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ABSTRACT

The use of coalgebras for the specification of dynamical systems with a hidden state space is receiving more and more attention in the years, as a valid alternative to algebraic methods based on observational equivalences. However, to our knowledge, the coalgebraic framework is still lacking a complete equational deduction calculus which enjoys properties similar to those stated in Birkhoff's completeness theorem for the algebraic case.

In this paper we present a sound and complete equational calculus for coalgebras of a restricted class of polynomial functors. This restriction allows us to borrow some "algebraic" notions for the formalization of the calculus. Additionally, we discuss the notion of *colours* as a suitable dualization of variables in the algebraic case. Then the completeness result is extended to the "non-ground" or "coloured" case, which is shown to be expressive enough to deal with equations of hidden sort. Finally we discuss some weaknesses of the proposed results with respect to Birkhoff's, and we suggest possible future extensions.

1 Introduction

In recent years there has been a growing interest in the theory of coalgebras, motivated by the fact that they are particularly suitable to specify, in an implementation independent way, a wide class of systems, typically discrete dynamical systems with a hidden state space [Rut96, Jac96b, Jac96a, Rei95, HS95]. Examples include various kinds of transition systems, deterministic and nondeterministic automata, (concurrent) objects, hybrid systems, and (possibly) infinite data structures like streams and trees.

In the theory of *algebraic specification*, an abstract data type is specified by a set of operations (constructors) which determine how values of the carrier are built up, and a set of formulas (in the simplest case equations) stating which values should be identified. In the standard initial semantics the defining equations impose a congruence on the initial algebra. Dually, the coalgebraic specification of a class of systems is characterized by a set of operations, sometimes called destructors, which tell us what can be observed out of a *state* (i.e., an element of the carrier), and how can a state be transformed to successor states. Also in this case it is often convenient to impose additional defining conditions which restrict the range of possible observations and transitions. In this paper we will stick to the case where such conditions are expressed by mean of defining equations only. Using the standard final semantics, defining equations determine a sub-coalgebra of the final coalgebra, containing only the behaviours of interest. In particular, as explained in [HR95], properties expressible by equations in this framework are *safety* or *invariant* properties, i.e., properties that must hold in any possible state of a coalgebra.

A natural question is whether for equations in a coalgebraic framework there is a complete calculus of deduction, enjoying properties similar to those of the classical rules of equational deduction in an algebraic framework, as stated by Birkhoff's completeness theorem. This is indeed the main topic of this paper.

Before summarizing the contents of the paper, it is worth recalling that many authors agree on the fact that it is useful to employ coalgebraic techniques for the specification of systems with a hidden state space, but to stick to *algebraic* techniques for the specification of the involved data. This is consistent with the purely algebraic approaches which use initial semantics for the specification of data structures, and final semantics (based on behavioural equivalences) for state spaces (see for example [ONS93, BHW95] and the references therein). The results in this paper are based instead on a purely coalgebraic approach: we leave as a topic of future research to investigate how far they can be generalized to a hybrid coalgebraic/algebraic framework.

We start introducing in Section 2 the class of coalgebras we are concerned with. They are presented in an algebraic style, by providing a *co-signature*, i.e., a signature satisfying strong constraints on sorts and operations. In particular, sorts include one single "hidden sort", corresponding to the carrier of the coalgebra, and other "visible" sorts for inputs and outputs, which are given a fixed interpretation. In order to understand better how wide (or restricted) this class of coalgebras is, in Section 2.1 we make use of the categorical definition of "algebras and coalgebras of a functor", showing that while usual algebras are algebras for polynomial functors (which are closed with respect to coproducts, products and exponents), our coalgebras are coalgebras for "restricted" polynomial functors (for which coproducts are not allowed). In transition system terminology, such coalgebras can model deterministic, non-terminating transition systems with inputs and outputs, but cannot model in a direct way possibly terminating or non-deterministic systems for which powersets or coproducts would be needed.

Clearly, to speak about "equations" in a coalgebraic setting one first needs to understand what "terms" are. In the related literature [HR95, Jac96b], the terms used in coalgebraic equations are standard "algebraic" terms built from constructors and destructors, with the restriction that only one variable of the hidden sort can appear in an equation. We are more strict in this respect, because of our commitment to a pure coalgebraic framework. Firstly, we only allow destructors and not constructors; thus visible sorts will be interpreted as sets without any algebraic structure defined on them. Coalgebraic terms, built only over destructors, have for us a precise interpretation as the basic *experiments* or *observations* that one can make on the states of a coalgebra. As such, they will be further constrained to include only constants of input sorts, and to be of visible sort,

because the result of an observation cannot be a hidden state.

In the general case, using an object-based terminology, an observation on a state of a coalgebra consists of performing a sequence of (possibly parametrized) methods (or transitions), followed by a (possibly parametrized) attribute which delivers the observed result. As an equation is just a pair of terms denoting observations, such equation is valid in a coalgebra if the two observations return the same result for all the states of the carrier. This notion of validity is used in a standard way in Section 4 to define the class of models of a *coalgebraic specification* (i.e., a co-signature and a set of equations). Next the first main result of the paper is presented, namely a set of equational deduction rules which is shown to be sound and complete, in the sense that an equation is entailed by a set of equations E if and only if it is valid in all models for E. For the proof of the theorem, it turned out to be useful to have at hand an explicit description of the final coalgebra for a given co-signature, which is introduced in Section 3. The leading idea there is that an element of the final coalgebra is characterized by all the experiments or observations one can perform on it, thus it is just a function from the set of possible observations to corresponding output values.

In Section 5 we compare the expressive power of our notion of coalgebraic equations with those proposed in the related literature. It is easily shown that our equations are less expressive, because of the strong restrictions we impose on terms, but we hint at possible generalizations of our main result that should allow us to recover greater expressive power. As a specific example, in some situations it is desirable to impose equations of hidden sort. Equality in the hidden sort has two possible interpretations (see, e.g., [HR95]): as true equality of states, or as bisimilarity. In our framework, a "bisimilarity equation" can be regarded as a shorthand for a (usually infinite) set of equations of visible sorts, stating that for each possible observation the two states deliver the same result; as such it is not really problematic. For "true equations" of hidden sort, instead, a proper generalization is needed. A bit surprisingly, we are able to present such a generalization of the completeness result (at the end of Section 7) through motivations and techniques which are apparently not very much related to the notion of "true equality in hidden sorts", but are instead related to the notion of variables and substitutions in algebraic specification. In fact in Sections 6 and 7 we first explain why it is reasonable to expect that *colouring functions* from the carrier of a coalgebra to a given set of *colours* could play for coalgebras a rôle dual to that of variables and assignments for algebras (following [Rut96]). Then we argue that the deduction rules of Section 4 are actually for *ground* equations, and we present new rules for the "non-ground" or *coloured* case, for which we prove again soundness and completeness. Quite interestingly, we show next that the expressive power gained with the introduction of colours is just what we need to deal with true equations of hidden sort.

Finally, in Section 8 we show that our completeness result is weaker than Birkhoff's in the following sense: every class of coalgebras determined by a set of equations forms a *covariety* (according to a recent definition by Jan Rutten [Rut96]), but there are covarieties which are not definable by a set of equations. We conclude with Section 9 where we hint at future extensions of this research.

2   Coalgebras

In this section we introduce the class of coalgebras which will be considered in the rest of the paper; next in Section 2.1 we relate the chosen class of coalgebras to the standard definition of algebras in Universal Algebra, explaining in which sense our class of interest is "restricted".

A class of coalgebras will be introduced in an "algebraic style", by providing a set of sorts and operator names satisfying suitable restrictions. This style of presentation is essentially borrowed from work by Bart Jacobs [Jac96b, Jac96a] (see also [HR95, Rei95]), and it will be related to the more usual functorial definition of coalgebras in Section 2.1.

Definition 1 (coalgebraic signatures and coalgebras). A *(one sorted) coalgebraic signature* or *co-signature* is a triple $\Pi = \langle S, OP, [\![\_]\!] \rangle$, where S, the *sorts*, OP, the *operators*, and $[\![\_]\!]$ the *interpretation of visible sorts* are as follows:

- S is a triple $S = \langle X, \{I_1, \ldots, I_k\}, \{O_1, \ldots, O_h\}\rangle$ where X is the *hidden sort*, I_j is an *input sort* for $j \in \underline{k}$,[1] and O_j is an *output sort* for $j \in \underline{h}$. The sets of input and output sorts do not need to be disjoint, and their elements are also called *visible sorts*.
- OP is a pair of sets $OP = \langle \{m_1, \ldots m_l\}, \{a_1, \ldots, a_n\}\rangle$, where $m_j : X \times I_{k_j} \to X$ is a *method* for $j \in \underline{l}, k_j \in \underline{k}$, and $a_j : X \times I_{k_j} \to O_{h_j}$ is an *attribute* for $j \in \underline{n}, k_j \in \underline{k}, h_j \in \underline{h}$.
- $\llbracket\_\rrbracket$ is a function mapping each visible sort to a non-empty set. For each visible sort V and each element $v \in \llbracket V \rrbracket$, $\underline{v} : V$ will be a constant denoting element v.

A *Π-coalgebra* A consists of a set X_A, the *carrier*, a function $m_{j_A} : X_A \times \llbracket I_{k_j} \rrbracket \to X_A$ for each method, and a function $a_{j_A} : X_A \times \llbracket I_{k_j} \rrbracket \to \llbracket O_{h_j} \rrbracket$ for each attribute. The class of all Π-coalgebras is denoted $Coalg(\Pi)$.

Thus all the sorts appearing in a co-signature have a fixed interpretation, but for the hidden sort. The separation between sorts and their interpretation is just for conceptual clarity, and we will simply ignore it in the following by denoting both the sort and the corresponding set by the same symbol, and dropping the third component of a co-signature. Such co-signatures are "one sorted" because only one hidden sort is allowed; in our opinion the notions and results presented in this paper should lift smoothly to the many sorted case, as in the algebraic setting, but we did not check the details yet.

Allowing exactly one input argument (or parameter) for methods and attributes is not a restriction, because the input set can be a cartesian product. For example, a method with n parameters $m : X \times I_1 \times \ldots \times I_n \to X$ will be denoted by $m : X \times \prod_{j \in \underline{n}} I_j \to X$, and set $\prod_{j \in \underline{n}} I_j$ will be listed as an input sort. If $n = 0$, since the empty product yields a one-element set $\mathbf{1}$, the parameterless method $m : X \times \mathbf{1} \to X$ will be denoted by $m : X \to X$.

In object-based terminology, the coalgebras just introduced are expressive enough to specify parametric methods and attributes. In transition system terminology, such coalgebras can model deterministic, non-terminating transition systems with inputs and outputs, but cannot model in a direct way possibly terminating or non-deterministic systems: this could be obtained for example by allowing methods to be relations instead of total functions.

Definition 2 (coalgebra homomorphisms, sub-coalgebras). Given two Π-coalgebras A and B, a *homomorphism* $f : A \to B$ is a function between their carriers $f : X_A \to X_B$ such that for each method $m : X \times I \to X$ it holds $m_B(f(x), v) = f(m_A(x, v))$ for all $x \in X_A$ and $v \in I$, and for each attribute $a : X \times I \to O$ it holds $a_B(f(x), v) = a_A(x, v)$ for all $x \in X_A$ and $v \in I$.

A Π-coalgebra Z is *final* if for each Π-coalgebra A there is exactly one homomorphism to Z, which we will denote $!_A : A \to Z$.

A coalgebra A is a *sub-coalgebra* of B if $X_A \subseteq X_B$, and the inclusion $A \hookrightarrow B$ is a homomorphism. A subset $S \subseteq X_B$ of the carrier of a Π-coalgebra B is the carrier of a sub-coalgebra of B iff for all methods $m : X \times I \to X$ and for all $x \in S, v \in I$, we have $m_B(x, v) \in S$. In this case the coalgebraic structure on S is obtained by restricting all functions of B to the subcarrier S.

Here are the running examples that we will use all along the paper.

Example 1 (deterministic transition systems). Let TS be the co-signature $TS = \langle \langle X, \mathbf{1}, O \rangle, \langle next : X \to X, val : X \to O \rangle \rangle$,[2] where $O = \{o_0, o_1, \ldots, o_n\}$ (we assume that $n > 1$). A TS-coalgebra can be interpreted as a non-terminating, deterministic transition system: the method *next* returns for each state in the carrier the successor state; the attribute *val* returns an observation in O for each state. As we have no way to fix an initial state, one should think of a TS-coalgebra as specifying the collection of all the transition sequences starting from all possible states. Here are some examples of TS-coalgebras:

1. $SW = \langle \{\texttt{ON}, \texttt{OFF}\}, next_{SW} : \{\texttt{ON} \mapsto \texttt{OFF}, \texttt{OFF} \mapsto \texttt{ON}\}, val_{SW} : \{\texttt{ON} \mapsto o_1, \texttt{OFF} \mapsto o_0\}\rangle$. This coalgebra represents a system which loops forever between the two states, producing an infinite

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[1] For a natural number n, by $\underline{n}$ we denote the set $\{1, \ldots, n\}$; thus $\underline{0} = \emptyset$.

