A COMPARISON OF POWER INDICES AND A NONSYMMETRIC GENERALIZATION

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CONTENTS

ABSTRACT

The first half of this paper describes and contrasts two well known measures of power in voting systems, introduced by Shapley and Shubik and by Banzhaf and Coleman, respectively. The second half develops an explanatory model for a generalization of the Shapley-Shubik measure, first proposed by Owen in a slightly different form, in which ideological differences among the voters can be taken into account. The purpose of the paper is mainly expository, but a number of new results and interpretations are included.

A COMPARISON OF POWER INDICES AND A NONSYMMETRIC GENERALIZATION*

1. Two Power Indices

One way of looking at how a voting system distributes power among its members is to suppose that each bill or issue will rank the members in order of the degree of their support--the most dedicated advocates first, the less fervid and more persuadable supporters next, and so on, down to the most stubborn opponent at the end of the list. In any such an ordering, one member will always play the role of pivot: he, in company with his more enthusiastic forerunners, can just barely pass the bill. Intuitively, the pivot is a very important person; he may be the one who decides just how strong a bill gets passes, or how much money gets appropriated, or how much advertising a ballot proposition or a candidate for office will need, etc.

For a measure of a priori power--by which we mean abstract power within the given voting system, not power with respect to any particular issue or goal that the system faces--we may as well assume that all orderings of members will occur equally often, i.e., be equally probable. We can then take the probability of being pivotal as a power index for each individual. This is known as the Shapley-Shubik (S-S) index [22], and we shall denote it here by φ_i, $i \in N$, where N represents the set of all individuals. Since each ordering has exactly one pivot, it is clear that $\Sigma_N \varphi_i = 1$.

For a simple example, consider the 3-player voting game with players $N = \{a, b, c\}$ in which the winning coalitions are just $\overline{ac}$,

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$\overline{bc}$, and $\overline{abc}$. In other words, it is majority rule--except that player "c" has a veto.* Considering the six possible permutations of players, we see that ...

in	a b <u>c</u>	the	pivot	is	c
"	a <u>c</u> b	"	"	"	c
"	b a <u>c</u>	"	"	"	c
"	b <u>c</u> a	"	"	"	c
"	c <u>a</u> b	"	"	"	a
"	c <u>b</u> a	"	"	"	b

So the S-S power indices are $\varphi_a = \varphi_b = 1/6$, $\varphi_c = 4/6$.

Another approach to the measurement of power is to have each voter i regard the others as random decisionmakers, voting "yes" or "no" at the toss of a coin. He then asks, "what chance do I have of being able to tip the scales and decide the outcome?" Let us call a pair of coalitions of the form $(S - \{i\}, S \cup \{i\})$ a <u>swing for i</u> if the former is losing and the latter winning, and let us denote by Y_i the subset of $N - \{i\}$ that votes yes. Then i's chance of being decisive is just the probability that $(Y_i, Y_i \cup \{i\})$ is a swing for him. This provides another index of the <u>a priori</u> "power" inherent in the given voting rule. It is called the Banzhaf or Banzhaf-Coleman (B-C) index [3,4,7] and will be denoted here by β_i, $i \in N$.

*Another way to describe this game is by the "weighted majority" symbol [3; 1, 1, 2], meaning that the players have 1, 1, and 2 votes, respectively, and it takes 3 votes to win.

Returning to the previous example, we see that there are four possibilities for the set Y_a, namely $\overline{bc}$, $\overline{b}$, $\overline{c}$, and ϕ. Player "a" swings only if $Y_a = \overline{c}$, however, so $\beta_a = 1/4$. Similarly, $\beta_b = 1/4$. But "c", the veto player, swings whenever $Y_c = \overline{a}$, $\overline{b}$, or $\overline{ab}$, so we have $\beta_c = 3/4$. Note that these numbers do not total 1; this is because a swing, unlike a pivot, is not an isolated occurrence--swings may come in bunches, or not at all. This becomes apparent if we adopt a more unified approach to the probability model, basing it on a random subset Y of all of N instead of dealing separately with $Y_i \subset N - \{i\}$ for each $i \in N$. Then, in our example,

if $Y = \overline{abc}$,	then	c is a swinger
" $Y = \overline{ab}$	"	c " "
" $Y = \overline{ac}$	"	a and c are swingers
" $Y = \overline{bc}$	"	b " c " "
" $Y = \overline{a}$	"	c is a swinger
" $Y = \overline{b}$	"	c " "
" $Y = \overline{c}$	"	a and b are swingers
" $Y = \phi$	"	nobody swings.*

In a larger game there would be more bunching, and more cases where nobody swings. For example, how often does an election depend on a single vote? In the game $\underset{\sim}{M}_9$ (directly majority voting on nine players), there are swings only when $|Y| = 4$ or 5, which happens

*In this tabulation each swing gets counted twice, since we are disregarding whether the player in question is a member of Y or not.

slightly less than half of the time. Asymptotically, the probability that there will be swingers in $\mathcal{M}_n$ goes to zero like $1/\sqrt{n}$. But when swings do occur there are $\approx n/2$ of them, so the <u>expected number</u> of swingers in these games grows like $\sqrt{n}$.

2. Additivity

"Power" in political science is a penumbral concept--a blur of largely overlapping but slightly inconsistent ideas. Even when the term is used carefully and logically in verbal discourse a variety of different formal properties may be suggested to the mathematical modeller.* Sometimes power is an ordinal affair, e.g., when control over other people is the issue, as in an organizational hierarchy or a "pecking order." Sometimes a kind of physical analogy with matter or energy is suggested, power being a mysterious substance that is conserved during certain operations or transformations, so that one person's loss of power is always another's gain. In another view, however, the total power can wax and wane, and a "power vacuum" or even "negative power" can appear. Sometimes power is talked about as if it inheres only in individuals, or in groups or parties congealed into well-defined voting blocs. But at other times it may make sense to speak of the power of arbitrary sets of individuals, not necessarily acting in concert. Such a set-functional power index may or may not be naturally <u>additive</u>--that is, there may or may not be an intuitive justification for

$$\text{power}(S) + \text{power}(T) = \text{power}(S \cup T),$$

whenever S and T are disjoint sets. Power indices are like other solution concepts in game theory. In choosing among them we must always consider the real-world context of the model that we are solving and

*For a broad sampling of views on the anatomy of political power, the reader is invited to browse through the collections edited by Shubik [23], Ulmer [29], Pennock and Chapman [19], Krimerman [15], Groennings, Kelley, and Leiserson [11], and Papayanopoulos [18].

the questions that we want the solution to answer. Small wonder, then, that such a variety of "power indices" have been defined in the literature, many of them similar in their general design but not at all equivalent when their ramifications are pursued.*

For the present, let us focus on additivity. In the S-S view of "power" there is a fixed total power, namely $\Sigma_N \varphi_i = 1$, which is distributed among the voters according to their constitutional prerogatives. The S-S power index is immediately seen to be additive if we make the obvious definitional extension to a set function, namely, let $\varphi(S)$ for any set $S \subset N$ be the probability that S includes the pivot. Then $\varphi(S)$ is simply $\Sigma_S \varphi_i$. With this natural additivity it would make sense, say, to add the numbers φ_a and φ_b in our 3-person example and, observing such an inequality as $\varphi_c > \varphi(\overline{ab})$, to make statements like "c is more powerful than a and b together," or "c has more than half the total power."

