397	4751	92.678	8	108.875	91.864	10
10	622	539	493	465	444	426	410	400	389	379	4567	93.732	5	100.688	93.934	5
11	631	539	493	464	444	426	410	400	389	379	4575	93.687	7	104.469	92.978	7
12	631	539	493	464	444	426	410	400	389	379	4575	93.687	7	104.766	92.903	8
13	559	486	451	428	412	400	390	381	374	368	4249	95.554	2	98.469	94.495	3
14	573	500	465	443	427	415	405	397	390	384	4399	94.695	4	102.563	93.46	6
15	627	650	668	681	691	697	694	708	700	719	6835	80.739	12	150.188	81.422	12
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 17.** The ranking of methods for the elliptical plate

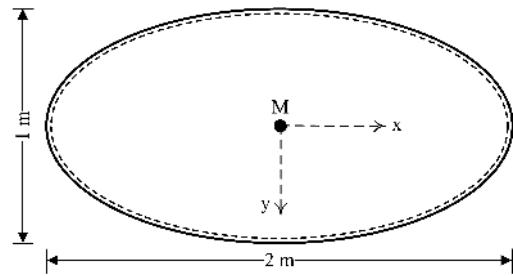
Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	49311	38371	26186	21391	19104	21316	19984	23577	21011	18562	258813	0	13	30012.974	0	14
2	1360	1506	1481	1323	1136	1292	1150	1231	1227	1114	12820	99.443	6	1738.703	99.468	4
3	2152	2129	2137	2131	2120	2109	2101	2098	2100	2105	21182	96.063	10	3466.157	93.391	10
4	1914	1484	1111	1111	1056	1039	1019	1002	985	966	11687	99.901	3	1587.594	100	1
5	2007	1557	1163	1163	1109	1089	1070	1048	1030	1015	12251	99.673	5	1664.296	99.73	3
6	3514	3234	3016	3072	2671	2889	2673	2937	2535	2449	28990	92.906	12	3941.5	91.719	11
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	2071	2038	2203	1945	1887	1754	1673	1648	1604	1688	18511	97.142	8	2609.766	96.404	9
9	1839	1477	1163	1163	1109	1089	1070	1048	1030	1015	12003	99.773	4	2096.891	98.208	8
10	1913	1484	1109	1108	1057	1038	1019	1002	985	966	11681	99.903	2	1591.25	99.987	2
11	1829	1332	1109	1108	1056	1038	1019	1000	985	966	11442	100	1	2002.344	98.541	6
12	1829	1332	1109	1108	1056	1038	1019	1000	985	966	11442	100	1	2002.985	98.539	7
13	2152	2129	2137	2131	2119	2109	2101	2098	2100	2105	21181	96.063	9	5271.782	87.039	12
14	2174	2148	2159	2154	2144	2135	2129	2127	2129	2136	21435	95.96	11	5336.594	86.811	13
15	1350	1392	1423	1434	1447	1462	1474	1492	1500	1507	14481	98.771	7	1970.563	98.653	5
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 18.** The ranking of methods for the donut-shaped plate

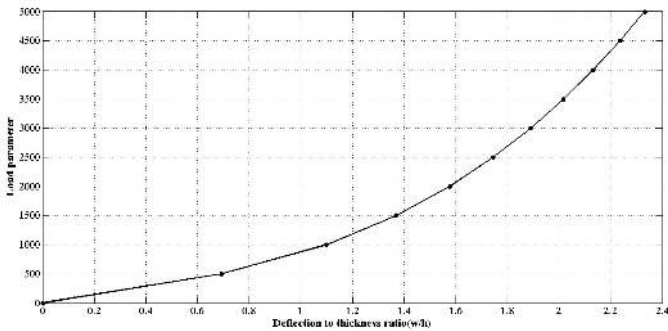
Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	9436	18643	12520	9765	10678	11020	11948	15918	9532	16255	125715	0	14	8381.609	0	14
2	1897	1931	1759	2181	1989	2319	2324	2315	1807	2353	20875	100	1	1181.453	100	1
3	5822	6820	7584	8330	9097	9895	10726	11590	12482	13398	95744	28.587	12	5696.875	37.287	11
4	2105	2220	2193	2154	2124	2139	2144	2113	2102	2168	21462	99.44	3	1213.422	99.556	2
5	2208	2324	2299	2263	2225	2235	2249	2212	2208	2271	22494	98.456	6	1268.437	98.792	4
6	2445	2411	2405	2416	2428	2239	2299	2354	2394	2394	23785	97.224	8	1349.375	97.668	6
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	2817	3361	3040	3482	3013	2904	2977	2501	2688	2700	29483	91.789	9	1709.859	92.661	9
9	2363	2324	2299	2263	2225	2235	2249	2212	2208	2271	22649	98.308	7	1404.421	96.903	8
10	2105	2215	2189	2155	2124	2128	2145	2112	2103	2168	21444	99.457	2	1214.391	99.543	3
11	2520	2215	2189	2155	2124	2128	2143	2112	2103	2168	21857	99.063	5	1348.953	97.674	5
12	2512	2215	2189	2155	2124	2128	2143	2112	2103	2168	21849	99.071	4	1350.641	97.65	7
13	5822	6820	7583	8329	9096	9894	10725	11589	12481	13396	95735	28.596	11	6920.922	20.287	12
14	5875	6892	7673	8435	9219	10034	10883	11766	12678	13614	97069	27.324	13	7026.344	18.823	13
15	5996	4969	4579	4430	4401	4441	4526	4642	4777	4926	47687	74.426	10	2697.516	78.944	10
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0



