A COMPARISON THEOREM
FOR CRYSTALLINE EVOLUTION
IN THE PLANE

BY

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1. Introduction. Crystal surfaces are often endowed with energies whose dependence on orientation $\mathbf{n}$ (outward unit normal) displays "low energy cusps" at a finite set $\mathcal{N}$ of orientations;[1] such energies lead to crystal shapes that are fully faceted, the orientations of the facets being the orientations $\mathbf{n} \in \mathcal{N}$. A possible model[2] for the *planar* evolution of such crystals is based on an evolution equation relating the normal velocity $V_i(t)$ and crystalline curvature

$$K_i(t) = \chi_i L_i(t)^{-1} \qquad (1.1)$$

of each facet F_i, where $L_i(t)$ is the length of F_i, while χ_i has constant value -1, $+1$, or 0 according as the crystal is strictly convex, strictly concave, or neither near F_i; this evolution equation has the form

$$\beta(\mathbf{n}_i)V_i(t) = l(\mathbf{n}_i)K_i(t) - U, \qquad (1.2)$$

where $\beta(\mathbf{n}_i) > 0$, the kinetic modulus, and $l(\mathbf{n}_i) > 0$, the Wulff modulus,[3] depend only on the (fixed) orientation $\mathbf{n}_i \in \mathcal{N}$ of F_i, where U is the constant bulk energy of the crystal relative to its exterior.

In many respects this evolution equation exhibits behavior typical of a parabolic PDE,[4] and it seems reasonable to ask whether it has an associated comparison principle. What

[1] $\mathcal{N}$ is the set of orientations that appear on the Wulff crystal (the unique crystal that minimizes total surface energy at fixed enclosed area (in the plane)).

[2] Proposed independently by Taylor [Ta] and Angenent and Gurtin [AG] (cf. [Gu]).

[3] $l(\mathbf{n})$ is the length of the facet on the Wulff crystal that has orientation $\mathbf{n}$. For all of our results, except those of Sec. 5, we use only the condition $l(\mathbf{n}) > 0$; the form of the underlying surface energy is irrelevant.

[4] For example, the curve-shortening equation $V = K$ (V = normal velocity, K = curvature) for smooth boundary curves; in fact, Girão and Kohn [GK] and Girão [Gi] use crystalline evolution as a method of approximating generalized curve-shortening equations.

makes this question especially important is that comparison can form the basis for weak formulations of the underlying evolution problem.[5]

We here establish such a comparison principle:[6] we show that if $\mathcal{C}$ and $\overline{\mathcal{C}}$ are admissible evolving crystals with $\mathcal{C}(0)$ contained in $\overline{\mathcal{C}}(0)$, then $\mathcal{C}(t)$ is contained in $\overline{\mathcal{C}}(t)$ as long as both evolutions are well defined.

2. Crystals. Let $\mathcal{C}$ be the closure of an open, possibly unbounded set in the plane. We then refer to $\mathcal{C}$ as a **crystal** if its boundary $\partial\mathcal{C}$ is a piecewise linear curve; that is, if $\partial\mathcal{C}$ is the finite union

$$\partial\mathcal{C} = \bigcup_{i=1}^{N} F_i \tag{2.1}$$

of (closed, maximally connected, possibly infinite) line segments F_i called **facets**. We use the following terminology: (2.1) is the **facet decomposition** of $\partial\mathcal{C}$; the point of intersection of adjacent facets is called an **edge**; the outward unit normal $\mathbf{n}$ to $\partial\mathcal{C}$ is the **orientation** of $\mathcal{C}$.

We will study crystals whose orientation belongs to a finite subset $\mathcal{N}$ of the unit circle S^1; $\mathcal{N}$ is related to the lattice structure of the crystal and should be envisaged as representing stable orientations of the crystal surface. The unit vectors $\mathbf{n} \in \mathcal{N}$ will be referred to as **admissible orientations**. The identification of admissible orientations $\mathbf{n}$ with their argument $\vartheta \in \mathbb{R}/(2\pi\mathbb{Z})$ through $\mathbf{n} = (\cos\vartheta, \sin\vartheta)$ renders meaningful the term adjacent orientations, or, more precisely, $\mathcal{N}$-**adjacent** orientations, as well as the assertion "$\mathbf{m}$ lies between $\mathbf{n}$ and $\mathbf{p}$" (for $\mathbf{m}, \mathbf{n}, \mathbf{p} \in \mathcal{N}$).