[2] To improve readability, a singleton set is denoted by its only element.

sequence (a *stream*) of alternating o_0 and o_1. Clearly, the first element of the stream depends on the state we choose to start with.

2. For every $0 < m \le n$, let TS_m be the coalgebra $\langle \mathbb{N}, next_m : i \mapsto i+1, val_m : i \mapsto o_{(i \bmod m)} \rangle$. As a system, TS_m never passes twice through the same state, and outputs a cyclic stream where the elements $o_0, o_1, \ldots, o_m$ are repeated forever in that order. Note that the possible output streams of TS_2 are the same as those of SW; actually it is easy to check that there is a homomorphism from TS_2 to SW mapping even numbers to `OFF` and odd numbers to `ON`.

3. Let $O^{\mathbb{N}}$ be the set of all streams of elements of O, w range over $O^{\mathbb{N}}$ and o range over O. Then the coalgebra Z_{TS} is defined as $\langle O^{\mathbb{N}}, next_Z : o \cdot w \mapsto w, val_Z : o \cdot w \mapsto o \rangle$. It is easy to see that Z_{TS} is a final TS coalgebra: if A is a TS-coalgebra, and $x \in X_A$, define $!_A(x) = w$, where $w \in O^{\mathbb{N}}$ is the stream of values returned in A starting from state x. In fact, this mapping satisfies the conditions of Definition 2 and it can be shown that it is the only one.

Example 2 (bank accounts). Let BA be the co-signature $\langle \langle X, \{\mathbb{Z}, \mathbf{1}\}, \mathbb{Z} \rangle, \langle ch : X \times \mathbb{Z} \to X, bal : X \to \mathbb{Z} \rangle \rangle$, where $\mathbb{Z}$ is the set of integers. A BA-coalgebra can be interpreted as a collection of (very rudimentary) *bank account* states. At a state x of the carrier, two operations are possible: to see the *bal*ance attribute of the state, $bal(x)$, which is an integer, or to *ch*ange the account state to a new state $ch(x, z)$ using an input integer z.

Here are some examples of BA-coalgebras.

1. $BA_1 = \langle \mathbb{Z}, ch_1 : \langle z, z' \rangle \mapsto z + z', bal_1 : z \mapsto z \rangle$. This coalgebra models a "correct" bank account, which records in the state the total amount of the deposited (or withdrawn) money, and returns it when bal is applied.

2. $BA_0 = \langle \mathbf{1} = \{*\}, ch_0 : \langle *, z' \rangle \mapsto *, bal_0 : * \mapsto 0 \rangle$. This models an account consisting of a single state, which always returns 0.

3. Let w range over $\mathbb{Z}^*$, the set of finite strings over $\mathbb{Z}$, and let $\Sigma(w)$ denote the sum of all integers in w. Then $BA_h = \langle \mathbb{Z}^*, ch_h : \langle w, z \rangle \mapsto w \cdot z, bal_h : w \mapsto \Sigma(w) \rangle$ is again a correct account, which also records in the state the history of all deposited sums.

4. Let $Z_{BA} = \langle \{\psi : \mathbb{Z}^* \to \mathbb{Z}\}, ch_Z : \langle \psi, z \rangle \mapsto \lambda w \cdot \psi(z \cdot w), bal_Z : \psi \mapsto \psi(\epsilon) \rangle$. We will see in Section 3 that this is a final BA-coalgebra.

2.1   Signatures, co-signatures and polynomial functors

The goal of this section is to relate the class of coalgebras just introduced to the standard class of algebras considered in Universal Algebra, usually introduced via signatures. By exploiting the categorical definition of algebras and coalgebras, we show that while signatures and *polynomial functors* determine essentially the same classes of algebras, the co-signatures of Definition 1 determine the class of coalgebras for *restricted polynomial functors*, i.e., where coproduct is not allowed.

The contents of this section is not strictly necessary for the rest of the paper, so the reader might decide to skip it at a first reading. Only, the categorical notion of *algebra* or *coalgebra for a functor* [ML71, Rut96] will be used in Section 6; we introduce it immediately, sticking to the case of endofunctors on the category of sets.

Definition 3 (algebras and coalgebras for a functor). Let $F : \mathbf{Set} \to \mathbf{Set}$ be a functor. An *algebra for F* or *F-algebra* is a pair $\langle A, f : F(A) \to A \rangle$, where A is a set, the carrier, and f is a function. Dually, an *F-coalgebra* is a pair $\langle C, g : C \to F(C) \rangle$ where again C is a set and g is a function.

It is well known that there is a close relationship between the usual notion of Σ-*algebra* for a signature Σ and the F-algebras, as just defined, for a polynomial functor F.

Definition 4 (polynomial functors). The class of *polynomial (endo)functors* over $\mathbf{Set}$ is the least class of functors $F : \mathbf{Set} \to \mathbf{Set}$ containing:

1. the identity functor $id : \mathbf{Set} \to \mathbf{Set}$;
2. the constant functor $A : \mathbf{Set} \to \mathbf{Set}$ (for each set A), mapping every set to A and every function to the identity;
3. the functor $F_1 \otimes F_2$ for functors F_1 and F_2 already belonging to the class, mapping each set X to $F_1(X) \times F_2(X)$;[3]
4. the functor $F_1 \oplus F_2$ for functors F_1 and F_2 already belonging to the class, mapping each set X to $F_1(X) + F_2(X)$;[4]
5. the exponent functor $F \uparrow A$ for each set A and for F already belonging to the class, mapping a set X to $F(X)^A$ (the set of functions from A to $F(X)$).

A set is an *output set* for a polynomial functor F if it is introduced in the definition of F by rule 2; it is an *input set* if it is introduced by rule 5. A *restricted* polynomial functor is a polynomial functor not including coproducts, i.e., which is built according to the above rules without using rule 4.

Proposition 5 (Σ-algebras are F-algebras for a polynomial F). *Let Σ be a (algebraic) one-sorted, finite signature, i.e., a finite set of operators equipped with a function arity $: \Sigma \to \mathbb{N}$, returning for each operator its arity. Let F_Σ be the polynomial functor defined as*

$$F_\Sigma(X) = \coprod_{\sigma \in \Sigma} X^{arity(\sigma)}$$

where $\coprod$ denotes coproduct in $\mathbf{Set}$. Then every Σ-algebra determines an F_Σ-algebra and, vice versa, every F_Σ-algebra determines a Σ-algebra, the correspondence being an isomorphism.

Furthermore, let F be a polynomial functor whose input and output sets are finite. Then there is a finite set J and natural numbers $\{k_j\}_{j \in J}$ such that F is naturally isomorphic to the functor

$$F'(X) = \coprod_{j \in J} X^{k_j}$$

Thus every F-algebra can be regarded as a $\Sigma_{F'}$-algebra, where $\Sigma_{F'} = \{\sigma_j\}_{j \leq n}$ and $arity(\sigma_j) = k_j$.

Proof. For the first point, the bijective correspondence between Σ-algebras and F_Σ-algebras is given by $\langle A, \{\sigma_A : A^{arity(\sigma)} \to A\}_{\sigma \in \Sigma} \rangle \longleftrightarrow \langle A, [\sigma_A]_{\sigma \in \Sigma} : F_\Sigma(A) \to A \rangle$[5].

Next we show the second point by structural induction on the polynomial functor F. If $F = id$, the functor is already in the desired form. If $F = A$, then F is naturally isomorphic to functor $X \mapsto \coprod_{a \in A} X^0$. Now suppose that F_1 and F_2 are naturally isomorphic to functors $F_1'(X) = \coprod_{j \in J_1} X^{k_j^1}$ and $F_2'(X) = \coprod_{j \in J_2} X^{k_j^2}$, respectively. If $F = F_1 \otimes F_2$, then it is naturally isomorphic to $F'(X) = \coprod_{\langle i,j \rangle \in J_1 \times J_2} X^{k_i^1 + k_j^2}$, because products distribute (up to natural isomorphism) over coproducts in $\mathbf{Set}$. If $F = F_1 \oplus F_2$ then it is naturally isomorphic to $F'(X) = \coprod_{j \in J_1 + J_2} X^{k_j}$, where $k_j = k_j^1$ if $j \in J_1$ and $k_j = k_j^2$ if $j \in J_2$. And finally, if $F = F_1 \uparrow A$, then F is naturally isomorphic to functor $F'(X) = \coprod_{f \in J_1^A} X^{\Sigma_{a \in A} f(a)}$ (where J_1^A is the set of all functions from A to J_1), again by distributivity of products over coproducts. $\square$

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[3] Formally, $F_1 \otimes F_2 \overset{def}{=} \Delta; F_1 \times F_2; \_ \times \_ : \mathbf{Set} \to \mathbf{Set}$, where $\Delta : \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$ is the diagonal functor $\Delta : X \mapsto \langle X, X \rangle$, $F_1 \times F_2 : \langle X, Y \rangle \mapsto \langle F_1(X), F_2(Y) \rangle$, and $\_ \times \_ : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is the product functor (for an arbitrary but fixed choice of products in $\mathbf{Set}$).

[4] Similarly, $F_1 \oplus F_2 \overset{def}{=} \Delta; F_1 \times F_2; \_ + \_ : \mathbf{Set} \to \mathbf{Set}$, where $\_ + \_ : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ is the coproduct functor.

[5] If $f_i : X_i \to Y$ and $g_i : Z \to X_i$ are functions for $i \leq n$, then $[f_i]_{i \leq n} : \coprod_{i \leq n} X_i \to Y$ denotes the *copairing* of the f_i's, i.e., the only arrow determined by the universal property of coproducts, and similarly $\langle g_i \rangle_{i \leq n} : Z \to \prod_{i \leq n} X_i$ denotes the *pairing* of the g_i's.

This correspondence between algebras for a functor and algebras with respect to a signature can be extended easily to signatures with an infinite number of operators (by allowing infinite coproducts in rule 4 of Definition 4 and infinite output sets in rule 2), and with operators of infinite arity (by allowing infinite input sets).

Proposition 6 (Π-coalgebras and restricted polynomial functors). *Let $\Pi = \langle\langle X, \{I_1, \ldots, I_k\}, \{O_1, \ldots, O_h\}\rangle, \langle\{m_1, \ldots m_l\}, \{a_1, \ldots, a_n\}\rangle\rangle$ be a co-signature, where $m_j : X \times I_{k_j} \to X$ and $a_j : X \times I_{k_j} \to O_{h_j}$ for each admissible j. Let F_Π be the (restricted) polynomial functor defined as*

$$F_\Pi(X) = \prod_{j \in \underline{l}} X^{I_{k_j}} \times \prod_{j \in \underline{n}} \left(O_{h_j}\right)^{I_{k_j}}$$

where $\prod$ denotes product in $\mathbf{Set}$. Then every Π-coalgebra determines a coalgebra for the functor F_Π and vice versa, the correspondence being an isomorphism.

Similarly, let F be a restricted polynomial functor. Then there exist two natural numbers l and n and suitable sets O_j, I_i such that F is naturally isomorphic to the functor

$$F'(X) = \prod_{j \in \underline{l}} X^{I_j} \times \prod_{i \in \underline{n}} (O_i)^{I_i}$$

Thus every F-coalgebra can be regarded as a $\Pi_{F'}$-coalgebra for a co-signature that can be extracted easily from F'.

Proof. Let $A = \langle X_A, \{m_{j_A}\}_{j \in \underline{l}}, \{a_{j_A}\}_{j \in \underline{n}}\rangle$ be a Π-coalgebra. For each binary function $f : X \times Y \to Z$, let $f' : X \to Z^Y$ be its curried form, defined as $\lambda x . \lambda y . f(x, y)$. Then clearly $\langle\langle m'_{j_A}\rangle_{j \in \underline{l}}, \langle a'_{j_A}\rangle_{j \in \underline{n}}\rangle$ is a function from X_A to $F_\Pi(X_A)$, which is the F_Π-coalgebra associated with A.

For the second statement, let us check that every restricted polynomial functor F is naturally isomorphic to one in the canonical form above, by induction on the structure of F. In fact if F is the identity functor or a constant functor, then it is already in the required form (just use $\mathbf{1}$ as exponent); if $F = F_1 \otimes F_2$ with both in canonical form, then the commutativity of products up to isomorphism can be used to turn F in such form as well; and, similarly, if $F = F_1 \uparrow A$, then we get the desired form by distributing the exponent A with respect to the product, and by taking as input sets the cartesian product of A with the exponents appearing in F_1. $\qquad\square$

Example 3 (co-signatures as functors). If TS and BA are the co-signatures of Examples 1 and 2, respectively, then the following are the corresponding functors:

$$F_{TS}(X) = X \times O \qquad\qquad F_{BA}(X) = X^{\mathbb{Z}} \times \mathbb{Z}$$

The compared analysis of the last two propositions shows that the class of coalgebras considered in the rest of the paper (i.e., those presented via co-signatures according to Definition 1) is somewhat restricted, because in the corresponding functors coproducts are not allowed. In "algebraic" terms, this could be understood as restricting to signatures where all operators have the same arity.

