A COMPLEMENTARY SET THEORY FOR QUATERNARY CODE DESIGNS

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Quaternary code (QC) designs form an attractive class of nonregular factorial fractions. We develop a complementary set theory for characterizing optimal QC designs that are highly fractionated in the sense of accommodating a large number of factors. This is in contrast to existing theoretical results which work only for a relatively small number of factors. While the use of imaginary numbers to represent the Gray map associated with QC designs facilitates the derivation, establishing a link with foldovers of regular fractions helps in presenting our results in a neat form.

1. Introduction. Fractional factorial designs have received much attention due to their theoretical elegance and practical applicability to such diverse fields as engineering, agriculture and medicine. While the literature on regular designs arising from defining equations is now quite rich, in recent years it has been increasingly recognized that nonregular designs can potentially perform even better. See [6, 12, 15] for detailed reviews and further references.

A significant development in nonregular two-level designs over the last few years has been the use of quaternary codes (QC) for construction of such designs, henceforth referred to as QC designs. Xu and Wong [16] pioneered work in this direction and this was followed up by [7–9, 18]. As noted by these authors, QC designs can have an edge over their regular counterparts under commonly used criteria. Moreover, these designs are relatively straightforward to construct and have simple design representation.

The present article aims at developing a theory for optimal QC designs which accommodate a large number of factors and hence are attractive from the practical viewpoint of experimental economy. Our theory covers, in particular, highly fractionated designs with relatively large run sizes. These have been hitherto unexplored in the context of QC designs but, as discussed in [14], can be of much use

in modern day applications. For instance, our findings are applicable to run sizes 128, 256, 512 and 1024 with 96–112, 224–240, 448–480 and 960–992 factors, respectively. No result, either computational or theoretical, is as yet available on optimal QC designs in these cases. For example, the 128- or 256-run design tables in [16] cover up to 64 factors. On the other hand, the results in [7, 9, 18] on 1/4th, 1/8th, 1/16th or 1/64th fractions, though theoretically appealing, are applicable only when the number of factors is small compared to the run size, for example, they together cater only to the cases of 10, 11, 12 or 14 factors for run size 256.

Indeed, the existing approaches for theoretical study of QC designs, such as those based on induction [9], trigonometric formulation [18] or code arithmetic [7], get increasingly involved and unmanageable as the degree of fractionation increases. To overcome this difficulty, we develop a complementary set theory, via the use of imaginary numbers, which considerably facilitates the task of finding optimal QC designs in such situations. While the use of complementary sets for the study of QC designs is inspired by the corresponding development in the regular case [2, 3, 11], neither our final results nor their method of derivation can be anticipated from the latter.

The commonly used optimality criteria in selecting factorial fractions are resolution, aberration and projectivity, as introduced briefly later in this section. In our setup where the number of factors exceeds half the run size, regular designs have resolution three and projectivity two, while following [5, 16], QC designs have resolution at least 3.5 and projectivity at least three. In other words, with regard to resolution and projectivity, QC designs have an edge over regular ones. So, there will be a strong case in favor of QC designs, justifying their use in practice, provided they compete well with regular designs under the minimum aberration (MA) criterion as well. This necessitates identification, in our highly fractionated setup, of minimum aberration quaternary code (MA QC) designs, which is precisely the focus of the present work. Thus, in addition to strengthening the currently available theory of QC designs, our results facilitate their comparison with regular MA designs and help in making a choice between the two.

Before concluding the Introduction, we briefly recall some definitions. A two-level design D in N runs and q factors is represented by an $N \times q$ matrix with elements ± 1, where the rows and columns are identified with the runs and factors, respectively. The *aliasing index* of any subset H of k columns of D is defined as $\rho_k(H; D) = |\text{mean}\{\text{Schur}(H)\}|$. Here $\text{Schur}(H)$ is the Schur product of the columns in H, that is, each element of $\text{Schur}(H)$ is the product of the corresponding elements of the columns in H, and mean $\{\text{Schur}(H)\}$ is the arithmetic mean of the elements of $\text{Schur}(H)$; see [4, 5]. Clearly, we have $0 \leq \rho_k(H; D) \leq 1$. The columns in S are fully aliased, partially aliased or unaliased according as $\rho_k(H; D)$ equals 1, lies strictly between 0 and 1, or equals 0, respectively. For $1 \leq k \leq q$, let

$$(1) \qquad \rho_{k,\max}(D) = \max \rho_k(H; D), \qquad A_k(D) = \sum \{\rho_k(H; D)\}^2,$$

the maximum and the sum being over all H of cardinality k. Write r for the smallest integer such that $\rho_{r,\max}(D) > 0$. Then the *resolution* of D is defined [5] as $R(D) = r + 1 - \rho_{r,\max}(D)$, while the MA criterion calls for sequential minimization of the components of the vector $(A_1(D), \ldots, A_q(D))$, known as the *wordlength pattern* (WLP) of D [10, 17]. These definitions apply to all two-level designs, regular or not, and reduce to the corresponding combinatorial definitions in the regular case. Finally, following [1], design D is said to have *projectivity* p if p is the largest integer such that every p-factor projection of D contains a complete 2^p factorial design, possibly with some points replicated.

2. Quaternary code designs in 2^{2n} runs and an even number of factors.

2.1. *Preliminaries.* Let C be the quaternary linear code given by the $n \times s$ generator matrix $G = [g_1 \cdots g_s]$ whose columns are $n \times 1$ vectors, $n \geq 2$, over the set of integers $Z_4 = \{0, 1, 2, 3\}$ (mod 4). The code C, consisting of $4^n \, (= 2^{2n})$ codewords, each of size s, can be described as

$$(2) \qquad C = \{(u'g_1, \ldots, u'g_s) : u = (u_1, \ldots, u_n)', u_j \in Z_4, 1 \leq j \leq n\},$$

where the primes stand for transposition and each of $u'g_1, \ldots, u'g_s$ is reduced mod 4. The *Gray map*, which replaces each element of Z_4 with a pair of two symbols according to the rule

$$(3) \quad 0 \to (1, 1), \qquad 1 \to (1, -1), \qquad 2 \to (-1, -1), \qquad 3 \to (-1, 1),$$

transforms C into a binary code D, called the binary image of C. With its codewords as rows, D is a $2^{2n} \times (2s)$ matrix having entries ± 1. In this sense, with columns and rows identified with factors and runs, respectively, D represents a QC design in $2s$ two-level factors and 2^{2n} runs.

In order to meet the essential design objective of keeping all main effects orthogonally estimable at least when interactions are absent, the QC design D constructed as above must be an orthogonal array of strength two. Xu and Wong [16] showed that D meets this basic requirement if and only if (a) none of $g_1, \ldots, g_s$ has each entry even, that is, 0 or 2, and (b) no two of these vectors are multiples of each other over Z_4. They also noted that if these conditions hold, then D has resolution at least 3.5 and hence projectivity at least three. So, hereafter we consider only those QC designs which satisfy (a) and (b). Then each $g_1, \ldots, g_s$ has an odd element and, without loss of generality, the first odd element in each of these is 1. Following [16] again, there are altogether $v = (4^n - 2^n)/2$ such vectors over Z_4. Let Ω be the collection of these v vectors. For example, if $n = 2$, then $\Omega = \{(0, 1)', (1, 0)', (1, 1)', (1, 2)', (1, 3)', (2, 1)'\}$. The task of obtaining an MA QC design then amounts to finding a subset $S = \{g_1, \ldots, g_s\}$ of Ω such that the design D arising from $G = [g_1 \cdots g_s]$ sequentially minimizes $A_3(D), \ldots, A_q(D)$, where $q = 2s$. The first two terms of the WLP, $A_1(D)$ and $A_2(D)$, are dropped

here since they equal zero for any such D. The case $s = v$ is trivial, for then $S = \Omega$ is the unique choice of S. Therefore, in the rest of this section, we consider $s < v$.

