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A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics

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Abstract. We introduce the first complete and approximatively universal diagrammatic language for quantum mechanics. We make the ZX-Calculus, a diagrammatic language introduced by Coecke and Duncan, complete for the so-called Clifford+T quantum mechanics by adding two new axioms to the language. The completeness of the ZX-Calculus for Clifford+T quantum mechanics was one of the main open questions in *categorical quantum mechanics*. We prove the completeness of the $\frac{\pi}{4}$-fragment of the ZX-Calculus using the recently studied ZW-Calculus, a calculus dealing with integer matrices. We also prove that the $\frac{\pi}{4}$-fragment of the ZX-Calculus represents exactly all the matrices over some finite dimensional extension of the ring of dyadic rationals.

1 Introduction

The ZX-Calculus is a powerful graphical language for quantum reasoning and quantum computing introduced by Bob Coecke and Ross Duncan [9]. The language comes with a way of interpreting any ZX-diagram as a matrix – called the *standard interpretation*. Two diagrams represent the same quantum evolution when they have the same standard interpretation. The language is also equipped with a set of axioms – transformation rules – which are sound, i.e. they preserve the standard interpretation. Their purpose is to explain how a diagram can be transformed into an equivalent one.

The ZX-calculus has several applications in quantum information processing [11] (e.g. measurement-based quantum computing [16,21,12], quantum codes [14,15,8,7], foundations [4,13]), and can be used through the interactive theorem prover Quantomatic [24,25]. However, the main obstacle to wider use of the ZX-calculus was the absence of a *completeness* result for a *universal* fragment of quantum mechanics, in order to guarantee that any true property is provable using the ZX-calculus. More precisely, the language would be complete if, given any two diagrams representing the same matrix, one could transform one diagram into the other using the axioms of the language. Completeness is crucial, it means in particular that all the fundamental properties of quantum mechanics are captured by the graphical rules.

ZX-Calculus has been proved to be incomplete in general [30], and despite the necessary axioms that have since been identified [28,23], the language remained incomplete. However, several fragments of the language have been proved to be complete ($\frac{\pi}{2}$-fragment [2]; π-fragment [17]; single-qubit $\frac{\pi}{4}$-fragment [3]), but none of them are *universal* for quantum mechanics, even approximatively. In particular all quantum algorithms expressible in these fragments are efficiently simulable on a classical computer.

As a consequence, most of the attention has been paid to find a complete axiomatisation of the $\frac{\pi}{4}$-fragment of the ZX-Calculus for the Clifford+T quantum mechanics, the simplest approximatively universal fragment of quantum mechanics, which is widely used in quantum computing.

Our Approach. In the following, we introduce the first complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics, thanks to the help of the ZW-Calculus, another graphical language – based on the interactions of the so-called GHZ and W states [10]. The ZW-Calculus has been proved to be complete [19] but its diagrams only represent matrices over $\mathbb{Z}$, and hence is not approximatively universal. We introduce the $\mathrm{ZW}_{1/2}$-calculus, a simple extension of the ZW-Calculus which remains complete and in which any matrix over the dyadic rational numbers can be represented. We then introduce two interpretations from the ZX-Calculus to the $\mathrm{ZW}_{1/2}$-calculus and back. Thanks to these interpretations, we derive the completeness of the $\frac{\pi}{4}$-fragment of the ZX-Calculus from the completeness of the $\mathrm{ZW}_{1/2}$-calculus. Notice that the interpretation of ZX-diagrams (which represent complex matrices) into $ZW_{1/2}$-diagrams (which represent dyadic rationals) requires a non trivial encoding. Notice also that this approach provides a completion procedure. Roughly speaking each axiom of the $ZW_{1/2}$-calculus generates an equation in the ZX-calculus: if this equation is not already provable using the existing axioms of the ZX-calculus one can treat this equality as an new axiom. A great part of the work has been to reduce all these equalities to only two additional axioms for the language.

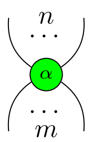
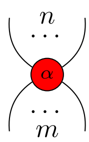
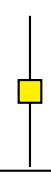
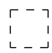
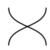
Related works. The first version of the present paper has been uploaded on Arxiv in May 2017. In the following weeks, Hadzihasanovic [20] independently introduced in his PhD thesis the $ZW_{\mathbb{C}}$-calculus, an extension of the ZW-Calculus, which is universal and complete for complex matrices. Notice that the $ZW_{\mathbb{C}}$-calculus does not capture the peculiar properties of Clifford+T quantum mechanics, and hence the use of the $ZW_{1/2}$ remains crucial in the proof of the completeness of the ZX-calculus for this fragment. Based upon our work and Hadzihasanovic's, Ng and Wang [26] have then introduced a complete axiomatisation of the ZX-calculus for the full quantum mechanics. Their approach consists in deriving the completeness of the ZX-calculus from the completeness of the $ZW_{\mathbb{C}}$-calculus, using a completion procedure based on the back and forth interpretations. In [22], we improved this result, showing that a single additional axiom is sufficient to make the ZX-calculus complete in general, whereas 22 new axioms together with two additional generators are used in [26]. Ng and Wang uploaded afterwards a note [27] on Arxiv providing an alternative complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics. Their axiomatisation is using two additional generators and significantly more axioms than the axiomatisation given in the present paper.

The paper is structured as follows: A ZX-Calculus augmented with two new axioms is presented in Section 2. Section 3 gives a general overview of the completeness proof. In Section 4, we introduce an extension of the ZW-Calculus that deals with matrices over dyadic rational numbers $\mathbb{D} = \mathbb{Z}[1/2]$ and show its completeness. Sections 5 and 6 are presenting a back and forth translation between the ZX- and ZW-calculi, from which we deduce the completeness of the ZX-Calculus for Clifford+T quantum mechanics in section 7. In Section 8, we characterise the exact expressive power of the $\frac{\pi}{4}$-fragment of the ZX-Calculus: the diagrams of this fragment represent exactly the matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$. In section 9, we briefly discuss the interpretation of the two new axioms of the language.

2 ZX-Calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \to l$ with k inputs and l outputs is generated by:

$R_Z^{(n,m)}(\alpha) : n \to m$		$R_X^{(n,m)}(\alpha) : n \to m$	
$H : 1 \to 1$		$e : 0 \to 0$	
$\mathbb{I} : 1 \to 1$		$\sigma : 2 \to 2$	
$\epsilon : 2 \to 0$		$\eta : 0 \to 2$	

where $n, m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and the generator e is the empty diagram.

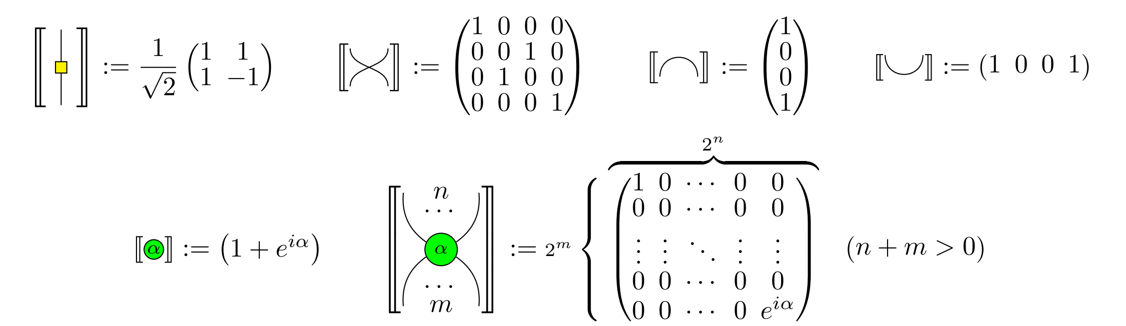
and the two compositions:

- Spacial Composition: for any $D_1 : a \to b$ and $D_2 : c \to d$, $D_1 \otimes D_2 : a + c \to b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1.
- Sequential Composition: for any $D_1 : a \to b$ and $D_2 : b \to c$, $D_2 \circ D_1 : a \to c$ consists in placing D_1 on the top of D_2, connecting the outputs of D_1 to the inputs of D_2.

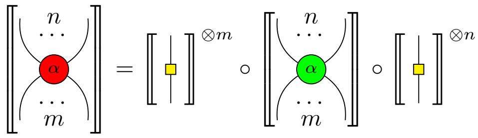
The standard interpretation of the ZX-diagrams associates to any diagram $D : n \to m$ a linear map $[\![D]\!] : \mathbb{C}^{2^n} \to \mathbb{C}^{2^m}$ inductively defined as follows:

$[\![.]\!]$

$$[\![D_1 \otimes D_2]\!] := [\![D_1]\!] \otimes [\![D_2]\!] \qquad [\![D_2 \circ D_1]\!] := [\![D_2]\!] \circ [\![D_1]\!]$$

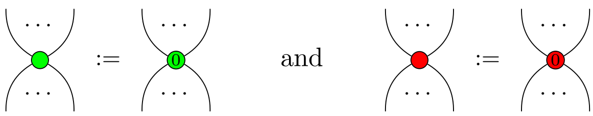


For any $n, m \geq 0$ and $\alpha \in \mathbb{R}$:



$\big($where $M^{\otimes 0} = (1)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*\big)$.

To simplify, the red and green nodes will be represented empty when holding a 0 angle:



Also in order to make the diagrams a little less heavy, when n copies of the same sub-diagram occur, we will use the notation $(.)^{\otimes n}$.

ZX-Diagrams are universal:

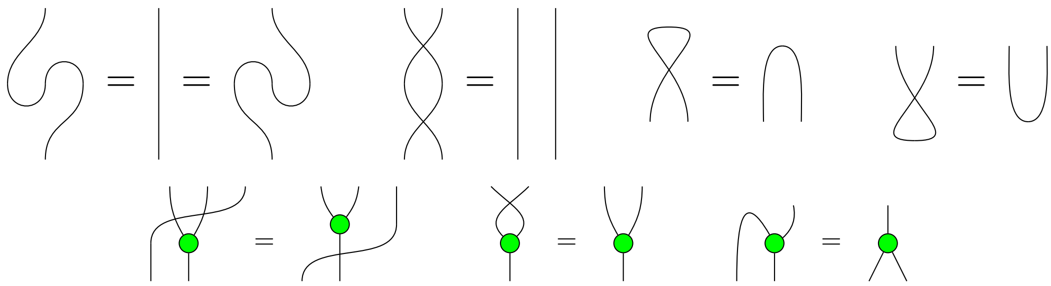
$$\forall A \in \mathbb{C}^{2^n} \times \mathbb{C}^{2^m}, \quad \exists D : n \to m, \quad [\![D]\!] = A$$

This implies dealing with an uncountable set of angles, so it is generally preferred to work with *approximate* universality – the ability to approximate any linear map with arbitrary accuracy – in which only a finite set of angles is involved. The $\frac{\pi}{4}$-fragment – ZX-diagrams where all angles are multiples of $\frac{\pi}{4}$ – is one such approximately universal fragment, whereas the $\frac{\pi}{2}$-fragment is not [1].

2.2 Calculus

The diagrammatic representation of a matrix is not unique in the ZX-Calculus. As a consequence the language comes with a set of axioms. Additionally to the axioms of the language described in Figure 1, one can:

– bend any wire of a ZX-diagram at will, without changing its semantics. This paradigm – the so-called **Only Topology Matters** – can be derived from the following axioms:



– apply the axioms to sub-diagrams. If ZX $\vdash D_1 = D_2$ then, for any diagram D with the appropriate number of inputs and outputs:

- ZX $\vdash D_1 \circ D = D_2 \circ D$
- ZX $\vdash D \circ D_1 = D \circ D_2$
- ZX $\vdash D_1 \otimes D = D_2 \otimes D$
- ZX $\vdash D \otimes D_1 = D \otimes D_2$

where ZX $\vdash D_1 = D_2$ means that D_1 can be transformed into D_2 using the axioms of the ZX-Calculus.

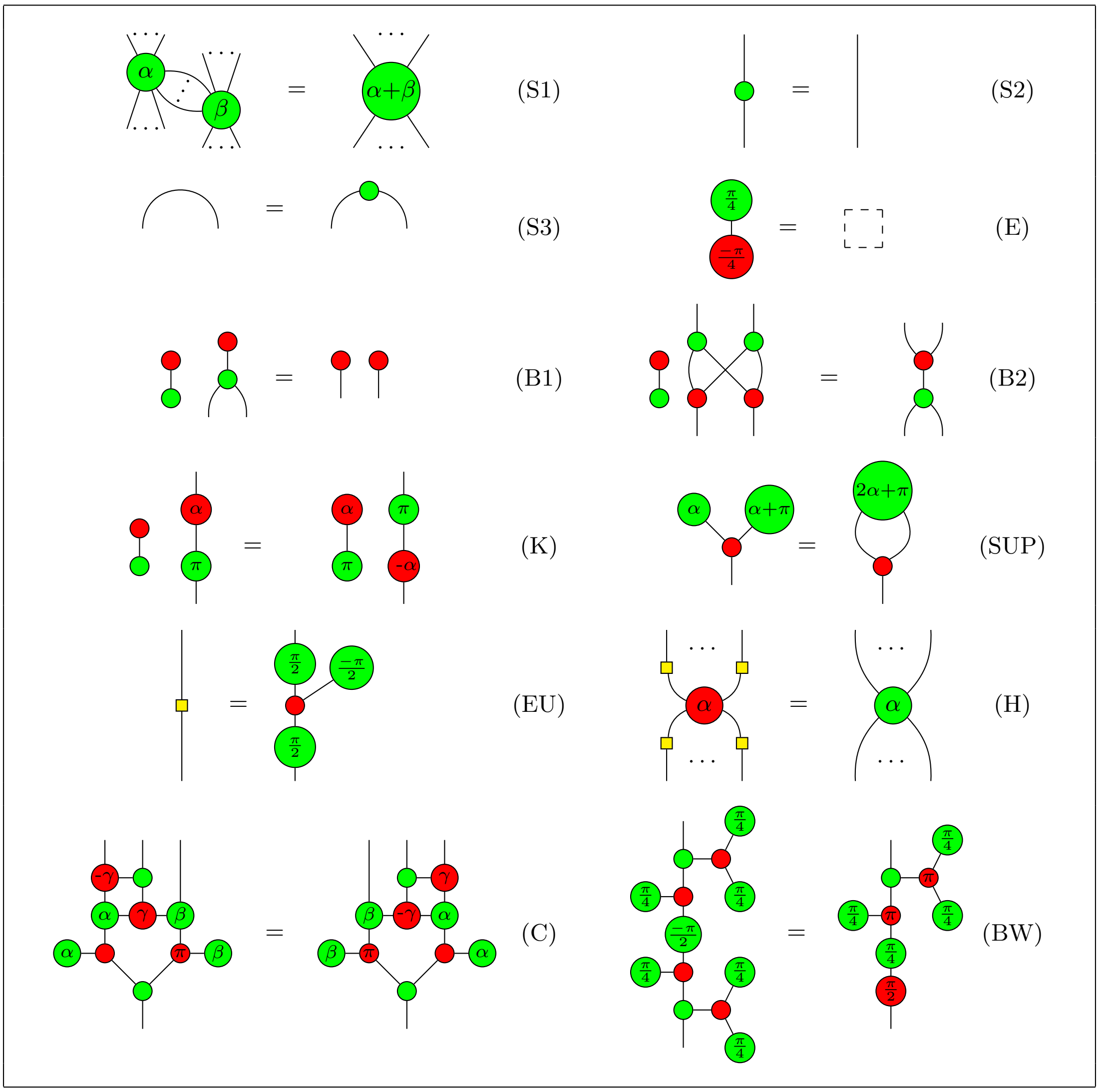


Fig. 1. Set of rules for the ZX-Calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (E) is an empty diagram. (...) denote zero or more wires, while (.·˙) denote one or more wires.

Equalities between ZX-diagrams have the two following interesting properties:

- "Colour-swapping" (exchanging red and green dots) preserves the equality.
- Multiplying all the angles by -1 preserves the equality (see Lemma 13).

In the following, $ZX_{\pi/4}$ will denote the entire $\frac{\pi}{4}$-fragment of the ZX-Calculus with the set of rules in figure 1.