The B-C index, in contrast, does not lead quite so naturally to an additive set function. If we try to define $\beta(S)$ as the probability that S includes a swinger, in analogy to $\varphi(S)$ above, then β is in general not additive; it is, however, <u>subadditive</u>: $\beta(S \cup T) \leqq \beta(S) + \beta(T)$ whenever $S \cap T = \emptyset$. As an alternative, we could try to define $\beta(S)$ as the probability that the members of S, voting as a bloc, are able to swing the election, the others voting at random. But this also leads to a nonadditive function--in this case neither

*For other definitions, not discussed in this paper, see Allingham [1], Blair [5], Dubey [9], Owen [17], Straffin [25], and Young [30]; see also Sec. 12 of [10] where further references will be found.

subadditive nor superadditive.* Of course, we can just go ahead and add up the individual powers β_i and hope to find some meaning in the sum. In fact, $\Sigma_S\ \beta_i$ represents the average or expected number of swinging players in S; but in what intuitive sense does this measure the "power" of S? Indeed, there is some philosophical inconsistency in talking about two or more voters simultaneously being swingers, since supposedly each person imagines that he alone can make a conscious decision, the others being mechanical actors. To put it another way, it is easier to explain the separate random variables Y_i than the "unified" random variable Y.

We should mention that nonadditive extensions of the S-S index can also be devised. For example, we could consider for each S the pivoting probability that would result if the members of S were fused into a single player. This approach will be discussed further in Sec. 4.

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*Direct majority voting $\underset{\sim}{M}_n$ yields a subadditive function under this definition, while the unanimity rule B_n yields a superadditive function. Intermediate rules, however, can yield a mixed result. For example, for the game $\underset{\sim}{M}_{13,9}$--i.e., thirteen equal players with nine votes needed to win--the swing probabilities for coalitions of size 2, 3, 5, and 8 are 990/K, 1500/K, 2608/K, and 3968/K, respectively, where $K = 2^{12} = 4096$, revealing nonadditivity in both directions since $990 + 1500 < 2608$ and $1500 + 2608 > 3968$.

3. An Example

The following nine-player example has been discussed several times in the literature.* The game, denoted conventionally by the symbol $\underset{\sim}{M}_5 \otimes \underset{\sim}{M}_3 \otimes \underset{\sim}{M}_1$, is the "product" of three separate direct majority games [21]. For a coalition to be winning it must contain at least three of the first five players, two of the next three, and always the ninth player. We may think of the latter as holding a sort of "Presidential veto" over acts passed by majority vote in a 5-person "House" and a 3-person "Senate."

In Table 1 we give the individual players power indices of both kinds, S-S and B-C, expressed also as percentages for readier comparison. Table 2 gives the same for the three component "chambers" of the game, using the five different set functions mentioned in Sec. 2: two extensions of the S-S index and three extensions of the B-C index. Note that the B-C extensions yield three qualitatively different comparisons of "cameral power." But the version that we would consider most plausible (line 3 of the table) is in fairly good agreement with the additive S-S definition (line 1).**

Of course, there is no particular reason, with any of these extensions, to confine our attention to just those three particular subsets of players. If we are to have a set-functional definition of power, then it should be applicable to all subsets. One could imagine, for example, that players 1, 3, 4, 6, 8, and 9 are "Democrats"

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*See especially Shapley and Shubik [22], p. 792, Brams [6], p. 193, and Straffin [24].

**Brams loc. cit. considers only the two additive extensions (lines 1 and 5), and makes much of the contrast between them.

Table 1

Players:	1 thru 5	6, 7, or 8	9 (veto)
S-S Index	.060 (6.0%)	.107 (10.7%)	.381 (38.1%)
B-C Index	.094 (8.6%)	.125 (11.4%)	.250 (22.9%)

Table 2

Sets:	{1,2,3,4,5} ("House")	{6,7,8} ("Senate")	{9} ("President")
prob. set contains the pivot	.298 (29.8%)	.321 (32.1%)	.381 (38.1%)
prob. set as a whole pivots	.350 (32.2%)	.357 (32.8%)	.381 (35.0%)
prob. set contains a swinger	.156 (26.3%)	.188 (31.6%)	.250 (42.1%)
prob. set as a whole swings	.250 (33.3%)	.250 (33.3%)	.250 (33.3%)
exp. no. of swingers in the set	.469 (42.9%)	.375 (34.3%)	.250 (22.9%)

while 1, 4, 6, 7, 8, and 9 are "Southerners." Then Table 3 reveals a curious paradox: the two additive measures (first and fifth lines) both say that the Southerners have more power than the Democrats. This is in spite of the fact that the Democrats constitute a winning coalition and the Southerners do not. It is not hard to see why. The Southern coalition has one more "Senator" and one less "Representative," and Senators are individually more powerful than Representatives.

Generalizing this observation, it can easily be shown that any additive measure of coalitional power, applied to any simple game that is not expressible as a weighted majority game, will necessarily exhibit a pair of coalitions, one winning and one losing, in which the losing coalition has at least as much power as the winning coalition. The proof is very simple: if there were an additive measure that did not exhibit such a pair of coalitions, then it could be used directly to define a threshold and a set of voting weights.

Table 3

Sets:	{1,3,4,6,8,9} ("Democrats")	{1,4,6,7,8,9} ("Southerners")
prob. set contains the pivot	.774 (77.4%)*	.821 (82.1%)*
prob. set as a whole pivots	1.000 (82.0%)	.750 (72.2%)
prob. set contains a swinger	.422 (66.7%)	.414 (72.6%)
prob. set as a whole swings	1.000 (84.2%)	.875 (77.8%)
exp. no. of swingers in the set	.781 (71.4%)	.812 (74.3%)

*The percentages are obtained by comparison with the "power" that the same definition would give to the complementary sets (i.e., {2, 5, 7} and {2, 3, 5}).

4. Merged Players

The "bloc voting" definitions illustrated on lines 2 and 4 of Tables 2 and 3 lead us naturally to consider the problem of merged or aggregated players. How should the "power" of a group of players be affected if they always vote the same way, i.e., if they totally submerge their individualities in a single personality? Modellers in the social sciences often have to deal with aggregations of individuals--e.g., firms, households, political parties, even associations of nations (e.g., "Benelux")--which must play the role of individual voters or decisionmakers in some larger system.* On the other hand, there are equally important areas of application where principles like "All men are created equal" or "One person one vote" rule supreme, making it inappropriate for the modeller to resort to techniques of aggregation, or at least for him to treat the aggregates as though they were individuals on an equal footing.