For highly fractionated designs which form our main focus, s is large. Hence, instead of considering $S = \{g_1, \ldots, g_s\}$ directly, it will be more convenient and insightful to work with $\bar{S}$, the complement of S in Ω. This is motivated by what one does in the study of regular designs, but there are major differences. For example, the counterpart of Ω in the regular case is a finite projective geometry, while the vectors in our Ω are not even on a finite field. This warrants the development of new techniques of proof for the present problem.

2.2. *MA criterion in terms of complementary set.* We now proceed to formulate the MA criterion in terms of the complementary set $\bar{S}$ introduced above. To that effect, from (2) note that the 2^{2n} rows of D can be indexed by the 2^{2n} vectors $u = (u_1, \ldots, u_n)'$ over Z_4. Let Δ be the collection of all such u. For any u ($\in \Delta$), write θ'_u for the corresponding row of D. Clearly, as D has $2s$ columns and elements ± 1, $\theta'_u \theta_w = 2c_{uw} - 2s$, where c_{uw} is the number of coincidences between θ'_u and θ'_w. In view of the equivalence between the MA and minimum moment aberration (MMA) criteria as established in [13], it is clear that sequential minimization of $A_3(D), \ldots, A_q(D)$ is equivalent to that of $M_3(D), \ldots, M_q(D)$, where

$$(4) \qquad M_k(D) = \sum_{u \in \Delta} \sum_{w \in \Delta} (\theta'_u \theta_w)^k, \qquad 3 \le k \le q.$$

In order to achieve further simplification, taking due cognizance of the structure of a QC design, let $i = \sqrt{-1}$ and, for any integer z, write

$$(5) \qquad \psi(z) = (i^z + i^{3-z})/(1 - i).$$

Since the Gray map (3) is equivalent to $z \to (\psi(-z), \psi(z))$, $z \in Z_4$, by (2), the row θ'_u of D can be written explicitly as

$$(6) \qquad \theta'_u = (\psi(-u'g_1), \psi(u'g_1), \ldots, \psi(-u'g_s), \psi(u'g_s)).$$

For any $u \in \Delta$, let σ_u be the sum of the elements of θ'_u. Then, from (5) and (6), after some algebra,

$$(7) \qquad \theta'_u \theta_w = \sigma_{u-w}, \qquad u, w \in \Delta,$$

where $u - w$ is reduced mod 4; cf. Theorem 3 in [16]. Now, for any fixed $w \in \Delta$, as u equals every member of Δ, so does $u - w$. Hence, by (4) and (7), $M_k(D) = 2^{2n} m_k(D)$, where

$$(8) \qquad m_k(D) = \sum_{u \in \Delta} \sigma_u^k, \qquad 3 \le k \le q,$$

and sequential minimization of $A_3(D), \ldots, A_q(D)$ reduces to that of $m_3(D), \ldots, m_q(D)$. Note that the $m_k(D)$ involve only the row totals of D rather than scalar products of rows, and hence are much simpler than the $M_k(D)$.

We next express the σ_u and hence the quantities $m_k(D)$ in terms of the complementary set $\bar{S}$. Recall that $S = \{g_1, \ldots, g_s\}$. Hence, if we write $\Omega = \{g_1, \ldots, g_v\}$, where $v = (4^n - 2^n)/2$ as before, then $\bar{S} = \{g_{s+1}, \ldots, g_v\}$. Analogously to D, let $\bar{D}$ be the QC design, in $2(v - s)$ two-level factors and 2^{2n} runs, arising from the generator matrix $\bar{G} = [g_{s+1} \cdots g_v]$. For $u \in \Delta$, write $\bar{\sigma}_u$ as the sum of elements of the row of $\bar{D}$ which is indexed by u. By (5) and the counterpart of (6) for $\bar{D}$,

$$(9) \qquad \bar{\sigma}_u = \sum_{j=s+1}^{v} \left(i^{u'g_j} + i^{-u'g_j} \right), \qquad u \in \Delta.$$

Also, let Δ_0 consist of the 2^n vectors in Δ which have all elements even, and define δ_u as 1 or 0 according to whether u belongs to Δ_0 or not. Trivially, by (5), (6) and (9),

$$(10) \qquad \sigma_u = 2s, \qquad \bar{\sigma}_u = 2(v - s) \qquad \text{if } u = (0, \ldots, 0)'.$$

The following lemma connects σ_u and $\bar{\sigma}_u$ for nonnull u.

LEMMA 1.   *Let $u(\in \Delta)$ be nonnull. Then $\sigma_u = -(2^n \delta_u + \bar{\sigma}_u)$.*

By (8), (10) and Lemma 1, $m_k(D) = \text{constant} + (-1)^k \bar{m}_k$ for each k, where

$$(11) \qquad \bar{m}_k = \sum_{u \in \Delta} \left(2^n \delta_u + \bar{\sigma}_u \right)^k,$$

and the constant does not depend on D. Hence, in the quest of an MA QC design, one needs to find $\bar{S}$ so as to maximize $\bar{m}_3$, then minimize $\bar{m}_4$, then maximize $\bar{m}_5$, and so on.

2.3. *Characterization of MA designs.*   Since each g_j has an odd element, from (9), (11) and the definitions of Δ_0 and δ_u, arguments similar to but simpler than those in the proof of Theorem 1 below show that $\bar{m}_3 = \text{constant} + F_3$, where the constant does not depend on $\bar{S}$ and

$$(12) \qquad F_3 = 3(2^n) \sum_{u \in \Delta_0} \bar{\sigma}_u^2 + \sum_{u \in \Delta} \bar{\sigma}_u^3.$$

We first explore $\bar{S}$ so as to maximize $\bar{m}_3$ or, equivalently, F_3. The set $\bar{S} = \{g_{s+1}, \ldots, g_v\}$ is called *even* if $g_j + g_k$ has all elements even for every $s + 1 \leq j, k \leq v$. Then the following theorem, which is a main result of this section, holds.

THEOREM 1.   (a) *The inequality $F_3 \leq 3(2^{2n+2})(v - s)^2$ holds for every $\bar{S}$.*
(b) *Equality holds in* (a) *if and only if $\bar{S}$ is even.*

In order to apply Theorem 1, one needs to know when an even set $\bar{S}$ exists. The next lemma settles this issue.

LEMMA 2. *There exists an even set $\bar{S}$ if and only if $v - s \le 2^{n-1}$.*

For $v - s \le 2^{n-1}$, Theorem 1 and Lemma 2 significantly reduce the problem of finding an MA QC design, since one needs to consider only sets $\bar{S} = \{g_{s+1}, \ldots, g_v\}$, which are even. Then each row of $\bar{G} = [g_{s+1} \cdots g_v]$ has either all elements odd or all elements even. Indeed, $\bar{G}$ has a row with all elements odd, because $\bar{S} \subset \Omega$. Since the row space of $\bar{G}$ remains invariant under elementary row operations, without loss of generality, let

$$(13) \qquad \bar{G} = \begin{bmatrix} 1 & 1'_{v-s-1} \\ 0_{n-1} & 2B \end{bmatrix},$$

where 0_{n-1} is the column vector of $n - 1$ zeros, $1'_{v-s-1}$ is the row vector of $v - s - 1$ ones, and B is an $(n - 1) \times (v - s - 1)$ binary matrix. Clearly, B can be interpreted as the generator matrix of a regular design, say, d, in $v - s - 1$ two-level factors and 2^{n-1} runs (these runs are not distinct in case B has less than full row rank). Our next theorem characterizes the MA property of $\bar{S}$ in terms of the much simpler regular design d. Here $A_k(d)$, $k \ge 1$, denotes the WLP of d and $A_1(d) = A_2(d) = 0$, because the columns of $\bar{G}$ are distinct and, as a result, those of B are distinct and nonnull.