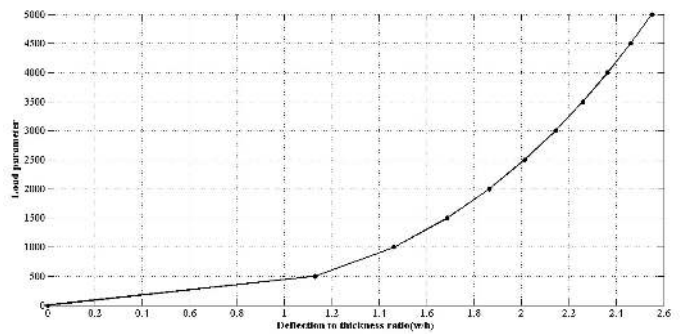
**Fig. 17.** The heterogeneous quadrilateral plate II



**Fig. 19.** The elliptical plate



**Fig. 18.** The load- maximum deflection curves for the heterogeneous quadrilateral plate II



**Fig. 20.** The load- maximum deflection curves for the elliptical plate

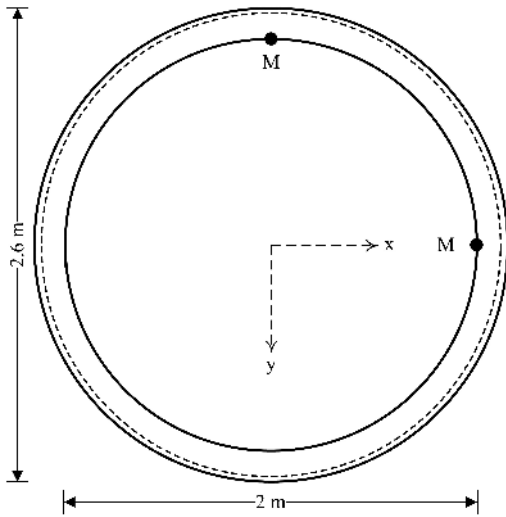


Fig. 21. The donut-shaped plate

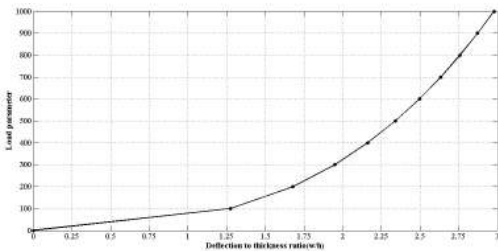


Fig. 22. The load- maximum deflection curves for the donut-shaped plate

### 3.11 The plate with arched edges

A quadrilateral plate with two arched edges is analyzed. Fig. 23 shows this structure. The right edge is a circle with a radius of 0.5 m, and the left boundary is a three-point arc. The number of used bending elements for this plate is 74. Fig. 23 shows the location of maximal deflection. The coordinates of this node are (-0.25, 0) m. Fig. 24 and Table 19 present the results. As it was the case for the previous samples, both RPTH methods are not able to converge to a proper answer. Moreover, the MRE technique diverges after the first increment. Underwood procedure and the kinetic dynamic relaxation process are the most efficient solutions for the analysis of this plate, while Papadrakakis algorithm is the worst method. It is worth emphasizing; Zhang1 and mdDR1 schemes as well as Qiang and RPS strategies have a similar behavior.