It is therefore worth asking to what extent our game solution concepts (i.e., power indices) are stable or invariant under merging of players. It is also worth searching for ways of introducing some systematic asymmetry into our solution concepts, in order to accommodate situations where the players may not be "created equal." The latter task will be addressed in the next section, but first let us look a bit further at the question of the power of merged players with the aid of a few simple examples.

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*A striking example of ambiguous identification of individuals is the provision in the United Nations charter whereby the Byelorussian and Ukrainian S.S.R.'s are given membership and votes in the General Assembly.

Consider first the very simple case of the unanimity game with, say, seven players; this is traditionally denoted $\underset{\sim}{B}_7$. Recalling our definitions of φ and β, we see without difficulty that $\varphi_i \equiv 1/7$ and $\beta_i \equiv 1/64$, identically for all players.* If two players are now combined, the new game is $\underset{\sim}{B}_6$ and we have $\varphi_i \equiv 1/6$, $\beta_i \equiv 1/32$. So in the S-S view, the merged players lose almost half their combined pivoting power, dropping from 2/7 = 28.57 percent to 1/6 = 16.67 percent, while the other players register a small gain. In the B-C view, the merged players maintain their combined (additive) swinging power at 1/32, but meanwhile the other players have doubled theirs. In either case it doesn't pay to merge, and this conclusion seems intuitively correct. When you merge in $\underset{\sim}{B}_7$, you are just giving up one way to cast a veto in a society where the veto power is everything.

For contrast, consider the game $\underset{\sim}{M}_7$ (i.e., simple majority voting on seven players). This time we have $\varphi_i \equiv 1/7$ and $\beta_i \equiv 5/16$, identically for all players.* Combining two players now produces the non-symmetric six-person weighted majority game [4; 2, 1, 1, 1, 1, 1], i.e., four votes to win, but one player has two votes. Since the two-vote player pivots in exactly two of the six positions in the ordering, his index is 2/6 = 1/3. The other five players all get 2/15, since the total must be 1. So this time the merged pair's S-S power <u>rises</u> from 28.57 percent to 33.33 percent, while the others drop from 14.29 percent to 13.33 percent. In this situation merging

*If the B-C indices are normalized to add up to 1 (as is often done in the literature, but with very little justification), they will of course coincide with the S-S indices in these symmetric games.

seems to pay. This is confirmed by the B-C indices, which are 5/8 for the merged pair against 2/8 apiece for the others. The two merging players again just break even in absolute terms, but this time they show a healthy relative gain over their rivals.*

In fact, it is a curious theorem that whenever two players combine into one, in any game, their new B-C index is exactly the sum of their old ones. (The proof is quite straightforward, and can safely be left as an exercise.) This little hint of additivity does not extend to mergers of more than two, however; indeed, additivity can fail in either direction. The "tricameral" game $\mathcal{M}_5 \otimes \mathcal{M}_3 \otimes \mathcal{M}_1$ of Sec. 3 provides an example: the B-C power of a merged set like {1, 6, 9} is 33/64 = .515625, which is greater than $\beta_1 + \beta_6 + \beta_9$ = .46875, whereas for the set {6, 7, 8} it is .250, which is less than $\beta_6 + \beta_7 + \beta_8$ = .375. The point of the matter is that combining two players, while conserving their own B-C power, will usually alter the B-C powers of the other players.** When we bring a third member into the club we add his new, altered power to the total, but this generally will be different from his original power.

These simple examples have shown that there can be sharp disturbances in the distribution of a priori power when players are aggregated

*Somewhat coincidentally, φ and β are proportional here. This happens to be true for all (2m)-person weighted majority games of the form [m + 1; 2, 1, ..., 1]; in fact, under either direction the advantage of the first player over the others is in the ratio (2m - 1)/(m - 1).

**This may seem strange, but it is not a quirk or defect in the particular B-C or S-S definitions. Consider the game $\mathcal{M}_3$. Merging two players creates a dictatorial situation, in which the "power" of the third player must drop to zero under any definition that recognizes that dummies are powerless.

or disaggregated, reflected even (or especially!) in the indices of individuals not directly involved in the aggregation. It seems inevitable that any theory that relies on the underlying symmetry of the individual voters should be unstable under aggregation, and should consequently be unreliable when applied to models where the basic decisionmaking units are not clearly identifiable in the real situation but are chosen more or less arbitrarily by the modeller.

5. An Asymmetric Generalization of the S-S Index

As we have just seen, asymmetry in a theory of power in political games may sometimes be desirable. We are speaking of course of asymmetry in the preferences or habits or outlooks of the players--not of differences that can be captured in the formal, constitutional rules of the game. We shall now proceed to sketch one way of differentiating among players in a natural and intuitive manner; it is in large measure inspired by a 1971 paper of Guillermo Owen [16], though there are several differences both in viewpoint and technique.

Since we want to be able to talk about similarities among players as well as differences, it seems reasonable to represent them by points in a topological space. We shall use the finite-dimensional Euclidean space R^m, as this seems to leave us ample scope for capturing many kinds of political or ideological parameters without an excess of abstraction and generality. In particular, the linear structure of R^m will help us represent geometrically the intuitive ideas of "moderation" and "extremism" that are so prominent in non-mathematical political analysis.*

We now recall the original S-S viewpoint (see Sec. 1), wherein "issues" arise out of some random process and align the players in order of their intensity of support or opposition. We shall represent issues in our present model as functions on R^m, using them to generate the desired order relations by the simple rule:

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*For a starter, the reader may set $m = 1$ and think of R^1 as the simplistic left/right spectrum of popular political analysis.

(1) $$i \underset{f}{<} j \Longleftrightarrow f(x^i) > f(x^j).$$

Here $x^i \in R^m$ represents an ideological description or "political profile" of voter i. Again trying to avoid uncalled-for generality, we shall restrict ourselves to linear functions $f(x) = \Sigma\ \xi_i x^i + \xi_0$, and hence (without further loss of generality) to homogeneous linear functions $f(x) = \Sigma\ \xi_i x^i$.* Thus, the political issues ξ and the political profiles x^i are both represented by m-dimensional vectors, but they should be regarded as belonging to different, dual spaces, not to the same space.** In the convenient inner product notation, (1) becomes

(2) $$i \underset{\xi}{<} j \Longleftrightarrow (\xi,\ x^i) > (\xi,\ x^j).$$

Figure 1 illustrates this graphically for m = 2; the picture in higher dimensions would be entirely analogous. The long arrows indicate the directions associated with two typical issues ξ. The

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*In fact, one could normalize by $\|\xi\| = 1$ and nothing would be lost but the degenerate case $\xi = 0$. In short, only the direction of the issue matters--at least for the present purpose. But there may be extended uses of this model--e.g., in treating the dynamics of coalition-forming--in which the magnitude of ξ could also play a role.

**A contrasting approach is often encountered where the players and issues occupy the same space. Tullock describes it this way:

> [the] situation can be represented by an ... issue space with each of the issues representing one dimension and an individual having some point which is for him optimal .... Presumably his level of satisfaction will fall off as the actual social choice moves away from his optimum. ... We assume this fall-off is uniform in every direction.