THEOREM 2. *Let $v - s \le 2^{n-1}$. Then $\bar{S}$ yields an MA QC design in $2s$ factors and 2^{2n} runs if and only if the matrix B is so chosen that the associated two-level regular design d sequentially minimizes $A_{2r-1}(d) + A_{2r}(d)$, for $r = 2, 3, \ldots$, etc.*

Theorem 2 suggests a connection with the full foldover of d, a point which is confirmed by its proof in the Appendix. To apply this theorem, one needs to consider all nonisomorphic choices of d and select one from among them meeting the condition of the theorem. The extensive tables of two-level regular designs available in the literature are very useful in this regard. We now present two illustrative examples followed by a design table where, for notational simplicity, any binary column vector with 1 in positions $h_1, \ldots, h_r$ and zeros elsewhere is represented by $h_1 \cdots h_r$. Thus, the binary matrix, with columns $(1, 0, 0)'$, $(0, 1, 0)'$, $(1, 1, 0)'$ and $(0, 0, 1)'$ is denoted by [1 2 12 3].

EXAMPLE 1. Let $n = 4$, that is, $v = 120$, and $v - s = 5$. Then there are two nonisomorphic choices of B, namely, [1 2 12 3] and [1 2 3 123]. Both entail $A_3(d) + A_4(d) = 1$ and hence satisfy the condition of Theorem 2, leading to MA QC designs in 230 factors and 256 runs.

EXAMPLE 2. Let $n = 5$, that is, $v = 496$, and $v - s = 10$. Table 3A.2 in [6] lists all nonisomorphic choices of B along with the WLPs $(A_3(d), \ldots, A_9(d))$ of the corresponding two-level regular designs d. These are as shown below:

(i) $B = [1\ 2\ 12\ 3\ 13\ 4\ 14\ 234\ 1234]$, WLP $= (4, 14, 8, 0, 4, 1, 0)$;
(ii) $B = [1\ 2\ 12\ 3\ 13\ 4\ 24\ 34\ 1234]$, WLP $= (6, 9, 9, 6, 0, 0, 1)$;
(iii) $B = [1\ 2\ 12\ 3\ 13\ 23\ 4\ 14\ 234]$, WLP $= (6, 10, 8, 4, 2, 1, 0)$;
(iv) $B = [1\ 2\ 12\ 3\ 13\ 23\ 4\ 14\ 24]$, WLP $= (7, 9, 6, 6, 3, 0, 0)$;
(v) $B = [1\ 2\ 12\ 3\ 13\ 23\ 123\ 4\ 14]$, WLP $= (8, 10, 4, 4, 4, 1, 0)$.

The choice in (ii) uniquely minimizes $A_3(d) + A_4(d)$ and hence yields an MA QC design in 972 factors and 1024 runs. Incidentally, the design d associated with (ii) does not itself have MA as a two-level regular fraction in $9\ (= v - s - 1)$ factors and $16\ (= 2^{n-1})$ runs.

For each $n = 3, 4, 5$ and $v - s \le 2^{n-1}$, Table 1 shows the columns of B so that the associated d satisfies the condition of Theorem 2 and hence yields an MA QC design in $2s$ factors and 2^{2n} runs. In the event of nonuniqueness as in Example 1, only one such B is shown. The cases $v - s = 0$ and 1 are not considered in Table 1 because if $v - s = 0$, then $\bar{S} = \Omega$ is the unique choice of $\bar{S}$, while if $v - s = 1$, then the matrix B does not arise in (13) and it suffices to take $\bar{S}$ as the singleton set consisting of $(1, 0, \ldots, 0)'$.

3. Quaternary code designs in 2^{2n} runs and an odd number of factors. A QC design with an odd number of factors is constructed as follows. First construct a QC design D_{even} in $2(s + 1)$ two-level factors and 2^{2n} runs as in Section 2 starting from the generator matrix $[g_1 \cdots g_s\ g_{s+1}]$, where $s + 1 \le v$. Each vector $g_1, \ldots, g_s, g_{s+1}$ contributes two columns to D_{even} via the Gray map (3). Delete a

TABLE 1
Choice of B leading to an MA QC design via Theorem 2

$n = 3$		$n = 5$	
$v - s$	Columns of B	$v - s$	Columns of B
2	1	2	1
3	1 2	3	1 2
4	1 2 12	4	1 2 3
		5	1 2 3 4
$n = 4$		6	1 2 3 4 1234
$v - s$	Columns of B	7	1 2 12 3 4 34
2	1	8	1 2 12 3 13 4 24
3	1 2	9	1 2 12 3 13 4 24 34
4	1 2 3	10	1 2 12 3 13 4 24 34 1234
5	1 2 12 3	11	1 2 12 3 13 23 4 14 24 34
6	1 2 12 3 13	12	1 2 12 3 13 23 123 4 14 24 34
7	1 2 12 3 13 23	13	1 2 12 3 13 23 123 4 14 24 124 34
8	1 2 12 3 13 23 123	14	1 2 12 3 13 23 123 4 14 24 124 34 134
		15	1 2 12 3 13 23 123 4 14 24 124 34 134 234
		16	1 2 12 3 13 23 123 4 14 24 124 34 134 234 1234

column of D_{even} to get a QC design D_{odd} in $2s + 1$ factors and 2^{2n} runs. Without loss of generality, suppose the second column contributed by g_{s+1} is deleted. Then as in (6), the rows of D_{odd} are given by

$$
\theta'_{\text{odd},u} = (\psi(-u'g_1), \psi(u'g_1), \ldots, \psi(-u'g_s), \psi(u'g_s), \psi(-u'g_{s+1})),
$$

(14)

$$
u \in \Delta.
$$

Let $\sigma_{\text{odd},u}$ be the sum of the elements of $\theta'_{\text{odd},u}$. By (5) and (14), analogously to (7), for $u, w \in \Delta$, the scalar product $\theta'_{\text{odd},u}\theta_{\text{odd},w}$ equals $\sigma_{\text{odd},u-w}$ or $\sigma_{\text{odd},w-u}$ according to whether $w'g_{s+1}$ is even or odd, respectively. Here $u - w$ and $w - u$ are reduced mod 4. Hence, arguing as in Section 2, finding an MA QC design calls for sequential minimization of $m_k(D_{\text{odd}})$, $3 \le k \le q$, where

$$
(15) \qquad m_k(D_{\text{odd}}) = \sum_{u \in \Delta} \sigma_{\text{odd},u}^k.
$$

It again helps to consider the set $\bar{S} = \{g_{s+1}, \ldots, g_v\}$ even though it is no longer a truly complementary set because of the partial contribution of g_{s+1} to D_{odd}. Let

$$
(16) \qquad \bar{\sigma}_{\text{odd},u} = \psi(u'g_{s+1}) + \sum_{j=s+2}^{v} \{\psi(-u'g_j) + \psi(u'g_j)\}, \qquad u \in \Delta.
$$