One main problem in dealing with coalgebras for arbitrary polynomial functors is that such functors do not have a canonical form as product of coproducts of the carrier, and therefore there is no natural definition of "co-signature" in this general case. This is due to the fact that in category $\mathbf{Set}$, products distribute over coproducts, but not vice versa; that is, for arbitrary sets A, B and C, it holds $(A + B) \times C \cong (A \times C) + (B \times C)$, but in general $(A \times B) + C \not\cong (A + C) \times (B + C)$. In our view this fact explains why in the literature on the one hand algebras are presented via signatures, on the other hand coalgebras are most often presented via functors. As far as the results of this paper are concerned, many of them have already been generalized to arbitrary polynomial functors, but their presentation would require to pass from the "algebraic-like" syntax used here to a much more unfamiliar syntax that will be introduced in a future paper.

3 On the Structure of Final Coalgebras

In Universal Algebra, although the initial algebra of a signature is only determined up to isomorphism, we usually have a concrete representation of its elements in mind which are the ground terms built over the signature. Furthermore in an *algebraic specification*, i.e., a pair $\langle \Sigma, E \rangle$ where E is a set of equations over Σ, terms (possibly non-ground) play a fundamental rôle, as they appear in equations.

Similarly, an explicit description of the elements of the final coalgebra of a given co-signature will provide the ingredients for defining equations and the background of the proofs of the main theorems in the next sections. The existence of final coalgebras is ensured by the fact that every polynomial functor is bounded [Bar93, Rut96] and by Proposition 6. The leading idea in what follows is that the elements of the hidden sort of the final coalgebra are functions from sets of contexts of visible sort to elements of the corresponding sorts. In other words, one determines the identity of a state of the final coalgebra by looking at all possible observations over that state.

Definition 7 (contexts, transitions and observations). Let Π be a co-signature. A *context c of sort Y over Π*, denoted $c : Y$, is a well-sorted term of sort Y built from the operators of Π and constants of input sorts, *containing exactly one occurrence of variable, which is denoted x, and which must be of hidden sort*.[6]

A context of hidden sort is also called a *transition sequence*; the *empty context* is $x : X$; and a *transition* is a transition sequence having only the empty context as proper sub-context. The set of transitions for Π will be denoted $\mathcal{T}rans(\Pi)$, and the set of transition sequences $\mathcal{T}rans^*(\Pi)$. An *observation $c : O$* is a context of output sort; the set of observations for Π will be denoted $\mathcal{O}bs(\Pi)$. If $c : Y$ is a context and $t : X$ is a transition sequence, by $c[t/x]$ we denote the context (of sort Y) obtained by replacing the only occurrence of x in c by t.[7]

Given a Π-coalgebra $A = \langle X_A, \{m_{j_A}\}_{j \in \underline{l}}, \{a_{j_A}\}_{j \in \underline{n}} \rangle$, every context $c : Y$ over Π determines a function $[c]_A$, the *interpretation of c in A*, having X_A as domain and defined in the following way:

1. $[x]_A : X_A \to X_A$ is the identity function;
2. If $c : X$ is a non-empty transition sequence, then it must be of the form $c = m(c', \underline{v})$, for a transition sequence $c' : X$, a method m, and a constant of input sort $\underline{v} : I$. In this case function $[c]_A : X_A \to X_A$ is inductively defined as $[c]_A(y) = m_A([c']_A(y), v)$ for all $y \in X_A$.
3. Similarly, if $c : O$ is an observation, then it must be of the form $c = a(c', \underline{v})$, for a transition sequence $c' : X$, an attribute a and a constant of input sort $\underline{v} : I$. In this case function $[c]_A : X_A \to O$ is defined as $[c]_A(y) = a_A([c']_A(y), v)$ for all $y \in X_A$.

Proposition 8 (properties of context interpretation). *The following useful properties hold:*

1. *If $c \in \mathcal{T}rans^*(\Pi)$ and $c' \in \mathcal{O}bs(\Pi)$, then $[c']_A([c]_A(y)) = [c'[c/x]]_A(y)$ for all $y \in X_A$.*
2. *If $f : A \to B$ is a homomorphism and $c \in \mathcal{T}rans^*(\Pi)$, then $f([c]_A(y)) = [c]_B(f(y))$ for all $y \in X_A$.*
3. *If $f : A \to B$ is a homomorphism and $c \in \mathcal{O}bs(\Pi)$, then $[c]_A(y) = [c]_B(f(y))$ for all $y \in X_A$.*

Proof. 1. Immediate from the last part of Definition 7.
2. If $c = x$ is the empty context, then $f([x]_A(y)) = f(y) = [x]_B(f(y))$.
   Otherwise, $c = m(c', \underline{v})$ for some transition sequence c', method m and input constant $\underline{v}$. In this case, $f([c]_A(y)) = f([m(c', \underline{v})]_A(y)) = $ [by Definition 7 (2)] $= f(m_A([c']_A(y), v)) = $ [because f

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[6] In an algebraic framework contexts are usually defined as terms "with a hole", i.e., with a single occurrence of a placeholder, denoted $[]$. Thus *ground* contexts where $[]$ is of hidden sort correspond exactly to those in our definition, if we regard variable x as playing the rôle of $[]$. Our notation is justified by the desire of being as much as possible consistent with the related literature (see the discussion in Section 5.)

[7] There is an obvious bijection between sequences of elements of $\mathcal{T}rans(\Pi)$ and elements of $\mathcal{T}rans^*(\Pi)$, mapping the empty sequence to the empty context, and sequence $c \cdot c'$ to $c'[c/x]$. Hence the name "transition sequences" for the elements of $\mathcal{T}rans^*(\Pi)$.

is a homomorphism] $= m_B(f([c']_A(y)), v) =$ [by induction hypothesis] $= m_B([c']_B(f(y)), v) =$ [again by Definition 7 (2)] $= [m(c', \underline{v})]_B(f(y)) = [c]_B(f(y))$.

3. Let $c = a(c', \underline{v})$ for some transition sequence c', attribute a and input constant $\underline{v}$. In this case, $[c]_B(f(y)) = [a(c', \underline{v})]_B(f(y)) =$ [by Definition 7 (3)] $= a_B([c']_B(f(y)), v) =$ [by the previous point] $= a_B(f([c']_A(y)), v) =$ [because f is a homomorphism] $= a_A([c']_A(y), v) = [a(c', \underline{v})]_A(y) = [c]_A(y)$. □

Example 4. Let TS be the co-signature of Example 1. Then the following are legal contexts, with the associated sort: $x : X$ (the empty context), $next(x) : X$ (a transition), $val(next(next(x))) : O$ (an observation). Making reference to the TS-coalgebra SW introduced in that example, we have for example that $[next(x)]_{SW} = next_{SW}$, and $[val(next(next(x)))]_{SW} = val_{SW} \circ next_{SW} \circ next_{SW} = \{\texttt{ON} \mapsto o_1, \texttt{OFF} \mapsto o_0\}$.

Let now BA be the co-signature of Example 2. We have the following:

– $bal(ch(x, z)) : \mathbb{Z}$ is a well-sorted term, but it is not a context because it contains two variables.
– $bal(x) + \underline{5} : \mathbb{Z}$ is not a context, because operator '+' does not belong to the co-signature.
– $bal(ch(x, \underline{5})) : \mathbb{Z}$ is a legal observation. Making reference to the BA-coalgebras of Example 2, we have $[bal(ch(x, \underline{5}))]_{BA_1} = z \mapsto z + 5$, $[bal(ch(x, \underline{5}))]_{BA_0} = * \mapsto 0$, $[bal(ch(x, \underline{5}))]_{BA_h} = w \mapsto \Sigma(w) + 5$, $[bal(ch(x, \underline{5}))]_{Z_{BA}} = \psi \mapsto \psi(5)$.
– $ch(x, bal(x)) : X$ is not a context because there are two occurrences of the variable x (see Section 5).

Theorem 9 (structure of the final coalgebra). *Let Π be a co-signature as in Definition 1, and let Z_Π be defined as[8]*

$$Z_\Pi = \prod_{(c:O) \in \mathcal{O}bs(\Pi)} O$$

Spelling out the definition of dependent product, we obtain the following equivalent definition of Z_Π, that we shall use along the paper:

$$Z_\Pi = \{\psi : \mathcal{O}bs(\Pi) \to \coprod_{i \in \underline{h}} O_i \mid \forall (c : O) \in \mathcal{O}bs(\Pi) \,.\, \psi(c) \in O\}$$

That is, Z_Π is the set of all functions having the set of observations as domain, and mapping each observation of sort O to a value in O.

Next, for each method $m : X \times I \to X$ in Π define $m_Z : Z_\Pi \times I \to Z_\Pi$ as $m_Z(\psi, v) = \lambda c \,.\, \psi(c[m(x, \underline{v})/x])$, and for each attribute $a : X \times I \to O$ in Π define $a_Z : Z_\Pi \times I \to O$ as $a_Z(\psi, v) = \psi(a(x, \underline{v}))$. These operations, that are clearly well-defined, turn Z_Π into the carrier of a Π-coalgebra (that will be denoted in the same way). Then

1. *Z_Π is a final Π-coalgebra;*
2. *For each observation $c : O \in \mathcal{O}bs(\Pi)$, the interpretation of c in Z_Π, $[c]_Z : Z_\Pi \to O$ is given by $[c]_Z : \psi \mapsto \psi(c)$.*

Proof. **(1)** Let A be a Π-coalgebra, and define $!_A : X_A \to Z_\Pi$ as $!_A(y) = \lambda c \,.\, [c]_A(y)$ for all $y \in X_A$. We have to show that $!_A$ is a well-defined homomorphism and that it is unique.

For well-definedness, let $y \in X_A$. To show that $!_A(y) \in Z_\Pi$, observe that for each $c : O \in \mathcal{O}bs(\Pi)$, $!_A(y)(c) = (\lambda c' \,.\, [c']_A(y))(c) = [c]_A(y) \in O$, because $[c]_A : X_A \to O$ by Definition 7 (3).

Now we show that $!_A$ is a homomorphism, i.e., by Definition 2, that $m_Z(!_A(y), v) = !_A(m_A(y, v))$ and $a_Z(!_A(y), v) = a_A(y, v)$ for each method m, attribute a, input value v and state $y \in X_A$.

For methods, we have $m_Z(!_A(y), v) =$ [by definition of m_Z] $= \lambda c \,.\, !_A(y)(c[m(x, \underline{v})/x]) =$ [by definition of $!_A$] $= \lambda c \,.\, (\lambda c' \,.\, [c']_A(y))(c[m(x, \underline{v})/x]) = \lambda c \,.\, [c[m(x, \underline{v})/x]]_A(y) =$ [by Proposition 8

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[8] This compact notation using dependent products was suggested by Bart Jacobs.

$(1)] = \lambda c . [c]_A([m(x,\underline{v})/x]_A(y)) = $ [by Definition 7 (2)] $= \lambda c . [c]_A(m_A(y,v)) = $ [by definition of $!_A] = !_A(m_A(y,v))$.

For attributes, similarly, $a_Z(!_A(y),v) = $ [by definition of $a_Z] = !_A(y)(a(x,\underline{v})) = $ [by definition of $!_A] = (\lambda c . [c]_A(y))(a(x,\underline{v})) = [a(x,\underline{v})]_A(y) = $ [by Definition 7 (3)] $= a_A(y,v)$.

For uniqueness, suppose that $f : A \to Z_\Pi$ is a homomorphism. Then for each $y \in X_A$ and for each $c \in \mathcal{O}bs(\Pi)$, we have $f(y)(c) = $ [by the next point] $= [c]_Z(f(y)) = $ [by Proposition 8.3] $= [c]_A(y)$, showing that $f = !_A$.

(2) Let us prove now that $[c]_Z(\psi) = \psi(c)$ for all $c \in \mathcal{O}bs(\Pi)$ and $\psi \in Z_\Pi$. We first show that for each transition sequence $c : X$ over Π, $[c]_Z(\psi) = \lambda c' . \psi(c'[c/x])$ (note that this definition is well-given because if $c \in \mathcal{T}rans^*(\Pi)$ and $c' \in \mathcal{O}bs(\Pi)$, then $c'[c/x] \in \mathcal{O}bs(\Pi)$).

If $c = x$, then $[x]_Z(\psi) = $ [by Definition 7 (1)] $= \psi = \lambda c' . \psi(c'[x/x])$.