([28], p. 419; see also [2, 12, 13, 14, 26, 27].) By a slight generalization, players in the Tullock model become identified with functions on the issue space, giving rise to a formulation that is in effect dual to our present one.

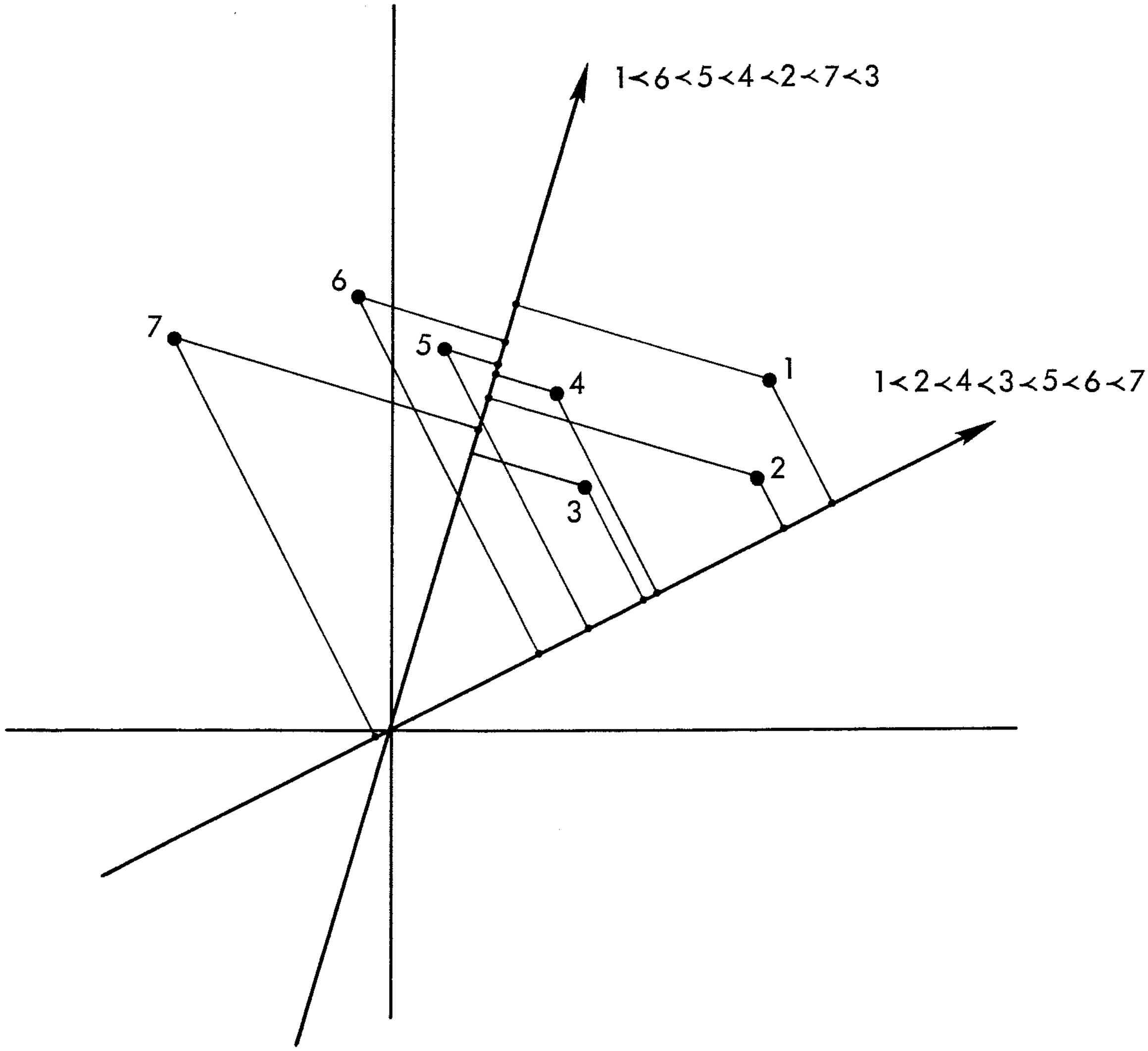


Fig. 1

corresponding player-orders are discovered by the simple process of dropping perpendiculars to the shafts of the arrows. As the arrow turns different orders are produced; thus, a 180° turn reverses the order. While ties are possible, they can be neglected if the points x^i are distinct and the distribution of issues is nonatomic, i.e., if no significant fraction of the issues have precisely the same direction.

It is interesting to observe that we may not be able to generate all possible orders by turning ξ even if the constellation of voter profiles is "in general position." To see this, consider a typical subset of points:

(3) $$C(S) = \{x^i : i \in S\}, \qquad S \subset N.$$

Unless we can separate C(S) from C(N - S) by a hyperplane in R^m, it will never be possible for the members of S to appear in some order as the initial or final segment. In particular, an individual whose position happens to be interior to the convex hull of C(N) can never be ranked first (or last). If the number of profiles is large compared to the dimension of R^m, the proportion of excluded orders will be very large (see Note 1 at the end of this paper). In other words, when m is small and $n = |N|$ is big, as will be the case in many electoral applications, then our theory envisages that only a very small fraction of the possible alignments of players will actually be ideologically consistent.

On the other hand, it is easy to conjure up situations in application where it is natural to have at least as many dimensions as players. For example, each person's private welfare might require a separate coordinate of R^m, making $m \geq n$. In this case, all alignments will be possible if the profiles are in general position.

In order to define the power indices associated with a given voting rule or simple game and a given constellation C(N) of voter profiles, we shall assume that the "political winds" blow across the ideological space in a perfectly random way--i.e., that all issue directions are equally likely. Since the magnitude of ξ (or "force of the wind," i.e., intensity or importance of the issue) does not matter to us at present, we can choose ξ according to the uniform probability distribution over the unit sphere $S^m = \{\xi : \|\xi\| = 1\}$. Then the probability that ξ lies in any given region in S^m is proportional to the area of that region.* For each player i, define

$$P_i(\xi) = \{j \in N : j \prec_{\xi} i\};$$

thus, $P_i(\xi)$ is the set of players that are more enthusiastic about the issue-direction ξ than i is. Define S_i^m to be the set of ξ in S^m for which $P_i(\xi)$ is losing and $P_i(\xi) \cup \{i\}$ is winning. Our generalized S-S power index is then given by

(4) $$\varphi_i = \frac{\text{area of } S_i^m}{\text{area of } S^m}.$$

*Of course, other probability distributions could be considered. But, as we remarked earlier, extra generality *per se* would serve little purpose. We already have more than enough flexibility in the model (e.g., in choosing and scaling the dimensions of our space; see Note 2 in the appendix), especially in view of the vagueness inherent in so many of the political measurements and classifications that must be made.

Examples of how this works out in some particular cases will be given in the next section.