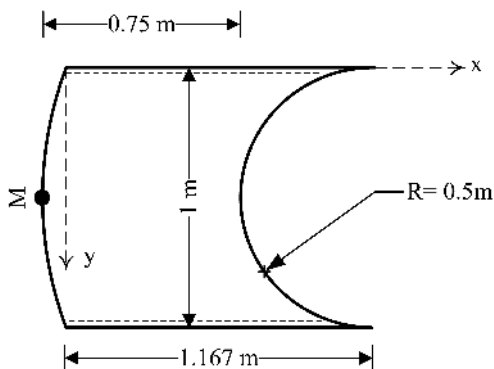


Fig. 23. The plate with arched edges

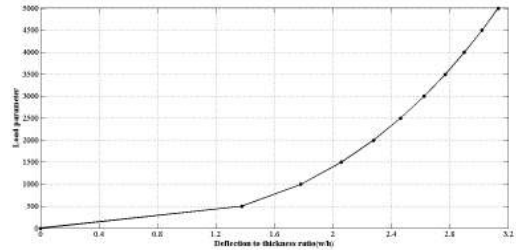


Fig. 24. The load- maximum deflection curves for the plate with arched-shape edges

### 3.12 The circular plate with rectangular opening

Dynamic relaxation processes are used for the analysis of circular plates shown in Fig. 25. This structure has a quadrilateral opening. The meshing is done in one-quarter of the plate due to the geometrical and loading symmetry. The number of elements used is 52. Fig. 26 portrays the load- maximum displacement curve. The number of iterations and the analysis time are inserted in Table 20. Underwood and mdDR1 methods are the most appropriate ones to solve this problem.

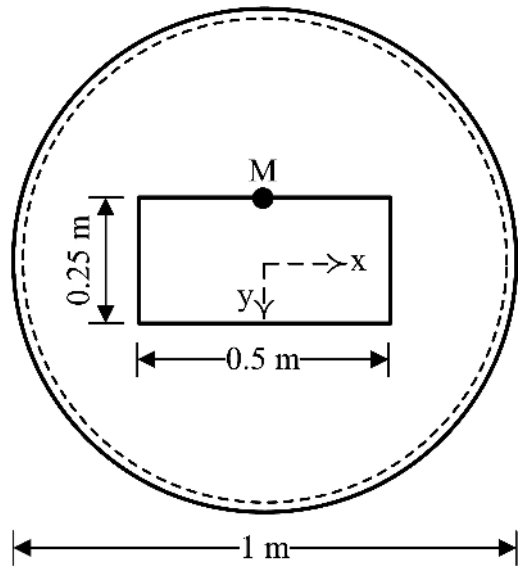


Fig. 25. The circular plate with rectangular opening

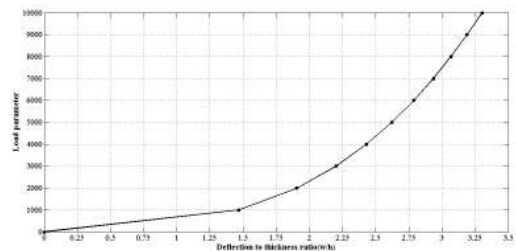


Fig. 26. The load- maximum deflection curves for the circular plate with a rectangular opening

### 3.13 The L-shaped plate with opening

Here, the plate shown in Fig. 27 is studied. Values of **a** and **b** are 0.3 and 0.2 m, respectively. The maximum deflection occurs at a node with coordinates (1.4, 0.5) m. This structure is modeled by 199 bending elements. The results are shown in