Let $\mathcal{C}$ be a crystal and let $\mathbf{a}$ be a point on $\partial\mathcal{C}$. Then $\mathcal{C}$ is **admissible** at $\mathbf{a}$ if
  (i) the orientation of each facet containing $\mathbf{a}$ is $\mathcal{N}$-admissible; and
  (ii) for $\mathbf{a}$ an edge, the orientations of the two facets that intersect at $\mathbf{a}$ are $\mathcal{N}$-adjacent.
$\mathcal{C}$ is **admissible** if $\mathcal{C}$ is admissible at each point of $\partial\mathcal{C}$. We will also use a local definition of admissibility: $\mathcal{C}$ is **admissible near** $\mathbf{a}$ if there is an open set W containing all facets containing $\mathbf{a}$ such that $\mathcal{C}$ is admissible at every point of $W \cap \partial\mathcal{C}$. Here it is important to note that, since facets are (relatively) closed, if an open set W contains a facet F, then W contains all facets adjacent to F.

For $\mathbf{a} \in \mathbb{R}^2$ and $\mathbf{n} \in S^1$, let

$$\mathcal{H}(\mathbf{a}, \mathbf{n}) = \{\mathbf{x} \in \mathbb{R}^2 : (\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} \leq 0\} \tag{2.2}$$

denote the halfspace whose boundary contains $\mathbf{a}$ and whose outward unit normal is $\mathbf{n}$; for $\mathbf{n}$ an admissible orientation, $\mathcal{H}(\mathbf{a}, \mathbf{n})$ furnishes a trivial example of an admissible crystal.

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[5]This issue will be addressed in a forthcoming paper of Giga. For smooth interfacial energies, comparison is a key tool in establishing global existence via the level-set method [CGG, ES]. An alternative approach to existence is that of Fukui and Giga [FG], who establish a weak formulation for motion by crystalline curvature using an adaptation of nonlinear semigroup theory; the study [FG] is limited to $U = 0$ and to an interface which is the graph of a (spatially periodic) function.

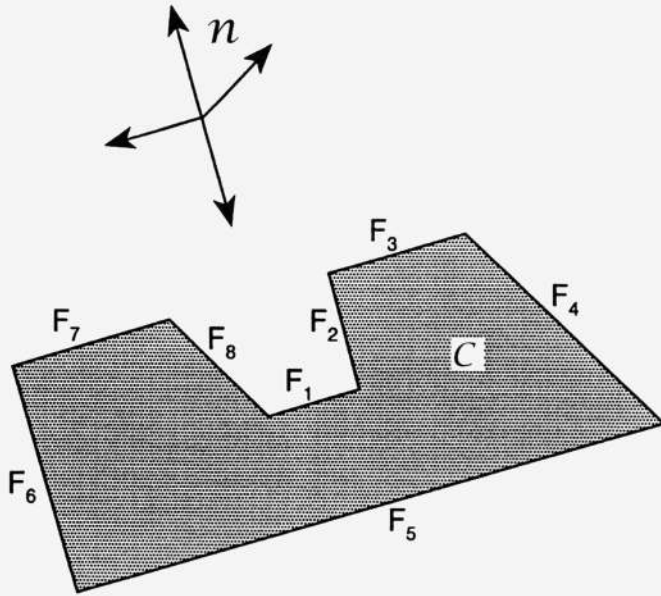[6]The comparison theorem (12H) of [Gu] is valid only when restricted to convex crystals.

FIGURE 1. The set $\mathcal{N}$ of admissible orientations and a corresponding crystal $\mathcal{C}$. The transition numbers are $\chi_1 = +1$, $\chi_2 = \chi_8 = 0$, $\chi_3 = \chi_4 = \chi_5 = \chi_6 = \chi_7 = -1$.

Given a crystal $\mathcal{C}$, let F be a facet of $\mathcal{C}$, let $\mathbf{n}$ be the orientation of F, and let L denote the **length** of F. The **transition number** χ for F is defined as follows: for $\mathbf{a} \in F$,

$$\chi = -1 \quad \text{if the facets adjacent to } F \text{ belong to } \mathcal{H}(\mathbf{a}, \mathbf{n});$$
$$\chi = +1 \quad \text{if the facets adjacent to } F \text{ belong to } \mathcal{H}(\mathbf{a}, -\mathbf{n});$$
$$\chi = 0 \quad \text{otherwise.}$$

Roughly speaking, χ is -1 or $+1$ according to $\mathcal{C}$ is convex or concave near F (Fig. 1). In fact, we say that $\mathcal{C}$ is **convex** if the transition number of each of its facets is -1.

Finally, the **crystalline curvature** K of F is defined by

$$K = \chi L^{-1} \quad \text{if } L < \infty;$$
$$K = 0 \qquad \text{if } L = \infty.$$

3. Maximum principle. A classical maximum principle—for smooth real functions—asserts that if $f \le \overline{f}$ in a neighborhood of x_0 and if $f(x_0) = \overline{f}(x_0) = y_0$, then
(i) $f'(x_0) = \overline{f}'(x_0)$,
(ii) $f''(x_0) \le \overline{f}''(x_0)$.
The first condition asserts that the graphs of f and $\overline{f}$ have a common normal at (x_0, y_0), while (ii) compares the curvatures of the two graphs.