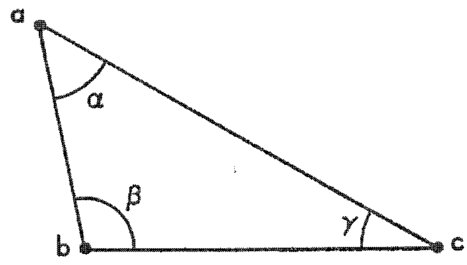
In what sense is the present set-up a generalization of the symmetric theory? In order for (4) to reduce to the original S-S indices, the profiles of the players must form the vertices of a regular simplex. No other arrangement will do. This means of course that m must be at least n - 1. For a canonical representation of the symmetric S-S theory, however, it is more convenient to waste a dimension and take $m = n$, letting the points x^i be the unit vectors of R^m. In this way, each coordinate can be identified with a different player's private interest.

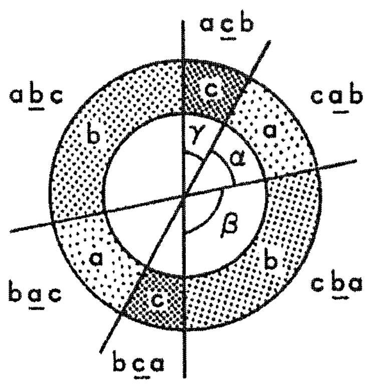
6. Examples

It is now time to illustrate our ideologically oriented power measure with some examples; for some comparable material see Owen [16].

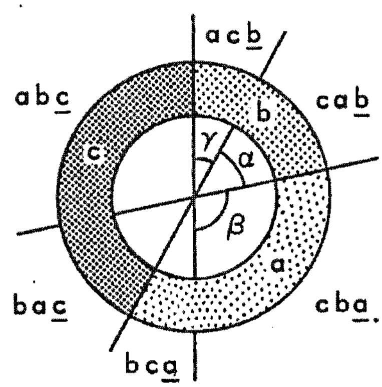
First let m = 1. Assuming that the points x^i are all distinct, there are just two relevant orders: left-to-right and right-to-left. Hence there are just two potential pivots. Under any voting rule, half the power goes to each; they may of course be the same player. An example would be a centrist party in a majority-rule ($\underset{\sim}{M}_n$) parliament. No matter how small this party, if it holds the "balance of power" between those to its left and those to its right, then it will be all-powerful. On the other hand, in a "$\underset{\sim}{B}_n$" parliament, acting only on unanimous consent, the central parties would have no influence; instead, the extremists of right and left would divide all the power. Other voting rules would give intermediate results, but in no case would there be more than two parties with any power, in this politically naive, one-dimensional country.

Next, let m = 2 and consider the case of three players. The points x^i, if distinct and noncolinear, form a triangle as in Fig. 2. The accompanying "cartwheel" diagrams show the orders induced by each issue-direction; the radial lines are perpendicular to the triangle sides and indicate where two players are tied. The probability of any particular permutation proves to be proportional to the angle associated with the middle player. Equipped with this little trigonometric insight, we can take any 3-person voting rule and immediately read off the power indices. The first cartwheel in Fig. 2 reveals that in $\underset{\sim}{M}_3$ a player's power is directly proportional to his angle. Moderation is rewarded: the more extreme or "pointed" a

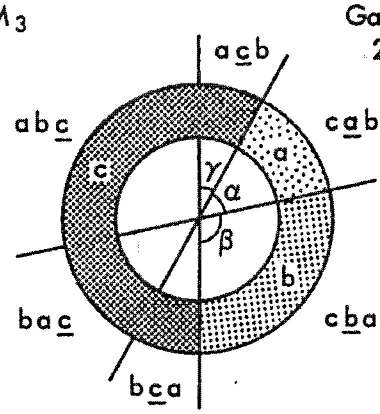




Game: [2; 1, 1, 1] = M_3
$2\pi\phi = (2\alpha,\ 2\beta,\ 2\gamma)$



Game: [3; 1, 1, 1] = B_3
$2\pi\phi = (\pi-\alpha,\ \pi-\beta,\ \pi-\gamma)$



Game: [3; 1, 1, 2]
$2\pi\phi = (\alpha,\ \beta,\ \pi+\gamma)$

Fig. 2

position the less influence it carries. But in $\underset{\sim}{B}_3$, the second case, the opposite is true: power is proportional to the exterior angle at the player's vertex--i.e., to the sum of the other two interior angles. Sharp extremists are now more likely to pivot. Finally, in the game $\underset{\sim}{V}_3(c)$ that we considered in Sec. 1, the veto player "c" gets half the total power plus his "angular" share of the rest; this is shown at the third cartwheel. Interestingly, $\underset{\sim}{V}_3(c)$ resembles $\underset{\sim}{M}_3$ more than $\underset{\sim}{B}_3$ in that extreme positions are penalized, even for the player with veto power.*

For further variety, let us look at a constellation of four players in R^2 (Fig. 3); note that of the possible 24 orders only 12 actually occur.** The first cartwheel depicts the game $\underset{\sim}{B}_4$, and we see that "c" has no power at all. He is not a dummy in the voting, but since he is never an extremist on any issue, he never turns up as the last holdout, controlling the outcome.***

The second cartwheel in Fig. 3 illustrated an asymmetric weighted majority game that has also received some attention in the literature.****

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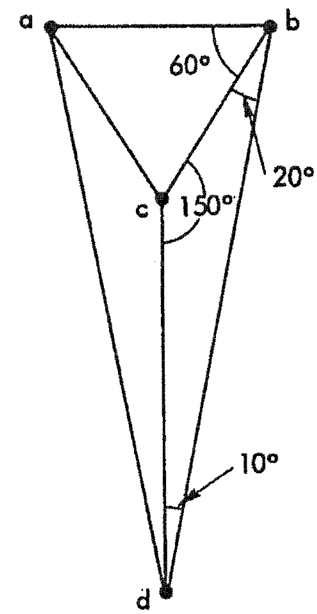
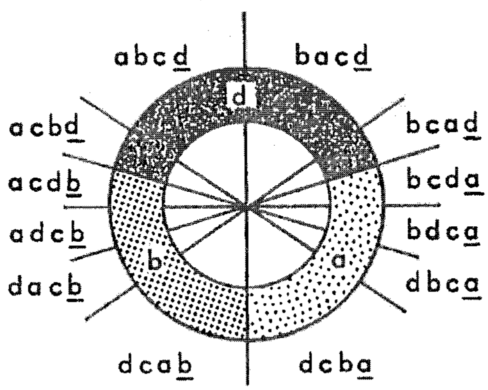
*This analysis for three players in R^2 extends to higher dimensions without change, since all that matters is the 2-plane in R^m determined by the three profiles. (Note that a uniform distribution of issue-directions in R^m, when projected on any subspace of R^m, results in a uniform distribution of directions in that subspace.)