**Tab. 19.** The ranking of methods for the plate with arched edges

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	14501	11512	10540	12149	8791	10330	8367	7959	9168	6890	100207	0	14	21430.75	0	14
2	1260	1052	962	903	834	810	781	760	763	639	8764	100	1	1385.687	100	1
3	1566	1352	1246	1170	1112	1065	1027	995	967	943	11443	97.07	9	2186.328	96.006	10
4	1208	1051	986	925	894	865	837	808	790	769	9133	99.596	2	1443.141	99.713	2
5	1267	1101	1035	971	939	906	878	849	828	804	9578	99.11	6	1512.156	99.369	4
6	1996	1735	1656	1218	1189	1136	1086	1067	1031	1017	13131	95.224	12	2063.937	96.616	8
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1768	1703	1477	1356	1245	1229	1114	1089	1069	1060	13110	95.247	11	2180.641	96.034	9
9	1281	1101	1035	971	939	906	878	849	828	804	9592	99.095	7	1942.312	97.223	7
10	1208	1049	986	929	894	865	837	808	789	769	9134	99.595	3	1468.672	99.586	3
11	1597	1016	986	929	894	864	837	808	789	769	9489	99.207	5	1917.828	97.345	6
12	1427	1012	986	929	894	864	837	808	789	769	9315	99.397	4	1887.047	97.499	5
13	1566	1352	1245	1169	1111	1065	1027	995	967	942	11439	97.075	8	3327.14	90.315	12
14	1584	1368	1262	1186	1129	1083	1045	1013	985	961	11616	96.881	10	3390.437	89.999	13
15	1732	1773	1791	1806	1811	1821	1829	1839	1833	1851	18086	89.806	13	2838.125	92.754	11
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 20.** The ranking of methods for the circular plate with rectangular opening

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	3285	4630	4162	4122	5389	2329	1627	2489	2451	2735	33219	0	14	2314.219	0	14
2	590	552	465	460	483	421	389	426	410	409	4605	100	1	277.469	100	1
3	693	630	594	568	548	533	521	512	504	498	5601	96.519	10	358.25	96.034	9
4	568	539	528	502	496	483	471	459	450	443	4939	98.833	3	296.797	99.051	3
5	595	566	551	527	520	507	494	481	471	460	5172	98.018	5	311.141	98.347	5
6	621	552	510	518	497	479	460	450	465	457	5009	98.588	4	301.516	98.819	4
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	920	899	846	753	785	708	692	682	689	675	7649	89.362	12	509.688	88.599	12
9	608	565	551	527	520	507	494	481	471	460	5184	97.977	6	342.078	96.828	6
10	567	539	524	503	495	483	471	459	450	438	4929	98.868	2	296.688	99.056	2
11	814	561	524	501	495	483	471	459	450	438	5196	97.935	7	344.031	96.732	7
12	814	561	527	501	495	483	471	459	450	438	5199	97.924	8	344.266	96.72	8
13	693	630	594	568	548	533	521	511	504	498	5600	96.523	9	425.719	92.721	10
14	709	644	608	582	563	548	536	527	520	514	5751	95.995	11	435.14	92.259	11
15	889	937	967	982	1001	1009	1027	1041	1052	1048	9953	81.31	13	599.344	84.197	13
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

Fig. 28. The ranking and scores of the methods are illustrated in Table 21. Both RPTH schemes are divergent. For this reason, the rank of these tactics is zero. Moreover, the MRE and mdDR2 processes have obtained the first rank from the point of view of the number of iterations. On the other hand, Zhang1 technique has the first rank considering the analysis time.

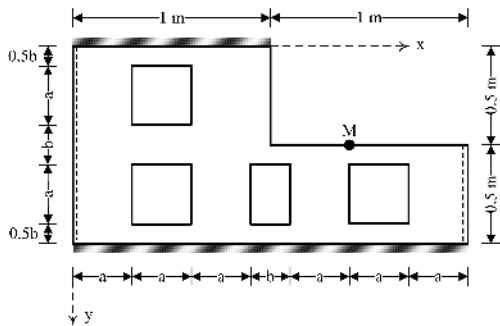


Fig. 27. The L-shaped plate with opening

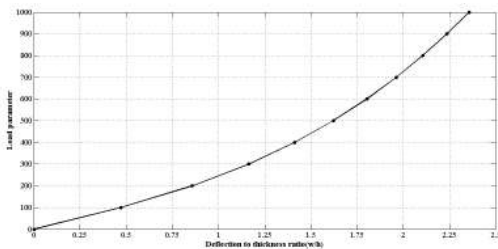


Fig. 28. The load- maximum deflection curves for the L-shaped plate with opening

### 3.14 The rectangular plate with the circular hole

In this section, the quadrilateral plate of Fig. 29 is analyzed. One-quarter of the structure is modeled with 76 elements. The results are shown in Fig. 30 and inserted in Table 22. According to Table 22, RPTH1 and RPTH2 algorithms are not able to find the responses. Moreover, Zhang1 and mdDR1 approaches had the same behavior. These two strategies provide the best solution for this plate. Moreover, Papadrakakis tactic is the worst one.