We now derive an analogous result for crystals. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be crystals. We say that $\partial\mathcal{C}$ **touches** $\partial\overline{\mathcal{C}}$ **at a from inside** if $\mathbf{a} \in \partial\mathcal{C} \cap \partial\overline{\mathcal{C}}$ and if there is an open neighborhood W of $\mathbf{a}$ such that

   (a) W contains all facets of $\mathcal{C}$ that contain $\mathbf{a}$ or all facets of $\overline{\mathcal{C}}$ that contain $\mathbf{a}$; and

   (b) $\mathcal{C} \cap W \subset \overline{\mathcal{C}} \cap W$.

There are four possibilities for the touching of $\mathcal{C}$ and $\overline{\mathcal{C}}$ (Figure 2):

   (1) **facet-facet**: $\mathbf{a}$ belongs to a facet interior of each crystal;

   (2) **edge-facet**: $\mathbf{a}$ belongs to an edge of one crystal and a facet interior of the other;

   (3) **proper edge-edge**: $\mathbf{a}$ belongs to an edge of each crystal and, near $\mathbf{a}$, $\mathcal{C}$ and $\overline{\mathcal{C}}$ intersect only at $\mathbf{a}$;

   (4) **improper edge-edge**: $\mathbf{a}$ belongs to an edge of each crystal, but the touching at $\mathbf{a}$ is not proper.

The next result is an analog of the classical maximum principle.

MAXIMUM PRINCIPLE FOR CRYSTALS. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be crystals with $\mathcal{C}$ and $\overline{\mathcal{C}}$ admissible near $\mathbf{a}$. Assume that $\partial\mathcal{C}$ touches $\partial\overline{\mathcal{C}}$ at $\mathbf{a}$ from inside. Then

   (i) There are facets F and $\overline{F}$ of $\mathcal{C}$ and $\overline{\mathcal{C}}$ such that $\mathbf{a} \in F \cap \overline{F}$ and such that the orientation of F equals that of $\overline{F}$. Moreover, if $\mathbf{a}$ is a proper edge-edge touching, then the set of orientations of facets of $\mathcal{C}$ meeting at $\mathbf{a}$ equals the corresponding set for $\overline{\mathcal{C}}$.

   (ii) If F and $\overline{F}$ are facets of $\mathcal{C}$ and $\overline{\mathcal{C}}$ with $\mathbf{a} \in F \cap \overline{F}$ and with the orientation of F equal to that of $\overline{F}$, then the curvatures K and $\overline{K}$ of F and $\overline{F}$ satisfy

$$K \leq \overline{K}. \tag{3.1}$$

If $K = \overline{K} \neq 0$, then $\partial\mathcal{C} = \partial\overline{\mathcal{C}}$ in some neighborhood of F and $\overline{F}$.

Proof. For a facet-facet touching, the definition

$$\text{“}\partial\mathcal{C} \text{ touches } \partial\overline{\mathcal{C}} \text{ at } \mathbf{a} \text{ from inside”} \tag{3.2}$$

implies that the facets of $\mathcal{C}$ and $\overline{\mathcal{C}}$ whose interiors intersect at $\mathbf{a}$ must have the same orientation, for otherwise they would cross. Consider next an edge-facet touching with F and $\mathbf{e}$, respectively, the facet and edge in question. Then (3.2) implies that the orientation of F must lie between those of the facets G and H of the other crystal that meet at $\mathbf{e}$. But, by admissibility, the orientations of G and H must be $\mathcal{N}$-adjacent. Thus the orientation of F must coincide with either that of G or that of H.

For an improper edge-edge touching there are facets F and $\overline{F}$ of $\mathcal{C}$ and $\overline{\mathcal{C}}$ such that $\mathbf{a} \in F \cap \overline{F}$ and such that, for some $\mathbf{b} \in F \cap \overline{F}$, $\partial\mathcal{C}$ touches $\partial\overline{\mathcal{C}}$ at $\mathbf{b}$ from inside, with the touching of facet-facet type; thus, as above, F and $\overline{F}$ must have the same orientation.