2.3 What's new?

We introduce in this paper a new axiomatisation of the ZX-Calculus. We briefly review here the differences with the previous version of the ZX-Calculus. Since we are only interested in the $\frac{\pi}{4}$-fragment of the ZX-Calculus in this paper, all the axioms which are not expressible with angles multiple of $\pi/4$, like the cyclotomic supplementarity [23], are ignored. However, the rule (E), also introduced in [23], is specific to the $\frac{\pi}{4}$-fragment, and hence appears in the set of rules. The two axioms ((C), (BW)) given in Figure 1 are new axioms, for which we do not know any derivation using the previous axiomatisations of the language.

The two rules are of a different kind. On the one hand, (BW) is specific to the $\frac{\pi}{4}$-fragment, and is hardly understandable as is. On the other hand, (C) is parametrised by 3 different angles, and the rule holds whatever these angles are.

2.4 Soundness and Completeness

It's routine to prove the soundness of the axioms of the ZX-Calculus given in Figure 1. The main result of the paper is the completeness of this axiomatisation for Clifford+T quantum mechanics:

Theorem 1. *The $\frac{\pi}{4}$-fragment of the ZX-Calculus as presented in Figure 1 is complete: for any two diagrams D_1, D_2 in the $\frac{\pi}{4}$-fragment of the ZX-Calculus, $[\![D_1]\!] = [\![D_2]\!]$ iff $ZX_{\pi/4} \vdash D_1 = D_2$.*

The five following sections of the paper are devoted to the proof of the theorem. A general overview of the proof is given in the next section.

3 A Bird's Eye View of the Proof of Theorem 1

The proof uses the completeness result of the ZW-Calculus, a calculus dealing with matrices with integer coefficients. The syntax and semantics of the ZW-Calculus are presented in section 4.

We start by slightly changing the ZW-Calculus to obtain a new language, the $ZW_{1/2}$-calculus, that is able to express matrices with dyadic rational coefficients, i.e. rational numbers of the form $p/2^q$. This is done merely by adding a symbolic inverse to the scalar 2. We then prove that this new language is complete:

Part 1 (Proposition 1) *The $ZW_{1/2}$-calculus is complete: for two diagrams D_1, D_2 of the $ZW_{1/2}$-calculus, we have $[\![D_1]\!] = [\![D_2]\!]$ iff $ZW_{1/2} \vdash D_1 = D_2$.*

This is done in subsection 4.3.

We now introduce two interpretations, from $ZX_{\pi/4}$ to $ZW_{1/2}$ and back.

First, we provide an interpretation $[\![\cdot]\!]_{XW}$ from $ZX_{\pi/4}$ to $ZW_{1/2}$ that transforms $ZX_{\pi/4}$-diagrams of type $k \to l$ to $ZW_{1/2}$-diagrams of type $k+2 \to l+2$. This interpretation is sound in the following sense:

Part 2 (Proposition 5) *Let D_1, D_2 be two diagrams of the $ZX_{\pi/4}$-calculus.*
Then $[\![D_1]\!] = [\![D_2]\!]$ iff $[\![[\![D_1]\!]_{XW}]\!] = [\![[\![D_2]\!]_{XW}]\!]$.

The encoding is nontrivial as the $ZX_{\pi/4}$-Calculus expresses matrices with complex coefficients, and the $ZW_{1/2}$-calculus is only able to express matrices with dyadic rational coefficients. It turns out that coefficients involved in matrices of the $ZX_{\pi/4}$-Calculus are actually in a vector space (more accurately a module) of dimension 4 over the set of dyadic rational numbers, so that every complex coefficient will be represented by a 4×4 matrix with dyadic rational coefficients. This encoding is done in section 5.

We then provide an interpretation $[\![\cdot]\!]_{WX}$ from $\mathrm{ZW}_{1/2}$ to $\mathrm{ZX}_{\pi/4}$. This interpretation preserves both semantics and provability:

Part 3 (Proposition 6 and 7) *Let D_1, D_2 be two diagrams of the $\mathrm{ZW}_{1/2}$-calculus.*
Then $[\![[\![D_i]\!]_{WX}]\!] = [\![D_i]\!]$.
Furthermore, if $\mathrm{ZW}_{1/2} \vdash D_1 = D_2$ then $\mathrm{ZX}_{\pi/4} \vdash [\![D_1]\!]_{WX} = [\![D_2]\!]_{WX}$

This is done in section 6.

The composition of the two interpretations does not give back the initial diagram (we obtain after all a diagram with two more inputs and outputs), but we can (provably) recover it. In fact

Part 4 (Corollary 1) *Let D_1, D_2 be a diagram of the ZX-Calculus. If $\mathrm{ZX}_{\pi/4} \vdash [\![[\![D_1]\!]_{XW}]\!]_{WX} = [\![[\![D_2]\!]_{XW}]\!]_{WX}$ then $\mathrm{ZX}_{\pi/4} \vdash D_1 = D_2$.*

Our main theorem is now obvious:

Proof (Proof of Theorem 1). Let D_1, D_2 be two diagrams of the $\mathrm{ZX}_{\pi/4}$-Calculus s.t. $[\![D_1]\!] = [\![D_2]\!]$.
By Part 2, $[\![[\![D_1]\!]_{XW}]\!] = [\![[\![D_2]\!]_{XW}]\!]$.
By Part 1, the $\mathrm{ZW}_{1/2}$-calculus is complete and therefore $\mathrm{ZW}_{1/2} \vdash [\![D_1]\!]_{XW} = [\![D_2]\!]_{XW}$.
By Part 3, $\mathrm{ZX}_{\pi/4} \vdash [\![[\![D_1]\!]_{XW}]\!]_{WX} = [\![[\![D_2]\!]_{XW}]\!]_{WX}$.
By Part 4 this implies $\mathrm{ZX}_{\pi/4} \vdash D_1 = D_2$. □

This approach gives a completion procedure. It gives a set of equalities between $\mathrm{ZX}_{\pi/4}$-diagrams whose derivability proves the completeness of the language. Hence, the new rules of the $\mathrm{ZX}_{\pi/4}$-Calculus we introduced have obviously been chosen for Parts 4 and 3 to hold. However they have been greatly simplified from what one can obtain using the approach naively.

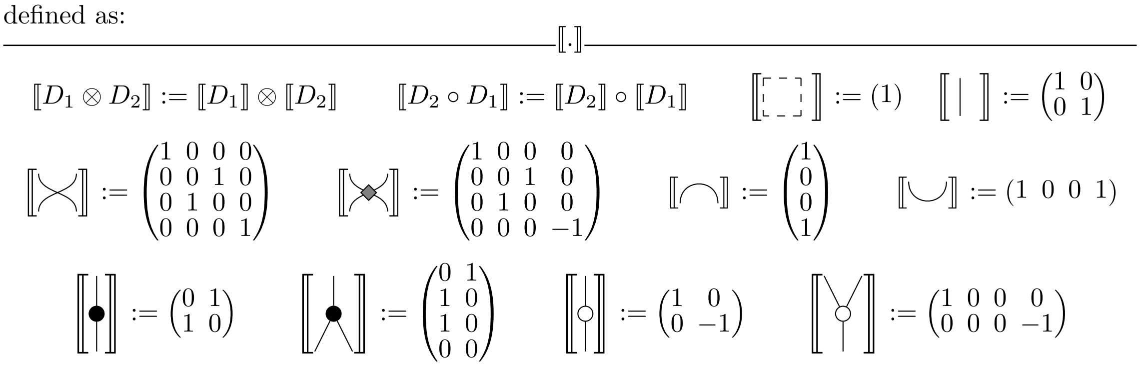
4 ZW-Calculus

4.1 Diagrams and Standard Interpretation

The ZW-Calculus has been introduced by Amar Hadzihasanovic in 2015 [19] and is based on the GHZ/W-Calculus [10]. We will present here the expanded version of this calculus. To stay coherent with the previous definition of the ZX-Calculus, we will assume that the time flows from top to bottom – which is the opposite of the original definition in the ZW-Calculus. It has the following finite set of generators:

and diagrams are created thanks to the same two – spacial and sequential – compositions.

As for the ZX-Calculus, we define a standard interpretation, that associates to any diagram of the ZW-Calculus D with n inputs and m outputs, a linear map $[\![D]\!] : \mathbb{Z}^{2^n} \to \mathbb{Z}^{2^m}$, inductively defined as:



This map is obviously different from the one of the ZX-Calculus– the domain is different – but we will use the same notation.

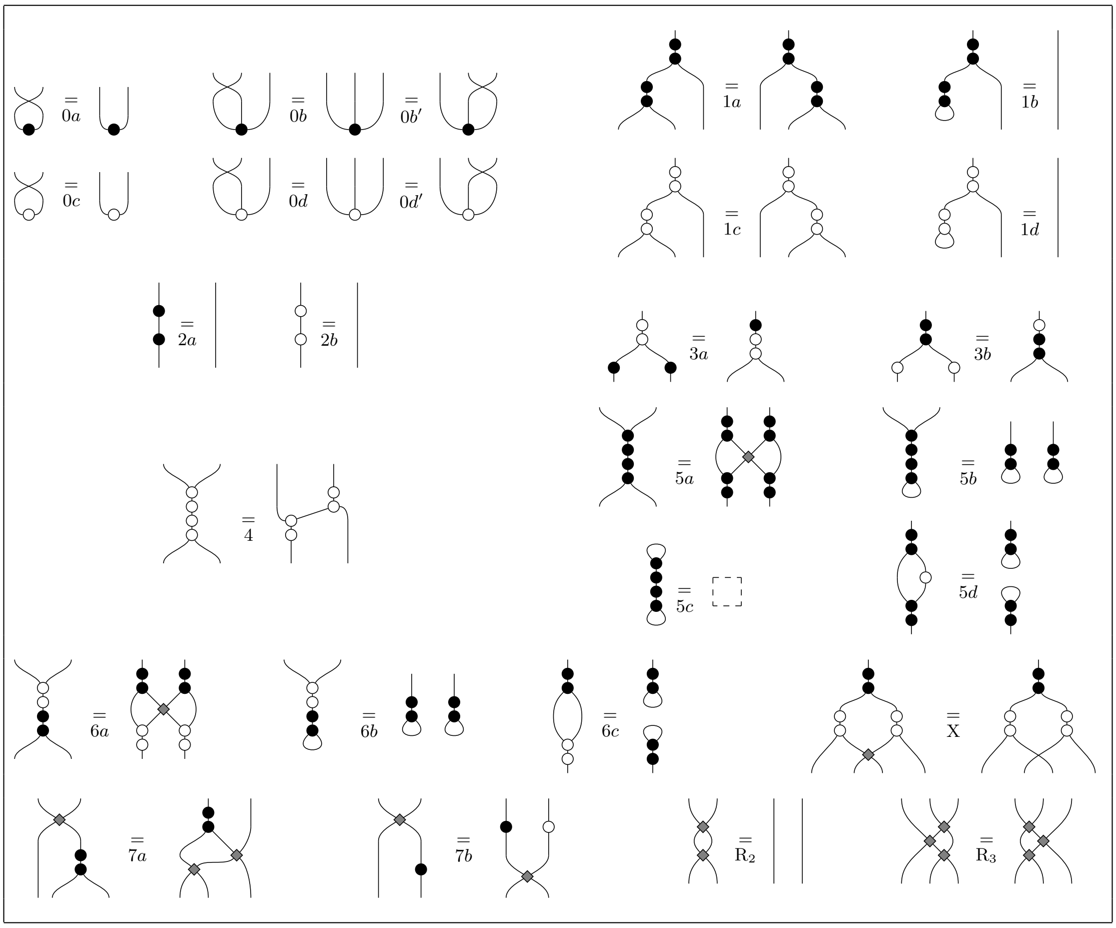


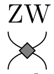
Fig. 2. Set of rules for the ZW-Calculus.

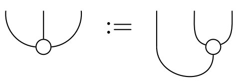
Remark 1. The symbols used for the generators have be altered from the original ZW-Calculus in order to make it more compatible with the ZX-Calculus.

Lemma 1 **([19]).** *ZW-Diagrams are universal for matrices of* $\mathbb{Z}^{2^n} \times \mathbb{Z}^{2^m}$*:*

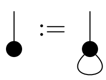
$$\forall A \in \mathbb{Z}^{2^n} \times \mathbb{Z}^{2^m}, \quad \exists D : n \to m, \quad [\![D]\!] = A$$

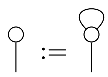
4.2 Calculus

The ZW-Calculus comes with a *complete* set of rules ZW that is given in figure 2. Here again, the paradigm *Only Topology Matters* applies except for  where the order of inputs and outputs is important. It gives sense to nodes that are not directly given in T_e, e.g.:



All these rules are sound. We use the same notation $\vdash$ as defined in section 2, and we can still apply the rewrite rules to subdiagrams. In the following we may use the shortcuts:



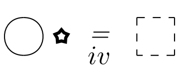
and


4.3 Extension to Dyadic Matrices

We define an extension of the ZW-Calculus by adding a new node that represents $\frac{1}{2}$ and binding it to the calculus with an additional rule.

Definition 1. *We define the $\mathrm{ZW}_{1/2}$-Calculus as the extension of the ZW-Calculus such as:*

$$\begin{cases} T_{1/2} = T_e \cup \{✿\} \\ \mathrm{ZW}_{1/2} = \mathrm{ZW} \cup \left\{ \quad \right\} \end{cases}$$



The standard interpretation of a diagram $D : n \to m$ is now a matrix $[\![D]\!] : \mathbb{D}^{2^n} \to \mathbb{D}^{2^m}$ over the ring $\mathbb{D} = \mathbb{Z}[1/2]$ of dyadic rationals and is given by the standard interpretation of the ZW-Calculus extended with $[\![✿]\!] := \left(\frac{1}{2}\right)$.

Proposition 1. *The* $\mathrm{ZW}_{1/2}$ *is sound and complete: For two diagrams D_1, D_2 of the* $\mathrm{ZW}_{1/2}$*-calculus, $[\![D_1]\!] = [\![D_2]\!]$ iff* $\mathrm{ZW}_{1/2} \vdash D_1 = D_2$.

Proof. Soundness is obvious.

Now let D_1 and D_2 be two diagrams of the $\mathrm{ZW}_{1/2}$-Calculus such that $[\![D_1]\!] = [\![D_2]\!]$. We can rewrite D_1 and D_2 as $D_i = d_i \otimes (✿)^{\otimes n_i}$ for some integers n_i and diagrams d_i of the ZW-Calculus that do not use the ✿ symbol.

From the new introduced rule, we get that $\mathrm{ZW}_{1/2} \vdash d_i = D_i \otimes \left(\bigcirc\right)^{\otimes n_i}$. W.l.o.g. assume $n_1 \leq n_2$. Then $\left[\!\!\left[d_1 \otimes \left(\bigcirc\right)^{\otimes n_2 - n_1} \right]\!\!\right] = 2^{n_2 - n_1} [\![d_1]\!] = 2^{n_2} [\![D_1]\!] = [\![d_2]\!]$. Since d_1 and d_2 are ZW-diagrams and have the same interpretation, thanks to the completeness of the ZW-Calculus, $\mathrm{ZW}_{1/2} \vdash d_1 \otimes \left(\bigcirc\right)^{\otimes n_2 - n_1} = d_2$, which means $\mathrm{ZW}_{1/2} \vdash D_1 = D_2$ by applying n_2 times the new rule on both sides of the equality. □

5 From $\mathrm{ZX}_{\pi/4}$ to $\mathrm{ZW}_{1/2}$-Diagrams

In this section we explain how to encode diagrams of the $\mathrm{ZX}_{\pi/4}$-Calculus into diagrams of the $\mathrm{ZW}_{1/2}$-Calculus. The main difficulty is of course that the former represents matrices with complex coefficients and the latter matrices with dyadic rational coefficients. We use for this classical results of algebra that we summarize in the next subsection.

5.1 From $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ to $\mathbb{Q}$

All results used in the next two sections are standard in field theory, see e.g. [29]. Let $R \subseteq \mathbb{C}$ be a (commutative) ring and $\alpha \in \mathbb{C}$. By $R[\alpha]$ we denote the smallest subring of $\mathbb{C}$ that contains both R and α.

Of primary importance will be the ring $\mathbb{Q}[e^{i\frac{\pi}{4}}]$, as all terms of the $\pi/4$ fragment of the ZX-Calculus have interpretations as matrices in this ring. This is clear for all terms except possibly for $\sqrt{2}$, but $\sqrt{2} = e^{i\frac{\pi}{4}} - (e^{i\frac{\pi}{4}})^3$.