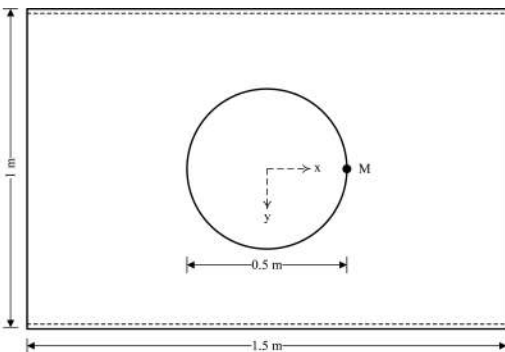


Fig. 29. The rectangular plate with circular hole

### 3.15 The sector donut

In this example, the structure shown in Fig. 31 is solved by using 90 elements. The maximum deflection occurs in the middle

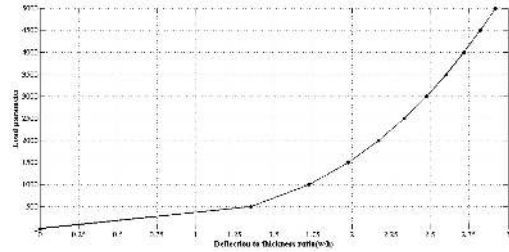


Fig. 30. The load- maximum deflection curves for the rectangular plate with the circular hole

of the inner edge. Fig. 32 demonstrates the load-displacement curve for this node. The number of iterations and the time taken for the analysis are inserted in Table 23. In this problem, Underwood method holds the first rank.

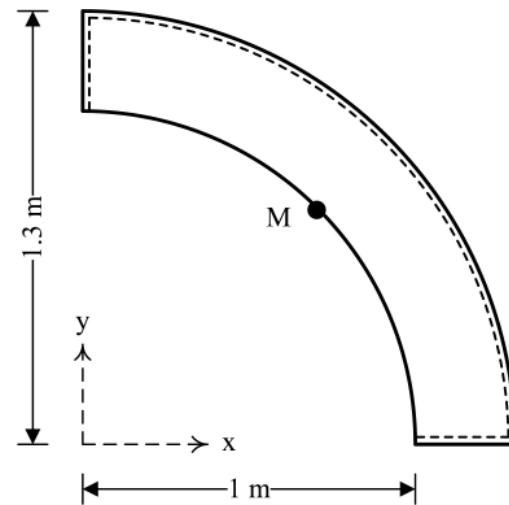


Fig. 31. The sector donut

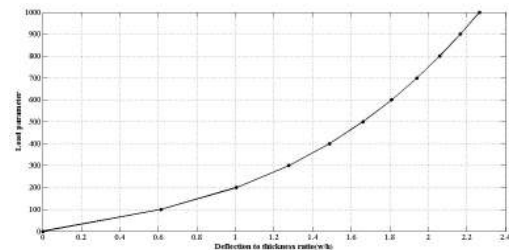


Fig. 32. The load- maximum deflection curves for the sector donut

### 3.16 The sector plate

The final sample is a quarter of a circular plate. Fig. 33 shows the geometry of this structure. Its finite element meshing is done by utilizing 113 bending elements. The maximum deflection is at the central node of the outer boundary. The results are presented in Fig. 34. Table 24 lists the ranking of procedures. RPTH1, RPTH2 and MFT approaches were not able to solve this plate. Moreover, Underwood technique is the best tactic to find the solution for this problem.

## 4 The ranking of methods

Based on the number of iterations and the required time for the analysis, the rank of each method in each sample was