For a proper edge-edge touching there are two subcases to consider; to state these let W be an open set that contains $\mathbf{a}$ and intersects only those facets of $\mathcal{C}$ and $\overline{\mathcal{C}}$ that contain $\mathbf{a}$. Case α : $W \cap \partial\mathcal{C}$ and $W \cap \partial\overline{\mathcal{C}}$ are contained in a halfspace $\mathcal{H}$ with $\mathbf{a} \in \partial\mathcal{H}$. Case β : there is a halfspace $\mathcal{H}$ with $\mathbf{a} \in \partial\mathcal{H}$ such that $W \cap \partial\mathcal{C}$ is contained in $\mathcal{H}$ and $W \cap \partial\overline{\mathcal{C}}$ is contained in the closure of the complement of $\mathcal{H}$. These two cases are shown in Fig. 3 (see p. 732) and in either case the requirement that $\mathbf{n}_1$ and $\mathbf{n}_2$ not lie between $\overline{\mathbf{n}}_1$ and $\overline{\mathbf{n}}_2$ and that $\overline{\mathbf{n}}_1$ and $\overline{\mathbf{n}}_2$ not lie between $\mathbf{n}_1$ and $\mathbf{n}_2$ implies that $\mathbf{n}_1 = \overline{\mathbf{n}}_1$ and $\mathbf{n}_2 = \overline{\mathbf{n}}_2$, so that the set of orientations of facets of $\mathcal{C}$ meeting at $\mathbf{a}$ equals the corresponding set for $\overline{\mathcal{C}}$.
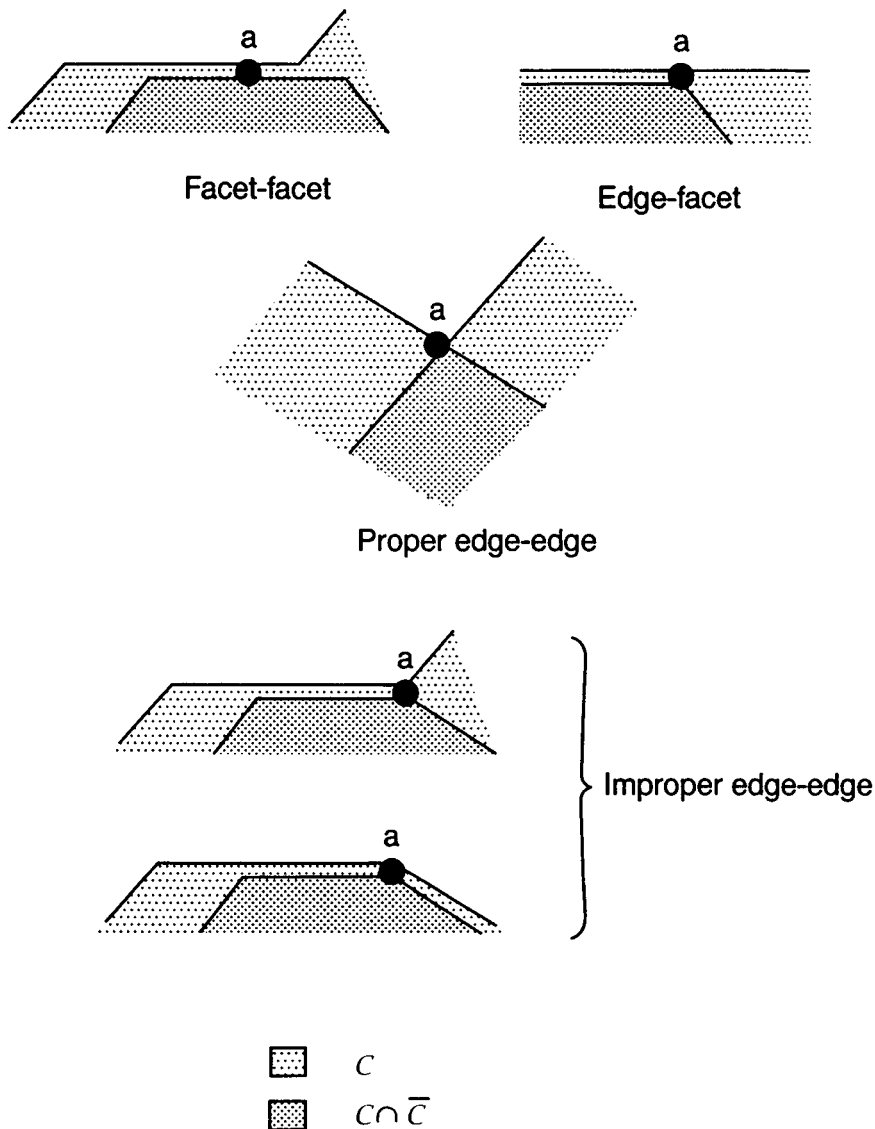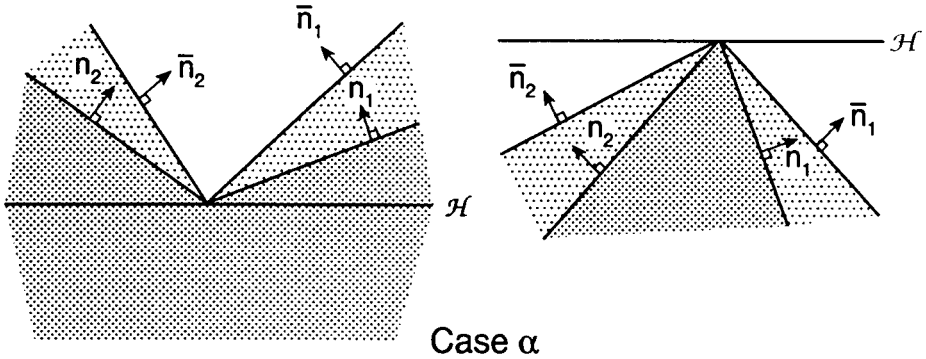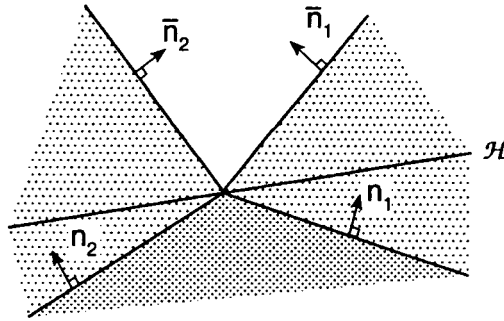
Facet-facet   Edge-facet