If $c = m(c'',\underline{v})$, then $[c]_Z(\psi) = $ [by Definition 7 (2)] $= m_Z([c'']_Z(\psi),v) = $ [by definition of $m_Z] = \lambda c' . ([c'']_Z(\psi)(c'[m(x,\underline{v})/x])) = $ [by induction hypothesis] $= \lambda c' . (\lambda c_1 . \psi(c_1[c''/x])(c'[m(x,\underline{v})/x])) = \lambda c' . \psi(c'[m(x,\underline{v})/x])[c''/x] = \lambda c' . \psi(c'[m(c'',\underline{v})/x])$, as desired.

Now, suppose that $c : O$ is the observation $c = a(c',\underline{v})$ for an attribute $a : X \times I \to O$, a transition sequence c', and a constant $\underline{v} : I$. Then for $\psi \in Z_\Pi$ we have $[a(c',\underline{v})]_Z(\psi) = a_Z([c']_Z(\psi),v) = $ [by definition of $a_Z] = [c']_Z(\psi)(a(x,\underline{v})) = $ [by the above] $= (\lambda c'' . \psi(c''[c'/x]))(a(x,\underline{v})) = \psi(a(x,\underline{v})[c'/x]) = \psi(a(c',\underline{v})) = \psi(c)$, as desired. $\qquad\square$

Example 5 (final coalgebras). For the co-signature TS of Example 1 the set of observations is clearly $\mathcal{O}bs(TS) = \{val(x) : O, val(next(x)) : O, \ldots, val(next^n(x)) : O, \ldots\}$ which is isomorphic to $\mathbb{N}$. Thus by the last result, the carrier of the final coalgebra is given by $\{\psi : \mathcal{O}bs(TS) \to O\} \cong \{\psi : \mathbb{N} \to O\} \cong O^\mathbb{N}$, thus as expected it is isomorphic to the carrier of Z_{TS}.

For the co-signature BA, the possible observations are $\mathcal{O}bs(BA) = \{bal(x)\} \cup \{bal(ch(x,z_1)) \mid z_1 \in \mathbb{Z}\} \cup \{bal(ch(ch(x,z_1),z_2)) \mid z_1, z_2 \in \mathbb{Z}\} \cup \ldots \cong \mathbb{Z}^*$, showing that coalgebra Z_{BA} of Example 2 is final.

4 Equational deduction in coalgebras

In this section we present the first main result of the paper, by introducing an equational calculus for coalgebras and proving its soundness and completeness. We use an algebraic terminology, avoiding to prefix all nouns to be introduced with "co-", even if this would be probably more correct.

Definition 10 (terms). For a co-signature Π, the set of Π-*terms* contains all observations for Π and all constants of output sort, formally: $Terms(\Pi) = \{t : O \mid (t : O) \in \mathcal{O}bs(\Pi)\} \cup \{\underline{o} : O \mid o \in O$ and O is an output sort of $\Pi\}$.

Given a Π-coalgebra A, every term $t : O$ induces a function $[t]_A : X_A \to O$, defined as in Definition 7 if $t : O$ is an observation, while if it is a constant $\underline{o} : O$, then $[\underline{o}]_A : y \mapsto o$ for all $y \in X_A$.

Definition 11 (equations and validity). An *equation over Π* is a pair $\langle t_1 : O, t_2 : O \rangle$ of terms of the same sort, usually written $t_1 =_O t_2$.

Given a Π-coalgebra A and an element of its carrier $y \in X_A$, we say that equation $t_1 =_O t_2$ *holds for y in A* if $[t_1]_A(y) = [t_2]_A(y)$, denoted $y, A \models t_1 =_O t_2$, or simply $y \models t_1 =_O t_2$ if A is clear from the context. The same equation is *valid* in A, denoted $A \models t_1 =_O t_2$, if it holds in A for all elements of the carrier, i.e., if $[t_1]_A$ and $[t_2]_A$ are the same function from X_A to O.

These notions extend in the expected way to a set of equations E: we write $A \models E$ if $A \models e$ for all $e \in E$. Also, if $\mathcal{A}$ is a class of algebras, we write $\mathcal{A} \models E$ if $A \models E$ for all $A \in \mathcal{A}$.

A *(ground)*[9] *coalgebraic specification* is a pair $\langle \Pi, E \rangle$, where E is a set of equations over the co-signature Π. Given a coalgebraic specification $\langle \Pi, E \rangle$, the class of its *models* $Coalg(\Pi, E)$ is defined as $Coalg(\Pi, E) = \{A \in Coalg(\Pi) \mid A \models E\}$

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[9] We shall explain in Section 6 the meaning of this qualification.

Definition 12 (rules of coalgebraic equational deduction). The *deduction rules of coalge-braic (ground) equational logic* are the following, where t, t', t'' range over $Terms(\Pi)$, O, O' over output sorts, and all such symbols are universally quantified if they don't appear in the premises; furthermore, all equations are assumed to be well-sorted:

[**reflexivity**]

$$\frac{}{\emptyset \vdash t =_O t}$$

[**symmetry**]

$$\frac{E \vdash t =_O t'}{E \vdash t' =_O t}$$

[**transitivity**]

$$\frac{E_1 \vdash t =_O t', E_2 \vdash t' =_O t''}{E_1 \cup E_2 \vdash t =_O t''}$$

[**unity**] For all output sorts O such that $card(O) = 1$,

$$\frac{}{\emptyset \vdash t =_O t'}$$

[**contradiction**] For all output sorts O such that $card(O) > 1$ and $o_1, o_2 \in O$ with $o_1 \neq o_2$,

$$\frac{E \vdash \underline{o_1} =_O \underline{o_2}}{E \vdash t =_{O'} t'}$$

[**forward closure**] For all transitions $c \in \mathcal{T}rans(\Pi)$,

$$\frac{E \vdash t =_O t'}{E \vdash t[c/x] =_O t'[c/x]}$$

where term $t[c/x] : O$ is defined as $\underline{o} : O$ if $t = \underline{o}$, and as the term obtained by substituting c for the only occurrence of x in t, if t is an observation.

Given a set of equations E and an equation e, we write $E \vdash e$ if there is a proof of $E' \vdash e$, with $E' \subseteq E$, using only the above rules and the following one:[10]

[**axiom**]

$$\frac{e \in E}{\{e\} \vdash e}$$

In the following we shall sometimes denote by $\hat{E}$ the *theory* of E, i.e., the set $\hat{E} = \{e \mid E \vdash e\}$.

Let us briefly comment on the rules just introduced. The first five rules express structural properties of equality: *reflexivity*, *symmetry* and *transitivity* are standard and do not need any comment; and obviously *unity* and *contradiction* are sound rules, due to the fixed interpretation of output sorts. In particular, *contradiction* allows one to entail any possible equation in the case of an "inconsistent" specification which equates two distinct constants.[11]

The last rule, *forward closure*, is the only one closely related to the coalgebraic structure: it states that if two observations deliver the same result when applied to each state, then they deliver the same result also when applied to a state reachable through a transition c, essentially because the carrier of the coalgebra is closed with respect to the application of methods. It is worth stressing that this rule can be considered as the dual of the *congruence* rule used in algebraic specification, stating that for every operator f of arity n, if $E \vdash t_i = t'_i$ for $i \in \underline{n}$, then $E \vdash f(t_1, \dots, t_n) = f(t'_1, \dots, t'_n)$.

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[10] Note that this definition of the entailment relation subsumes both weakening and contraction.

[11] The *soundness* of a rule equivalent to *contradiction* is remarked also by Lawrence Moss in [Mos97] in a much richer logical framework. Theorem 15 below shows that, as far as an equational calculus is concerned, such a rule contributes in an essential way to completeness. A similar rule was suggested in a private discussion by Bart Jacobs.

Example 6. Let TS be the co-signature of Example 1, and let $E_1 = \{val(x) =_O val(next(next(x)))\}$ and $E_2 = \{val(next(x)) =_O \underline{o_0}\}$ be two sets of equations.

By repeated use of rules *forward closure*, *transitivity* and *reflexivity* it is easy to see that $E_1 \vdash val(next^n(x)) =_O val(next^m(x))$ if and only if $|n-m|$ is even. Furthermore, $E_2 \vdash val(next^n(x)) =_O \underline{o_0}$ for all $n \geq 1$. Note that rule *unity* cannot be used because we assumed that $card(O) > 1$.

Theorem 13 (soundness of coalgebraic equational deduction). *Let Π be a co-signature and let E be a set of equations over Π. Then for all equations $t =_O t'$ over Π,*

$$E \vdash t =_O t' \quad \Longrightarrow \quad Coalg(\Pi, E) \models t =_O t'$$

Proof. We proceed by induction on the depth of the proof that $E \vdash t =_O t'$. If the last rule used is *axiom*, the statement holds by definition of $Coalg(\Pi, E)$. If it is *reflexivity*, *symmetry* or *transitivity*, then the statement follows by the standard properties of equality.

If the rule applied is *unity*, then set O is a singleton, and for all $A \in Coalg(\Pi, E)$ the fact that $[t]_A = [t']_A$ as functions from X_A to O follows by finality of O.

If the last rule applied is *contradiction*, then we first show that $Coalg(\Pi, E)$ contains only the empty coalgebra. In fact, suppose that $A \in Coalg(\Pi, E)$ is such that $A \models \underline{o_1} =_O \underline{o_2}$ and $y \in X_A$; this implies that $[\underline{o_1}]_A(y) =_O [\underline{o_2}]_A(y)$, i.e., $o_1 = o_2$, contradicting the premise of the rule. Now, from the definition of validity it follows immediately that every equation is valid in the empty coalgebra; thus it is the only coalgebra in $Coalg(\Pi, E)$, an it follows that $Coalg(\Pi, E) \models e$ for every equation e, showing the soundness of the rule.

If the last rule is *forward closure*, suppose by induction hypothesis that $A \models t =_O t'$, and that $c \in \mathcal{T}rans(\Pi)$ is a transition. By definition, for all $y \in X_A$ we have $y \models t =_O t'$, which implies that $[c]_A(y) \models t =_O t$, i.e., $[t]_A([c]_A(y)) = [t']_A([c]_A(y))$ for all $y \in X_A$. Then the statement follows by Proposition 8 (2), because for each term t, $[t]_A([c]_A(y)) = [t[c/x]]_A(y)$. □

Next we show that the class of models of a coalgebraic specification $Coalg(\Pi, E)$ has a final object, which is a sub-coalgebra of the final Π-coalgebra, and will play a central role in the proof of the completeness result below.

Proposition 14 (final model of a coalgebraic specification). *Let Π be a co-signature and let E be a set of equations over Π. Let Z_Π be the final coalgebra for Π (as for Theorem 9) and let*

$$Z_{\Pi \downarrow E} = \{\psi \in Z_\Pi \mid \psi \models \hat{E}\}$$

where $\hat{E}$ is the theory of E. Then $Z_{\Pi \downarrow E}$ is the carrier of a final coalgebra in $Coalg(\Pi, E)$.

Proof. We first show that $Z_{\Pi \downarrow E}$ is the carrier of a sub-coalgebra of Z_Π, then that it belongs to $Coalg(\Pi, E)$, and lastly that it is a final coalgebra there.

For the first point, let $m : X \times I \to X$ be a method in Π, m_Z be its interpretation in Z_Π, $\psi \in Z_{\Pi \downarrow E}$ and $v \in I$. By Definition 2 we have to show that $m_Z(\psi, v) \in Z_{\Pi \downarrow E}$, i.e., that $m_Z(\psi, v) \models e$ for all e such that $E \vdash e$. In fact, suppose that $E \vdash t =_O t'$; then $[t]_Z(m_Z(\psi, v)) = [$using Definition 7$] = [t[m(\psi, \underline{v})/x]]_Z(\psi) = [$by *forward closure*$] = [t'[m(\psi, \underline{v})/x]]_Z(\psi) = [t']_Z(m_Z(\psi, v))$.

Next, for all $e \in E$, $E \vdash e$ holds by rule *axiom*, thus by the very definition of $Z_{\Pi \downarrow E}$ it belongs to $Coalg(\Pi, E)$. Finally, let A be a coalgebra in $Coalg(\Pi, E)$, let $!_A : X_A \to Z_\Pi$ be the unique homomorphism to the final Π-coalgebra, and let $t =_O t'$ be an equation in $\hat{E}$. By the soundness result above we have that for all $y \in X_A$ it holds that $[t]_A(y) = [t']_A(y)$, thus $[t]_Z(!_A(y)) = [t']_Z(!_A(y))$ by point 3 of Proposition 8, and since this holds for any equation in $\hat{E}$, we have $!_A(y) \models \hat{E}$, and thus $!_A(y) \in Z_{\Pi \downarrow E}$. Therefore $!_A : X_A \to Z_{\Pi \downarrow E}$ is a well-defined homomorphism. Its uniqueness follows from the finality of Z_Π.