**Tab. 21.** The ranking of methods for the L-shaped plate with opening

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	13799	9244	13535	24567	16894	11845	9314	8244	7222	12870	127534	0	13	6156.813	0	14
2	1626	1492	1218	1156	1119	1178	1100	1095	898	1084	11966	98.734	6	515.671	98.869	6
3	3510	2956	2791	2683	2602	2537	2483	2438	2398	2362	26760	86.095	11	1223.234	86.468	11
4	1408	1281	1154	1092	1007	976	935	900	884	848	10485	99.999	2	451.125	100	1
5	1476	1347	1211	1146	1063	1024	975	945	920	890	10997	99.562	4	473.719	99.604	3
6	2385	1770	1613	1518	1446	1391	1346	1310	1279	1252	15310	95.877	8	661.656	96.31	9
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	2221	2169	1854	1740	1557	1467	1424	1530	1372	1430	16764	94.635	9	754.922	94.676	10
9	1686	1350	1211	1140	1063	1024	975	945	920	890	11204	99.385	5	527.109	98.668	7
10	1407	1281	1154	1092	1007	976	935	900	884	848	10484	100	1	452.078	99.983	2
11	1617	1281	1154	1092	1007	976	935	900	884	848	10694	99.821	3	503.063	99.09	5
12	1617	1281	1154	1092	1007	976	935	900	884	848	10694	99.821	3	503.047	99.09	4
13	3510	2956	2791	2683	2602	2537	2483	2437	2397	2362	26758	86.097	10	1377.844	83.758	12
14	3550	2994	2832	2726	2646	2582	2529	2484	2445	2410	27198	85.721	12	1426.109	82.912	13
15	2894	1684	1103	1031	1007	1029	1025	1028	1032	1046	12879	97.954	7	553.485	98.206	8
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 22.** The ranking of methods for the rectangular plate with the circular hole

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	17362	17257	14189	7727	6789	7717	6612	6460	5516	5360	94989	0	12	5881.203	0	14
2	1313	1356	1284	1082	1089	1041	1060	970	1002	961	11158	100	1	573.125	100	1
3	2846	3127	3108	3035	2959	2891	2831	2778	2731	2689	28995	78.723	10	1601.782	80.621	11
4	1166	1363	1336	1270	1225	1210	1196	1181	1140	1119	12206	98.75	2	626	99.004	2
5	1223	1424	1404	1326	1279	1270	1255	1235	1194	1174	12784	98.06	4	655.844	98.442	4
6	1588	1446	1318	1356	1293	1249	1195	1165	1123	1098	12831	98.004	5	658.984	98.382	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	1878	1792	1637	1767	1740	1539	1532	1582	1447	1242	16156	94.038	8	867.172	94.46	10
9	1610	1424	1404	1326	1279	1270	1255	1235	1194	1174	13171	97.599	6	755.516	96.564	9
10	1166	1362	1337	1268	1225	1210	1197	1181	1140	1120	12206	98.75	2	627.234	98.981	3
11	1454	1362	1337	1268	1225	1210	1197	1181	1140	1119	12493	98.408	3	717.422	97.282	7
12	1454	1362	1337	1268	1225	1210	1197	1181	1140	1119	12493	98.408	3	720.843	97.217	8
13	2846	3127	3108	3034	2959	2890	2830	2777	2730	2689	28990	78.729	9	1969.406	73.695	12
14	2874	3164	3151	3081	3008	2941	2882	2830	2785	2744	29460	78.168	11	1999.375	73.131	13
15	3058	2050	1408	1022	1023	1052	1041	1070	1055	1099	13878	96.755	7	711.765	97.388	6
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 23.** The ranking of methods for the sector donut

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	7352	10009	9798	5434	7229	8374	6113	17509	6242	5213	83273	0	12	6733.235	0	13
2	1296	1106	1296	1027	1051	1161	1149	1139	1027	1018	11270	100	1	748.234	100	1
3	3472	3189	3090	3032	2995	2970	2953	2941	2933	2928	30503	73.289	10	2243.969	75.009	10
4	1724	1442	1328	1275	1223	1192	1162	1121	1094	1075	12636	98.103	2	841.375	98.444	2
5	1808	1512	1393	1335	1282	1250	1219	1175	1147	1125	13246	97.256	5	882.563	97.756	4
6	2087	1629	1529	1449	1309	1279	1256	1239	1219	1203	14199	95.932	6	950.766	96.616	5
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	2645	2000	1832	1635	1764	1609	1542	1659	1562	1545	17793	90.941	8	1240.703	91.772	9
9	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
10	1724	1442	1330	1275	1222	1192	1162	1121	1095	1075	12638	98.1	3	846.953	98.351	3
11	1739	1442	1330	1277	1222	1192	1162	1121	1095	1075	12655	98.076	4	978.125	96.159	7
12	1739	1442	1330	1277	1222	1192	1162	1121	1095	1075	12655	98.076	4	980.828	96.114	8
13	3471	3189	3089	3031	2994	2969	2952	2940	2932	2927	30494	73.301	9	2873.031	64.498	11
14	3522	3240	3143	3087	3052	3028	3013	3002	2996	2992	31075	72.494	11	2946.594	63.269	12
15	1349	1409	1425	1462	1472	1467	1510	1513	1514	1516	14637	95.324	7	976.141	96.192	6
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0