Evidently, $\sigma_{\text{odd},u} = \sigma_u + \psi(-u'g_{s+1})$ and $\bar{\sigma}_{\text{odd},u} = \bar{\sigma}_u - \psi(-u'g_{s+1})$ for each u, recalling the definitions of σ_u and $\bar{\sigma}_u$. Hence, by (5) and (10), if $u = (0, \ldots, 0)'$, then $\sigma_{\text{odd},u} = 2s + 1$ and $\bar{\sigma}_{\text{odd},u} = 2(v - s) - 1$, while by Lemma 1, if $u(\in \Delta)$ is nonnull, then $\sigma_{\text{odd},u} = -(2^n \delta_u + \bar{\sigma}_{\text{odd},u})$. As a result, by (15), $m_k(D_{\text{odd}}) = $ constant $+ (-1)^k \bar{m}_{\text{odd},k}$ for each k, where

$$
(17) \qquad \bar{m}_{\text{odd},k} = \sum_{u \in \Delta} (2^n \delta_u + \bar{\sigma}_{\text{odd},u})^k,
$$

and the constant does not depend on D_{odd}. Hence, as before, in order to obtain an MA QC design, one needs to find $\bar{S}$ so as to maximize $\bar{m}_{\text{odd},3}$, then minimize $\bar{m}_{\text{odd},4}$, then maximize $\bar{m}_{\text{odd},5}$, and so on. In particular, analogously to (12), $\bar{m}_{\text{odd},3} = $ constant $+ F_{\text{odd},3}$, where

$$
(18) \qquad F_{\text{odd},3} = 3(2^n) \sum_{u \in \Delta_0} \bar{\sigma}_{\text{odd},u}^2 + \sum_{u \in \Delta} \bar{\sigma}_{\text{odd},u}^3.
$$

We now have the following counterpart of Theorem 1 for an odd number of factors.

THEOREM 3. (a) *The inequality* $F_{\text{odd},3} \le 3(2^{2n})\{2(v - s) - 1\}^2$ *holds for every* $\bar{S}$.

(b) *Equality holds in* (a) *if and only if* $\bar{S}$ *is even.*

For $v - s \leq 2^{n-1}$, by Theorem 3 and Lemma 2, only even sets $\bar{S} = \{g_{s+1}, \ldots, g_v\}$ need to be considered in order to find an MA QC design. Then the matrix $\bar{G} = [g_{s+1} \cdots g_v]$ can be represented as in (13) via a binary matrix B and the link with the associated regular design d in $v - s - 1$ two-level factors and 2^{n-1} runs is again useful. As usual, denote the WLP of d by $A_k(d), k \geq 1$, where $A_1(d) = A_2(d) = 0$. Also, write $A_0(d) = 1$ and $A_k(d) = 0$ for $k > v - s - 1$. For $r = 2, 3, \ldots$, define

$$(19) \qquad E_{2r}(d) = \sum_{k=0}^{2r} \binom{v - s - 1 - k}{\langle r - k/2 \rangle} 2^k A_k(d),$$

where the combination is interpreted as zero if $\langle r - k/2 \rangle$, which is the largest integer in $r - k/2$, exceeds $v - s - 1 - k$.

THEOREM 4. *Let $v - s \leq 2^{n-1}$. Then $\bar{S}$ yields an MA QC design in $2s + 1$ factors and 2^{2n} runs if and only if the matrix B is so chosen that the associated two-level regular design d sequentially minimizes $E_{2r}(d)$, for $r = 2, 3, \ldots$, etc.*

Since $\bar{\sigma}_{\mathrm{odd},u}$ and $\bar{m}_{\mathrm{odd},k}$ are more involved than their counterparts in Section 2, it is natural that in general the $E_{2r}(d)$ in Theorem 4 look more complicated than the $A_{2r-1}(d) + A_{2r}(d)$ in Theorem 2. By (19), however, $E_4(d) = \mathrm{constant} + 8A_3(d) + 16A_4(d)$, where the constant does not depend on d. So, minimization of $E_4(d)$ simply calls for that of $A_3(d) + 2A_4(d)$ and, as shown below, this alone is often helpful.

EXAMPLE 1 (continued).   Let $n = 4$, that is, $v = 120$, and $v - s = 5$. Then out of the two nonisomorphic choices of B, namely, [1 2 12 3] and [1 2 3 123], the first one uniquely minimizes $A_3(d) + 2A_4(d)$ and hence, by Theorem 4, yields an MA QC design in 231 factors and 256 runs.

EXAMPLE 2 (continued).   Let $n = 5$, that is, $v = 496$, and $v - s = 10$. Then among the five nonisomorphic choices of B shown earlier, the one in (ii) uniquely minimizes $A_3(d) + 2A_4(d)$ and hence, by Theorem 4, yields an MA QC design in 973 factors and 1024 runs.

Indeed, for each $n = 3, 4, 5$ and $v - s \leq 2^{n-1}$, one can check that either (a) there is a unique B up to isomorphism or (b) the B shown in Table 1 uniquely minimizes $A_3(d) + 2A_4(d)$ and hence, by Theorem 4, leads to an MA QC design in $2s + 1$ factors and 2^{2n} runs. For $v - s = 1$, the matrix B does not arise in (13) and one only has to take $\bar{S}$ as the singleton set consisting of $(1, 0, \ldots, 0)'$. Therefore, in conjunction with what was found in Section 2, we get, in particular, MA QC designs in (i) 64 runs and 48–56 factors, (ii) 256 runs and 224–240 factors, and (iii) 1024 runs and 960–992 factors. While the MA designs in (i) can be seen to agree with those reported in [16], the ones in (ii) and (iii) are new.

4. Quaternary code designs in 2^{2n-1} runs. Consider again the 2^{2n}-run QC designs D and D_{odd}, in $2s$ and $2s+1$ factors, introduced in Sections 2 and 3. Suppose the last rows of the generator matrices, $[g_1 \cdots g_s]$ for D and $[g_1 \cdots g_s \ g_{s+1}]$ for D_{odd}, have all elements even. Then by (5), (6) and (14), the row indexed by $u = (u_1, \ldots, u_n)'$ in either design remains the same if u_n is replaced by $u_n + 2$ (mod 4), that is, the 2^{2n} runs of the design can be split into two identical halves. Following [16], any one of these halves represents a 2^{2n-1}-run QC design. It has the same number of factors and, by (1), the same WLP as the corresponding original design D or D_{odd}, and hence has MA if and only if so does the original design. Therefore, depending on whether the number of factors is even or odd, it suffices to find an MA QC design D or D_{odd} based on a generator matrix as stated above and then take one of its two identical halves as the final design in 2^{2n-1} runs.

To adapt the complementary set theory for this purpose, observe that not all vectors in the reference set Ω now qualify as columns of the generator matrix, but that only the ones with last element even do so. Write $\Omega_0 = \{g_1, \ldots, g_{v_0}\}$ for the collection of these qualifying vectors, where $v_0 = 4^{n-1} - 2^{n-1}$. Let $\bar{S} = \{g_{s+1}, \ldots, g_{v_0}\}$ be the complement of $S = \{g_1, \ldots, g_s\}$ in Ω_0. Define σ_u, $\bar{\sigma}_u$, $m_k(D)$ and $\bar{m}_k$ as in Section 2, and $\sigma_{\text{odd},u}$, $\bar{\sigma}_{\text{odd},u}$, $m_k(D_{\text{odd}})$ and $\bar{m}_{\text{odd},k}$ as in Section 3, with v replaced by v_0 in $\bar{\sigma}_u$ and $\bar{\sigma}_{\text{odd},u}$; cf. (9) and (16). Then one can check that $\sigma_u = 2s$, $\bar{\sigma}_u = 2(v_0 - s)$, $\sigma_{\text{odd},u} = 2s + 1$ and $\bar{\sigma}_{\text{odd},u} = 2(v_0 - s) - 1$ if u has first $n - 1$ elements 0 and last element 0 or 2, and that the conclusion of Lemma 1 remains unaltered for every other u. Therefore, despite working with Ω_0 rather than Ω, we still have $m_k(D) = \text{constant} + (-1)^k \bar{m}_k$ and $m_k(D_{\text{odd}}) = \text{constant} + (-1)^k \bar{m}_{\text{odd},k}$ for each k, where the constants do not depend on D or D_{odd}. Furthermore, in the representation (13) for an even set $\bar{S}$ via $\bar{G}$, each vector in $\bar{S}$ has last element even and hence $\bar{S} \subset \Omega_0$, as required here.