Alternative characterizations of the carrier of the final coalgebra satisfying a set of equations are proposed in [HR95] and [Jac96b]. In [HR95] the set $Z_{\Pi \downarrow E}$ is determined as the collection of all elements $!_A(y) \in Z_\Pi$ for $A \in Coalg(\Pi, E)$ and $y \in X_A$; in [Jac96b], instead, the same set is determined as the carrier of the largest subcoalgebra contained in the set $\{\psi \in Z_\Pi \mid \psi \models E\}$, which in general is not the carrier of a coalgebra.

Theorem 15 (completeness of coalgebraic equational deduction). *Let Π be a co-signature and let E be a set of equations over Π. Then for all equations $t =_O t'$ over Π,*

$$Coalg(\Pi, E) \models t =_O t' \quad \Longrightarrow \quad E \vdash t =_O t'$$

Proof. Let $Z_{\Pi \downarrow E}$ be the final coalgebra in $Coalg(\Pi, E)$ (as for Proposition 14). We prove by contradiction that for each equation $t =_O t'$, $Z_{\Pi \downarrow E} \models t =_O t'$ implies that $E \vdash t =_O t'$; clearly this implies completeness.

Suppose, by absurd, that $Z_{\Pi \downarrow E} \models t =_O t'$ and $E \not\vdash t =_O t'$. We can immediately deduce that (1) O is not a singleton (otherwise $E \vdash t =_O t'$ would hold by *unity*), and that (2) for each output sort O', it does not hold that $E \vdash \underline{o} =_{O'} \underline{o'}$ for two distinct elements $o, o' \in O'$ (otherwise $E \vdash t =_O t'$ would hold by *contradiction*).

Next we show that $Z_{\Pi \downarrow E}$ is not empty. For this, we use the explicit structure of Z_Π provided by Theorem 9 to determine one of its elements for which all equations derivable from E hold. For each output sort O, let $\hat{o} \in O$ be an arbitrarily chosen but fixed element (which exists by non-emptyness). Now, let function $\hat{\psi} : \mathcal{O}bs(\Pi) \to \coprod_{j \in \underline{h}} O_j$ be defined as follows:

$$\hat{\psi}(c) = \begin{cases} o & \text{if } c : O \text{ and } \underline{o} : O \in [c]_{\hat{E}} \\ \hat{o} & \text{if } c : O \text{ and there is no constant in } [c]_{\hat{E}} \end{cases}$$

Function $\hat{\psi}$ is a well-defined element of Z_Π: in fact, all terms in $[c]_{\hat{E}}$ are of the same sort, because the equations in E are well-sorted and the rules of deduction preserve this property; furthermore, it is not possible that two distinct constants are in $[c]_{\hat{E}}$, because this would contradict (2) above. Next, it is immediate to see that function $\hat{\psi}$ actually belongs to $Z_{\Pi \downarrow E}$, because its definition is consistent over all $\hat{E}$-equivalence classes. Thus $Z_{\Pi \downarrow E}$ is not empty.

Let us now proceed by case analysis on the structure of t and t', i.e., the left- and right-hand side of the equation. If both are constants of sort O, then they must be the same because $Z_{\Pi \downarrow E}$ is not empty, but then $E \vdash t =_O t'$ by reflexivity, yielding a contradiction.

Suppose now that $t : O$ is an observation, while $t' = \underline{o} : O$ is a constant, and consider the equivalence class $[t]_{\hat{E}}$. Obviously, such class cannot contain $\underline{o}$ (otherwise $E \vdash t =_O t'$), neither can it contain a constant $\underline{o'} \neq \underline{o}$. In this last case, in fact, we would have $E \vdash t =_O \underline{o'}$, which implies (by soundness) $Z_{\Pi \downarrow E} \models t =_O \underline{o'}$, and thus $Z_{\Pi \downarrow E} \models \underline{o} =_O \underline{o'}$, which is absurd by non-emptyness. Now let ψ be an arbitrary element of $Z_{\Pi \downarrow E}$ (which exists by non-emptyness), let o' be an element of O different from o (which exists by (1) above), and define function ψ' as follows:

$$\psi'(c) = \begin{cases} \psi(c) & \text{if } c \notin [t]_{\hat{E}} \\ o' & \text{if } c \in [t]_{\hat{E}}. \end{cases}$$

By construction we have that $\psi' \in Z_{\Pi \downarrow E}$, and clearly $\psi' \not\models t =_O \underline{o}$, contradicting the hypothesis that $Z_{\Pi \downarrow E} \models t =_O t'$.

As for the last case, suppose that both t and t' are observations, and consider the equivalence classes $[t]_{\hat{E}}$ and $[t']_{\hat{E}}$. By arguments similar as above, we can deduce that both classes contain at most one constant, which is necessarily of sort O. Furthermore, it is not possible that both contain a constant. For, if the two constant were equal, we could infer that $E \vdash t =_O t'$ (by reflexivity and transitivity); and if they were distinct, we could obtain that $Z_{\Pi \downarrow E} \models \underline{o} =_O \underline{o'}$ for $o \neq o'$, which is impossible by non-emptyness. Now using the same technique as in the last case, we can easily find an element $\psi' \in Z_{\Pi \downarrow E}$ such that $\psi' \models t =_O o$ and $\psi' \models t' =_O o'$ for $o \neq o'$, which contradicts the hypothesis that $Z_{\Pi \downarrow E} \models t =_O t'$. $\square$

Example 7. Let E_1 and E_2 be the equations of Example 6. Making reference to the TS-coalgebras of Example 1, it is easy to check, for example, that

- $SW \models E_1$
- $TS_1 \models E_1$ and $TS_2 \models E_1$

– $TS_m \not\models E_1$ for $m > 2$
– $TS_1 \models E_2$ and $TS_m \not\models E_2$ for $m > 1$

Furthermore, if Z_{TS} is the final TS-coalgebra having carrier $O^{\mathbb{N}}$, then its sub-coalgebra $Z_{TS\downarrow E_1}$, as for Proposition 14, has as carrier the set $\{(o_i \cdot o_j)^\omega \mid i, j \in \{0, \ldots, n\}\}$,[12] while $Z_{TS\downarrow E_2}$ has as carrier $\{o \cdot o_0^\omega \mid o \in O\}$.

5 Coalgebraic specifications in the related literature

We discuss here how our approach is related to some simple examples of coalgebraic specifications taken from the literature. In general, we will see that less restrictive kinds of equations are often used, for which, however, the topic of finding a complete calculus of deduction has not been addressed. So we shall hint at possible generalizations of our approach which should allow us to capture such more general formats of equations.

Equations as introduced in Definition 11, i.e., pairs of observations, can be found for example in [Rei95, Jac96b]. But in the same papers one finds immediately other equations which are not legal according to our definition. This can happen for various reasons, that we shall discuss in turn.

Equations with variables of input sorts. As an example, in [Jac96a] the co-signature[13] BA of Example 2 is extended with attribute $name : X \to String$ and with method $ch\text{-}name$: $X \times String \to X$ with the obvious meaning. Then, among others, equation $bal(ch\text{-}name(s, x)) = bal(x)$ is considered, stating that the change of the owner's name should not affect the balance of the account. In this equation s is a variable of type $String$, which is not allowed according to our definition. This fact is not really problematic because such an equation can safely be considered as an *equation scheme*, representing the set of equations obtained by replacing s with all possible constants of type $String$.

Equations containing algebraic operators. In [Jac96b] the following equation over BA is considered: $bal(ch(x, z)) = bal(x) + z$. Even replacing the variable $z : \mathbb{Z}$ with a constant, such an equation would not be legal because "$+$" is not an operator of the co-signature, but it belongs to the algebraic structure of $\mathbb{Z}$. This is a quite a common situation in literature, as mentioned in the Introduction, where algebraic and coalgebraic aspects are mixed in a specification. Sticking to our pure coalgebraic setting, the equation above could be replaced by an equivalent, infinite set of *conditional* equations like

$$bal(x) = \underline{z'} \quad \Longrightarrow \quad bal(ch(x, \underline{z})) = \underline{z' + z}$$

where $\underline{z'}, \underline{z}$, and $\underline{z' + z}$ are three constants of sort $\mathbb{Z}$ related in the expected way.

Terms with multiple occurrences of state variable. As stressed in Example 4, a term like $bal(ch(x, bal(x))) : X$ is not a context, according to our definition, because it contains two occurrences of variable x. Our intuitive reason for this restriction is that contexts should be basic experiments, and $bal(ch(x, bal(x)))$ is not basic, as it consists of two experiments: first observe $bal(x)$, then change state according to the result and observe again the balance. The study of a generalization of the calculus to admit such terms is left as future work. As for the previous point, one possibility would be to use conditional equations. For example, equation $bal(ch(x, bal(x))) = 2 * bal(x)$ would become the conditional equation scheme

$$bal(x) = \underline{z} \quad \Longrightarrow \quad bal(ch(x, \underline{z})) = \underline{2 * z}.$$

Conditional equations. As soon as the examples become a bit larger, conditional equations are needed: see for example, besides of the considerations in the last two points, the specification

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[12] If w is a sequence, w^ω denotes the stream obtained by concatenating infinitely many times w.

[13] The examples from other papers we refer to in this section are often presented with a different terminology. We take the freedom of recasting them according to our syntax.

of a simple database system in [Rei95], or that of a memory in [Jac97]. Our hope is that the results presented in this paper generalize smoothly to the conditional case, as it happens in the algebraic case (see [Sel72]): this is a topic for future work.

Equations of hidden sort. In [HR95] the authors consider an equation of hidden sort which we can recast as equation $e \equiv (next(next(x)) =_X x)$ over co-signature TS. Such an equation is said to be *valid* in a TS-coalgebra A if $[next(next(x))]_A = [x]_A$, and *behaviourally valid* if for all $y \in A$ we have $!_A([next(next(x))]_A(y)) = !_A([x]_A(y))$, i.e., the interpretations in A of the two transition sequences $next(next(x))$ and x map each state to *bisimilar* states.[14] Making reference to Example 1, we have for example that $SW \models e$, $TS_2 \models_{beh} e$, but $TS_2 \not\models e$. In [Jac97] a different syntax is used for equations that are required to be behaviourally valid, namely $t \leftrightarrow t'$.

As far as behavioural validity is concerned, the above equation e is equivalent to the infinite set of equations $E = \{t[next(next(x))/x] =_O t \mid (t : O) \in \mathcal{O}bs(TS)\}$, which are legal according to Definition 11. Therefore under this interpretation an equation of hidden sort can be regarded as a syntactic representation of an infinite set of equations.

If instead the above equation is interpreted as true equality of elements of the carrier of a coalgebra, then it does not fit in the formal framework introduced in the previous section. Nevertheless, as shown in the next sections, the calculus of deduction can be extended (starting from apparently unrelated motivations) to a calculus for *generalized* equations, i.e., equations as in Definition 11 but possibly of hidden sort as well.

6 Colours as dual to variables

The aim of this and of the following section is to generalize the completeness result of Section 4 to the case of "non-ground" equations. Obviously, we first have to explain in which sense the equations considered till now are "ground", and we will do this by reasoning by analogy to the case of algebraic specification. Once a convincing duality is established between algebraic and coalgebraic concepts, we will use this relationship to determine what are reasonable coalgebraic notions dualizing the algebraic notions of *variables*, *assignments* and *substitutions*. The main source of inspiration for this is [Rut96].

Let us recall, in a very informal and incomplete way, some basic notions of *algebraic* specification that we shall use as guidelines for setting up the coalgebraic counterpart. In this paragraph we use standard algebraic terminology (e.g., [MT92]); for example, by *equation* we mean the traditional ones (that we assume familiar to the reader) and not those of Definition 11. Every algebraic specification $\langle \Sigma, E \rangle$ where E is a set of equations over Σ, possibly with variables in X, determines a class of Σ-algebras, denoted $Alg(\Sigma, E)$, consisting of all algebras where the equations of E are valid. Such class can also be defined in a diagrammatic way as follows (see the left diagram of Figure 1): *a Σ-algebra A belongs to $Alg(\Sigma, E)$ iff for each* assignment $\alpha : X \to A$, *the unique extension $\overline{\alpha} : T_\Sigma(X) \to A$ (by the freeness of $T_\Sigma(X)$) factorizes via the natural mapping $nat_E : T_\Sigma(X) \to T_\Sigma(X)/{\equiv_E}$,* where $\equiv_E$ denotes the least congruence induced by E. In the case where the equations in E are ground, the definition of $Alg(\Sigma, E)$ can be simplified as follows (see the right diagram of Figure 1): *a Σ-algebra A belongs to $Alg(\Sigma, E)$ iff the unique homomorphism $0_A : T_\Sigma \to A$ (by initiality of T_Σ) factorizes via the natural mapping $nat_E : T_\Sigma \to T_\Sigma/{\equiv_E}$.*

For the case of coalgebraic specification introduced in the previous section, we have a similar situation illustrated by the left diagram of Figure 2.