**Tab. 24.** The ranking of methods for the sector plate

Method	Number of iterations in each loading step										Total Iterations	Score	Grade	Time (Second)	Score	Grade
	1	2	3	4	5	6	7	8	9	10						
1	2232	1736	2335	2513	2471	2621	2119	2097	3604	2781	24509	0	14	813.672	0	14
2	374	302	286	246	262	235	212	220	231	223	2591	100	1	83.328	100	1
3	382	336	317	305	298	293	291	290	289	290	3091	97.719	9	102.469	97.379	7
4	344	329	313	296	285	278	271	264	259	255	2894	98.618	3	93.078	98.665	3
5	361	346	326	310	299	292	284	277	271	267	3033	97.983	6	97.297	98.087	4
6	521	546	537	526	515	503	490	477	476	461	5052	88.772	12	162.188	89.202	11
7	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0
8	588	518	449	461	461	450	416	426	395	420	4584	90.907	11	173.75	87.619	12
9	375	343	325	308	299	289	284	277	271	267	3038	97.961	7	103.047	97.3	8
10	344	329	310	296	288	278	270	264	258	255	2892	98.627	2	92.796	98.704	2
11	357	329	310	296	288	278	270	264	258	255	2905	98.567	4	98.188	97.965	5
12	365	329	310	296	288	278	270	264	258	255	2913	98.531	5	98.657	97.901	6
13	382	336	316	305	297	293	291	289	289	289	3087	97.737	8	106.516	96.825	9
14	397	350	331	319	313	309	307	306	306	307	3245	97.016	10	111.141	96.192	10
15	713	758	778	799	812	823	834	848	853	857	8075	74.979	13	257.641	76.133	13
16	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail	Fail			0			0



**Tab. 25.** The ranking of methods based on the number of iteration

Grade	Score ( $S_i$ )	Method	$Q_{ij}$															
			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	94.5652	2	0	17	0	2	2	0	2	0	0	0	0	0	0	0	0	0
2	84.5109	10	0	1	9	3	2	6	0	2	0	0	0	0	0	0	0	0
3	83.6957	4	0	1	5	8	3	2	3	0	1	0	0	0	0	0	0	0
4	76.087	11	0	2	1	3	5	2	5	2	3	0	0	0	0	0	0	0
5	68.75	12	2	2	0	3	5	3	2	3	2	1	0	0	0	0	0	0
6	60.0543	5	0	0	0	0	2	5	3	4	1	3	2	1	1	1	0	0
7	58.6957	13	0	0	5	0	0	0	1	1	2	7	4	3	0	0	0	0
8	55.9783	6	0	0	0	0	1	3	3	3	5	2	2	0	3	1	0	0
9	55.163	9	2	0	0	1	1	2	5	3	3	2	1	2	1	0	0	0
10	52.9891	3	0	0	1	4	0	0	0	1	1	2	8	3	3	0	0	0
11	45.6522	14	0	0	0	1	4	0	0	0	1	0	2	8	3	4	0	0
12	43.2065	8	0	0	1	0	0	0	0	0	6	4	1	5	2	0	3	1
13	42.1196	15	0	0	0	0	2	1	0	5	0	1	2	0	4	4	1	3
14	23.0978	1	0	0	0	0	0	0	0	0	0	0	1	1	5	7	5	0
15	14.6739	16	19	2	0	1	0	0	0	0	0	1	0	0	0	0	0	0
16	14.1304	7	19	0	3	0	0	0	0	0	0	0	1	0	0	0	0	0