If α is algebraic, it is well known that $\mathbb{Q}[\alpha]$ is a field. When $F \subseteq F'$ are two fields, F' can be seen as a vector space (actually an algebra) over F. Its dimension is denoted $[F' : F]$ and we say that F' is an extension of F of degree $[F' : F]$. In the specific case of $\mathbb{Q}[\alpha]$, its dimension over $\mathbb{Q}$ is exactly the degree of the minimal polynomial over $\mathbb{Q}$ of α. Notice that the minimal polynomial of a $n-$th primitive root of the unity is $\phi(n)$ where ϕ is Euler's totient function.

In our case, $e^{i\frac{\pi}{4}}$ is a eighth primitive root of the unity, so that $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ is a vector space of dimension 4, one basis being given by $1, e^{i\frac{\pi}{4}}, (e^{i\frac{\pi}{4}})^2, (e^{i\frac{\pi}{4}})^3$. In particular:

Proposition 2. *Every element of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some rationals numbers a, b, c, d.*

For $x \in \mathbb{Q}[e^{i\frac{\pi}{4}}]$, let $\psi(x)$ be the function defined by $\psi(x) = y \mapsto xy$. For each x, $\psi(x)$ is a linear map and therefore can be given by a 4×4 matrix in the basis $(e^{i\frac{\pi}{4}})^3, (e^{i\frac{\pi}{4}})^2, e^{i\frac{\pi}{4}}, 1$. $\psi(1)$ is of course the identity matrix and

$$\psi(e^{i\frac{\pi}{4}}) = M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Notice that M^t is the companion matrix of the polynomial X^4+1 which characterises $e^{i\frac{\pi}{4}}$ as an algebraic number.

Proposition 3. *The map:*

$$\psi : a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3 \mapsto aI_4 + bM + cM^2 + dM^3$$

is a homomorphism of $\mathbb{Q}$-algebras from $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ to $M_4(\mathbb{Q})$

This homomorphism has a left-inverse. Indeed, let

$$\theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix}$$

Then $\psi(x)\theta = x\theta$.

With this morphism, we can see elements of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ as matrices over $\mathbb{Q}$.

Of course we can do the same with matrices over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$.

Definition 2. *Define:*

$$\psi : A + Be^{i\frac{\pi}{4}} + C(e^{i\frac{\pi}{4}})^2 + D(e^{i\frac{\pi}{4}})^3 \mapsto A\otimes I_4 + B\otimes M + C\otimes M^2 + D\otimes M^3$$

ψ is injective and maps a matrix over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ of dimension $n\times m$ to a matrix over $\mathbb{Q}$ of dimension $4n\times 4m$.

We use the same notation ψ as before, as the definitions are equivalent for one-by-one matrices (i.e. scalars).

It is easy to see that Proposition 3 holds for the extended ψ in the sense that $\psi(qA) = q\psi(A)$ for q rational, $\psi(A+B) = \psi(A)+\psi(B)$, $\psi(AB) = \psi(A)\psi(B)$ whenever this makes sense.

Notice however that $\psi(A\otimes B)$ is not $\psi(A)\otimes\psi(B)$.

As before, ψ has a left inverse, as evidenced by:

Proposition 4. *For all matrices X of dimension $n\times m$, $\psi(X)(I_m\otimes\theta) = X\otimes\theta$*

While it is true that all coefficients of the standard interpretation of the $\pi/4$ fragment are in $\mathbb{Q}[e^{i\frac{\pi}{4}}]$, we can be more precise.

Let $\mathbb{D} = \mathbb{Z}[1/2]$ be the set of all dyadic rational numbers, i.e. rational numbers of the form $p/2^n$.

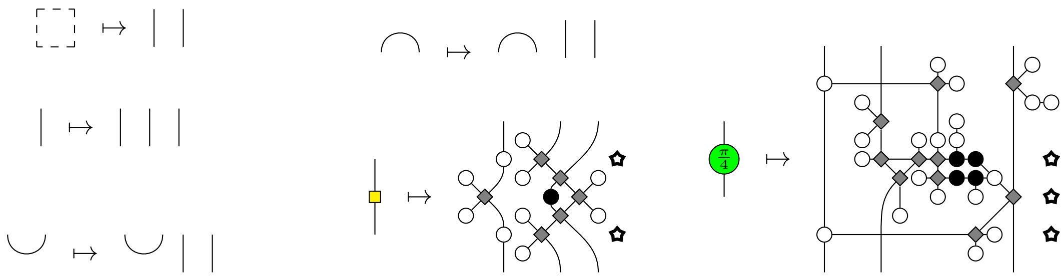
It is easy to see that any element of $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some dyadic rational numbers a, b, c, d. (It is NOT a consequence of the similar statement for $\mathbb{Q}$. We have to use here the additional property that $e^{i\frac{\pi}{4}}$ is not only an algebraic number, but also an algebraic *integer*).

Then it is clear that actually all coefficients of the $\pi/4$ fragment of the ZX-Calculus are in $\mathbb{D}[e^{i\frac{\pi}{4}}]$. As $\mathbb{D}\subset\mathbb{Q}$ all we said before still holds, and we actually obtain with ψ a map from matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ to matrices over $\mathbb{D}$.

5.2 Interpretation

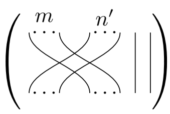
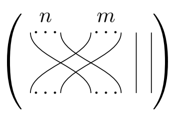
Based on the previous discussion, we define an interpretation $[\![.]\!]_{XW}$ from ZX$_{\pi/4}$-diagrams to ZW$_{1/2}$-diagrams as follows:

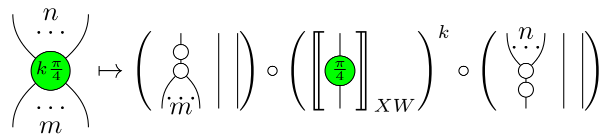
$[\![\cdot]\!]_{XW}$

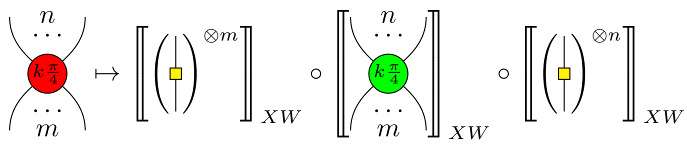


$\forall D_1 : n \to n',\ \forall D_2 : m \to m' :$

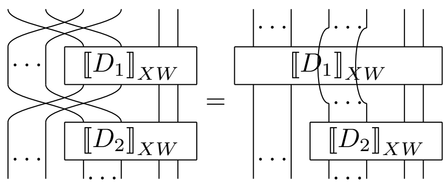
$D_1 \circ D_2 \mapsto [\![D_1]\!]_{XW} \circ [\![D_2]\!]_{XW} \qquad (\text{if } m' = n)$

$D_1 \otimes D_2 \mapsto \left(\mathbb{I}^{\otimes n'} \otimes [\![D_2]\!]_{XW}\right) \circ$  $\circ \left(\mathbb{I}^{\otimes m} \otimes [\![D_1]\!]_{XW}\right) \circ$ 

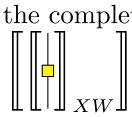
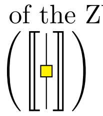
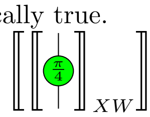
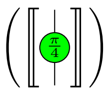




The interpretation of the spacial composition $\otimes$ might seem a tad cryptical. It is in fact a way of putting "side-by-side" the interpretations of D_1 and D_2, while at the same time making them share the two control wires. We can see it as:



It order for this to make sense, D_1 and D_2 should be able to commute on the control wires. This property is provided by the completeness of the ZW-Calculus, since it is semantically true.

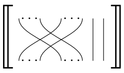
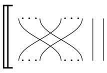
One can check that  $= \psi$  $= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \otimes (M - M^3)$ and  $=$ ψ  $= \begin{pmatrix} I_4 & 0 \\ 0 & M\end{pmatrix}$. More generally:

Proposition 5. *Let D be a diagram of the ZX$_{\pi/4}$-Calculus. Then*

$$[\![[\![D]\!]_{XW}]\!] = \psi([\![D]\!])$$

In particular, if $[\![D_1]\!] = [\![D_2]\!]$ then $[\![[\![D_1]\!]_{XW}]\!] = [\![[\![D_2]\!]_{XW}]\!]$

The proof is a straightforward induction using the fact that ψ is an homomorphism. Slight care has to be taken to treat the case of $D_1 \otimes D_2$:
Suppose $[\![D_1]\!] = \sum_{k=0}^{3} A_k e^{i\frac{k\pi}{4}}$ and $[\![D_2]\!] = \sum_{k=0}^{3} B_k e^{i\frac{k\pi}{4}}$ are their *unique* decomposition, and that $[\![[\![D_i]\!]_{XW}]\!] = \psi([\![D_i]\!])$. Then:

$$[\![[\![D_1 \otimes D_2]\!]_{XW}]\!] = (I \otimes \psi([\![D_1]\!])) \circ \left[\!\!\left[\right.\right.$$

$$\left.\left.\right]\!\!\right] \circ \left(I \otimes \sum_{k=0}^{3} A_k \otimes M^k\right) \circ \left[\!\!\left[\right.\right.$$

$$\left.\left.\right]\!\!\right]$$

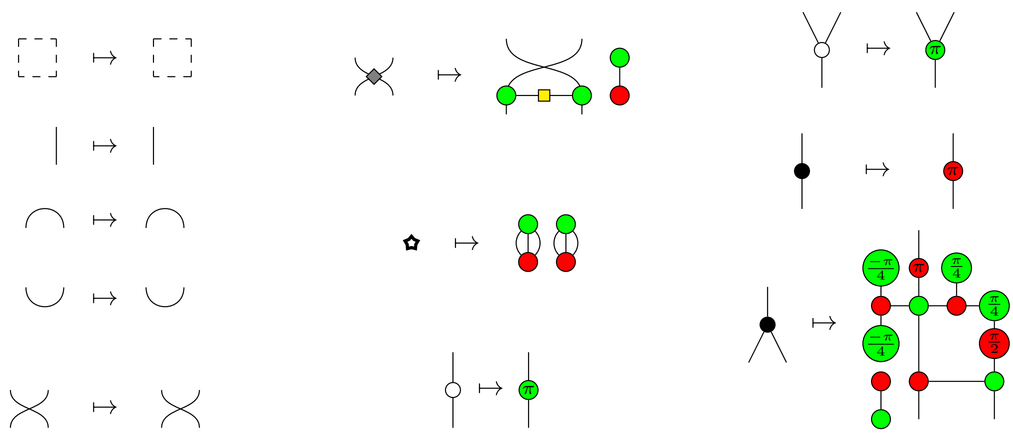
$$= \left(\sum_{l=0}^{3} I \otimes B_l \otimes M^l\right) \circ \left(\sum_{k=0}^{3} A_k \otimes I \otimes M^k\right) = \sum_{k,l} A_k \otimes B_l \otimes M^{k+l}$$

$$= \psi\left(\sum_{k,l} (A_k \otimes B_l) e^{i\frac{(k+l)\pi}{4}}\right) = \psi([\![D_1 \otimes D_2]\!])$$

6 From $\mathrm{ZW}_{1/2}$ to $\mathrm{ZX}_{\pi/4}$-Diagrams

We define here an interpretation $[\![.]\!]_{WX}$ that transforms any diagram of the $\mathrm{ZW}_{1/2}$-Calculus into a $\mathrm{ZX}_{\pi/4}$-diagram, which is easy to do since $\mathbb{D} \subset \mathbb{D}[e^{i\frac{\pi}{4}}]$:

$[\![\cdot]\!]_{WX}$



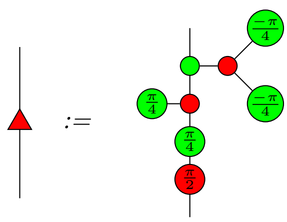
$$D_1 \circ D_2 \mapsto [\![D_1]\!]_{WX} \circ [\![D_2]\!]_{WX} \qquad D_1 \otimes D_2 \mapsto [\![D_1]\!]_{WX} \otimes [\![D_2]\!]_{WX}$$

Proposition 6. *Let D be a diagram of the $\mathrm{ZW}_{1/2}$ calculus. Then $[\![[\![D]\!]_{WX}]\!] = [\![D]\!]$*

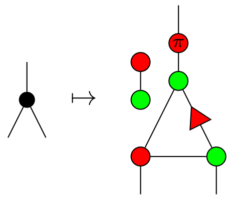
The proof is by induction on D.

This interpretation $[\![.]\!]_{WX}$ from the ZW-Calculus to the ZX-Calculus is pretty straightforward, except for the three-legged black node. This is where some syntactic sugar can come in handy.

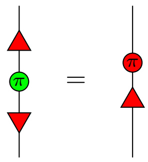
Definition 3. *We define the "triangle node" as:*



One can check that $\left[\!\left[\right.\right.$$\left.\left. \right]\!\right] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the interpretation of the three-legged black dot is simplified:



as is the rule (BW) (see Lemma 31):



(BW')

This shortcut will be very useful in the technical proof of the completeness of the language for Clifford+T with the set of rules in Figure 1.

Proposition 7. *The interpretation* $[\![.]\!]_{WX}$ *preserves all the rules of the* $\mathrm{ZW}_{1/2}$*-Calculus:*

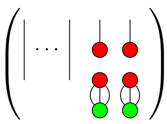
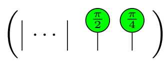
$$\mathrm{ZW}_{1/2} \vdash D_1 = D_2 \qquad \Longrightarrow \qquad \mathrm{ZX}_{\pi/4} \vdash [\![D_1]\!]_{WX} = [\![D_2]\!]_{WX}$$

The proof is in appendix at Section A.3.

7 Completeness of the $\frac{\pi}{4}$-fragment of the ZX-Calculus

To finish the proof it remains to compose the two interpretations:

Proposition 8. *We can retrieve any* $ZX_{\pi/4}$*-diagram from its image under the composition of the two interpretations:*

$$\forall D \in \mathrm{ZX}_{\pi/4}, \quad \mathrm{ZX}_{\pi/4} \vdash D =$$

$$\circ\, [\![[\![D]\!]_{XW}]\!]_{WX} \circ$$


The proof is in appendix at Section *viii)*.

Corollary 1. *If* $\mathrm{ZX}_{\pi/4} \vdash [\![[\![D_1]\!]_{XW}]\!]_{WX} = [\![[\![D_2]\!]_{XW}]\!]_{WX}$ *then* $\mathrm{ZX}_{\pi/4} \vdash D_1 = D_2$.

8 Expressive power of the $\mathrm{ZX}_{\pi/4}$-diagrams

The ZW-Calculus is complete, and additionally any integer matrix can be represented in the ZW-Calculus [19]. A similar result follows immediately for the $\mathrm{ZW}_{1/2}$-calculus.

Proposition 9. $\mathrm{ZW}_{1/2}$*-Diagrams are universal for matrices of* $\mathbb{D}^{2^n} \times \mathbb{D}^{2^m}$*:*

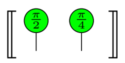
$$\forall A \in \mathbb{D}^{2^n} \times \mathbb{D}^{2^m}, \quad \exists D \in \mathrm{ZW}_{1/2}, \quad [\![D]\!] = A$$

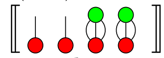
Regarding the expressive power of $\mathrm{ZX}_{\pi/4}$-diagrams, since the unitary matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ are representable with Clifford+T circuits [18], so are they with $\mathrm{ZX}_{\pi/4}$-diagrams. We actually show that any matrix over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be represented by a $\mathrm{ZX}_{\pi/4}$-diagram:

Proposition 10. *The* $\frac{\pi}{4}$*-fragment of the ZX-Calculus represents exactly matrices over* $\mathbb{D}[e^{i\frac{\pi}{4}}]$*:*

$$\forall A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}, \quad \exists D \in \mathrm{ZX}_{\pi/4}, \quad [\![D]\!] = A$$

Proof. Let $A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}$. We define $A' = \psi(A) \in \mathbb{D}^{2^{n+2} \times 2^{m+2}}$. Since $\mathrm{ZW}_{1/2}$-diagrams are universal for matrices over dyadic rationals: $\exists D \in \mathrm{ZW}_{1/2},\ \llbracket D \rrbracket = A'$. Since $\llbracket . \rrbracket_{WX}$ preserves the semantics, we can define a ZX-diagram of the $\frac{\pi}{4}$-fragment $D' = \llbracket D \rrbracket_{WX}$ such that $\llbracket D' \rrbracket = A'$.