From the above, it is evident that the findings in Sections 2 and 3 as well as Table 1 continue to remain valid in the present setup, with v simply replaced by v_0, thus leading to 2^{2n-1}-run MA QC designs in $2s$ factors, $0 \le v_0 - s \le 2^{n-1}$, and $2s + 1$ factors, $1 \le v_0 - s \le 2^{n-1}$. As a result, we get, in particular, MA QC designs in (i) 128 runs and 96–112 factors, and (ii) 512 runs and 448–480 factors.

EXAMPLE 3. Let $n = 4$, that is, $v_0 = 56$, and $v_0 - s = 6$. Then from Table 1, $B = [1 \ 2 \ 12 \ 3 \ 13]$ and, hence, by (13),

$$\bar{S} = \{(1, 0, 0, 0)', (1, 2, 0, 0)', (1, 0, 2, 0)', (1, 2, 2, 0)', (1, 0, 0, 2)', (1, 2, 0, 2)'\}.$$

If one (i) finds the complement $S = \{g_1, \ldots, g_s\}$ of $\bar{S}$ in Ω_0 and (ii) constructs a design D in 100 factors and 256 runs from the generator matrix $[g_1 \cdots g_s]$, then any one of the two identical halves of D is an MA QC design in 100 factors and 128 runs.

EXAMPLE 4.   Let $n = 5$, that is, $v_0 = 240$, and $v_0 - s = 7$. Then from Table 1, $B = [1\ 2\ 12\ 3\ 4\ 34]$ and, hence, by (13),

$$\bar{S} = \{(1, 0, 0, 0, 0)', (1, 2, 0, 0, 0)', (1, 0, 2, 0, 0)', (1, 2, 2, 0, 0)', (1, 0, 0, 2, 0)',$$

$$(1, 0, 0, 0, 2)', (1, 0, 0, 2, 2)'\}.$$

Now, if one (i) finds the complement $S = \{g_1, \ldots, g_s\}$ of $\bar{S}$ in Ω_0, (ii) constructs a design D_{even} in 468 factors and 1024 runs from the generator matrix $[g_1 \cdots g_s\ g_{s+1}]$, where $g_{s+1} = (1, 0, 0, 0, 0)'$, and (iii) deletes the last column of D_{even} (i.e., the second column contributed by g_{s+1}) to get a design D_{odd} in 467 factors and 1024 runs, then any one of the two identical halves of D_{odd} is an MA QC design in 467 factors and 512 runs.

5. Comparison with regular MA designs and concluding remarks.
In a highly fractionated setup, our results explore the best that can be achieved by QC designs under the MA criterion and hence facilitate comparison with their regular counterparts. Indeed, as seen below, MA QC designs obtained here compete very well with MA regular designs. For illustration, we consider the cases of $N = 128$ and 256 runs and recall that for these N, our results yield MA QC designs for $96 \leq q \leq 112$ and $224 \leq q \leq 240$, where q is the number of factors.

For $N = 128$, MA QC designs have (i) the same WLP as MA regular designs if $96 \leq q \leq 99$ or $109 \leq q \leq 112$, and (ii) the same A_3 but a little larger A_4 if $100 \leq q \leq 108$. In the first case, MA QC designs clearly outperform MA regular designs because of higher resolution and projectivity. For most practical purposes, in the second case too the same features of MA QC designs far outweigh their slightly higher aberration. For instance, if $q = 103$, then the A_4 values for the MA QC and MA regular designs are 35,707 and 35,705, respectively, while both have $A_3 = 1360$. Thus, the marginally larger A_4 for the MA QC design is more than compensated by the fact that it has projectivity at least three, while the MA regular design has as many as 1360 three-factor projections which do not contain a complete 2^3 factorial.

For $N = 256$, MA QC designs have (i) the same WLP as MA regular designs if $224 \leq q \leq 227$ or $237 \leq q \leq 240$, and (ii) the same A_3 but marginally larger A_4 if $228 \leq q \leq 236$, for example, if $q = 228$, then the A_4 values for the MA QC and MA regular designs are 434,057 and 434,056, respectively, while both have $A_3 = 7616$. This has the same implications as before in favor of MA QC designs. The same pattern is seen to persist for larger N.

It will be of interest to extend the present results to QC designs which are less highly fractionated than the ones considered here, that is, for which the size of $\bar{S}$ exceeds 2^{n-1}. In view of Lemma 2, then the bounds in Theorems 1 and 3 are not attainable and, therefore, one cannot have neat results in terms of even sets. However, Lemma 1 as well as Lemma 3 and equations (23), (24), (34), (35) in the Appendix continue to hold and should be useful in replacing the bounds in Theorems 1 and 3 by sharper, attainable ones. We conclude with the hope that the present endeavor will generate more interest in this area.

APPENDIX: PROOFS

PROOF OF LEMMA 1. For $u \in \Delta$, consider the row of $[D\bar{D}]$ which is indexed by u. This row is of the form (6) with s there replaced by v. Hence, by (5), analogously to (9),

$$(20) \qquad \sigma_u + \bar{\sigma}_u = \sum_{j=1}^{v} (i^{u'g_j} + i^{-u'g_j}).$$

By the definitions of Ω, Δ and Δ_0, the union of the sets $\{g_j, -g_j\}$, $1 \le j \le v$, equals $\Delta - \Delta_0$. Hence by (20), $\sigma_u + \bar{\sigma}_u = \sum_{w \in \Delta} i^{u'w} - \sum_{w \in \Delta_0} i^{u'w}$. The lemma now follows, noting that (i) $\sum_{w \in \Delta} i^{u'w} = 0$ for nonnull u, (ii) $\sum_{w \in \Delta_0} i^{u'w} = 0$ if u has an odd element, and (iii) $\sum_{w \in \Delta_0} i^{u'w} = 2^n$ if u has all elements even, since then $u'w = 0 \bmod 4$, for every $w \in \Delta_0$. $\square$

In order to prove Theorem 1, we require some notation and another lemma. For any $n \times 1$ vector g with integer elements, define $\alpha(g)$ as 1 or 0 according to whether g is null (mod 4) or not. Recall that $\bar{S} = \{g_{s+1}, \ldots, g_v\}$. For $s + 1 \le j, k, h \le v$, let

$$(21) \qquad \begin{aligned} \beta_{jkh} = {} &\alpha(g_j + g_k + g_h) + \alpha(g_j + g_k - g_h) + \alpha(g_j - g_k + g_h) \\ &+ \alpha(g_j - g_k - g_h). \end{aligned}$$

LEMMA 3. (a) *For each* j, k, h, $\beta_{jkh} = 0$ *or* 1.
(b) *For any fixed* j, k, (i) $\beta_{jkh} = 0$ *for all* h, *if* $g_j + g_k$ *has all elements even*, (ii) $\beta_{jkh} = 1$ *for at most two choices of* h, *otherwise*.