Proper edge-edge

Improper edge-edge

$\mathcal{C}$

$\mathcal{C} \cap \overline{\mathcal{C}}$

FIGURE 2.   The four types of touching at **a**

(ii) We assume first that both F and $\overline{F}$ are finite. Let χ and L denote the transition number for—and length of—F, and let $\overline{\chi}$ and $\overline{L}$ denote the corresponding quantities for $\overline{F}$. Suppose that $\overline{K} > 0$ so that $\overline{\chi} = 1$. By (3.2), either $\chi \le 0$ or $\chi = 1$ and $L \ge \overline{L}$. In both cases $K \le \overline{K}$ and in the latter case equality holds only if $L = \overline{L}$, so that $\partial \mathcal{C} = \partial \overline{\mathcal{C}}$ in some neighborhood of $F = \overline{F}$. The case $\overline{K} < 0$ is treated analogously. If $\overline{K} = 0$, then $\overline{\chi} = 0$ and $\chi \le 0$, so that $K \le \overline{K}$. Finally, if F and $\overline{F}$ are infinite, then $K = \overline{K} = 0$; if $\overline{F}$ is infinite and F finite, then $\overline{K} = 0$ and $\chi \le 0$, so that $K \le \overline{K}$, and similarly when F is infinite and $\overline{F}$ is finite. $\square$
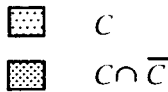
Case α

Case β

FIGURE 3.   Proper edge-edge touching

REMARKS.

1. If in (i) of the Maximum Principle, **a** is an improper edge-edge touching, then the set of orientations of facets of C meeting at **a** may not equal the corresponding set for $\overline{C}$ (Fig. 2).

2. The edge-facet and improper edge-edge touchings are essentially facet-facet touchings; that is, there are points **b** arbitrarily close to **a** such that ∂C touches $\partial \overline{C}$ at **b** from inside with **b** a facet-facet touching.

4. Evolving crystals comparison theorem. Our final step is to establish a comparison theorem for crystals $C(t)$ that evolve with time t. The set $\mathcal{N}$ of admissible orientations is fixed throughout the discussion.

By an **admissible evolving crystal** $\mathcal{C}$ (of **duration** $T = T_{\mathcal{C}}$) we mean a one-parameter family $\mathcal{C}(t)$, $0 \leq t < T$, of *admissible* crystals such that, writing

$$\partial \mathcal{C}(t) = \bigcup_{i=1}^{N} F_i(t) \tag{4.1}$$

for the facet decomposition of $\partial \mathcal{C}(t)$:
  (i) N is independent of t,
  (ii) each facet $F_i(t)$ has orientation $\mathbf{n}_i$ that is independent of t,
  (iii) the position vector of each edge varies smoothly in t.
$\mathcal{C}$ is **convex** if $\mathcal{C}(t)$ is convex for $0 \leq t < T$. Implicit in the definition of an admissible evolving crystal is the requirement that a facet present at some $t \in [0, T)$ is present for all $t \in [0, T)$, so that facets are neither created nor destroyed.

We say that an admissible evolving crystal $\mathcal{C}$ **corresponds to data** $\{\beta, l, U\}$ if
  (i) $\beta(\mathbf{n}) > 0$ and $l(\mathbf{n}) > 0$ for every admissible orientation $\mathbf{n} \in \mathcal{N}$,
  (ii) each facet $F_i(t)$ evolves on $(0, T_{\mathcal{C}})$ according to the evolution equation

$$\beta(\mathbf{n}_i) V_i(t) = l(\mathbf{n}_i) K_i(t) + U, \tag{4.2}$$

with $\mathbf{n}_i$ the orientation, $V_i(t)$ the normal velocity in the direction $\mathbf{n}_i$, and $K_i(t)$ the crystalline curvature of $F_i(t)$. Here β is the *kinetic modulus*, l is the *Wulff modulus*, and U is the *bulk energy*.