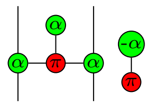
Now, notice that $\theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix} =$  ,

and $e_1 = (1\ \ 0\ \ 0\ \ 0) =$ , so if we apply the second diagram at the two bottom right wires, and the first state on the two top right wires of D', we end up with D'' such that $\llbracket D'' \rrbracket = A$. Indeed:

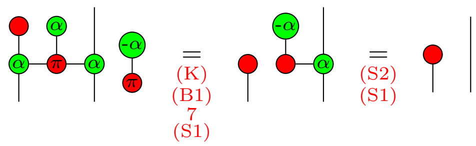
$\llbracket D'' \rrbracket = (I \otimes e_1) \circ \llbracket D' \rrbracket \circ (I \otimes \theta) = (I \otimes e_1) \circ \psi(A) \circ (I \otimes \theta) = (I \otimes e_1) \circ (A \otimes \theta) = A \otimes (e_1 \circ \theta) = A$ □

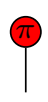
9 Discussion on the New Rules

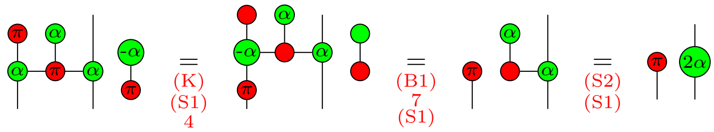
We can try and give an explanation of the rule (C), in terms of commutation of controlled operations. Consider the following diagram:



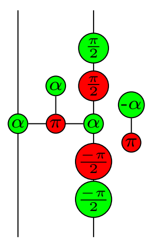
It is a *controlled operation*. Indeed, if  is plugged on the left wire, we obtain the identity on the right one:

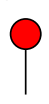
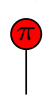


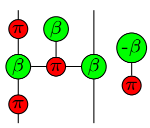
and if  is plugged:



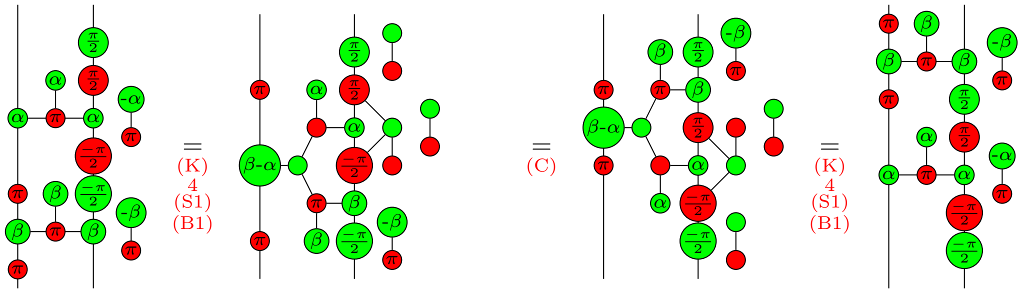
Hence, the diagram is actually the controlled Z-rotation $\Lambda R_Z(2\alpha)$. Similarly, one can show that the following diagram represents the controlled X-rotation $\Lambda R_X(2\alpha)$:



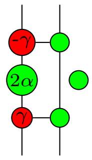
A controlled operation ΛU can be turned into an *"anti-controlled" operation* $\overline{\Lambda} U$ (where the role of  and  are exchanged) by adding two red π-nodes on the first wire, one on the input and one on the output. For instance, $\overline{\Lambda} R_Z(2\beta)$ is expressed as:



Controlled and anti-controlled operations obviously commute: for any operations U and V of the same size, $\Lambda U \circ \overline{\Lambda} V = \overline{\Lambda} V \circ \Lambda U$. To derive this result in the ZX-calculus for $\Lambda R_X(2\alpha)$ and $\overline{\Lambda} R_Z(2\beta)$ the use of rule (C) seems to be required:

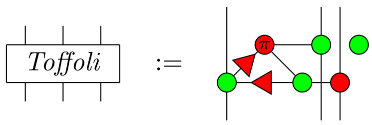


The rule (C) itself actually goes a step further and directly expresses the equality $\Lambda U(\alpha,\gamma) \circ \overline{\Lambda}(R_Z(2\beta) \otimes \mathbb{I}) = \overline{\Lambda}(R_Z(2\beta) \otimes \mathbb{I}) \circ \Lambda U(\alpha,\gamma)$ for

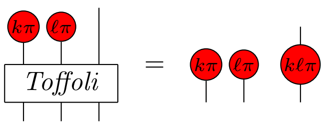
$U(\alpha,\gamma) =$ 

$\Lambda U(\alpha,\gamma)$ is a control of a 2-qubits operation, such as the Toffoli gate, which controls CNOT. Because the operation it controls (CNOT) is also a controlled operation, it can also be seen as an operation on one wire, controlled by two others.

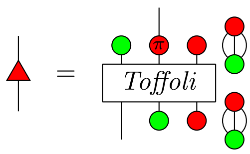
Since the Toffoli gate is represented by a matrix over $\mathbb{Z}$, it is possible to express it in the $\frac{\pi}{4}$-fragment of the ZX-Calculus. Indeed, using the  -notation:

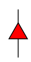


One can check that this diagram actually represents the Toffoli gate, for instance by deriving, for any $k, \ell \in \{0,1\}$:



More surprisingly, by plugging particular states, operation and projectors on the Toffoli gate, one can recover the triangle node:



This shows a connection between the diagram denoted  and the Toffoli gate. As pointed out in section 6, the rule (BW) can be greatly simplified when using the  -notation. Hence, (BW) might help to find rules for a potential axiomatisation of the quantum circuits, or in fact any graphical language that contains the Toffoli gate.

We leave open the question of minimality of the axiomatisation in Figure 1. Many of the previous axioms were proven to be necessary (not derivable from the others) [6,28], but some proofs may not hold with the addition of the two new axioms, and there is currently no known proof of the necessity of (C) and (BW).

Acknowledgements

The authors acknowledge support from the projects ANR-17-CE25-0009 SoftQPro, ANR-17-CE24-0035 VanQuTe, PIA-GDN/Quantex, and STIC-AmSud 16-STIC-05 FoQCoSS. All diagrams were written with the help of TikZit.

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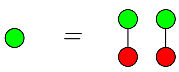
A Appendix

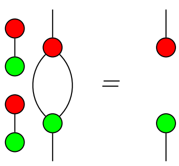
In this appendix (A.3, *viii*)) are the proofs of Propositions 7 and 8. To simplify the following work, we use the new node introduced as a notation in Section 6, and give a few lemmas in Section A.1, and prove them in Section A.2. Keep in mind that for any provable equation, its upside down version, its colour-swapped version, and (after Lemma 13) its version with opposed angles are all provable.

A.1 Lemmas

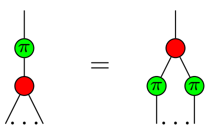
Lemma 2.



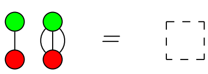
Lemma 3.



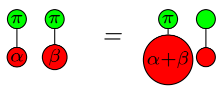
Lemma 4.



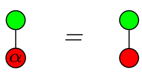
Lemma 5.



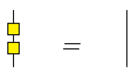
Lemma 6.



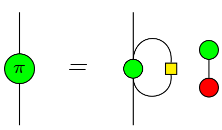
Lemma 7.



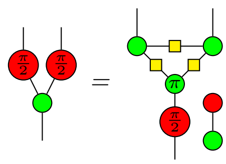
Lemma 8.



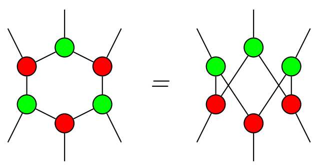
Lemma 9.



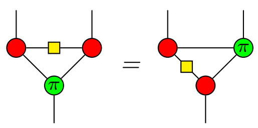
Lemma 10.



Lemma 11.



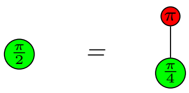
Lemma 12.



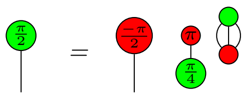
Lemma 13. *Let* $[\![.]\!]_{-1}$ *be the interpretation that multiplies all the angles by* -1*. Then:*

$$\mathrm{ZX} \vdash D_1 = D_2 \iff \mathrm{ZX} \vdash [\![D_1]\!]_{-1} = [\![D_2]\!]_{-1}$$

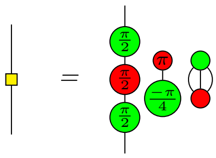
Lemma 14.



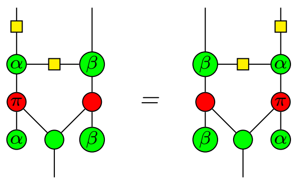
Lemma 15.



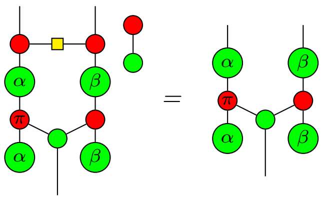
Lemma 16.



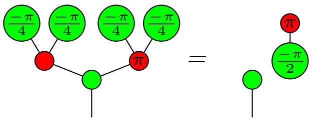
Lemma 17.



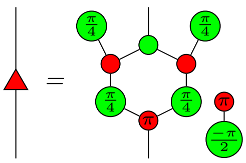
Lemma 18.



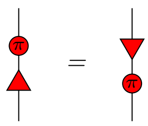
Lemma 19.



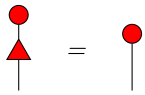
Lemma 20.



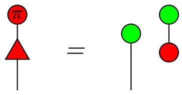
Lemma 21.



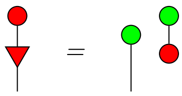
Lemma 22.



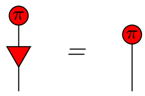
Lemma 23.



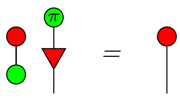
Lemma 24.



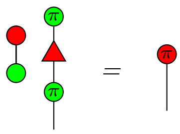
Lemma 25.



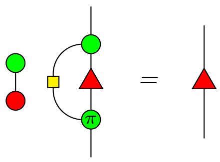
Lemma 26.



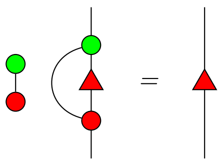
Lemma 27.



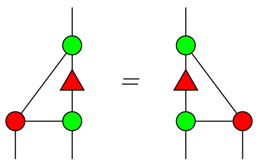
Lemma 28.



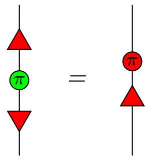
Lemma 29.



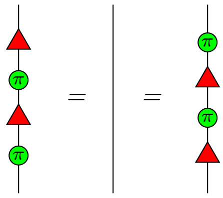
Lemma 30.



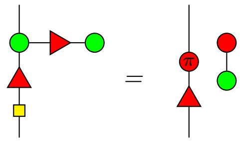
Lemma 31.



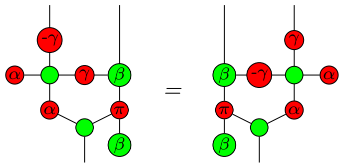
Lemma 32.



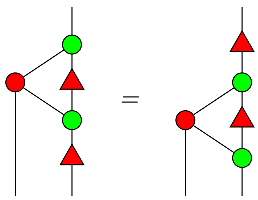
Lemma 33.



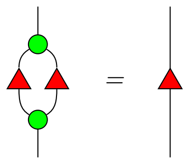
Lemma 34.



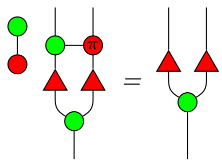
Lemma 35.



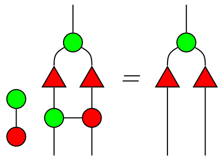
Lemma 36.



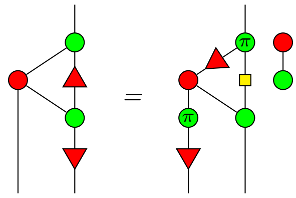
Lemma 37.



and



Lemma 38.

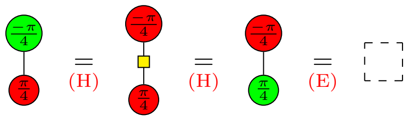


A.2 Proof of Lemmas

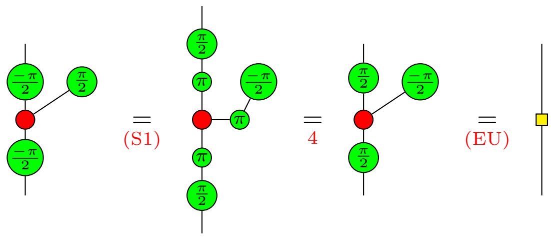
Proof (Lemmas 2 to 12).
Lemmas 6 and 7 are proven in [5,23]. The other lemmas are in the $\frac{\pi}{2}$-fragment and hence are derivable by completeness of this fragment. □

Proof (Lemma 13). The result is quite obvious for all rules except maybe for (E), (EU) and (BW).

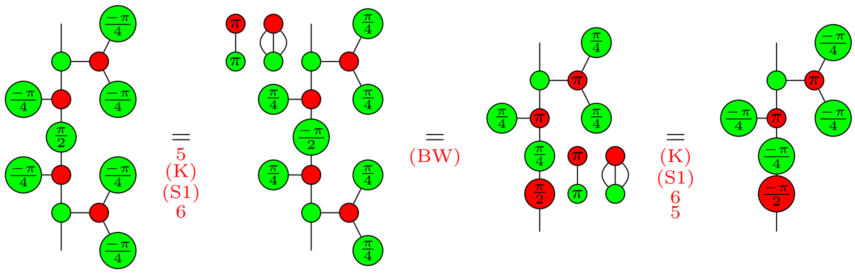
– (E):

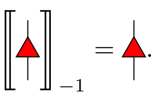


– (EU):

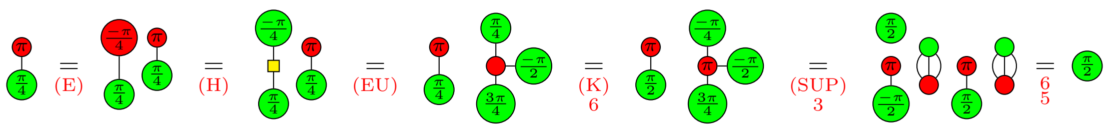


– (BW):



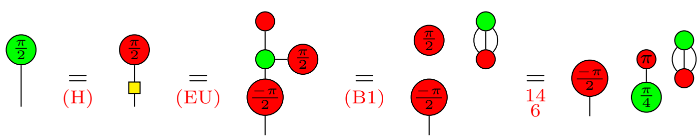
Moreover, it is to be noticed that  □

Proof (Lemma 14).



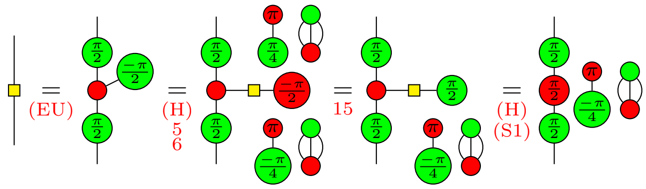
□

Proof (Lemma 15).



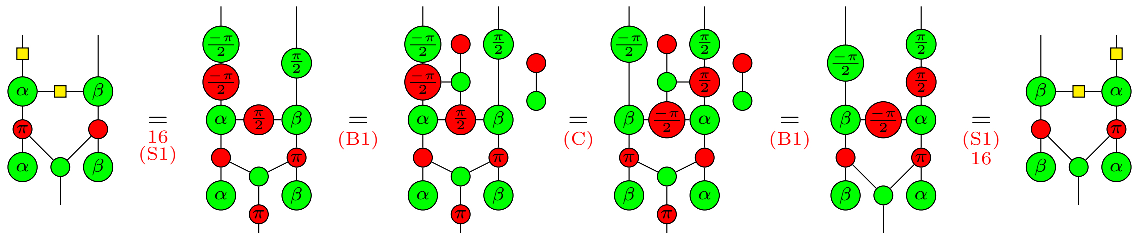
□

Proof (Lemma 16).



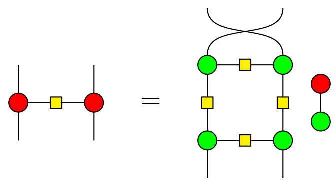
□

Proof (Lemma 17).