Proposition 16 (diagrammatic characterization of $Coalg(\Pi, E)$). *Let $\langle \Pi, E \rangle$ be a ground coalgebraic specification, let $Coalg(\Pi, E)$ be as in Definition 11, and let $Z_{\Pi \downarrow E}$ be as in Proposition 14.*

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[14] For the class of coalgebras considered in this paper, we can safely define as "bisimilar" two states which are mapped, by the unique homomorphism, to the same element of the final coalgebra. As a consequence, since homomorphisms preserve observations, bisimilar states cannot be distinguished through observations. For the more general definition of bisimulation of coalgebras and the relationship with the notion of bisimulation in process algebra we refer the reader to [Rut96].
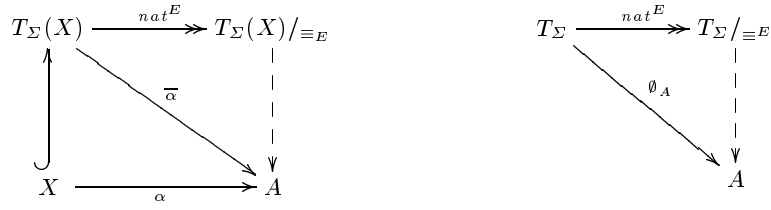
Fig. 1. Characterization of class $Alg(\Sigma, E)$ for a set E of equations with variables in X (left) and of ground equations (right).

Then a Π-coalgebra A belongs to $Coalg(\Pi, E)$ iff the only homomorphism to the final coalgebra Z_Π factorizes through the inclusion $in_E : Z_{\Pi\downarrow E} \hookrightarrow Z_\Pi$.

Proof. We have $A \in Coalg(\Pi, E)$ iff $\forall y \in X_A \,.\, y, A \models E$ iff [using Proposition 8 (3)] $\forall y \in X_A \,.\, !_A(y), Z_\Pi \models E$ iff [by soundness and since $E \subseteq \hat{E}$] $\forall y \in X_A \,.\, !_A(y), Z_\Pi \models \hat{E}$ iff [by definition of $Z_{\Pi\downarrow E}$] $\forall y \in X_A \,.\, !_A(y) \in Z_{\Pi\downarrow E}$. $\qquad\qquad\square$
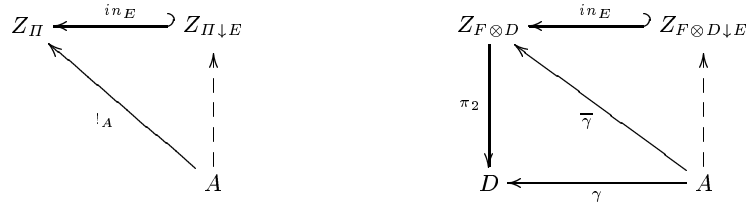

Fig. 2. (Left) Characterization of class $Coalg(\Pi, E)$ for a set E of equations over Π. (Right) Dual of the left diagram of Figure 1.

Now comparing the left diagram of Figure 2 and the right diagram of Figure 1, it should be quite evident in which sense they are dual to each other. The following table relates some concepts to their dual: this correspondence could be made more formal using category theory in terms of free and cofree constructions (or left and right adjoints), but we shall not need this.

Initial algebra	Final coalgebra
Unique homomorphism from the initial algebra	Unique homomorphism to the final coalgebra
Quotient algebra	Sub-coalgebra
Surjective homomorphism	Injective homomorphism
Natural mapping	Inclusion

This precise correspondence explains why we call the calculus of Section 4 *ground*: because it corresponds to the algebraic ground case. And it also provides a guideline, that we will immediately follow, to generalize coalgebraic specification to the "non-ground" case: that is, we will consider the dual of the left diagram of Figure 1, that we depicted in the right part of Figure 2.

Given a functor F, the idea is to fix a set D, whose elements we shall call *colours*, and to consider the final coalgebra with respect to functor $F \otimes D$ (this exists if F has a final coalgebra). As shown, e.g., in [Rut96], the final coalgebra $Z_{F \otimes D}$ is *cofree* over D, i.e., if A is an F-coalgebra, every function $\gamma : X_A \to D$ (which we call *colouring*) determines a unique cofree extension

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$\overline{\gamma} : X_A \to Z_{F \otimes D}$ making the lower triangle commuting. If F is a restricted polynomial functor (the case we are interested in), then so is $F \otimes D$, and as we will see the co-signature associated with $F \otimes D$ "extends" the co-signature of F. Therefore we can consider a set of equations E over this extended co-signature, which will determine a sub-coalgebra $Z_{F \otimes D \downarrow E}$ of $Z_{F \otimes D}$. And we can define $Coalg(F, D, E)$ as the class of all F-coalgebras A such that for each colouring $\gamma : X_A \to D$ its cofree extension $\overline{\gamma}$ factorizes through the inclusion $in_E : Z_{F \otimes D \downarrow E} \hookrightarrow Z_{F \otimes D}$.

Summarizing, the above correspondence between algebraic and coalgebraic concepts can be extended as follows:

Free algebra	Cofree coalgebra
Variables	Colours
Assignment	Colouring

It is worth stressing that the definition of class $Coalg(F, D, E)$ just given is an example of *covariety* as introduced in [Rut96]. The relationship between our classes of coalgebras and covarieties will be commented further in Section 8. In the next section we will actually redefine class $Coalg(\Pi, D, E)$ (for a co-signature Π) in a more standard, though equivalent, way, via a notion of validity.

7 Equational deduction with colouring

Here we generalize the completeness result of Section 4 to the case of equations over a fixed sets of "colours", by extending the calculus of Definition 12 with two additional rules. Making reference to the discussion at the end of the last section, it is easily seen that if Π_F is the co-signature associated with a restricted polynomial functor F (see Proposition 6), then co-signature $\Pi_{F \otimes D}$ of functor $F \otimes D$ is obtained simply by adding to Π_F a new attribute of output sort D, that we will denote $col : X \to D$ (assuming, without loss of generality, that col does not clash with names in Π). Here are the definitions and results that generalize the ground case considered in Section 4.

Definition 17 (equations and validity, the coloured case). Let Π be a co-signature and D be a set of *colours*. By $\Pi[D]$ we denote the co-signature which extends Π by adding D to the set of output sorts, and $col : X \to D$ to the attributes.

Given a Π-coalgebra A and a *colouring* $\gamma : X_A \to D$, every context $c : Y$ over $\Pi[D]$ determines a function $[c]_{\langle A, \gamma \rangle}$ having X_A as domain, in the following way.

1. If c is a transition sequence or c is an observation not containing attribute col, then $[c]_{\langle A, \gamma \rangle} = [c]_A$, as given in Definition 7.
2. If $c = col(c') : D$ where c' is a transition sequence, then $[c]_{\langle A, \gamma \rangle} = \gamma \circ [c']_A$.

Given a Π-coalgebra A, an element of its carrier $y \in X_A$, a colouring $\gamma : X_A \to D$, and an equation $t_1 =_O t_2$ over $\Pi[D]$, we say that:

- $t_1 =_O t_2$ *holds for y in* $\langle A, \gamma \rangle$ if $[t_1]_{\langle A, \gamma \rangle}(y) = [t_2]_{\langle A, \gamma \rangle}(y)$, denoted $y, \langle A, \gamma \rangle \models t_1 =_O t_2$.
- $t_1 =_O t_2$ *holds for y in A, denoted $y, A \models t_1 =_O t_2$*, if for all $\gamma : X_A \to D$ we have $y, \langle A, \gamma \rangle \models t_1 =_O t_2$
- $t_1 =_O t_2$ *is valid in A*, denoted $A \models t_1 =_O t_2$, if for all $y \in X_A$ we have $y, A \models t_1 =_O t_2$.

The notion of validity extends in the expected way to sets of equations and to classes of Π-coalgebras. A *coalgebraic specification* is a triple $\langle \Pi, D, E \rangle$, where E is a set of equations over $\Pi[D]$. Its class of *models* is given by $Coalg(\Pi, D, E) = \{A \in Coalg(\Pi) \mid A \models E\}$.

Definition 18 (extending the rules of equational deduction). Let Π be a co-signature and D be a set of colours, *whose cardinality is greater than 1*. The *deduction rules of coalgebraic equational logic* are given by the rules of Definition 12 (i.e., *reflexivity, symmetry, transitivity, unity, contradiction* and *forward closure*, where now t, t', t'' are intended to range over $Terms(\Pi[D])$), plus the following ones, where additionally c, c' are intended to range over transition sequences; as usual, all symbols are universally quantified if they don't appear in the premises, and all equations are assumed to be well-sorted.

[contradiction-2]

$$\frac{E \vdash col(c) =_D \underline{d}}{E \vdash t' =_O t''}$$

[bisimilarity] For all $t : O \in \mathcal{O}bs(\Pi[D])$,

$$\frac{E \vdash col(c) =_D col(c')}{E \vdash t[c/x] =_O t[c'/x]}$$

Given a set of equations E and an equation e, both over $\Pi[D]$, the meaning of $E \vdash e$ is as at the end of Definition 12, but now also allowing in a proof the two rules just introduced.

Even if the soundness of the above rules will be proved in the next result, a preliminary intuitive explanation may be helpful. As far as rule *contradiction-2* is concerned, an equation like $col(c) = \underline{d}$ would hold for an element y of the carrier of a Π-coalgebra A if for every possible colouring the element $[c]_A(y)$ had colour d. But this clearly does not hold for a colouring which maps $[c]_A(y)$ to an element $d' \neq d$, which exists because D has at least two elements. Thus such an equation can be valid only in the empty coalgebra, in which all equations are valid.

For the *bisimilarity* rule, $col(c) =_D col(c')$ holds for an element y of the carrier of a Π-coalgebra A only if for every colouring, the element $[c]_A(y)$ has the same colour as $[c']_A(y)$. But this is only possible if this two elements coincide: in this case they are obviously bisimilar, i.e., indistinguishable by any observation, as stated by the rule. Note that the premise of the rule actually implies that $[c]_A(y) = [c']_A(y)$, which is a stronger property than bisimilarity, but which is not expressible by an equation because equations of hidden sorts are not allowed, yet: they will be considered later in this section.

Let us present now the main result.

Theorem 19 (soundness and completeness, the coloured case). *Let Π be a co-signature, let D be a set of colours, and let E be a set of equations over $\Pi[D]$. Then for all equations $t =_O t'$ over $\Pi[D]$,*

$$E \vdash t =_O t' \quad \Longleftrightarrow \quad Coalg(\Pi, D, E) \models t =_O t'$$

Proof. **(Soundness, $\Longrightarrow$)** We proceed by induction on the depth of the proof that $E \vdash t =_O t'$. If the last rule applied is one introduced in Definition 12, then the argument is the same as in the proof of Theorem 13.

If the last rule applied is *contradiction-2*, then we show that $Coalg(\Pi, D, E)$ contains only the empty coalgebra. In fact, suppose that $A \in Coalg(\Pi, D, E)$ is such that $A \models col(c) =_D \underline{d}$ and $y \in X_A$. By definition of validity, this implies that for every colouring $\gamma : X_A \to D$, $[col(c)]_{\langle A, \gamma \rangle}(y) = d$, i.e., by Definition 17, $\gamma([c]_A(y)) = d$. But let $d' \neq d$ be an element of D (which exists because $card(D) > 1$), and let γ' be defined by $\gamma'(y) = d'$ for all $y \in X_A$. Clearly, the equation does not hold for y in $\langle A, \gamma' \rangle$, contradicting the assumption. Furthermore, from the definition of validity it follows immediately that every equation is valid in the empty coalgebra; thus it is the only coalgebra in $Coalg(\Pi, D, E)$, and it follows that $Coalg(\Pi, D, E) \models e$ for every equation e, showing the soundness of the rule.

Suppose now that the last rule applied is *bisimilarity*, and, by induction hypothesis, that $A \models col(c) =_D col(c')$. If $y \in X_A$, it follows that for every colouring $\gamma : X_A \to D$, $\gamma([c]_A(y)) = \gamma([c']_A(y))$. But this is only possible if $[c]_A(y) = [c']_A(y)$, i.e., they are the same element in X_A,

because otherwise we could easily define a colouring which maps them to two distinct colours (which exist by the hypothesis on D). Now let γ be a colouring and t be an observation: we have two cases, namely either t is an observation over Π, or $t = col(c'') : D$. In the first case, $[t[c/x]]_{\langle A,\gamma \rangle}(y) = [t]_A([c]_A(y)) = [t]_A([c']_A(y)) = [t[c'/x]]_{\langle A,\gamma \rangle}(y)$; in the second case, similarly, $[col(c'')[c/x]]_{\langle A,\gamma \rangle}(y) = \gamma([c'']_A([c]_A(y))) = \gamma([c'']_A([c']_A(y))) = [col(c'')[c'/x]]_{\langle A,\gamma \rangle}(y)$. Since this holds for all γ and for all $y \in X_A$, we have $A \models t[c/x] =_O t[c'/x]$, showing the soundness of the rule.