**See (A.3) in the appendix.

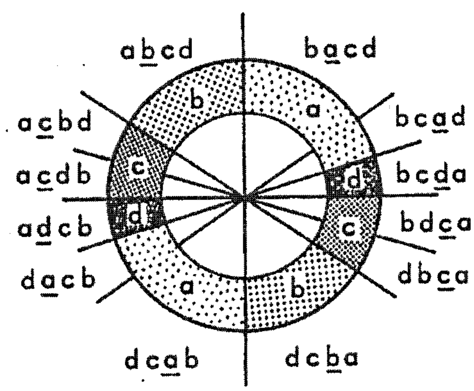
***Though it may seem unrealistic to declare that "c" is totally powerless--since, after all, he has a veto--one may regard this situation as a kind of "dimensional degeneracy," akin to the naive one-dimensionalism we spoke of before. If our friend "c" were not so average--if he would only develop some distinct opinions or interests of his own, then his profile might move out of R^2 into a third dimension and he could begin to wield some power.

****See Owen [16] and Davis and Maschler [8].

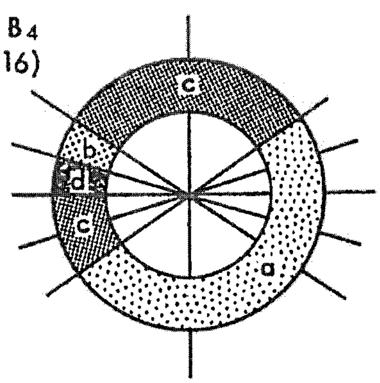




Game: [4; 1, 1, 1, 1] = B_4
Power: $36\phi = (10, 10, 0, 16)$



Game: [3; 2, 1, 1, 1]
Power: $36\phi = (16, 12, 6, 2)$



Game: [4; 2, 1, 1, 1] = $M_1 \otimes M_3$
Power: $36\phi = (18, 2, 15, 1)$

Fig. 3

It takes three out of five votes to win, with player "a" having two votes. Note that this does not automatically give him twice the power of "b", who occupies a similar geometrical position. One explanation might be that "a"'s double vote helps "b" as well, putting him in a better position to pivot despite his outlying political stance. If the double vote were given to "d" instead (not illustrated, but easily calculated), then the power indices would be $36\,\varphi = (4, 4, 24, 4)$, with the double vote now just happening to compensate exactly for the sharper angle at x^d. In this case, however, "c" is the dominant player; this is not surprising in view of the fact that decisive voting rules tend to favor central positions.* Indeed, if "c" had the double vote he would get <u>all</u> the power, like his centrist counterpart in the one-dimensional example.

In the third game illustrated in Fig. 3 we keep the double vote but raise the winning threshold to 4. This gives "a" a veto, though he still needs the support of two others to win. He gets precisely half the power, for the simple reason that in each pair of contrary issues ξ, $-\xi$ he pivots exactly once. Thus his power stems directly from the voting rule, and does not depend at all on his profile. (Rather curiously, the other players' indices are also independent of x^a; this will be explained below.) The rest of the power goes mostly to "c", because of the obtuse angle his position makes with those of "b" and "d", with whom he is engaged in a kind of $\underset{\sim}{M}_3$ game.

*"Decisive" means that no deadlock can arise, which in turn means that there are as many winning as losing coalitions and that the pivot of any order is the pivot of the reverse order. This decisiveness (also called "self-duality") is responsible for the 180° symmetry of the second cartwheel in Fig. 3 and the first cartwheel in Fig. 2.

The key to understanding this example comes from the fact that it can be represented as a product: $M_1 \otimes M_3$ (compare Sec. 3), both factors of which are decisive. It is a general theorem that whenever we form the product of two decisive games $\underset{\sim}{G}'$ and $\underset{\sim}{G}''$ having disjoint sets of players N' and N", then the power indices of the players in $\underset{\sim}{G}' \otimes \underset{\sim}{G}''$ are exactly half what they were in $\underset{\sim}{G}'$ or $\underset{\sim}{G}''$. (The proof is left as an exercise.) This means that the relative placement in R^m of the two "constellations" C(N') and C(N") does not matter at all. Each contingent of players takes half the total power and divides it among its members according to its own voting rule and ideological configuration.

* * * * * * * *

Some of the attractive features of the present model are revealed when we start moving players about and merging them. The power indices themselves are obviously continuous as functions of the ideological profiles, so long as they remain distinct. If two or more of them collide, or pass through each other, there may indeed be a discontinuity in the power distribution, but its effect is limited to the colliding players.* In fact, the <u>sum</u> of their indices behaves continuously if at the point of collision it is defined to be the power index of a single player enjoying their combined voting strength. The powers of the other players are not affected by such a merger, in contrast to what we observed in Sec. 4. Only when we try to merge "strange

*When two or more of the x^i coincide the power indices (4) may not be well-defined, since the issues can then only induce weak orderings which may fail to determine a pivot.

bedfellows," i.e., players with substantially different ideological profiles, do we find an abrupt change in the overall distribution of power.

This smooth behavior of our model under mergers of like-minded players opens the door to useful techniques of aggregation and approximation. When the number of players is large we can take clusters of like-minded voters that we do not care to study individually and approximate them by voting blocs, thereby reducing the size and complexity of the game while we concentrate on the powers of other voters that we _do_ care about as individuals.

We shall not attempt to illustrate this method on a large system, but our final illustration may serve to give some feeling for how it would work. As shown in Fig. 4, reading from left to right, three players in R^2 form first a 36-54-90 triangle, then an 18-72-90 triangle, and finally a line segment obtained by merging the two players on the left. Five different voting rules are considered (heavy dots denote veto players), and the resulting power indices (x 100) are written beside the corresponding vertices. Of course, the two-person games in the limit are pretty trivial, but the validity of the approximation process is nevertheless suggested quite clearly.