PROOF. (a) This follows noting that the right-hand side of (21) cannot have a pair of terms both of which equal 1. For instance, if the first two terms equal 1, then $g_j + g_k + g_h = 0 \pmod 4$ and $g_j + g_k - g_h = 0 \pmod 4$. So $2g_h = 0 \pmod 4$, which is impossible as g_h has an odd element. The same argument applies to any other pair of terms.

(b) If $g_j + g_k$ has all elements even, then the same holds for $g_j - g_k$. As g_h has an odd element, then all terms on the right-hand side of (21) vanish, that is, $\beta_{jkh} = 0$. Next, let $g_j + g_k$ have an odd element. Then, by (21), $\beta_{jkh} = 1$ if and only if $g_h = \pm(g_j + g_k) \pmod 4$ or $g_h = \pm(g_j - g_k) \pmod 4$. Since $-(g_j + g_k) = 3(g_j + g_k) \pmod 4$ and $-(g_j - g_k) = 3(g_j - g_k) \pmod 4$, and no two vectors in $\bar{S}$ can be multiples of each other, it follows that $\beta_{jkh} = 1$ for at most two choices of h. $\square$

PROOF OF THEOREM 1. (a) As in the proof of Lemma 1 but using a more formal notation,

$$(22) \qquad \sum_{u \in \Delta_0} i^{u'g} = \sum_{u \in \Delta_0} i^{-u'g} = 2^n \alpha(2g), \qquad \sum_{u \in \Delta} i^{u'g} = \sum_{u \in \Delta} i^{-u'g} = 2^{2n} \alpha(g).$$

Let $\sum^{(2)}$ and $\sum^{(3)}$ denote double and triple sums on j, k and j, k, h over the ranges $s + 1 \leq j, k \leq v$ and $s + 1 \leq j, k, h \leq v$, respectively. By (9) and (22),

$$
\sum_{u \in \Delta_0} \bar{\sigma}_u^2 = \sum^{(2)} \sum_{u \in \Delta_0} (i^{u'g_j} + i^{-u'g_j})(i^{u'g_k} + i^{-u'g_k})
$$

(23)

$$
= 2^{n+2} \sum^{(2)} \alpha(2g_j + 2g_k),
$$

because $2g_j - 2g_k = 2g_j + 2g_k \pmod 4$. Similarly,

$$
\sum_{u \in \Delta} \bar{\sigma}_u^3 = \sum^{(3)} \sum_{u \in \Delta} (i^{u'g_j} + i^{-u'g_j})(i^{u'g_k} + i^{-u'g_k})(i^{u'g_h} + i^{-u'g_h})
$$

(24)

$$
= 2^{2n+1} \sum^{(3)} \beta_{jkh},
$$

where β_{jkh} is given by (21).

Observe that $\bar{S} = \{g_{s+1}, \ldots, g_v\}$ can be partitioned into $t \ (\geq 1)$ mutually exclusive and exhaustive nonempty subsets such that any $g_j + g_k$, $s + 1 \leq j, k \leq v$, has all elements even if and only if g_j and g_k belong to the same subset. Denote the cardinalities of these subsets by $f_1, \ldots, f_t$. Then

$$
(25) \qquad f_1 + \cdots + f_t = v - s,
$$

since $\bar{S}$ has cardinality $v - s$. Clearly, $\bar{S}$ is even if and only if $t = 1$. Now by (23),

$$
(26) \qquad \sum_{u \in \Delta_0} \bar{\sigma}_u^2 = 2^{n+2}(f_1^2 + \cdots + f_t^2),
$$

because $\alpha(2g_j + 2g_k) = 1$ if and only if $g_j + g_k$ has all elements even, that is, g_j and g_k belong to the same subset of $\bar{S}$ as described above. Similarly, by (24), (25) and Lemma 3,

$$
(27) \quad \sum_{u \in \Delta} \bar{\sigma}_u^3 \leq 2^{2n+2} \sum_{l=1}^{t} \sum_{r(\neq l)=1}^{t} f_l f_r = 2^{2n+2}\{(v - s)^2 - (f_1^2 + \cdots + f_t^2)\}.
$$

By (12) and (25)–(27),

$$
(28) \qquad F_3 \leq 2^{2n+2}\{(v - s)^2 + 2(f_1^2 + \cdots + f_t^2)\} \leq 3(2^{2n+2})(v - s)^2,
$$

which proves (a).

(b) By (25) and (28), equality holds in (a) only if $f_1^2 + \cdots + f_t^2 = (f_1 + \cdots + f_t)^2$, which holds only if $t = 1$, that is, the set $\bar{S}$ is even. On the other hand, if $\bar{S}$ is even, then by (23), (24) and Lemma 3,

$$
\sum_{u \in \Delta_0} \bar{\sigma}_u^2 = 2^{n+2}(v - s)^2, \qquad \sum_{u \in \Delta} \bar{\sigma}_u^3 = 0.
$$

Therefore, (12) yields $F_3 = 3(2^{2n+2})(v - s)^2$, and equality holds in (a). $\square$

PROOF OF LEMMA 2. Only if: For an even set $\bar{S} = \{g_{s+1}, \ldots, g_v\}$, the vectors $g_{s+1} + g_j \pmod 4$, $s + 1 \le j \le v$, satisfy the following: (i) each of them has all elements even, and (ii) no two of them add up to $2g_{s+1} \pmod 4$. Here (ii) is due to the fact that no two vectors in $\bar{S}$ are multiples of each other over Z_4. Since there are at most 2^{n-1} distinct $n \times 1$ vectors over Z_4 satisfying (i) and (ii), the only if part follows.

If: There are 2^{n-1} distinct $(n-1) \times 1$ vectors over Z_4, each of which has all elements even. For $v - s \le 2^{n-1}$, consider any $v - s$ of these $(n-1) \times 1$ vectors, say, $\tilde{g}_{s+1}, \ldots, \tilde{g}_v$. Then the set $\bar{S}$, consisting of the vectors $(1, \tilde{g}_j')'$, $s + 1 \le j \le v$, is even and the if part follows. $\square$

PROOF OF THEOREM 2. Let $b_{s+2}, \ldots, b_v$ denote the columns of B. For any $u = (u_1, \ldots, u_n)' \in \Delta$, writing $u(2) = (u_2, \ldots, u_n)'$, from (9) and (13),

$$
\bar{\sigma}_u = (i^{u_1} + i^{-u_1}) \left\{ 1 + \sum_{j=s+2}^{v} (-1)^{u(2)'b_j} \right\}. \tag{29}
$$

Therefore, considering $u_1 = 0, 1, 2, 3$ separately, for any positive integer k,

$$
\sum_{u \in \Delta} \bar{\sigma}_u^k = 2^k \{1 + (-1)^k\} \sum_{u_2=0}^{3} \cdots \sum_{u_n=0}^{3} \left\{ 1 + \sum_{j=s+2}^{v} (-1)^{u(2)'b_j} \right\}^k. \tag{30}
$$