(Completeness, $\Longleftarrow$) The proof of completeness follows essentially the same outline as that of Theorem 15. Let $Z_{\Pi[D]}$ be the final coalgebra for $\Pi[D]$ (as for Theorem 9) and let

$$Z_{\Pi[D]\downarrow E} = \{\psi \in Z_{\Pi[D]} \mid \psi \models \hat{E}\}$$

where $\hat{E}$ is the theory of E. It is easy to show that $Z_{\Pi[D]\downarrow E}$ is the carrier of a sub-coalgebra of $Z_{\Pi[D]}$, by using *forward closure* as in the proof of Proposition 14. Now, as a sub-coalgebra of $Z_{\Pi[D]}$, $Z_{\Pi[D]\downarrow E}$ comes equipped with functions realizing all the operators in $\Pi[D]$ (which, by the way, are suitable restrictions of the corresponding functions in $Z_{\Pi[D]}$). But since the operators in Π are a proper subset of those in $\Pi[D]$, we have that $Z_{\Pi[D]\downarrow E}$ is also the carrier of a Π-coalgebra. In the rest of the proof, we will denote $Z_{\Pi[D]\downarrow E}$ by Z when it is regarded as a $\Pi[D]$-coalgebra, and by S when regarded as a Π-coalgebra.

Now we show that $S \in Coalg(\Pi, D, E)$, i.e., that for each $\psi \in S$, for each colouring $\gamma : S \to D$ and for each equation $t_1 =_O t_2$ over $\Pi[D]$ in E, we have $[t_1]_{\langle S,\gamma \rangle}(\psi) = [t_2]_{\langle S,\gamma \rangle}(\psi)$. If the output sort O is different from D, then $[t_1]_{\langle S,\gamma \rangle}(\psi) = $ [by point 1 of Definition 17] $= [t_1]_S(\psi) = $ [since Z extends S] $= [t_1]_Z(\psi) = $ [by the definition of $Z_{\Pi[D]\downarrow E}$] $= [t_2]_Z(\psi) = [t_2]_S(\psi) = [t_2]_{\langle S,\gamma \rangle}(\psi)$. If instead $O = D$, then we have three cases depending on the structure of t_1 and t_2. (a) If both t_1 and t_2 are constants, then either they are equal (and the statement is trivial), or they are distinct, and then S is empty and the statement holds because all equations are valid in the empty coalgebra. (b) If $t_1 = col(c)$ for a transition sequence c and $t_2 = \underline{d}$, then we can deduce that $S = \emptyset$ by the same argument as in the *soundness* part, and $S \in Coalg(\Pi, D, E)$ follows immediately. (c) Suppose now that $t_1 = col(c_1)$ and $t_2 = col(c_2)$ for $c_1, c_2 \in \mathcal{T}rans^*(\Pi[D])$. By point 2 of Definition 17 we have $[t_1]_{\langle S,\gamma \rangle}(\psi) = [col(c_1)]_{\langle S,\gamma \rangle}(\psi) = \gamma([c_1]_S(\psi))$, and similarly $[t_2]_{\langle S,\gamma \rangle}(\psi) = \gamma([c_2]_S(\psi))$. Therefore $\psi, \langle S, \gamma \rangle \models t_1 =_O t_2$ holds if we can prove that $[c_1]_S(\psi) = [c_2]_S(\psi)$. For this, note that since $\psi \in Z$ by hypothesis, we know that $\psi, Z \models col(c_1) =_D col(c_2)$, and, since $\hat{E}$ is closed under rule *bisimilarity*, that $\psi \models t[c_1/x] =_O t[c_2/x]$ for every observation $t : O \in \mathcal{O}bs(\Pi[D])$. Therefore $[c_1]_Z(\psi)$ and $[c_2]_Z(\psi)$ must be the same element of Z, because every function in $Z_{\Pi[D]}$ is uniquely determined by its action on $\mathcal{O}bs(\Pi[D])$. More formally, we have $[c_1]_Z(\psi) = \lambda t \in \mathcal{O}bs(\Pi[D]) . ([c_1]_Z(\psi))(t) = $ [by Theorem 9 (2)] $= \lambda t . [t]_Z([c_1]_Z(\psi)) = $ [by Proposition 8 (1)]$= \lambda t . [t[c_1/x]]_Z(\psi) = $ [by *bisimilarity*] $= \lambda t . [t[c_2/x]]_Z(\psi) = [c_2]_Z(\psi)$. The fact that $[c_1]_S(\psi) = [c_2]_S(\psi)$ follows by observing that S and Z have the same carrier, and for each $c \in \mathcal{T}rans^*(\Pi[D])$, $[c]_S = [c]_Z$. This completes the proof that $S \in Coalg(\Pi, D, E)$.

Now we show that for every equation $t_1 =_O t_2$ over $\Pi[D]$, $S \models t_1 =_O t_2 \Longrightarrow E \vdash t_1 =_O t_2$, from which the completeness follows immediately. Suppose, by absurd, that $S \models t_1 =_O t_2$ and $E \nvdash t_1 =_O t_2$. We can immediately deduce that O is not a singleton, that for each output sort O' and two distinct elements $o, o' \in O'$, $E \nvdash \underline{o} =_{O'} \underline{o'}$, and, for analogous reasons, that (†) for each transition sequence c and color $d \in D$, $E \nvdash col(c) =_D \underline{d}$. Furthermore, it can be shown that S is not empty, using exactly the same technique as in the proof of Theorem 15.

Let us now proceed by case analysis on the structure of the equation $t_1 =_O t_2$. If the output sort O is different from D, then a contradiction to the assumptions can be obtained as in the corresponding part of the proof of Theorem 15. If $O = D$, then we have three cases. (a) Both t_1 and t_2 are constants; then either they are equal, and this contradicts $E \nvdash t_1 =_O t_2$ by *reflexivity*, or they are distinct, and this contradicts $S \models t_1 =_O t_2$ and non-emptyness of S. (b) Only one among t_1 and t_2 is a constant, for example, $t_1 = col(c)$ and $t_2 = \underline{d}$; this contradicts the non-emptyness of S, by the same argument as in the proof of soundness of rule *contradiction-2*. (c) Both t_1 and t_2 are observations, say $t_1 = col(c_1) : D$ and $t_2 = col(c_2) : D$. Then consider the equivalence classes

$[col(c_1)]_{\hat{E}}$ and $[col(c_2)]_{\hat{E}}$: these classes cannot contain any constant of sort D (by (†)). Now let $\underline{d}, \underline{d'}$ be two different constants of sort D (which exist because $card(D) > 1$ by hypothesis), let ψ be an arbitrary element of Z, and define ψ' as follows:

$$\psi'(c) = \begin{cases} \underline{d} & \text{if } c \in [col(c_1)]_{\hat{E}} \\ \underline{d'} & \text{if } c \in [col(c_2)]_{\hat{E}} \\ \psi(c) & \text{otherwise.} \end{cases}$$

It is not difficult to show that $\psi', Z_{\Pi[D]} \models \hat{E}$, using the fact that $\psi, Z_{\Pi[D]} \models \hat{E}$ and that ψ' modifies ψ consistently with respect to $\hat{E}$-equivalence classes. Therefore $\psi' \in Z_{\Pi[D] \downarrow E}$. Now we show that $\psi', S \not\models col(c_1) =_D col(c_2)$, which contradicts $S \models t_1 =_O t_2$. In fact, let $\gamma : S \to D$ be any colouring such that $\gamma([c]_S(\psi')) = [col(c)]_Z(\psi')$ for each $c \in \mathcal{T}rans^*(\Pi[D])$; Then we have $[col(c_1)]_{\langle S, \gamma \rangle}(\psi') = \gamma([c_1]_S(\psi')) = [col(c_1)]_Z(\psi') = \psi'(col(c_1)) = d$, while, similarly $[col(c_2)]_{\langle S, \gamma \rangle}(\psi') = d'$. □

Example 8 ([Rut96]). Let $Triv$ be the co-signature $Triv = \{next : X \to X\}$. A $Triv$-coalgebra A is given simply by a set X_A and a function $next_A : X_A \to X_A$. The set of observations $\mathcal{O}bs(Triv)$ is empty, and the final coalgebra Z_{Triv} has a singleton as carrier (which is the empty function, according to Theorem 9).

Now let $\mathbf{2}$ be the two-element set $\mathbf{2} = \{0, 1\}$, let $Triv[\mathbf{2}]$ be the co-signature $\{next : X \to X, col : X \to \mathbf{2}\}$, and let $E = \{col(next(next(x))) =_{\mathbf{2}} col(x)\}$ be a set of equations over $Triv[\mathbf{2}]$.

It is easy to check that $Coalg(Triv, \mathbf{2}, E)$ contains all the $Triv$-coalgebras A such that for each $y \in X_A$, $next_A(next_A(y)) = y$. In fact, if this holds, then for each $y \in X_A$ and colouring $\gamma : X_A \to \mathbf{2}$ we have $[col(next(next(x)))]_A(y) = \gamma(next_A(next_A(y))) = \gamma(y) = [col(x)]_A(y)$, and thus $A \models E$. If instead there is a $y \in X_A$ such that $next_A(next_A(y)) \neq y$, then we can find a colouring γ such that, for example, $\gamma(next_A(next_A(y))) = 0$ and $\gamma(y) = 1$, showing that $A \not\models E$.

Alternatively, $Coalg(Triv, \mathbf{2}, E)$ can be characterized in a diagrammatic way, as suggested at the end of the last section (this is the approach followed in [Rut96]). First notice that $Z_{Triv[\mathbf{2}]}$, the final $Triv[\mathbf{2}]$-coalgebra, is isomorphic to $\{0, 1\}^{\mathbb{N}}$; in fact, using Theorem 9 we have that $\mathcal{O}bs(Triv[\mathbf{2}]) = \{col(x), col(next(x)), \ldots, col(next^n(x)), \ldots\} \cong \mathbb{N}$, thus $Z_{Triv[\mathbf{2}]} \cong \{\psi : \mathbb{N} \to \mathbf{2}\} \cong \{0, 1\}^{\mathbb{N}}$. Next, it is easy to check that $Z_{Triv[\mathbf{2}] \downarrow E}$ is the sub-coalgebra of $Z_{Triv[\mathbf{2}]}$ containing elements $\{0^{\omega}, 1^{\omega}, (0 \cdot 1)^{\omega}, (1 \cdot 0)^{\omega}\}$ (these are the only elements of $\{0, 1\}^{\mathbb{N}}$ for which the equations in $\hat{E}$ are valid). Finally, it is shown in [Rut96] that the $Triv$-coalgebras such that for each colouring γ the cofree extension $\overline{\gamma}$ factorizes through the inclusion are exactly the coalgebras where each element of the carrier belongs to a one- or two-loop,[15] and this is equivalent to $next_A(next_A(y)) = y$.

The next result shows that as far as the classes of coalgebras determined by coalgebraic specifications are concerned, it is sufficient to consider only a set of colours containing two elements.

Proposition 20 (two colours are enough). *Let $Coalg(\Pi, D, E)$ be the class of Π-coalgebras determined, according to Definition 17, by a set of equations E over $\Pi[D]$. Then there is a set of equations E_2 over $\Pi[\mathbf{2}]$ such that $Coalg(\Pi, D, E) = Coalg(\Pi, \mathbf{2}, E_2)$.*

Proof. Define E_2 as the set of equations over $\Pi[\mathbf{2}]$ containing:

1. all equations in E of sort different from D;
2. equation $\underline{0} =_{\mathbf{2}} \underline{1}$ if E includes an equation of sort D where the left- and right-hand sides are different, and at least one is a constant.
3. $col(c) =_{\mathbf{2}} col(c')$ if E contains $col(c) =_D col(c')$.