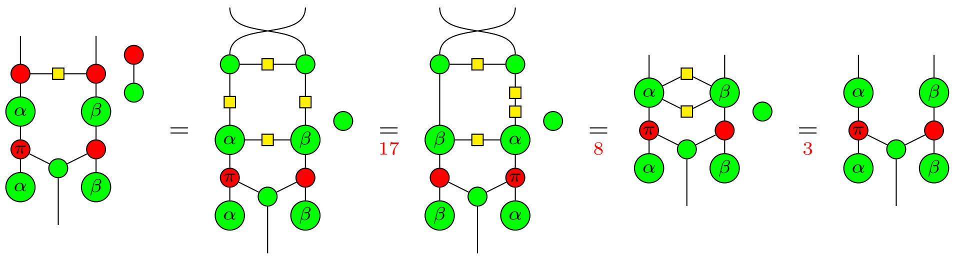


□

Proof (Lemma 18). By completeness of the $\frac{\pi}{2}$-fragment:

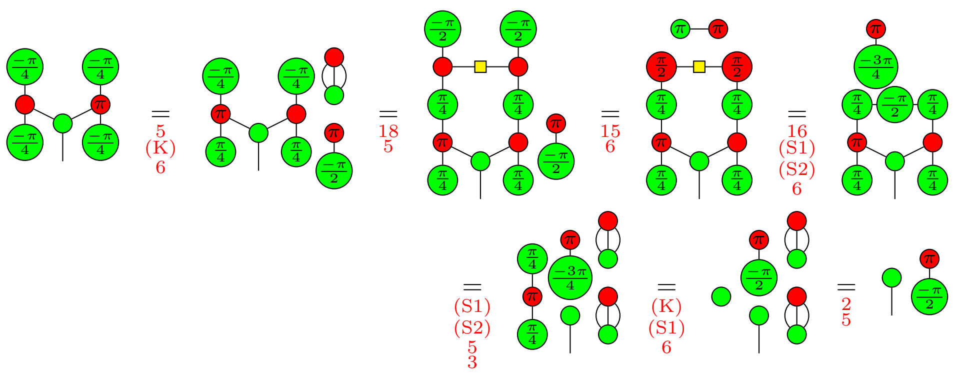


Then:



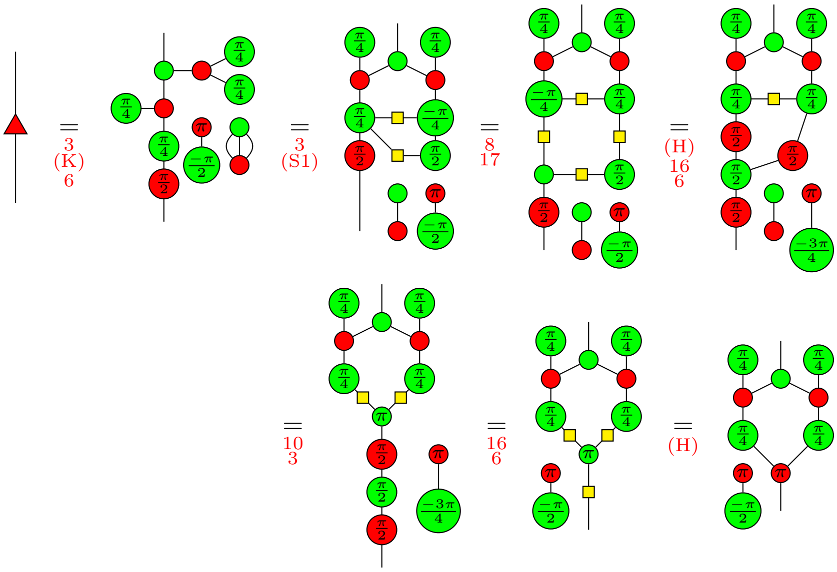
□

Proof (Lemma 19).



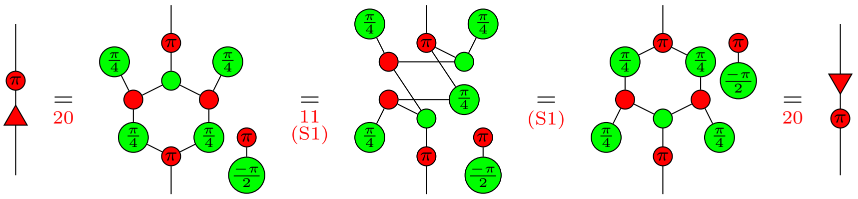
□

Proof (Lemma 20).



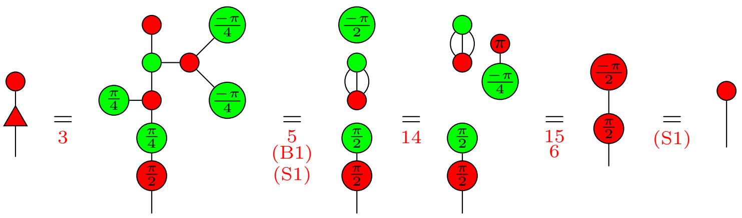
□

Proof (Lemma 21).



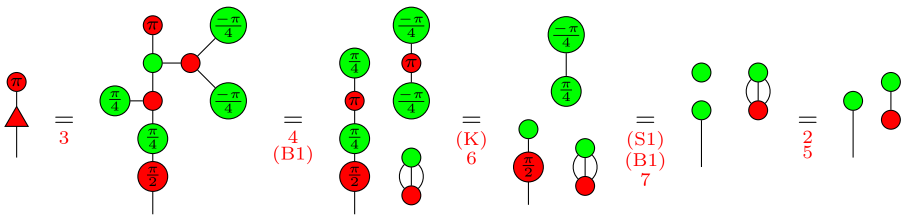
□

Proof (Lemma 22).



□

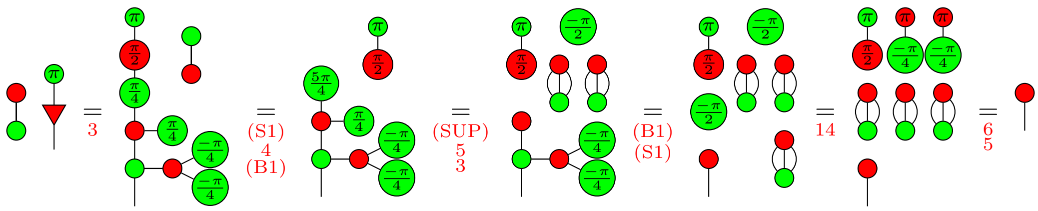
Proof (Lemma 23).



□

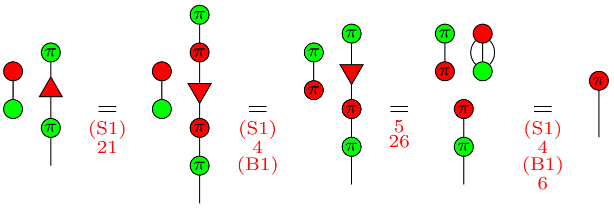
Proof (Lemmas 24 and 25). The result comes naturally from 21, 23 and 22. □

Proof (Lemma 26).



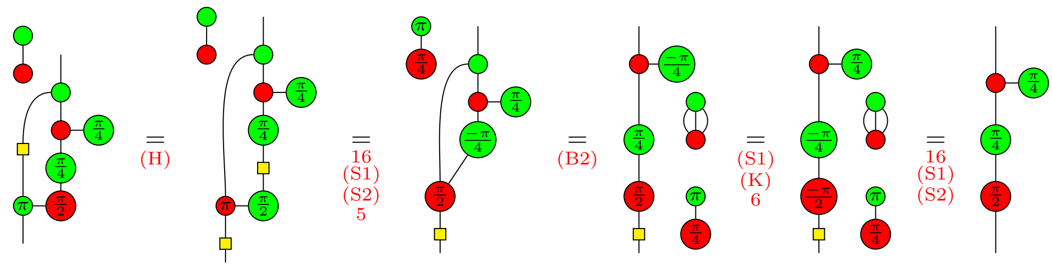
□

Proof (Lemma 27).

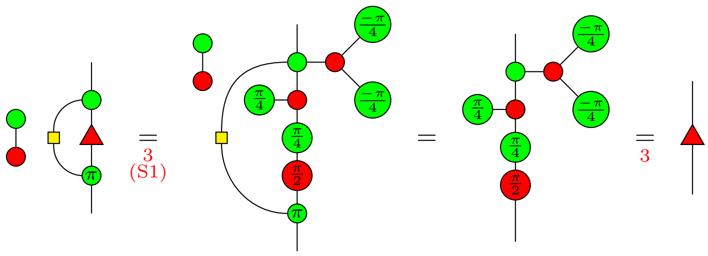


□

Proof (Lemma 28).

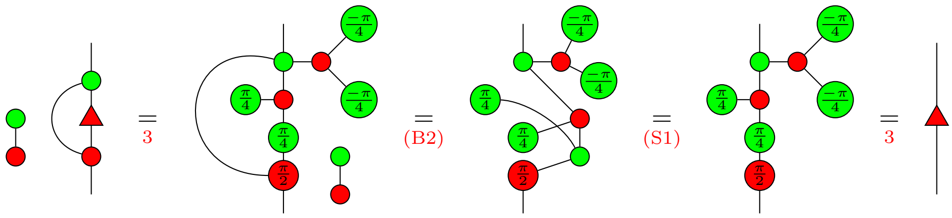


Then:



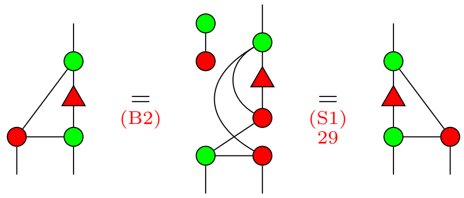
□

Proof (Lemma 29).



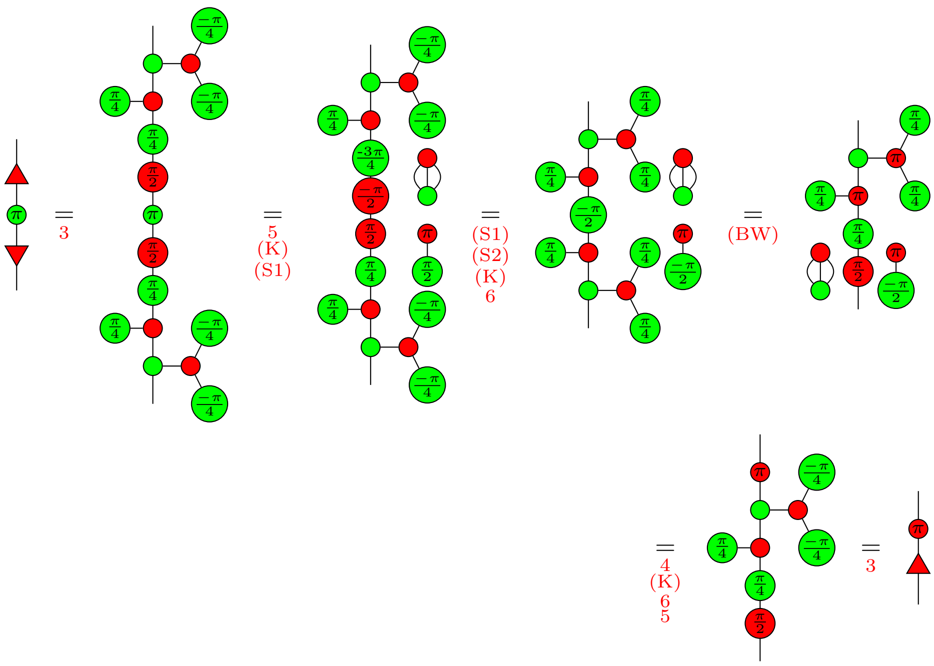
□

Proof (Lemma 30).



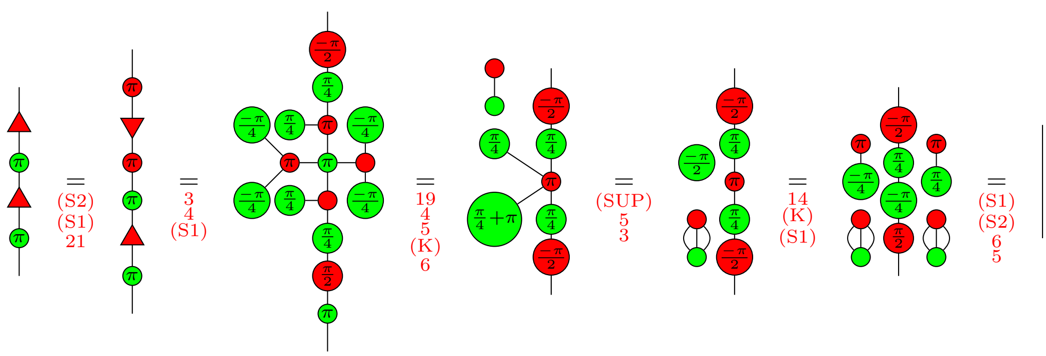
□

Proof (Lemma 31).



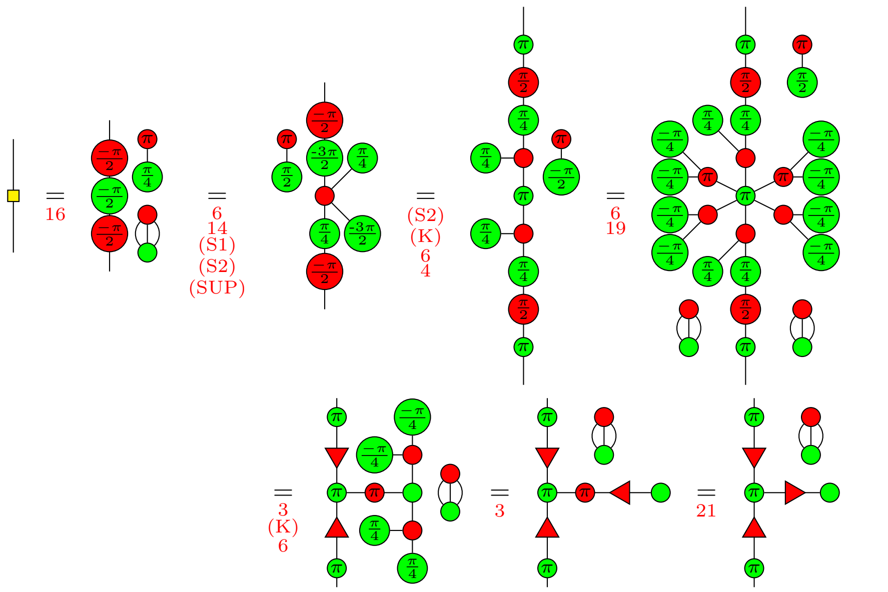
□

Proof (Lemma 32).