**Tab. 26.** The ranking of methods based on the analysis time

Grade	Score ( $S_i$ )	Method	$Q_{ij}$															
			0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	95.1087	2	0	17	0	3	1	1	1	0	0	0	0	0	0	0	0	0
2	88.0435	4	0	2	9	5	5	1	0	1	0	0	0	0	0	0	0	0
3	86.413	10	0	2	6	9	1	4	0	0	1	0	0	0	0	0	0	0
4	71.4674	5	0	0	0	2	8	3	4	2	1	2	0	1	0	0	0	0
5	65.7609	6	0	0	1	0	2	7	6	0	1	3	0	3	0	0	0	0
6	62.2283	11	0	0	0	0	1	3	5	8	1	3	0	2	0	0	0	0
7	53.2609	3	0	0	3	0	1	0	1	2	0	3	5	8	0	0	0	0
8	51.6304	12	2	0	0	0	1	1	1	4	9	1	2	1	1	0	0	0
9	46.4674	9	2	0	0	1	0	0	1	3	5	3	4	1	1	2	0	0
10	45.9239	15	0	0	0	0	2	3	2	1	2	0	1	2	3	4	1	2
11	42.9348	8	0	0	1	0	0	0	1	0	0	7	5	0	7	1	0	1
12	40.4891	13	0	0	0	2	0	0	0	1	1	1	4	1	9	3	1	0
13	32.337	14	0	0	0	0	0	0	1	1	1	0	1	4	2	10	2	1
14	18.4783	1	0	0	0	0	0	0	0	0	0	0	1	0	0	3	15	0
15	16.3043	16	19	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0
16	14.1304	7	19	0	2	1	0	0	0	0	0	1	0	0	0	0	0	0

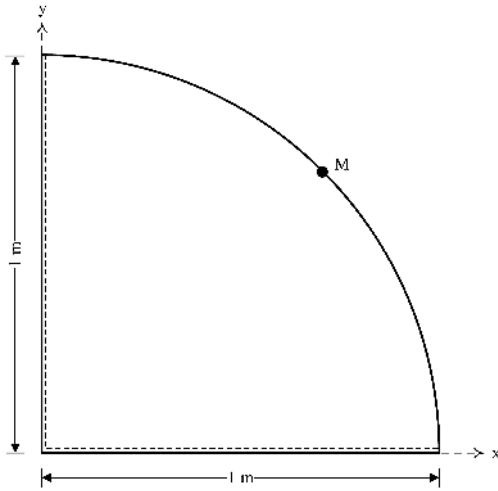


Fig. 33. The sector plate

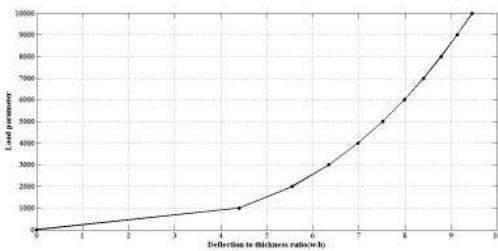


Fig. 34. The load- maximum deflection curves for the sector plate

achieved. Rank number one indicates the best process and rank number sixteen indicates the worst approach. Here, the  $j$ -th rank of the  $i$ -th scheme is shown by  $Q_{ij}$ . For example, Underwood solution has ranked first for seventeen times. So,  $Q_{i1}$  for this tactic is seventeen.  $Q_{i0}$  is the number of structures that the approach  $i$  has not been able to analyze. The rating of algorithm  $i$  is obtained using Eq. (48). It is worth emphasizing; if the procedure is not able to reach the proper response, then it is not entered into the Eq. (48):

$$S_i = 100 \times \sum_{j=1}^{16} Q_{ij} \times (17 - j)/368 \quad (48)$$

It should be noted that if a technique has acquired the number of 368, then that strategy has first rank in all 23 numerical samples. As a result, the rating  $S_i$  will be 100 for that solution. The rating and the final ranking of each method are shown in Tables 25 and 26.

## 5 Conclusion

In this paper, sixteen well-known dynamic relaxation methods were used for solving bending plate. First, the basis of dynamic relaxation approach and the related relationships were presented. Then, the previous procedures reviewed for estimating the fictitious parameters needed for each technique. Afterwards, several bending plate samples with geometrically nonlinear behavior were analyzed. The criteria chosen were the number of iterations and total analysis time. On this basis, various processes were ranked. The results of this study revealed that

Underwood solution took the lowest number of iterations to give the answer. Moreover, Zhang1 and mdDR1 tactics occupy the next-best subsequent ranks. Moreover, these procedures are the best techniques from the point of view analysis time. Rezaiee-Pajand and Taghavian Hakkak scheme was the worst strategy due to the usage of second-order approximations. Because, this process diverged in 19 examples. Among the methods that converged, Papadrakakis algorithm needed the largest number of iterations and the longest time to reach the assumed accuracy. In most samples, the MRE and mdDR2 processes as well as Qiang and RPS approaches had a similar behavior. Such features were also true for Zhang1 and mdDR1 as well as the MFT and Zhang2 techniques. In other words, the number of iterations required to run these procedures were almost the same.

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