Now suppose that $A \in Coalg(\Pi, \mathbf{2}, E_2)$ is a Π-coalgebra. We have to show that for all $\gamma : X_A \to D$, for all $y \in X_A$, and for all $t_1 =_O t_2 \in E$, we have $y, \langle A, \gamma \rangle \models t_1 =_O t_2$. For all equations of E of sort different from D this is obvious, because they belong to E_2 as well and the colouring does not

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[15] For a $Triv$-coalgebra A and $y \in X_A$, we say that y belongs to an n-loop if $y = next_A^n(y)$ and $y \neq next_A^m(y)$ for $m < n$

affect the evaluation of terms not containing attribute col. Thus suppose that t_1 and t_2 are of sort D. If they are equal, the statement is obvious by *reflexivity*. If at least one of t_1, t_2 is a constant, then A must be the empty coalgebra, and $y, \langle A, \gamma \rangle \models t_1 =_D t_2$ holds because all equations are valid in the empty coalgebra. Finally, if $t_1 = col(c)$ and $t_2 = col(c')$, then by point 3 we know that $y, \langle A, \gamma' \rangle \models col(c) =_\mathbf{2} col(c')$ for all $\gamma' : X_A \to \mathbf{2}$, which implies that $[c]_A(y) = [c']_A(y)$, from which $y, \langle A, \gamma \rangle \models t_1 =_D t_2$ follows. Therefore $A \in Coalg(\Pi, D, E)$. The opposite inclusion is similar. $\square$

From the proofs of the last two results we know that the validity of equation $col(c) =_D col(c')$ in a coalgebra A implies that for each element $y \in X_A$ the states $[c]_A(y)$ and $[c']_A(y)$ coincide. But, quite reasonably, we could think of expressing this fact in a more direct way by saying that the equation of hidden sort $c =_X c'$ is valid in A. In the rest of this section we will follow this intuition, by allowing for equations of hidden sort, and showing that we can easily get a sound and complete equational calculus for such generalized equations, which is a slight modification of that introduced in Definition 18.

Definition 21 (generalized equations). A *generalized equation* over a co-signature Π is either an equation as in Definition 11, or an *equation of hidden sort*, i.e., a pair $\langle c : X, c' : X \rangle$ of transition sequences over Π, usually written $c =_X c'$. The notion of validity of equations in a coalgebra of Definition 11 applies as it is to equations of hidden sort as well, interpreting equality in the hidden sort as true equality of states.

Given a set of generalized equations E, the class of models $Coalg(\Pi, E)$ is defined in the usual way. The *calculus of equational deduction for generalized equations* is given by all the rules of Definition 12 (where it is intended that the equations appearing in rules *reflexivity*, *symmetry*, *transitivity*, *forward closure* and in the consequence of *contradiction* can be generalized equations), plus the following one:

[substitution] For all context $c : Y$ over Π (i.e., observation or transition sequence),

$$\frac{E \vdash c_1 =_X c_2}{E \vdash c[c_1/x] =_Y c[c_2/x]}$$

We will write $E \vdash e$ in the usual way to say that e can be deduced from E using the rules for generalized equations.

Lemma 22. *Let A be a Π-coalgebra, let $c_1 =_X c_2$ be an equation of hidden sort over Π, and let $col(c_1) =_\mathbf{2} col(c_2)$ be a corresponding equation over $\Pi[\mathbf{2}]$. Then $A \models c_1 =_X c_2$ (according to Definition 21) if and only if $A \models col(c_1) =_\mathbf{2} col(c_2)$ (according to Definition 17).*

Proof. From the proof of Theorem 19 we know that for each $y \in X_A$, we have $y, A \models col(c_1) =_\mathbf{2} col(c_2)$ if and only if $[c_1]_A(y) = [c_2]_A(y)$, if and only if $y, A \models c_1 =_X c_2$. $\square$

Theorem 23 (completeness of the calculus for generalized equations). *The calculus of equational deduction for generalized equations is sound and complete, i.e., if Π is a co-signature, E is a set of generalized equations and e is a generalized equation over Π, then*

$$E \vdash e \quad \Longleftrightarrow \quad Coalg(\Pi, E) \models e$$

Proof. Consider the set E_2 of equations over $\Pi[\mathbf{2}]$ defined as $E_2 = \{e \mid e \in E$ and e is not of hidden sort$\} \cup \{col(c_1) =_\mathbf{2} col(c_2) \mid (c_1 =_X c_2) \in E\}$. It is easy to see, using Lemma 22 that $Coalg(\Pi, E) = Coalg(\Pi, \mathbf{2}, E_2)$.

Then the statement follows from the soundness and completeness of the calculus of Definition 17, by observing that an equation $col(c_1) =_\mathbf{2} col(c_2)$ can be deduced using that calculus if and only if equation $c_1 =_X c_2$ can be deduced using the rules for generalized equations. $\square$

The last result allows to deal in completely satisfactory way with the equations of hidden sort interpreted as true equality which often arise in the literature (see the discussion in Section 5). Actually, Lemma 22 also implies that as far as one is interested in characterizing (non-trivial) classes of coalgebras, equations with colourings and generalized equations have the same expressive power.

Corollary 24 (getting rid of colours). *Let $\langle \Pi, D, E \rangle$ be a coalgebraic specification such that the class $Coalg(\Pi, D, E)$ is not trivial, i.e., it contains a non-empty coalgebra. Then there is a set E' of generalized equations over Π such that $Coalg(\Pi, E') = Coalg(\Pi, D, E)$.*

A simple example of equation of hidden sort has been considered in Section 5. Another one is the following equation over co-signature BA:

$$ch(x, \underline{0}) =_X x$$

This equation is valid in coalgebras where the state is not changed by a method ch with input 0. Making reference to the coalgebras in Example 2, we have that this equation is valid in BA_1 and BA_0, but it is not valid in BA_h, nor in Z_{BA}.

Equations of hidden sort can be used also for the specification of *undo* methods which, when executed after another method, restore the orginal state. In our bank account example, we obtain this by extending the BA signature with a new method $undo : X \to X$ and with the set of equations $E_{undo} = \{ undo(ch(x, \underline{z})) =_X x \mid z \in \mathbb{Z} \}$.

8 Coalgebraic Specifications and Covarieties

In the theory of Universal Algebras [MT92], a *variety* of Σ-algebras is a class of algebras closed with respect sub-algebras, homomorphic images, and direct products. In the well-known *Variety Theorem*, Birkhoff proved that for each algebraic specification $\langle \Sigma, E \rangle$, the class $Alg(\Sigma, E)$ is a variety, and, vice versa, that for each variety $\mathcal{K}$ there is a set of equations $E_{\mathcal{K}}$ such that $\mathcal{K} = Alg(\Sigma, E_{\mathcal{K}})$.

The goal of this section is to explore how far the mentioned result can be dualized to a coalgebraic framework. We will show that the class of models of a coalgebraic specification is a "covariety", but that not all covarieties for restricted polynomial functors are equational. The next definition and result are borrowed from [Rut96].

Definition 25 (Covarieties). Let F be a bounded functor which preserves weak pullbacks. A class $\mathcal{K}$ of F-coalgebras is a *covariety* if it is closed under sub-coalgebras, homomorphic images and sums.

Theorem 26 (Covariety Theorem). *Let F be a bounded functor which preserves weak pullbacks, let D be a set, and let $Z_{F \otimes D}$ be a final $(F \otimes D)$-coalgebra (which exists because $F \otimes D$ is bounded). Furthermore, let $in_S : S \hookrightarrow Z_{F \otimes D}$ be a sub-coalgebra (see the diagram below), and let $\mathcal{K}(S)$ be the class of all F-coalgebras A such that for each function $\gamma : X_A \to D$, the cofree extension $\overline{\gamma} : X_A \to Z_{F \otimes D}$ factorizes through the inclusion in_S. Then $\mathcal{K}(S)$ is a covariety.*

Vice versa, Let $\mathcal{K} \subseteq Coalg(F)$ be a covariety. Then there is a set D and a sub-coalgebra $S \hookrightarrow Z_{F \otimes D}$ such that $\mathcal{K} = \mathcal{K}(S)$.

Proposition 27 (models of coalgebraic specifications are covarieties). *Let $\langle \Pi, D, E \rangle$ be a coalgebraic specification. Then $Coalg(\Pi, D, E)$ is a covariety.*

Proof. It is easy to check that the definition of $Coalg(\Pi, D, E)$ in Definition 17 is equivalent to the diagrammatic one at the end of Section 6, along the lines of the proof of Proposition 16, and this last one clearly defines a covariety by Theorem 26. $\square$

Unfortunately, we have the following negative result.

Proposition 28 (some covarieties are not equational). *There is a co-signature Π and a co-variety of Π-coalgebras $\mathcal{K}$ such that for each set D and for each set of equations E over $\Pi[D]$ we have $Coalg(\Pi, D, E) \neq \mathcal{K}$.*

Proof. Let $Triv$ be the co-signature of Example 8, and let $Z_{Triv[\mathbf{2}]} \cong \{0, 1\}^{\mathbb{N}}$ be the final $Triv[\mathbf{2}]$-coalgebra. Let S be its sub-coalgebra having as carrier $\{0^{\omega}, 1^{\omega}, (0 \cdot 1)^{\omega}, (1 \cdot 0)^{\omega}, (0 \cdot 0 \cdot 1)^{\omega}, (0 \cdot 1 \cdot 0)^{\omega}, (0 \cdot 1 \cdot 1)^{\omega}, (1 \cdot 0 \cdot 0)^{\omega}, (1 \cdot 0 \cdot 1)^{\omega}, (1 \cdot 1 \cdot 0)^{\omega}\}$; that is, S contains all one-, two-, and three-loops of $Z_{Triv[\mathbf{2}]}$. It is easy to check that S is the carrier of a sub-coalgebra of $Z_{Triv[\mathbf{2}]}$, and that, denoting S_{Triv} the $Triv$-coalgebra with carrier S, that S_{Triv} belongs to the covariety $\mathcal{K}(S)$.

Now we show that the covariety $\mathcal{K}(S)$ is not equational. By Proposition 20 it is sufficient to consider only equations over $Triv[\mathbf{2}]$. Suppose by absurd that $\hat{E}$ is a set of equations (closed under the rules of deduction) over $Triv[\mathbf{2}]$ such that $\mathcal{K}(S) = Coal(Triv, \mathbf{2}, \hat{E})$. Then from soundness we deduce that $\hat{E} \subseteq \{e \mid S_{Triv} \models e\}$. Clearly, all (non-trivial) equations over $Triv[\mathbf{2}]$ have the shape $col(next^{n}(x)) = col(next^{m}(x))$, and by inspecting the carrier S we easily deduce that $S_{Triv} \models col(next^{n}(x)) = col(next^{m}(x))$ if and only if $|n - m| \bmod 6 = 0$. Therefore $\hat{E} \subseteq \{col(next^{n}(x)) = col(next^{m}(x)) \mid |n - m| \bmod 6 = 0\}$. Now it is sufficient to show that there is a $Triv$-coalgebra T that does not belong to $\mathcal{K}(S)$, but where the above equations are valid, and therefore, *a fortiori*, which belongs to $Coalg(Triv, \mathbf{2}, \hat{E})$, contradicting the hypothesis. In fact, let $T = \langle \{0, 1, 2, 3, 4, 5\}, next_{T} : x \mapsto (x + 1) \bmod 6$. Then $T \notin \mathcal{K}(S)$ because, given the colouring $\gamma = \{0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0, 3 \mapsto 0, 4 \mapsto 1, 5 \mapsto 1\}$, $\overline{\gamma}$ does not factorizes through S; but clearly $T \models col(next^{n}(x)) = col(next^{m}(x))$ if $|n - m| \bmod 6 = 0$. $\square$

9 Conclusions

In this paper we presented a sound and complete equational calculus for a class of coalgebras, and we generalized it to the case with colours, which was shown to be expressive enough to handle equations of hidden sort interpreted as true equality.

Taking into account also the considerations in Section 5, we see various directions for possible generalizations of the results presented here. Firstly, we would like to consider *conditional* equations as well, looking for a complete calculus in the line of [Sel72]: conditional equations seem sufficiently expressive to capture most kinds of coalgebraic equations we have found in literature (always sticking to a purely coalgebraic setting).

Secondly, one could consider a wider class of coalgebras, for example those for functors closed under coproducts and powerset. Some preliminary efforts to consider the whole class of polynomial functors (thus including coproducts) showed that the main difficulties should not be in finding the rules of the calculus, but instead in designing a satisfactory syntax for introducing such coalgebras and for presenting the structure of the final coalgebra along the line of Section 3.

Thirdly, we would like to allow for equations between terms containing algebraic components as well, like constructors or variables. However, it is not clear at all whether the result presented here could be extended in a meaningful way to a hybrid algebraic and coalgebraic framework.

Finally, let us comment on the significance of the negative result presented in Proposition 28, showing that there exist covarieties which are not equational. This fact, together with the observation that coalgebraic equations only allow to express safety or invariant properties, suggest that the expressive power of equations for coalgebras is weaker than that for algebras (and the same can be said about conditional equations as well).

Thus on the one hand a natural question arises: which class of formulas has for coalgebras the same defining power that equations have for algebras, that is, is able to characterize *all* covarieties? An answer to this can be found, most probably, by adapting results recently presented in [Mos97], using some fragment of infinitary modal logic. On the other hand, if the emphasis is placed on the possibility of mechanizing an effective coalgebraic calculus of deduction, then one might prefer to avoid infinitary formulas at all, trying to find out which interesting properties of systems can be described by other kinds of finitary formulas, and whether some sound and complete calculus can be found for them.

Acknowledgements

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