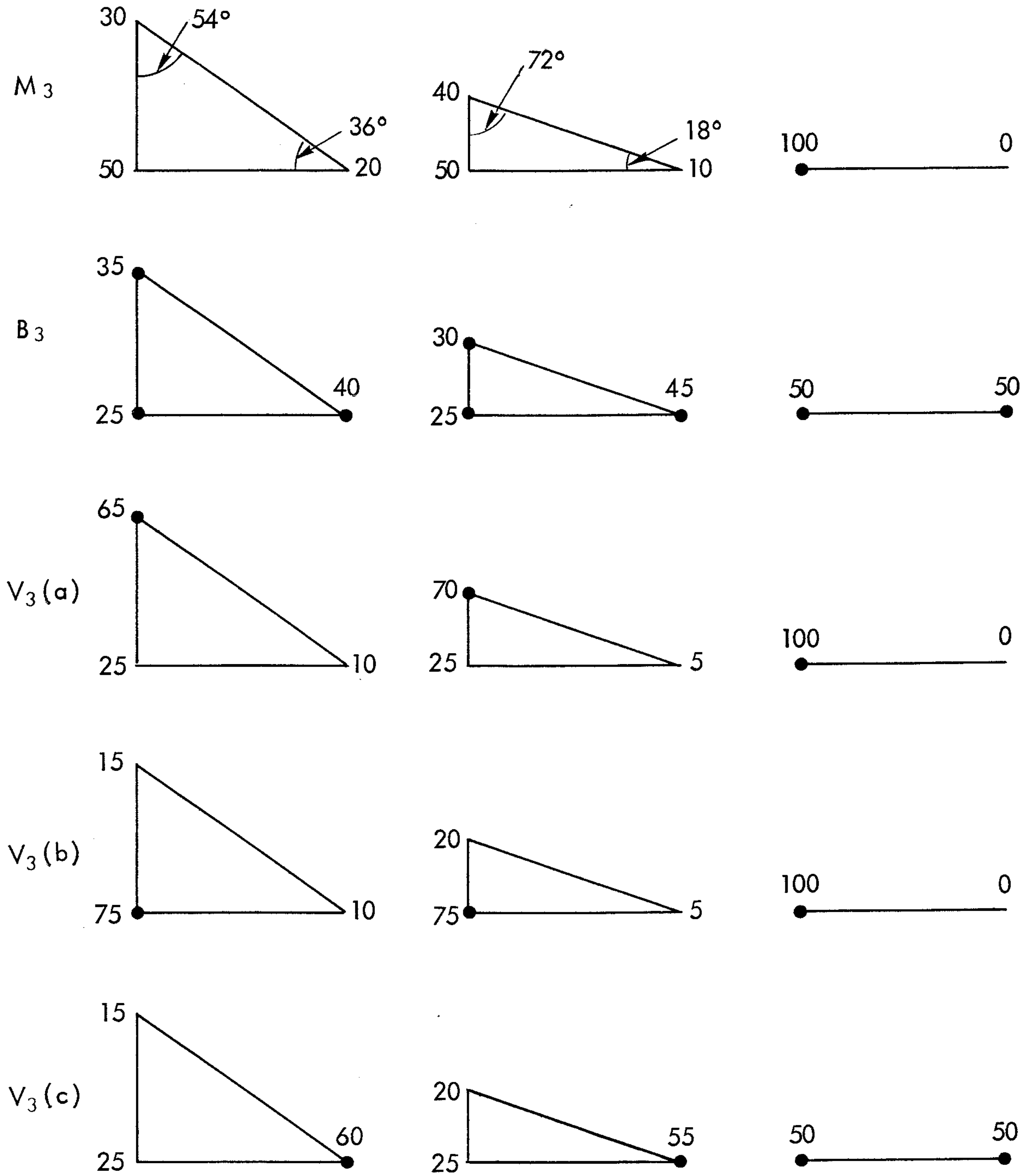

M 3
30
54°
36°
50
20
72°
40
18°
50
10
100
0
B 3
35
40
25
30
45
25
50
50
V3 (a)
65
25
10
70
25
5
100
0
V3 (b)
15
75
10
20
75
5
100
0
V3 (c)
15
60
25
20
55
25
50
50

Fig. 4

APPENDIX

We discuss here two mathematical problems incidental to the political model in Secs. 5 and 6. Note 1 considers the problem of counting the number of different orderings of n points in R^m that are produced when the points are ranked according to all possible directions in R^m. Note 2 considers the problem of subjecting a given set of directions in R^m to linear transformations in order to distribute them more uniformly over the unit sphere.

Note 1

Call an order $\succ$ [or weak order $\succsim$] on a finite set $X \subset R^m$ realizable if there is a vector $\xi \in R^m$ such that

$$x \succ y \Longleftrightarrow (x, \xi) < (y, \xi), \qquad \text{all } x, y \in X.$$

Let J(X) denote the number of distinct realizable orders on X, and let J(m, n) denote the maximum of J(X) for $X \subset R^m$ with $|X| = n$. We assert that this maximum is in fact achieved whenever X is "in general position,"* and that, moreover, the function J is given recursively by

$$\text{(A.1)} \qquad J(m, n + 1) = J(m, n) + nJ(m - 1, n),$$

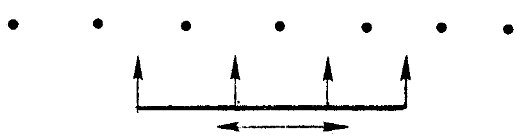
for $m \geqq 2$, $n \geqq 1$, with initial conditions

*i.e., achieved whenever certain determinants derived from the coordinates of the points in X do not vanish. We spare the reader a detailed accounting of all the degeneracies that must be guarded against.

(A.2) $J(m, 1) = 1, \quad \text{and} \quad J(1, n) = 2,$

for $m \geqq 1$ and $n \geqq 2$ (see the table below).*

This may be proved by induction. Take $X = \{x^{(1)}, \ldots, x^{(n)}\}$ to be in general position in R^m and add a new point $x^{(n+1)}$ so that the set $X^+ = X \cup \{x^{(n+1)}\}$ is also in general position. Having verified (A.2), we assume that $m \geqq 2$ and $n \geqq 1$ and consider (A.1). By assumption, there are exactly $J(m, n)$ realizable orders on X, each produced by an open cone of direction vectors ξ. (The closures of these cones fill R^m.) As we vary ξ continuously within one of these open cones, we can watch the position of $x^{(n+1)}$ relative to the fixed order that ξ induces on the elements of X. Whether $x^{(n+1)}$ remains with a particular "slot" in that order or travels back and forth through a series of adjacent slots:



(order of the first n players)

(position of the n + 1th player)

the number of slots it visits will always be <u>one more</u> than the number of different members of X it can be tied with. But the number of slots visited is just the number of realizable orders on X^+ that are consistent with the given order on X, so summing these slot counts over the different realizable orders on X gives us a count of all the realizable orders on X^+. This number is therefore equal to $J(m, n)$

*It is interesting to note that the function n! also satisfies (A1), though of course not (A2). Hence, if we write $E(m, n) = n! - J(m, n)$ for the number of <u>unrealizable</u> orders, then E also satisfies (A.1).

plus the total number of realizable <u>weak</u> orders on X^+ in which the members of X are strictly ordered but $x^{(n+1)}$ is tied with one of them.

These weak orders can be counted in another way. Indeed, we assert that for any $i = 1, \ldots, n$, the number of them in which $x^{(n+1)}$ ties $x^{(i)}$ is just $J(m - 1, n)$. To show this, let H^{m-1} denote the hyperplane through the origin that is orthogonal to the vector $x^{(i)} - x^{(n+1)}$. Drop perpendiculars from X to H^{m-1} and call the set of projected points X'. Since H^{m-1} is just a replica of R^{m-1}, and since X' will be in general position in H^{m-1}, we know by the inductive hypothesis that there are $J(m - 1, n)$ orders on X' produced by the direction vectors ξ that lie in H^{m-1}. But these vectors generate the same order on X as on X' (because the projection was perpendicular), and they are also precisely the vectors in R^m that cause $x^{(i)}$ and $x^{(n+1)}$ to be tied, proving our assertion.