For our first comparison theorem the two crystals correspond to the same moduli, but the bulk energies are different. In stating this theorem note that, for $\mathcal{C}$ and $\overline{\mathcal{C}}$ admissible evolving crystals, the distance

$$\text{dist}(\partial \mathcal{C}(t), \partial \overline{\mathcal{C}}(t)) = \inf\{|\mathbf{x} - \overline{\mathbf{x}}|; \mathbf{x} \in \partial \mathcal{C}(t), \overline{\mathbf{x}} \in \partial \overline{\mathcal{C}}(t)\} \tag{4.3}$$

is actually attained, since each crystal has a finite number of facets.

FIRST COMPARISON THEOREM. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be admissible evolving crystals of equal duration T. Let $\{\beta, l, U\}$ and $\{\beta, l, \overline{U}\}$ denote the data corresponding to $\mathcal{C}$ and $\overline{\mathcal{C}}$, with $U \geq \overline{U}$. Assume that

$$\mathcal{C}(0) \subset \overline{\mathcal{C}}(0). \tag{4.4}$$

Then
  (a) $\mathcal{C}(t) \subset \overline{\mathcal{C}}(t)$ for all $t \in [0, T)$,
  (b) $\text{dist}(\partial \mathcal{C}(t), \partial \overline{\mathcal{C}}(t))$ is a nondecreasing function of t.
  Proof. The proof will proceed in a series of steps.
  1. Admissible evolving crystals are *translation invariant*: if $\mathcal{C}(t)$, $0 \leq t < T$, is an admissible evolving crystal corresponding to data $\{\beta, l, U\}$ and $\mathbf{r}$ is a vector, then $\mathcal{C}(t) + \mathbf{r}$, $0 \leq t < T$, is an admissible evolving crystal corresponding to the same data.
  2. An admissible evolving crystal $\mathcal{C}$ corresponding to $\{\beta, l, U\}$ has its evolution governed by a system of coupled ODE's[7] (one equation for each facet) and from the theorem

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[7]Cf. (10.18) of [AG], (12.29) of [Gu].

on smooth dependence on data for this ODE system one can show that for each ε there is an admissible evolving crystal $\mathcal{C}_\varepsilon$ that satisfies $\mathcal{C}_\varepsilon(0) = \mathcal{C}(0)$, corresponds to the data $\{\beta, l, U + \varepsilon\}$, and is such that (i) its duration T_ε satisfies $T_\varepsilon \to T$ as $\varepsilon \to 0$; and (ii)

$$\text{dist}(\partial\mathcal{C}_\varepsilon(t), \partial\mathcal{C}(t)) \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly for t in any closed interval of $[0, T)$.

3. Our next step is to show that, granted (a), $d(t) := \text{dist}(\partial\mathcal{C}(t), \partial\overline{\mathcal{C}}(t))$ is nondecreasing. It suffices to show that $d(0) \leq d(t)$, $0 \leq t < T$. There is a vector $\mathbf{r}$ with length $d(0)$ such that $\mathcal{C}(0) + \mathbf{r}$ is contained in $\overline{\mathcal{C}}(0)$. By Step 1, $\mathcal{C}(t) + \mathbf{r}$ is an admissible evolving crystal and, appealing to (a) for this translated crystal, we conclude that $\mathcal{C}(t) + \mathbf{r}$ is contained in $\overline{\mathcal{C}}(t)$. Thus $d(0) \leq d(t)$.

4. We turn now to the proof of (a). Assume that $\mathcal{C}(t)$ is not contained in $\overline{\mathcal{C}}(t)$ at some $t \in (0, T)$. Then

$$0 \leq t_0 := \sup\{t; \mathcal{C}(t) \subset \overline{\mathcal{C}}(t)\}$$

satisfies $t_0 < T$. Further, in view of the Maximum Principle and Remark 2 of Sec. 3, $\mathcal{C}(t_0) \subset \overline{\mathcal{C}}(t_0)$ and there is a point $b \in \mathcal{C}(t_0) \cap \overline{\mathcal{C}}(t_0)$ such that $\partial\mathcal{C}(t_0)$ touches $\partial\overline{\mathcal{C}}(t_0)$ at $\mathbf{b}$ from inside with $\mathbf{b}$ either a facet-facet touching or a proper edge-edge touching; and such that:

(α) for $\mathbf{b}$ a facet-facet touching, if F and $\overline{F}$ are the relevant facets of $\mathcal{C}$ and $\overline{\mathcal{C}}$, then F and $\overline{F}$ have common orientation, $\mathbf{n}$ say, and the normal velocities V and $\overline{V}$ of F and $\overline{F}$ in the direction $\mathbf{n}$ satisfy

$$V(t_0) \geq \overline{V}(t_0); \tag{4.6}$$

(β) for $\mathbf{b}$ a proper edge-edge touching, $\mathbf{b}$ is an edge of adjacent facets F_1 and F_2 of $\mathcal{C}$ and an edge of adjacent facets $\overline{F}_1$ and $\overline{F}_2$ of $\overline{\mathcal{C}}$, and (renumbering if necessary), for $i = 1, 2$, F_i and $\overline{F}_i$ have the same orientation $\mathbf{n}_i$ and

$$V_1(t_0) \geq \overline{V}_1(t_0) \quad \text{or} \quad V_2(t_0) \geq \overline{V}_2(t_0), \tag{4.7}$$

where V_i and $\overline{V}_i$ are the velocities of F_i and $\overline{F}_i$ in the direction $\mathbf{n}_i$.