Note that the quantities $(-1)^{u(2)'b_j}$, $s + 2 \le j \le v$, remain unaltered if any element u_h of $u(2)$ is replaced by $u_h + 2 \pmod 4$. Hence, (30) yields

$$
\begin{aligned}
\sum_{u \in \Delta} \bar{\sigma}_u^k &= 2^{n+k-1} \{1 + (-1)^k\} \sum_{u_2=0}^{1} \cdots \sum_{u_n=0}^{1} \left\{ 1 + \sum_{j=s+2}^{v} (-1)^{u(2)'b_j} \right\}^k \\
&= 2^{n+k-1} \{1 + (-1)^k\} \sum_{x \in \Gamma} (1 + \lambda_x)^k,
\end{aligned} \tag{31}
$$

where Γ is the collection of the 2^{n-1} binary column vectors of order $n - 1$, and for any $x \in \Gamma$,

$$
\lambda_x = \sum_{j=s+2}^{v} (-1)^{x'b_j}. \tag{32}
$$

Also, by (29), for every $u = (u_1, \ldots, u_n)' \in \Delta_0$, $\bar{\sigma}_u$ equals $2(v - s)$ or $-2(v - s)$, according to whether $u_1 = 0$ or 2, respectively. For any nonnegative integer r, therefore, $\sum_{u \in \Delta} \delta_u \bar{\sigma}_u^r$ is a constant which does not depend on the choice of $\bar{S}$. Hence, by (11) and (31),

$$
\bar{m}_k = \text{constant} + 2^{n+k} \sum_{x \in \Gamma} (1 + \lambda_x)^k \qquad \text{if } k \text{ is even,}
$$

$$
= \text{constant} \qquad \text{if } k \text{ is odd,}
$$

where the constants do not depend on $\bar{S}$. Therefore, $\bar{S}$ yields an MA QC design if and only if the matrix B is so chosen as to sequentially minimize $\sum_{x \in \Gamma}(1 + \lambda_x)^{2r}$, for $r = 2, 3, \ldots$, etc.

Now from (32), observe that λ_x, $x \in \Gamma$, are the row (run) sums of the two-level regular design d generated by $B = [b_{s+2} \cdots b_v]$. Recall that d involves $v - s - 1$ two-level factors and 2^{n-1} runs. Write $\tilde{d}$ for the full foldover of d and note the following:

(i) $\tilde{d}$ involves $v - s$ two-level factors and 2^n runs,

(ii) the row (run) sums of $\tilde{d}$ are $\pm(1 + \lambda_x)$, $x \in \Gamma$, because those of d are λ_x, $x \in \Gamma$,

(iii) in any two-level regular N-run design with rows (runs) $\xi'_1, \ldots, \xi'_N$, the N row sums occur equally often among the N^2 scalar products $\{\xi'_j \xi_k : 1 \le j, k \le N\}$ of the rows.

So, as in the passage from (4) to (8) in Section 2, sequential minimization of $\sum_{x \in \Gamma}(1 + \lambda_x)^{2r}$, for $r = 2, 3, \ldots$, etc. amounts to choosing d so that its foldover $\tilde{d}$ has MMA, and hence MA, among all such foldovers. Now the result follows if we denote the WLP of $\tilde{d}$ by $A_k(\tilde{d})$, $k \ge 1$, and note that $A_1(\tilde{d}) = A_2(\tilde{d}) = 0$, while $A_{2r-1}(\tilde{d}) = 0$, $A_{2r}(\tilde{d}) = A_{2r-1}(d) + A_{2r}(d)$, for $r = 2, 3, \ldots$, etc. $\square$

We indicate only the key steps in the proofs of Theorems 3 and 4 which are similar to but more elaborate than those of Theorems 1 and 2.

PROOF OF THEOREM 3.   (a) By (5) and (16), analogously to (9),

$$(33) \quad \bar{\sigma}_{\text{odd},u} = \left(i^{u'g_{s+1}} + i^{3-u'g_{s+1}}\right)/(1-i) + \sum_{j=s+2}^{v} \left(i^{u'g_j} + i^{-u'g_j}\right), \qquad u \in \Delta.$$

Thus, using (22), along the lines of (23) and (24), but with heavier algebra,

$$(34) \quad \sum_{u \in \Delta_0} \bar{\sigma}_{\text{odd},u}^2 = 2^n + 2^{n+2}\Big\{\widetilde{\sum}^{(1)} \alpha(2g_{s+1} + 2g_j) + \widetilde{\sum}^{(2)} \alpha(2g_j + 2g_k)\Big\},$$

$$(35) \quad \sum_{u \in \Delta} \bar{\sigma}_{\text{odd},u}^3 = 3(2^{2n})\widetilde{\sum}^{(2)} \beta_{s+1jk} + 2^{2n+1}\widetilde{\sum}^{(3)} \beta_{jkh},$$

where $\widetilde{\sum}^{(1)}$ denotes the sum on j over $s + 2 \le j \le v$, while $\widetilde{\sum}^{(2)}$ and $\widetilde{\sum}^{(3)}$ denote double and triple sums on j, k and j, k, h over $s + 2 \le j, k \le v$ and $s + 2 \le j, k, h \le v$, respectively.

Partition $\bar{S} = \{g_{s+1}, \ldots, g_v\}$ into t (≥ 1) mutually exclusive and exhaustive nonempty subsets as specified in the proof of Theorem 1. The cardinalities $f_1, \ldots, f_t$ of these subsets satisfy (25). Without loss of generality, let g_{s+1} belong to the first of these subsets. Then $\widetilde{\sum}^{(1)} \alpha(2g_{s+1} + 2g_j) = f_1^*$, while, analogously

to (26), $\tilde{\sum}^{(2)} \alpha(2g_j + 2g_k) = \sum_{l=1}^{t}(f_l^*)^2$, where $f_1^* = f_1 - 1$ and $f_l^* = f_l$ for $l \geq 2$. Hence, by (34),

$$(36) \qquad \sum_{u \in \Delta_0} \bar{\sigma}_{\mathrm{odd},u}^2 = 2^n \left\{ 1 + 4f_1^* + 4\sum_{l=1}^{t}(f_l^*)^2 \right\}.$$

Also, by Lemma 3, $\tilde{\sum}^{(2)} \beta_{s+1\,jk} \leq 2\sum_{l=2}^{t} f_l^*$, while, analogously to (27), $\tilde{\sum}^{(3)} \beta_{jkh} \leq 2\{(\sum_{l=1}^{t} f_l^*)^2 - \sum_{l=1}^{t}(f_l^*)^2\}$, so that by (35),

$$(37) \qquad \sum_{u \in \Delta} \bar{\sigma}_{\mathrm{odd},u}^3 \leq 2^{2n+1} \left\{ 3\sum_{l=2}^{t} f_l^* + 2\left(\sum_{l=1}^{t} f_l^*\right)^2 - 2\sum_{l=1}^{t}(f_l^*)^2 \right\}.$$

By (25), the sum of the nonnegative integers $f_1^*, \ldots, f_t^*$ equals $v - s - 1$. Therefore,

$$f_1^* \leq v - s - 1, \qquad \sum_{l=1}^{t}(f_l^*)^2 \leq (v - s - 1)^2,$$

and, hence, from (18), (36) and (37), on simplification

$$F_{\mathrm{odd},3} \leq 2^{2n} \left\{ 3 + 6(v - s - 1) + 4(v - s - 1)^2 + 6f_1^* + 8\sum_{l=1}^{t}(f_l^*)^2 \right\}$$

$$\leq 3(2^{2n})\{2(v - s) - 1\}^2,$$

which proves (a).