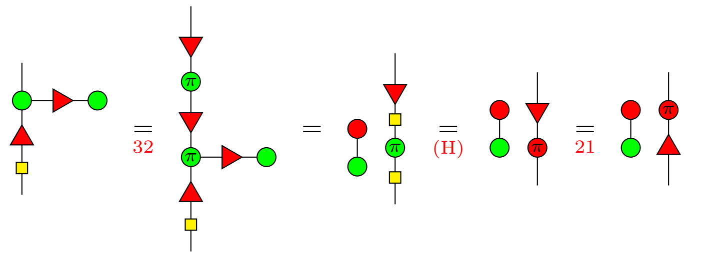


□

Proof (Lemma 33). First:

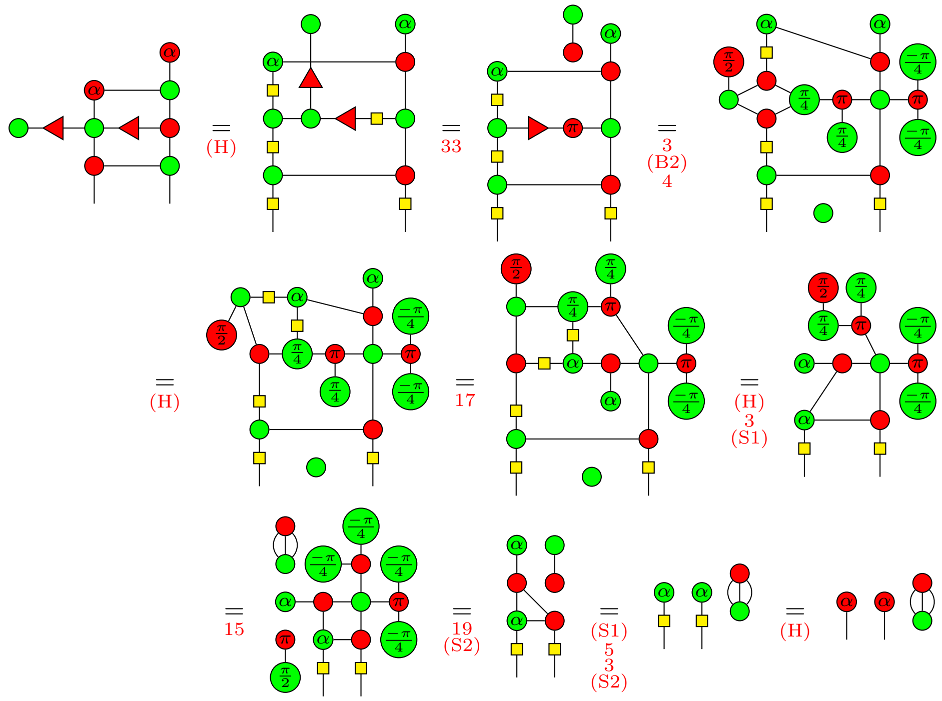


Then:

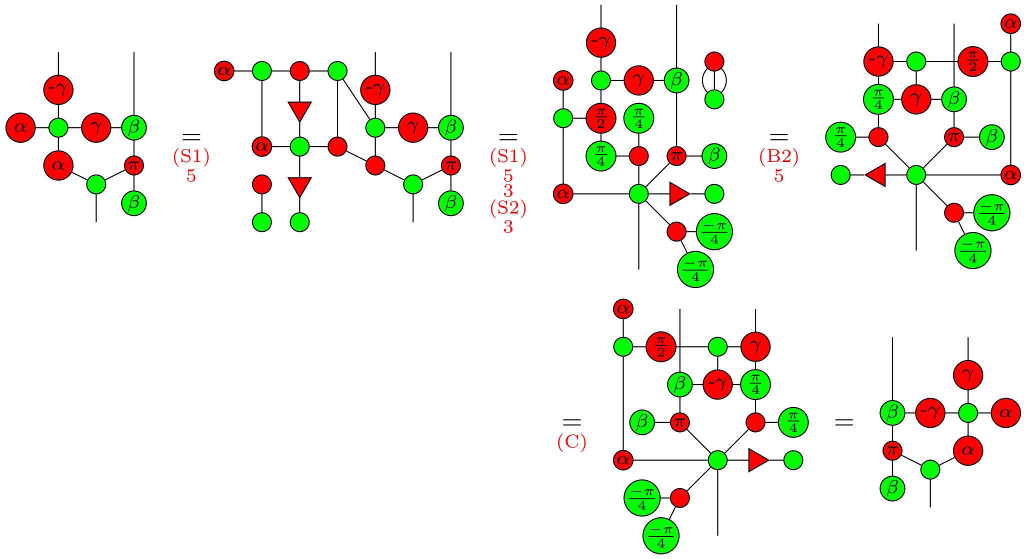


□

Proof (Lemma 34). First:

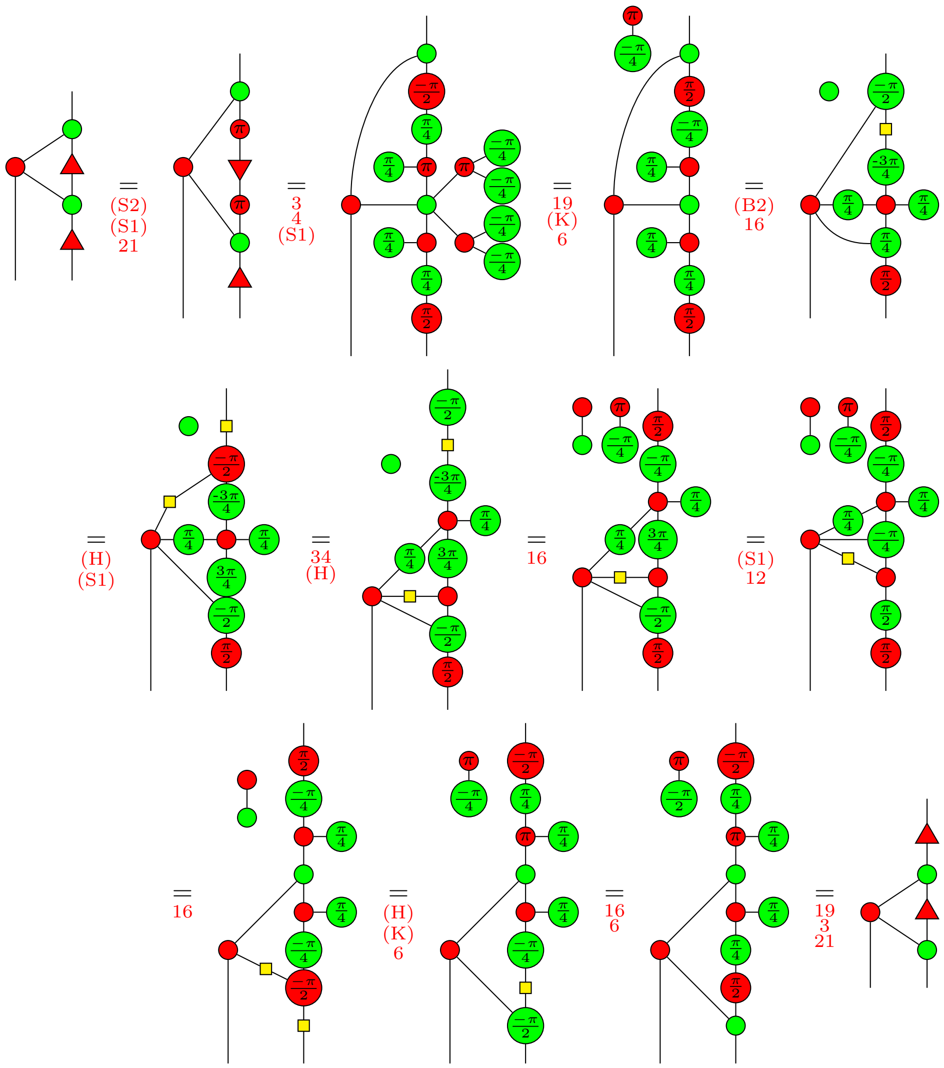


then:



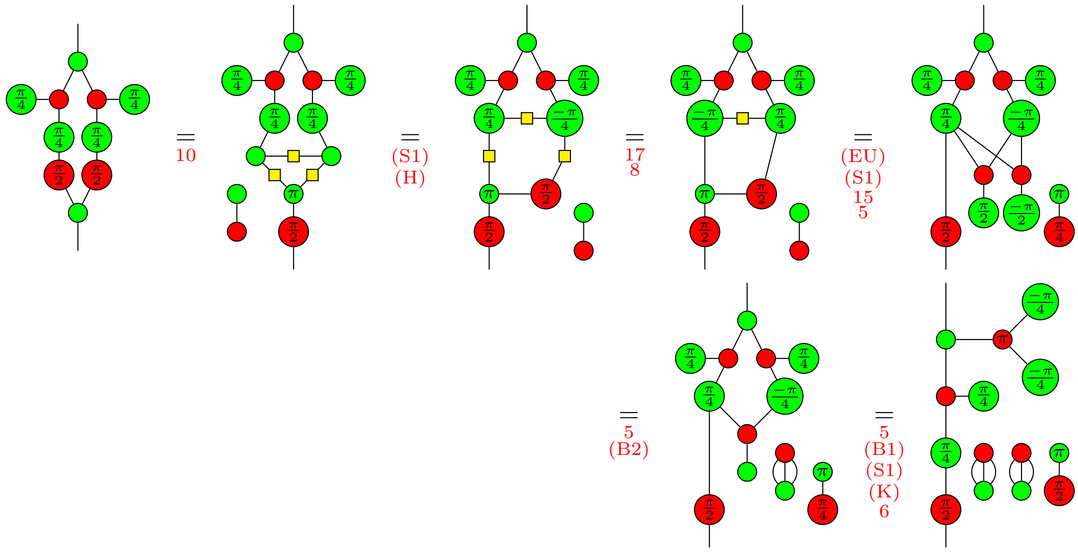
□

Proof (Lemma 35).

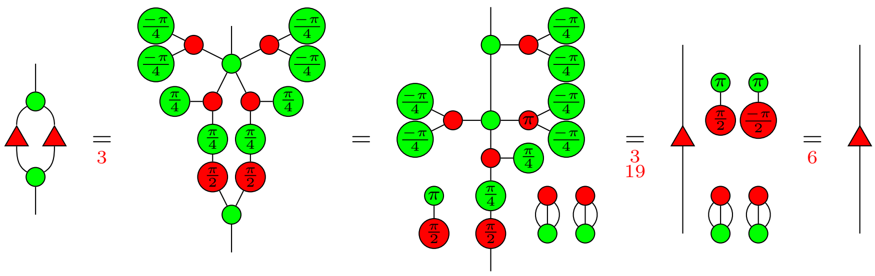


□

Proof (Lemma 36). First:

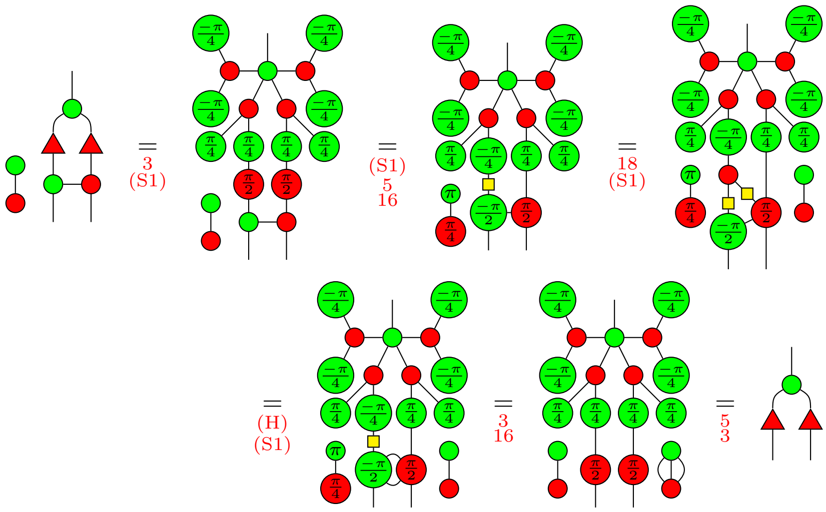


Then:

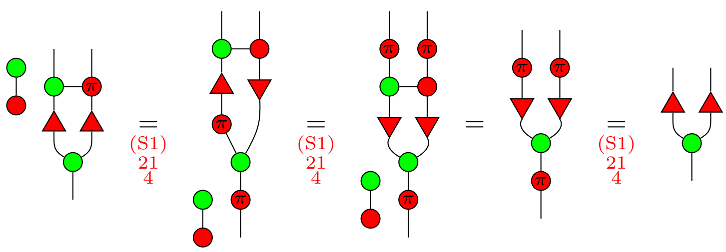


□

Proof (Lemma 37). First:

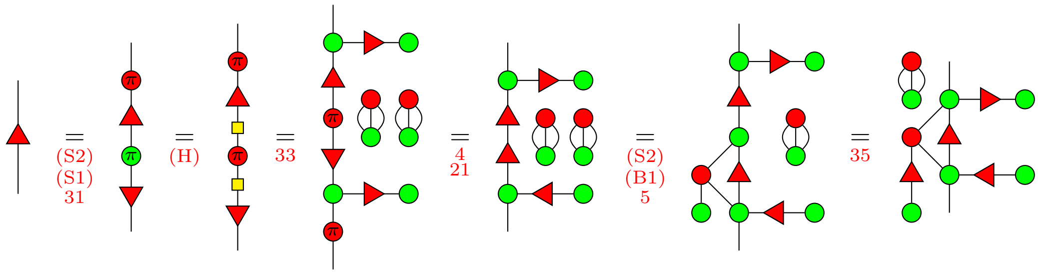


Then:

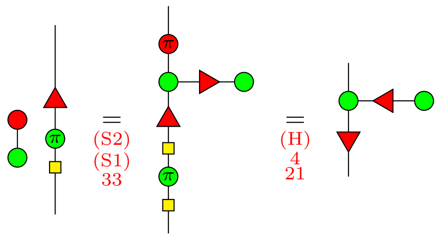


□

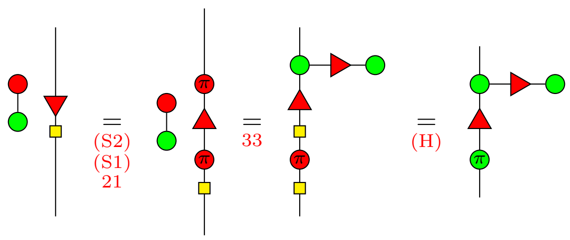
Proof (Lemma 38). First:



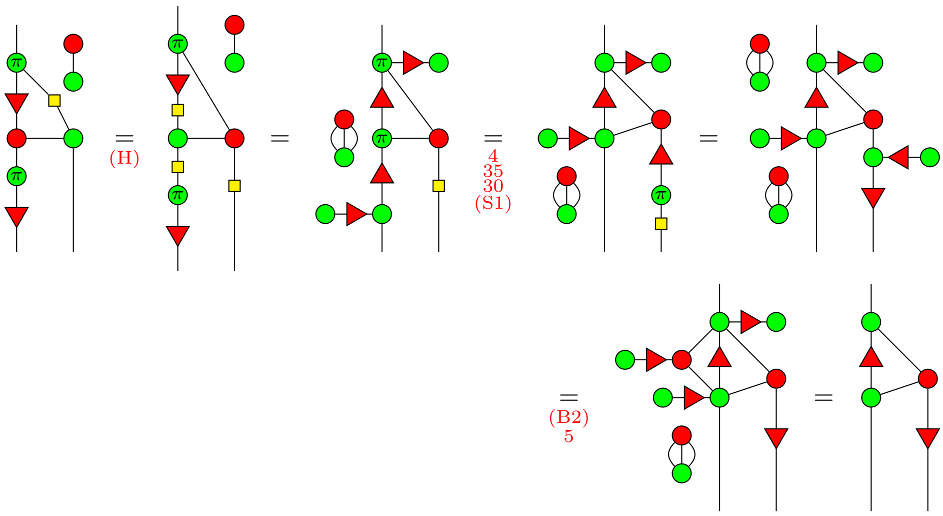
Moreover, from 33, we can easily derive:



and:



Finally:

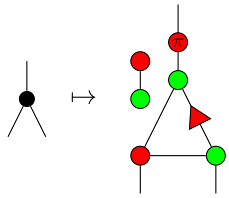


□

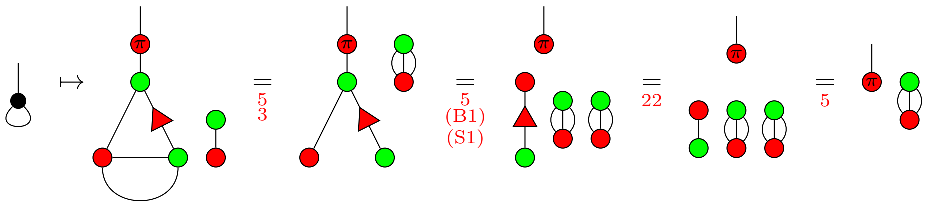
A.3 Proof of Propositions 7 and 8

We first derive an easy but useful lemma for the following:

Lemma 39. *As shown in Section 6:*

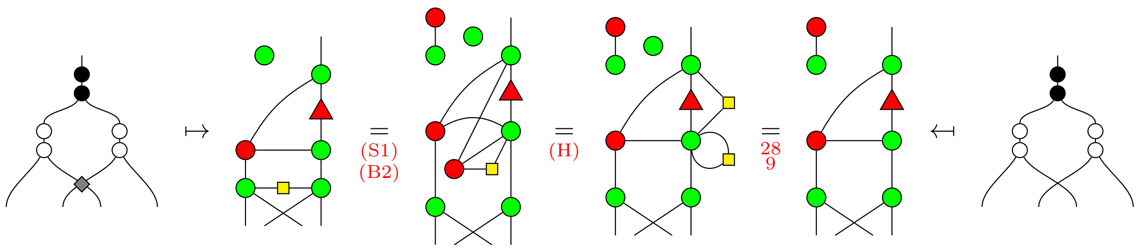


Then:



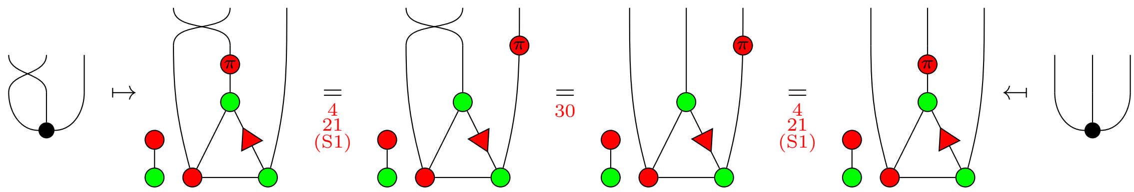
Proof of Proposition 7 We prove here that all the rules of the ZW-Calculus are preserved by $[\![\cdot]\!]_{WX}$.