The total number of weak orders on X^+ of the kind we want is therefore exactly $nJ(m - 1, n)$, giving us

$$J(m, n) + nJ(m - 1, n)$$

as the total number of realizable orders on X^+ when X^+ is in general position. That this is indeed the maximum number of realizable orders, which is what (A.1) claims, follows from the observation that wherever in the proof we relied on points being in "general position," any degeneracy would only have served to reduce the number of orders. Q.E.D.

Although we have not found a simple closed form for J(m, n), it is easy to write out a table for small values of m and n using (A.1) and (A.2):

m \ n	1	2	3	4	5	6	7	8
1	1	2	2	2	2	2	2	2
2	1	2	6	12	20	30	42	56
3	1	2	6	24	72	172	352	646
4	1	2	6	24	120	480	1512	3976

For m = 2, 3 and n > 1 we have

(A.3)
$$J(2, n) = n(n - 1),$$
$$J(3, n) = \frac{1}{12} n(n - 1)(n - 2)(3n - 1) + 2,$$

and in general J(m, n) is a polynomial in n of degree 2m - 2. Hence, the fraction of orders that are realizable, i.e., the ratio J(m, n)/n!, goes to zero as $n \to \infty$ for fixed m.

Note 2

In this note we discuss the problem of adjusting the coordinate system in R^m so as to make the assumption of a uniform distribution of issue-directions as unobjectionable as possible, given that we may have started with definite evidence or expectations as to the actual distribution of directions. For convenience, we shall work in the issue space, that is, in the adjoint or dual space of the space of voter-profiles. Of course, any linear transformation we make on the coordinates in one of these spaces will have to be accompanied by the inverse transformation applied to the coordinates in the other.

To simplify the discussion, we shall assume that the actual or observed distribution of issues is given by a finite set of equiprobable directions, which could be represented by specific vectors $y^{(\nu)}$ in R^m. Since linear transformations do not preserve distances or spheres, there is no point in assuming that the original $y^{(\nu)}$ are on the unit sphere. One may think of them as stars in the sky, as viewed from the origin, and the object is to find a linear transformation of R^m that will distribute them over the sky as uniformly or unbiassedly as possible.

First we consider a related problem. Assume that the set $Y = \{y^{(\nu)}\}$ is "centered," i.e., that $\Sigma_\nu y^{(\nu)} = 0$. Such a set will be called <u>spatially uncorrelated</u> if

$$\Sigma_\nu y_j^{(\nu)} y_k^{(\nu)} = 0, \tag{A.4}$$

for $1 \leq j < k \leq m$, and

$$\Sigma_\nu (y_j^{(\nu)})^2 = \Sigma_\nu (y_1^{(\nu)})^2, \tag{A.5}$$

for $1 < j \leq m$. Here (A.4) tells us that the different components of the vectors $y^{(\nu)}$ are uncorrelated in the usual sense, while (A.5) ensures that this remains true if the coordinate system is rotated or reflected. To see this, let $u^1, \ldots, u^n$ be a new basis of mutually orthogonal unit vectors for R^m. Then the new coordinates of any vector y are given by $\tilde{y}_j = (y, u^j)$, $j = 1, \ldots, m$. If Y is centered and spatially uncorrelated, then we have

$$\Sigma_\nu \tilde{y}^{(\nu)} = \tilde{0} = 0,$$

and

$$\Sigma_\nu \tilde{y}_j^{(\nu)} \tilde{y}_k^{(\nu)} = \sum_{r=1}^{m} \sum_{\substack{s=1 \\ s \neq r}}^{m} u_r^j u_s^k \, \Sigma_\nu y_r^{(\nu)} y_s^{(\nu)} + \sum_{t=1}^{m} u_t^j u_t^k \, \Sigma_\nu (y_t^{(\nu)})^2$$

$$= 0 + (u^j, u^k) \, \Sigma_\nu (y_1^{(\nu)})^2$$

$$= \begin{cases} 0 \text{ if } j \neq k, \\ \Sigma_\nu (y_1^{(\nu)})^2 \text{ if } j = k. \end{cases}$$

So $\tilde{Y}$ is also centered and spatially uncorrelated.

If Y is <u>not</u> spatially uncorrelated, then it is not difficult to see that a linear transformation T can always be found such that the set $TY = \{Ty^{(\nu)}\}$ <u>is</u> spatially uncorrelated, provided only that Y is not degenerately contained within some (m - 1) or lower-dimensional subspace of R^m. Moreover, this transformation will be unique up to a Euclidean similarity, i.e., up to a rotation, reflection, expansion, or contraction.*

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*Precisely, T, expressed as a matrix, is unique up to left-multiplication by an orthogonal matrix and multiplication by a positive scalar (see, e.g., Birkhoff and MacLane, <u>A Survey of Modern Algebra</u>, 1950, pp. 222-226).

We now return to the original problem. If Y is a set of non-zero vectors in R^m let $\bar{\bar{Y}}$ denote its projection on the unit sphere: $\bar{\bar{Y}} = \{y/\|y\| : y \in Y\}$, and call a finite set $Y \subset R^m \backslash \{0\}$ <u>directionally uncorrelated</u> if the set $Z = \bar{\bar{Y}} \cup -\bar{\bar{Y}}$ is spatially uncorrelated.* This is equivalent to demanding that $\bar{\bar{Y}}$ (which in general may not be centered) satisfy (A.4) and (A.5). The algebraic conditions on Y, however, are now non-linear because of the denominators $\|y^{(\nu)}\|^2$ that appear.

Having formulated this definition, we have for the present only conjectures to offer:

<u>Conjecture 1</u>. If Y is in "general position," in some appropriate sense, then there is a linear transformation U such that UY is directionally uncorrelated.

<u>Conjecture 2</u>. Such a transformation U, if it exists, is unique up to a Euclidean similarity.

<u>Conjecture 3</u>. Under suitable conditions on $Y \subset R^m \backslash \{0\}$, the sequence of sets $Z_1, Z_2, \ldots$ (on the unit sphere) defined by

$$Z_1 = \bar{\bar{Y}} \cup -\bar{\bar{Y}}, \quad Z_{n+1} = \overline{\overline{T_n Z_n}}, \quad n = 1, 2, \ldots,$$

where each T_n is chosen to make $T_n Z_n$ spatially uncorrelated, converges

*Thus, we are actually considering just sets of undirected lines rather than directions. In our political model, this means that the distribution of issues is assumed to be symmetric as between issues and their negations. (Compare Owen [16], p. 346).

(modulo rotations and reflections) to a limit Z_∞ that is directionally uncorrelated and satisfies $Z_\infty = UZ_1$ for some linear transformation U.

For a simple example to illustrate Conjecture 3 the reader is invited to take $Y = \{(1, 10), (1, 0), (1, -10)\} \subset R^2$. For an instructive counter example that limits Conjecture 1, take $Y = \{(1, 0, 0), (0, 1, 0), (0, 1, 1), (0, 1, -1)\} \subset R^3$.

Addendum

The problem considered in Note 1 was previously solved by T. M. Cover, using a somewhat different proof from the one sketched here. See "The number of linearly inducible orderings of points in d-space," SIAM J. Appl. Math. 15 (1967), 434-439.

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