(b) It is easily seen that equality holds in (a) only if $f_1^* = v - s - 1$, that is, $f_1 = v - s$, which holds only if $\bar{S}$ is even. On the other hand, if $\bar{S}$ is even, then by (34), (35) and Lemma 3,

$$\sum_{u \in \Delta_0} \bar{\sigma}_{\mathrm{odd},u}^2 = 2^n + 2^{n+2}\{v - s - 1 + (v - s - 1)^2\} = 2^n\{2(v - s) - 1\}^2,$$

and $\sum_{u \in \Delta} \bar{\sigma}_{\mathrm{odd},u}^3 = 0$, so that by (18), equality holds in (a). $\square$

Some notation and a lemma are needed for proving Theorem 4. With B as in (13), the binary matrix $[B \ B]$ generates a regular design, say, d_0, in $2(v - s - 1)$ two-level factors and 2^{n-1} runs. Denote the full foldover of d_0 by $\tilde{d}_0$. Let $A_k(d_0)$ and $A_k(\tilde{d}_0)$, $k \geq 1$, be the WLPs of d_0 and $\tilde{d}_0$, respectively. Clearly, $A_1(d_0) = 0$ and $A_2(d_0) = v - s - 1$, since the columns of $[B \ B]$ are nonnull but identical in pairs.

LEMMA 4. (a) $A_k(\tilde{d}_0) = 0$, *for every odd* k, (b) $A_2(\tilde{d}_0) = v - s - 1$, (c) $A_{2r}(\tilde{d}_0) = E_{2r}(d)$, *for* $r = 2, 3, \ldots$, *where* $E_{2r}(d)$ *is given by* (19).

PROOF. While (a) holds for any full foldover design, (b) is obvious. To prove (c), denote the columns of B by $b_{s+2}, \ldots, b_v$ and write $[B \ B] = [b_{s+2}^{(1)} \cdots b_v^{(1)} \ b_{s+2}^{(2)} \cdots b_v^{(2)}]$, where $b_j^{(1)} = b_j^{(2)} = b_j$, $s + 2 \leq j \leq v$. Then any set of h columns of $[B \ B]$, forming a word of length h of the design d_0, has the structure $\{b_{j_1}^{(l_1)}, \ldots, b_{j_k}^{(l_k)}\} \cup \{b_j^{(1)}, b_j^{(2)} : j \in J\}$, where k is such that $h - k$ is a nonnegative even integer, the columns $b_{j_1}, \ldots, b_{j_k}$ constitute a word of length k of d, each of $l_1, \ldots, l_k$ is either 1 or 2, and J is any subset, with cardinality $(h - k)/2$, of the complement of $\{j_1, \ldots, j_k\}$ in $\{s + 2, \ldots, v\}$. So

$$(38) \qquad A_h(d_0) = \sum \binom{v - s - 1 - k}{(h - k)/2} 2^k A_k(d),$$

where the sum ranges over $k = h, h - 2, \ldots$, etc. Now, $A_{2r}(\tilde{d}_0) = A_{2r-1}(d_0) + A_{2r}(d_0)$, since $\tilde{d}_0$ is the full foldover of d_0. Hence, (c) follows from (38), recalling the definition of $E_{2r}(d)$ from (19). □

PROOF OF THEOREM 4. By (13) and (33), using the same notation as in (29),

$$\bar{\sigma}_{\text{odd},u} = (i^{u_1} + i^{3-u_1})/(1 - i) + (i^{u_1} + i^{-u_1}) \sum_{j=s+2}^{v} (-1)^{u(2)'b_j}, \qquad u \in \Delta.$$

Hence, by (17), arguing as in the proof of Theorem 2, for any positive integer k,

$$\bar{m}_{\text{odd},k} = \text{constant} + 2^n \sum_{x \in \Gamma} (1 + 2\lambda_x)^k \qquad \text{if } k \text{ is even,}$$

$$= \text{constant} \qquad \text{if } k \text{ is odd,}$$

where λ_x is given by (32) and the constants do not depend on $\bar{S}$. Therefore, $\bar{S}$ yields an MA QC design if and only if the matrix B is so chosen as to sequentially minimize $\sum_{x \in \Gamma} (1 + 2\lambda_x)^{2r}$, for $r = 2, 3, \ldots$, etc. Again, as with Theorem 2, this happens if and only if the full foldover design $\tilde{d}_0$ in Lemma 4 has MA among all such foldovers, because the row (run) sums of $\tilde{d}_0$ are $\pm(1 + 2\lambda_x)$, $x \in \Gamma$, as those of d are λ_x, $x \in \Gamma$. The result is now immediate from Lemma 4. □

REFERENCES

[1] BOX, G. and TYSSEDAL, J. (1996). Projective properties of certain orthogonal arrays. *Biometrika* **83** 950–955. MR1440059

[2] CHEN, H. and HEDAYAT, A. S. (1996). 2^{n-l} designs with weak minimum aberration. *Ann. Statist.* **24** 2536–2548. MR1425966

[3] CHENG, C.-S. and MUKERJEE, R. (1998). Regular fractional factorial designs with minimum aberration and maximum estimation capacity. *Ann. Statist.* **26** 2289–2300. MR1700232

[4] CHENG, S.-W., LI, W. and YE, K. Q. (2004). Blocked nonregular two-level factorial designs. *Technometrics* **46** 269–279. MR2082497

[5] DENG, L.-Y. and TANG, B. (1999). Generalized resolution and minimum aberration criteria for Plackett–Burman and other nonregular factorial designs. *Statist. Sinica* **9** 1071–1082. MR1744824

[6] MUKERJEE, R. and WU, C. F. J. (2006). *A Modern Theory of Factorial Designs*. Springer, New York. MR2230487

[7] PHOA, F. K. H. (2012). A code arithmetic approach for quaternary code designs and its application to (1/64)th-fractions. *Ann. Statist.* **40** 3161–3175. MR3097973

[8] PHOA, F. K. H., MUKERJEE, R. and XU, H. (2012). One-eighth- and one-sixteenth-fraction quaternary code designs with high resolution. *J. Statist. Plann. Inference* **142** 1073–1080. MR2879752

[9] PHOA, F. K. H. and XU, H. (2009). Quarter-fraction factorial designs constructed via quaternary codes. *Ann. Statist.* **37** 2561–2581. MR2543703

[10] TANG, B. and DENG, L.-Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Ann. Statist.* **27** 1914–1926. MR1765622

[11] TANG, B. and WU, C. F. J. (1996). Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs. *Ann. Statist.* **24** 2549–2559. MR1425967

[12] WU, C. F. J. and HAMADA, M. S. (2009). *Experiments*: *Planning*, *Analysis*, *and Optimization*, 2nd ed. Wiley, Hoboken, NJ. MR2583259

[13] XU, H. (2003). Minimum moment aberration for nonregular designs and supersaturated designs. *Statist. Sinica* **13** 691–708. MR1997169

[14] XU, H. (2009). Algorithmic construction of efficient fractional factorial designs with large run sizes. *Technometrics* **51** 262–277. MR2562273

[15] XU, H., PHOA, F. K. H. and WONG, W. K. (2009). Recent developments in nonregular fractional factorial designs. *Stat. Surv.* **3** 18–46. MR2520978

[16] XU, H. and WONG, A. (2007). Two-level nonregular designs from quaternary linear codes. *Statist. Sinica* **17** 1191–1213. MR2397390

[17] XU, H. and WU, C. F. J. (2001). Generalized minimum aberration for asymmetrical fractional factorial designs. *Ann. Statist.* **29** 1066–1077. MR1869240

[18] ZHANG, R., PHOA, F. K. H., MUKERJEE, R. and XU, H. (2011). A trigonometric approach to quaternary code designs with application to one-eighth and one-sixteenth fractions. *Ann. Statist.* **39** 931–955. MR2816343

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