5. Assume that $U > \overline{U}$. Consider (α) of Step 4. By the Maximum Principle, the crystalline curvatures corresponding to F and $\overline{F}$ satisfy $K(t_0) \leq \overline{K}(t_0)$. Thus, applying the evolution equation (4.2) to each of the crystals, we find, with the aid of (4.6), that $U \leq \overline{U}$, a contradiction. An analogous argument yields a contradiction for (β) of Step 4. Thus (a) is proved for $U > \overline{U}$.

6. Our final step is to prove (a) for $U = \overline{U}$. In place of $\mathcal{C}$ we use the admissible evolving crystal $\mathcal{C}_\varepsilon$ corresponding to $U + \varepsilon$ ($> \overline{U}$) discussed in Step 2. By Step 5, $\mathcal{C}_\varepsilon(t) \subset \overline{\mathcal{C}}(t)$ for $0 \leq t < T_\varepsilon$ and letting $\varepsilon \to 0$ we conclude that $\mathcal{C}(t) \subset \overline{\mathcal{C}}(t)$ for $0 \leq t < T$.

REMARKS.

1. If $U > \overline{U}$, $\mathcal{C}(t)$ is actually contained in the interior of $\overline{\mathcal{C}}(t)$ for $0 < t < T$. Indeed, if not, there is a $t_1 > 0$ such that $\partial\mathcal{C}(t_1)$ touches $\partial\overline{\mathcal{C}}(t_1)$ at some $\mathbf{b}$ from inside. Since $\mathcal{C}(t) \subset \overline{\mathcal{C}}(t)$ for all $t > 0$ and since $t_1 > 0$, we have the same speed relations of facets near $\mathbf{b}$ as described in Step 4 of the proof of the First Comparison Theorem, an observation that yields, via the argument of Step 5, a contradiction to $U > \overline{U}$.

2. The standard proof of comparison for the heat equation does not require differentiability of the initial data. Within our framework differentiability is analogous to admissibility. Our assumption of admissibility of the initial crystal allows for the simple argument given in Steps 4 and 5. If we drop this assumption (but continue to assume that the crystal is admissible for $t > 0$), then, letting t_0 be as defined in (4.5), there is a $t_1 > t_0$ and a point $\mathbf{b} \in \mathcal{C}(t_1) \backslash \overline{\mathcal{C}}(t_1)$ such that

$$\text{dist}(\mathbf{b}, \partial \overline{\mathcal{C}}(t_1)) \geq \text{dist}(\mathbf{x}, \partial \overline{\mathcal{C}}(t_1)) \quad \text{for all } \mathbf{x} \in \mathcal{C}(t), \ 0 \leq t \leq t_1.$$

One can compare orientations, curvatures, and speed around $\mathbf{b}$ and reach a contradiction to $U > \overline{U}$ as in Step 5. This argument furnishes a proof for nonadmissible initial data but requires $U > \overline{U}$ as well as continuity of $\mathcal{C}(t)$ and $\overline{\mathcal{C}}(t)$ (as sets) up to $t = 0$. For the case $U = \overline{U}$, our argument of Step 6, which utilizes the corresponding ODE system, does not apply directly, since this system is not well defined for facets of zero length. We will not consider this question in the present paper.

A useful extension of the comparison theorem allows the moduli of the crystals to differ.

SECOND COMPARISON THEOREM. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be admissible evolving crystals of equal duration T, with either $\mathcal{C}$ or $\overline{\mathcal{C}}$ convex. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ correspond to the data $\{\beta, l, U\}$ and $\{\overline{\beta}, \overline{l}, \overline{U}\}$ with

$$U/\beta \geq \overline{U}/\overline{\beta}, \qquad l/\beta \geq \overline{l}/\overline{\beta}. \tag{4.7}$$

Assume that

$$\mathcal{C}(0) \subset \overline{\mathcal{C}}(0). \tag{4.8}$$

Then
 (a) $\mathcal{C}(t) \subset \overline{\mathcal{C}}(t)$ for all $t \in [0, T)$,
 (b) $\text{dist}(\partial \mathcal{C}(t), \partial \overline{\mathcal{C}}(t))$ is a nondecreasing function of t.
 We omit the proof, which is similar to the proof of the first comparison theorem.