• X:

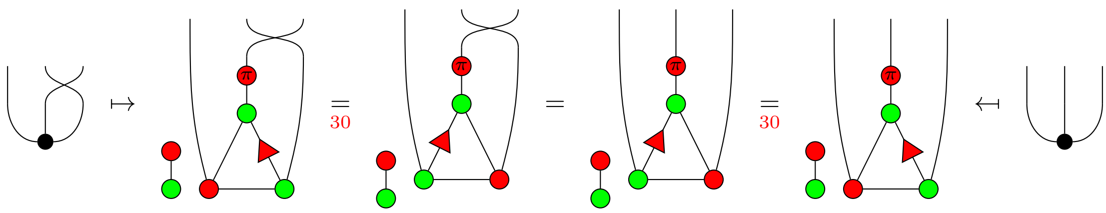


• $0a$, $0c$, $0d$ and $0d'$ come directly from the paradigm *Only Topology Matters.*

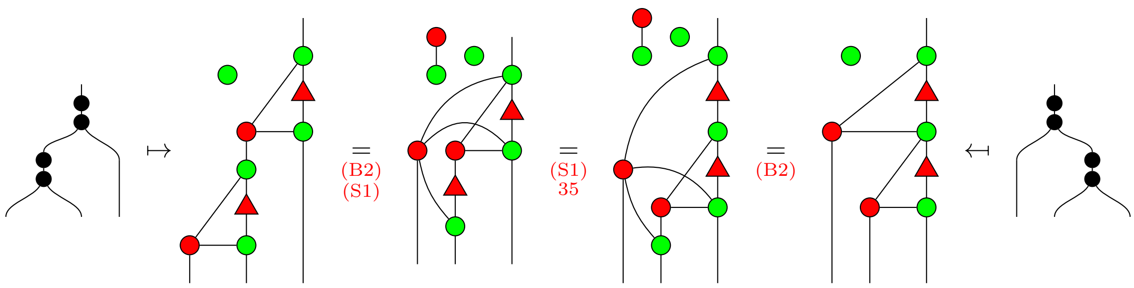
• $0b$:



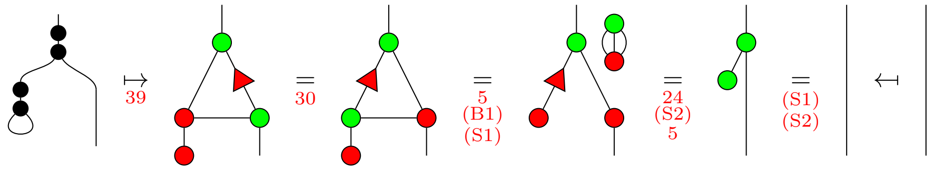
• $0b'$: Using the result for rule $0b$,



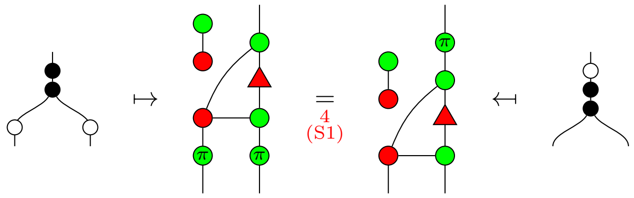
• $1a$:



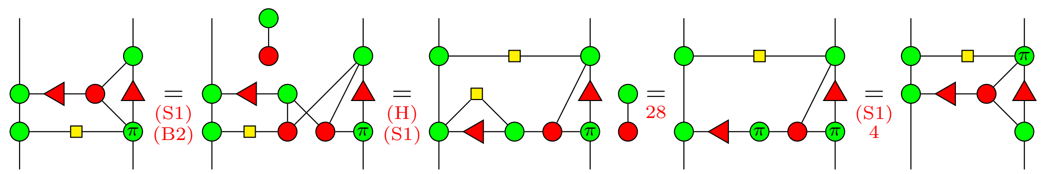
• 1b:



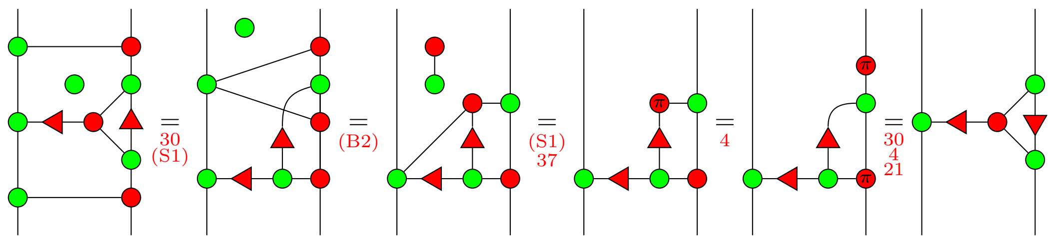
• 1c, 1d, 2a and 2b come directly from the spider rules (S1) and (S2).
• 3a is the expression of the colour-swapped version of Lemma 4.
• 3b:



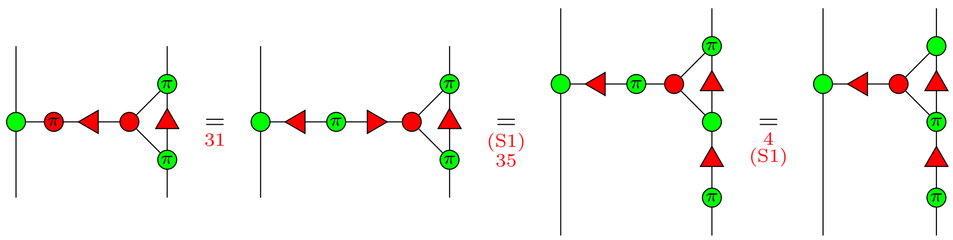
• 4 comes from the spider rule (S1).
• 5a: We will need a few steps to prove this equality.
i)



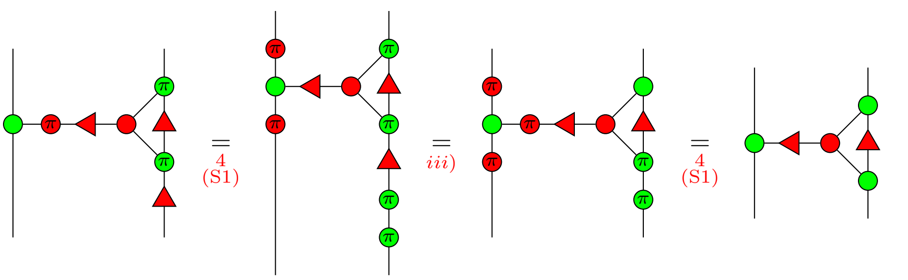
ii)



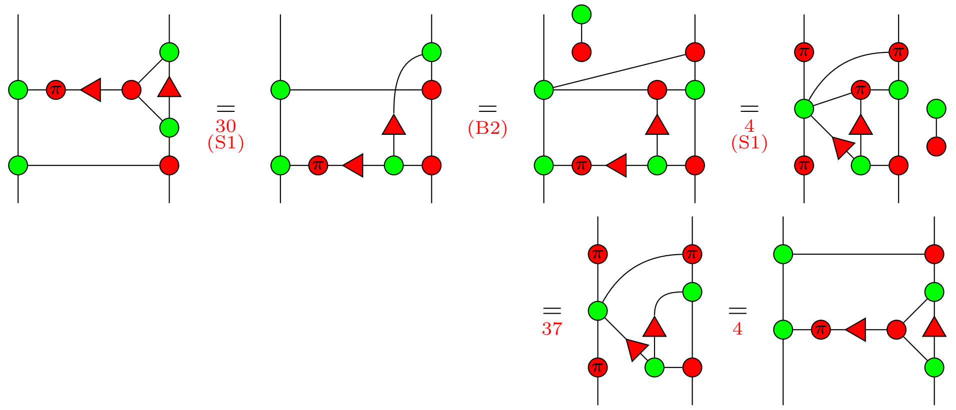
iii)



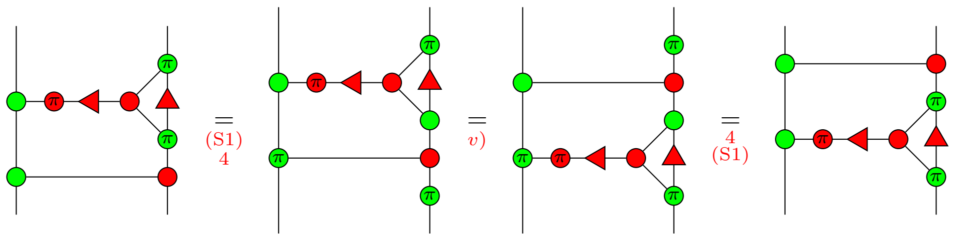
iv)



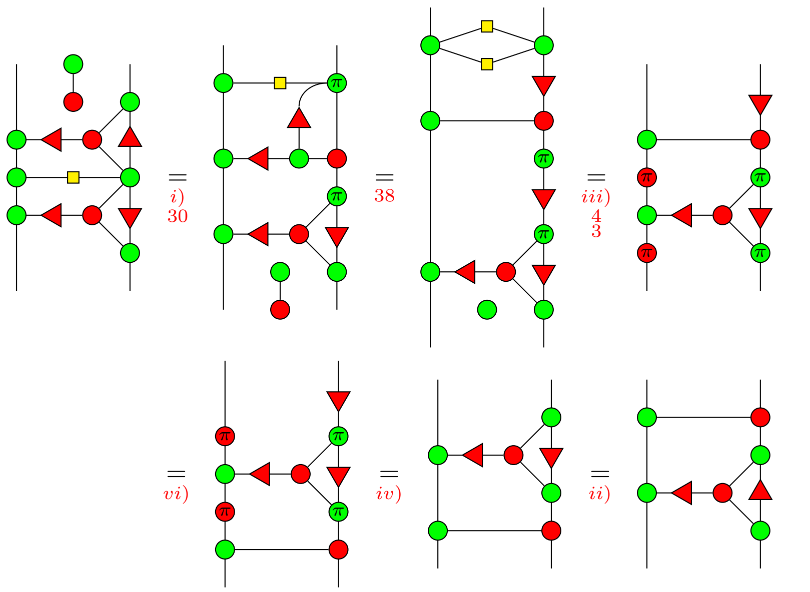
v)



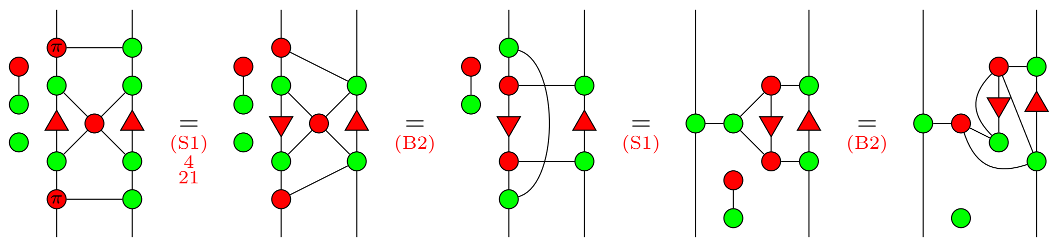
vi)

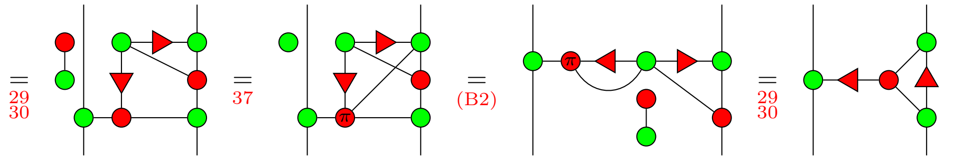


vii)

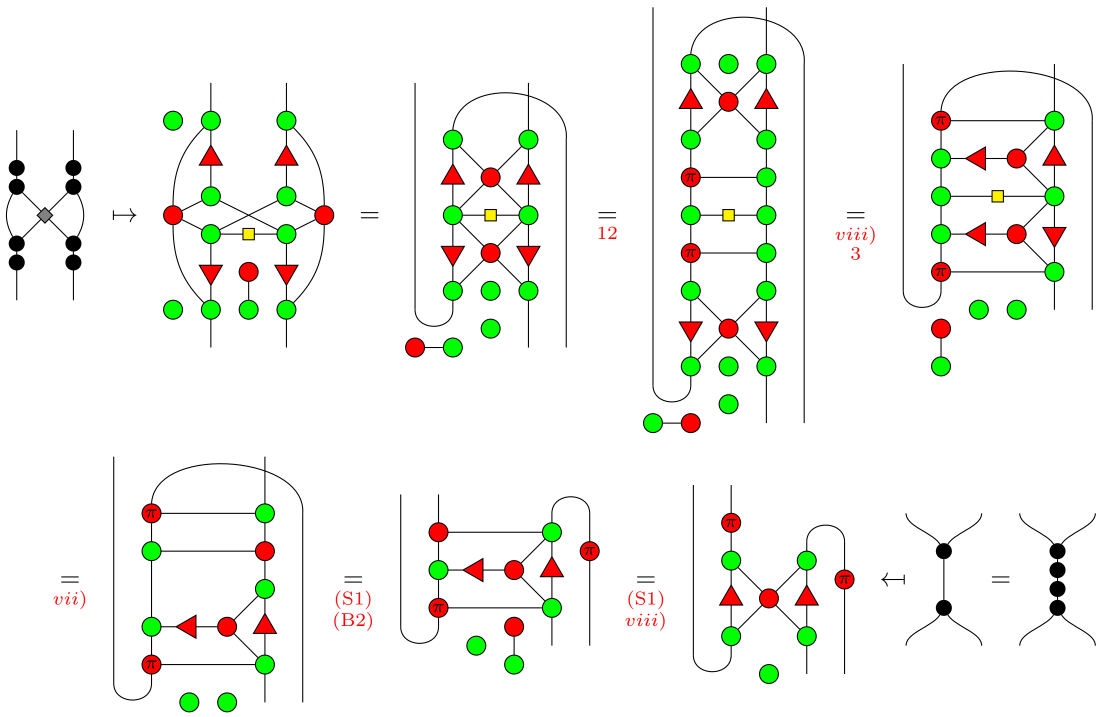


viii)

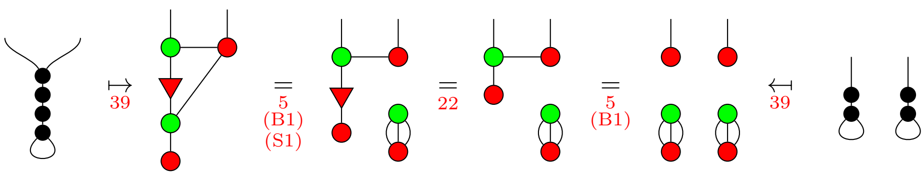




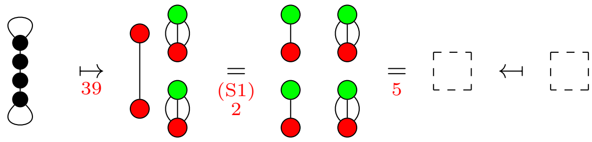
Finally,



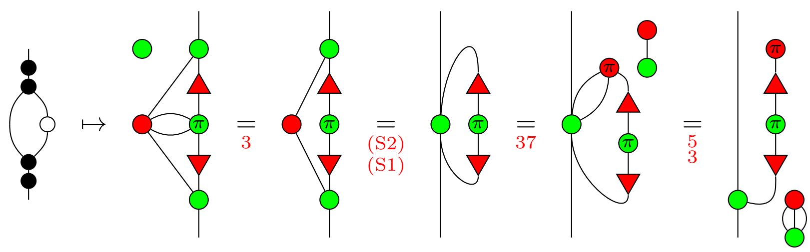
- $5b$:

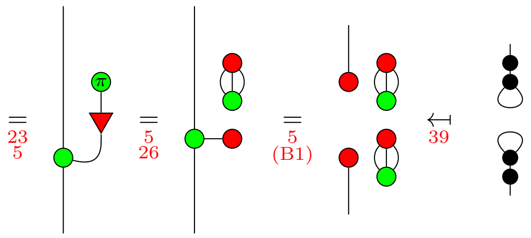


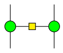
- $5c$:

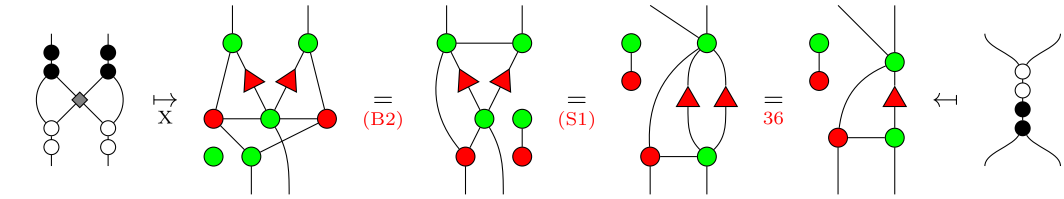


- $5d$:



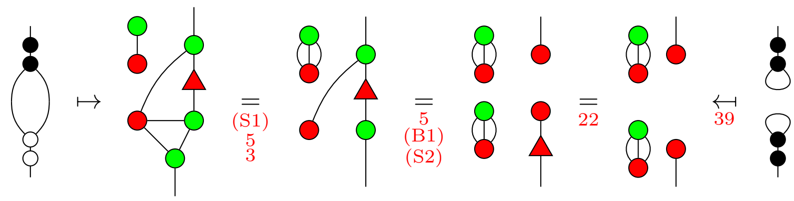


• 6*a*: Thanks to the rule X we can get rid of  induced by the crossing. Then,

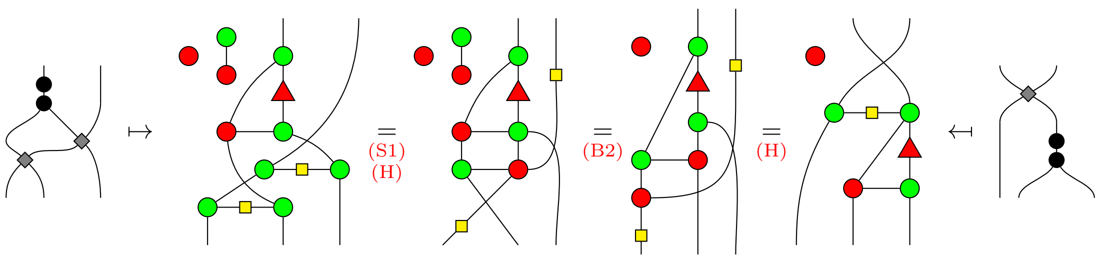


• 6*b* is exactly the copy rule (B1).

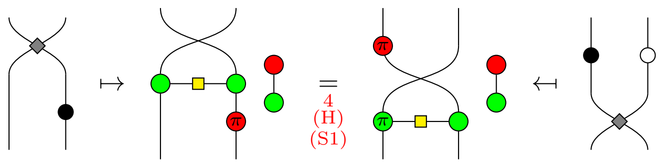
• 6*c*:



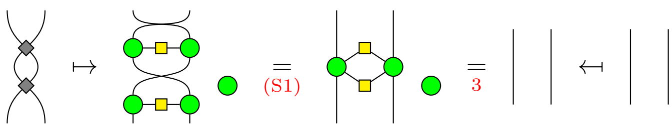
• 7*a*:



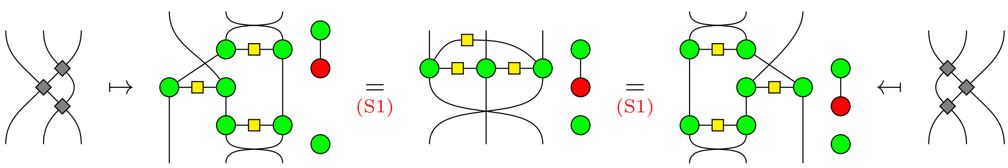
• 7*b*: using 4, (H) and (S1):



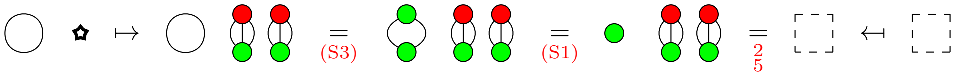
• R_2:



• R_3:

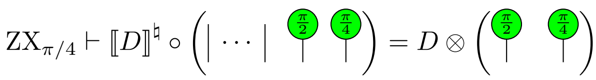


- *iv*: using (S3), (S1), 2 and 5,

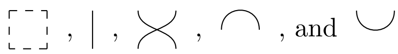


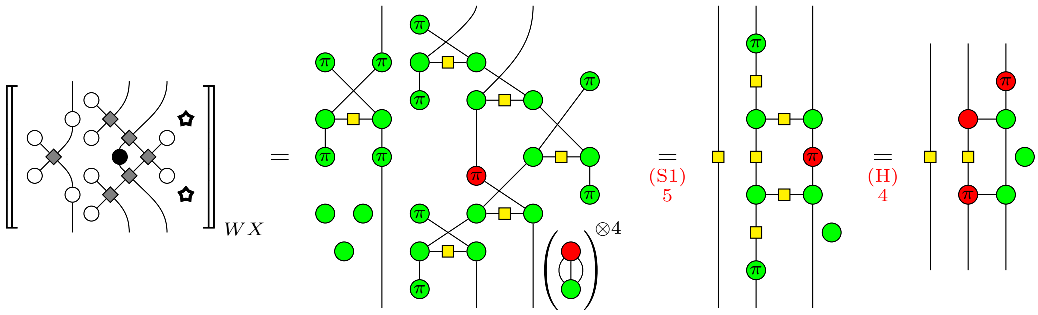
□

Proof of Proposition 8 Let us write $[\![.]\!]^\natural = [\![[\![.]\!]_{XW}]\!]_{WX}$. We can show inductively that:

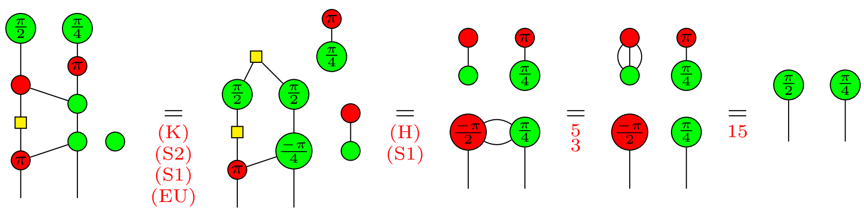


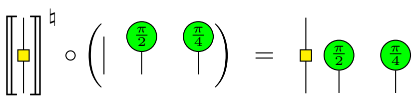
which is the expression of Proposition 4.

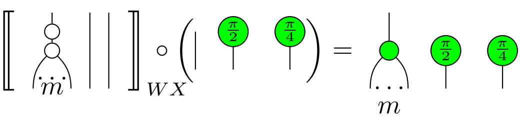
- The result is obvious for the generators  .
-  :

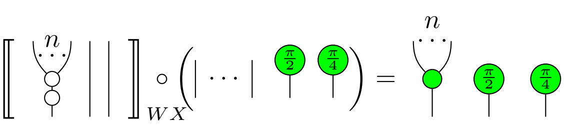


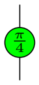
and, using (S1), (EU), 5, (H), 3 and 15:

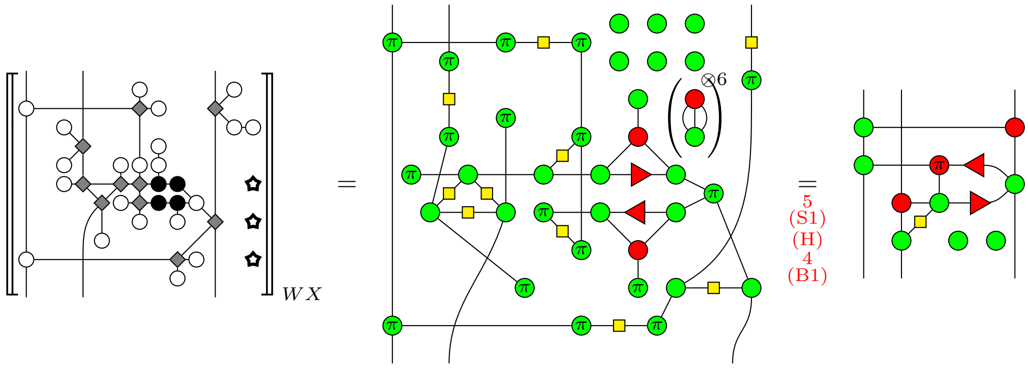


Hence $ZX \vdash$ 

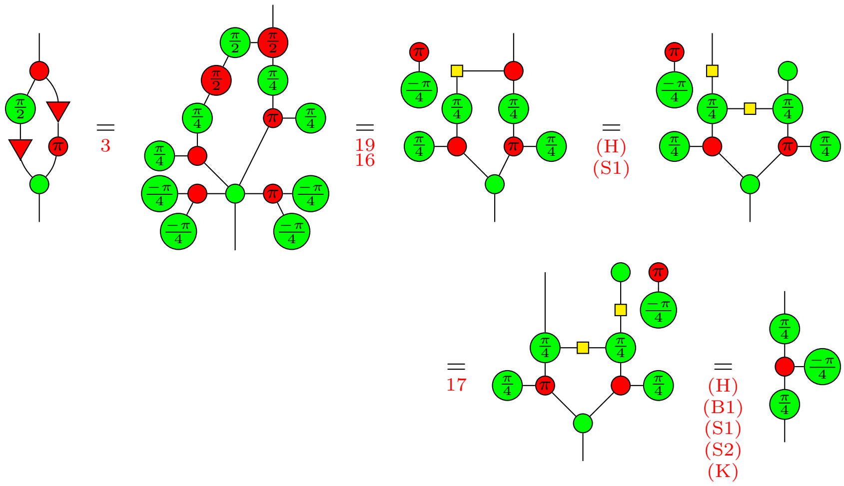
- 

- 

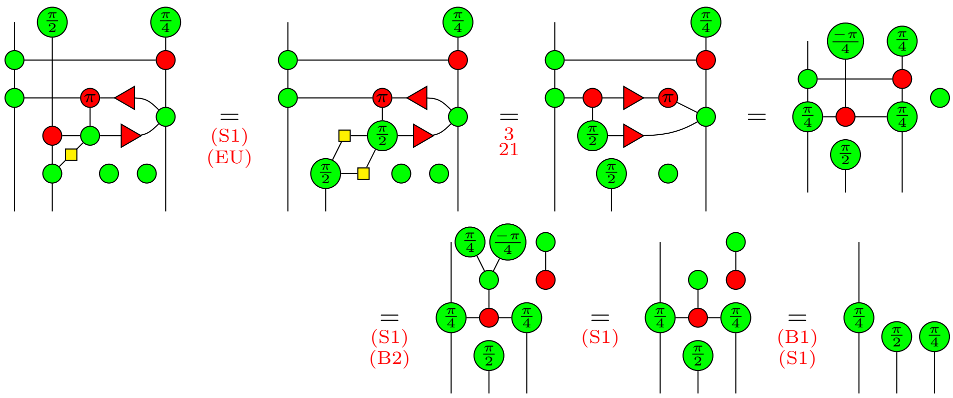
•  :

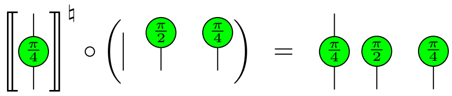


But:



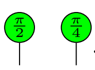
So that:



which means $\mathrm{ZX}_{\pi/4} \vdash$ 

• $D_1 \circ D_2$:

It is to be noticed that $[\![D_1 \circ D_2]\!]_{WX} = [\![D_1]\!]_{WX} \circ [\![D_2]\!]_{WX}$ and $[\![D_1 \otimes D_2]\!]_{WX} = [\![D_1]\!]_{WX} \otimes [\![D_2]\!]_{WX}$.

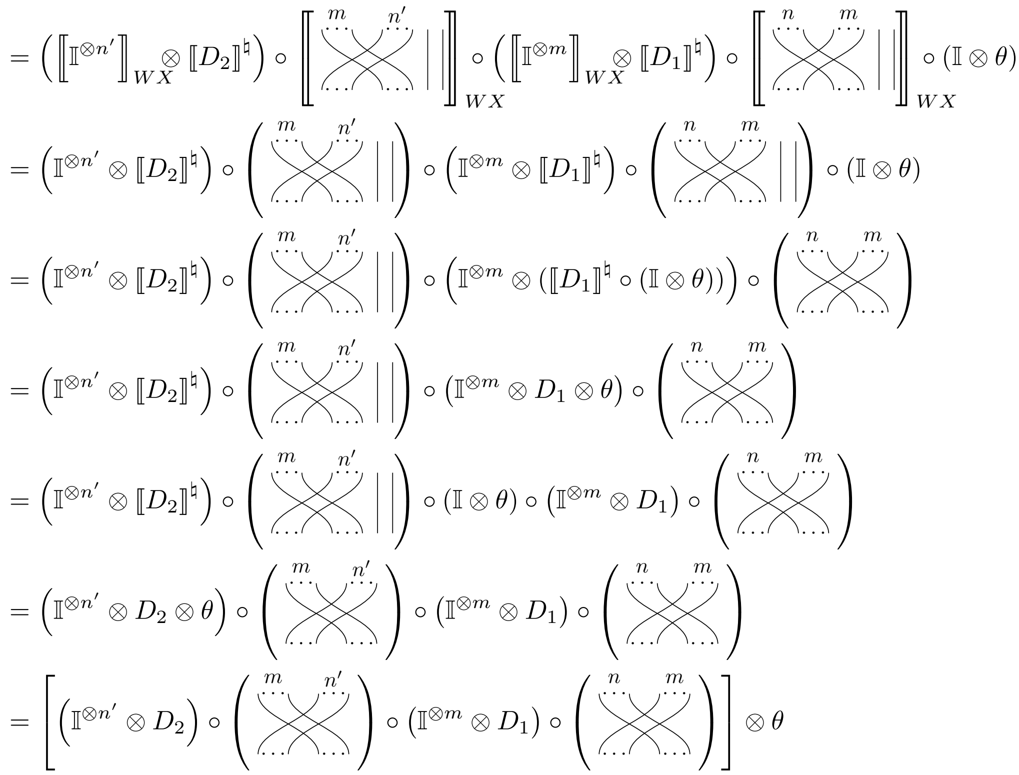
Let us write $\theta =$ . Then:

$$\mathrm{ZX}_{\pi/4} \vdash [\![D_1 \circ D_2]\!]^\natural \circ (\mathbb{I} \otimes \theta) = [\![D_1]\!]^\natural \circ [\![D_2]\!]^\natural \circ (\mathbb{I} \otimes \theta) = [\![D_1]\!]^\natural \circ (D_2 \otimes \theta)$$

$$= [\![D_1]\!]^\natural \circ (\mathbb{I} \otimes \theta) \circ D_2 = (D_1 \otimes \theta) \circ D_2$$
$$= (D_1 \circ D_2) \otimes \theta$$

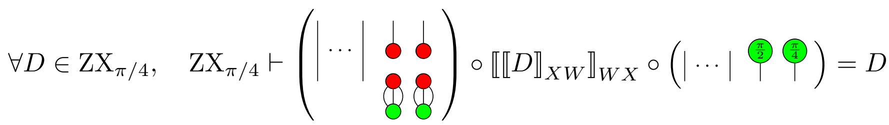
- $D_1 \otimes D_2$:

$$\mathrm{ZX}_{\pi/4} \vdash [\![D_1 \otimes D_2]\!]^\natural \circ (\mathbb{I} \otimes \theta)$$



$$= D_1 \otimes D_2 \otimes \theta$$

By compositions, for any diagram D, $\mathrm{ZX}_{\pi/4} \vdash [\![D]\!]^\natural \circ (\mathbb{I} \otimes \theta) = D \otimes \theta$. Then, using Lemmas 7 and 5:



□