We now give a weaker notion of an evolving crystal. By a **weakly admissible evolving crystal** $\mathcal{C}$ **corresponding to the data** $\{\beta, l, U\}$ we mean a one-parameter family $\mathcal{C}(t)$, $0 \leq t < T_\mathcal{C}$, of crystals such that
 (i) $\mathcal{C}(t)$ is continuous in t on $[0, T_\mathcal{C})$ as a set-valued function [AF];
 (ii) there are finitely many times $t_0 = 0 < t_1 < t_2 < \cdots < t_M < T_\mathcal{C}$ such that $\mathcal{C}$ is an admissible evolving crystal corresponding to $\{\beta, l, U\}$ on each $[t_j, t_{j+1})$, $j = 0, 1, 2, \ldots, M - 1$.
This definition allows for the disappearance of facets at the times $t_1, t_2, \ldots, t_{M-1}$.

THEOREM. The two comparison theorems hold without change for weakly admissible evolving crystals.

5. Bounds on crystal growth.

We now use the second comparison theorem to establish bounds on crystal growth.[8] We assume we are given a bulk energy U, a kinetic modulus $\beta(\mathbf{n}) > 0$, and a Wulff modulus $l(\mathbf{n}) > 0$, with both moduli defined for every

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[8]Cf. Soner [So], who uses analogous ideas to bound the growth of smooth crystals.

$\mathbf{n} \in \mathcal{N}$. We assume further that the Wulff modulus corresponds to the interfacial energy $f(\mathbf{n}) > 0$, $\mathbf{n} \in \mathcal{N}$. Precisely, we assume that the discrete set $f(\mathbf{n})^{-1}\mathbf{n}$, $\mathbf{n} \in \mathcal{N}$, lies on the boundary of its convex hull, so that the Wulff crystal

$$\Lambda = \{\mathbf{x} : \mathbf{x} \cdot \mathbf{n} \leq f(\mathbf{n}) \text{ for all } \mathbf{n} \in \mathcal{N}\} \tag{5.1}$$

is convex and fully faceted with $\mathcal{N}$ as its set of orientations; $l(\mathbf{n})$ is then the length of the facet on Λ with orientation $\mathbf{n} \in \mathcal{N}$.[9] Granted this, if $\overline{\beta}(\mathbf{n})$ satisfies

$$f(\mathbf{n})\overline{\beta}(\mathbf{n}) = \kappa = \text{constant} \tag{5.2}$$

and if $z(t)$ is a solution of the differential equation

$$\kappa \dot{z}(t) = -[U + z(t)^{-1}] \tag{5.3}$$

on a maximal time-interval $[0, T)$ with $z(t) \geq 0$, then[10]

$$\begin{aligned} \overline{\mathcal{C}}(t) = z(t)\Lambda \text{ is an admissible evolving crystal} \\ \text{corresponding to the data } \{\overline{\beta}, l, U\}. \end{aligned} \tag{5.4}$$

For $U \geq 0$, $\overline{\mathcal{C}}(t)$ shrinks to a point in finite time; for $U < 0$, $\overline{\mathcal{C}}(t)$ will shrink to a point or grow without bound according as $z(0) < |U|$ or $z(0) > |U|$.

Let $\mathcal{C}$ be an admissible evolving crystal corresponding to the data $\{\beta, l, U\}$, and let $z_0^- > 0$ and $z_0^+ > 0$ denote the largest and smallest numbers such that, for some vectors $\mathbf{b}^-$ and $\mathbf{b}^+$,

$$z_0^-(\Lambda + \mathbf{b}^-) \subset \mathcal{C}(0) \subset z_0^+(\Lambda + \mathbf{b}^+). \tag{5.5}$$

Further, let $\kappa^+ > 0$ denote the smallest constant such that $\overline{\beta}^+(\mathbf{n}) := \kappa^+ f(\mathbf{n})^{-1} \geq \beta(\mathbf{n})$ for all $\mathbf{n} \in \mathcal{N}$, let $\kappa^- > 0$ denote the largest constant such that $\overline{\beta}^-(\mathbf{n}) := \kappa^- f(\mathbf{n})^{-1} \leq \beta(\mathbf{n})$ for all $\mathbf{n} \in \mathcal{N}$, and let $z^{\pm}(t)$ with $z^{\pm}(0) = z_0^{\pm}$ denote the solution of (5.3) with $\kappa = \kappa^{\pm}$. Then, by (5.4) and the second comparison theorem,

$$z^-(t)(\Lambda + \mathbf{b}^-) \subset \mathcal{C}(t) \subset z^+(t)(\Lambda + \mathbf{b}^+). \tag{5.6}$$

Thus for $U \geq 0$ or for $U < 0$ and $|U|$ sufficiently small, $\mathcal{C}(t)$ lies between convex crystals that each shrink to a point in finite time; for $U < 0$ and $|U|$ sufficiently large, $\mathcal{C}(t)$ lies between convex crystals that grow without bound.

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[9]Cf. §12 of [Gu].

[10][Ta] for $U = 0$; (12G) of [Gu] for the general case.

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