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A complete Riemann zeta distribution and the Riemann hypothesis

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Let $\sigma, t \in \mathbb{R}$, $s = \sigma + \mathrm{i}t$, $\Gamma(s)$ be the Gamma function, $\zeta(s)$ be the Riemann zeta function and $\xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the complete Riemann zeta function. We show that $\Xi_\sigma(t) := \xi(\sigma - \mathrm{i}t)/\xi(\sigma)$ is a characteristic function for any $\sigma \in \mathbb{R}$ by giving the probability density function. Next we prove that the Riemann hypothesis is true if and only if each $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function, which is defined in this paper, for each $1/2 < \sigma < 1$. Moreover, we show that $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function when $\sigma = 1$. Finally we prove that the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible for any $\sigma > 1$.

Keywords: characteristic function; Lévy–Khintchine representation; Riemann hypothesis; zeta distribution

1. Introduction and main results

1.1. Riemann zeta function and distribution

The famous Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + \mathrm{i}t$, for $\sigma > 1$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the letter p is a prime number, and the product of $\prod_p$ is taken over all primes. The Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ and the Euler product $\prod_p (1-p^{-s})^{-1}$ converges absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this half-plane. The Riemann zeta function is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. Denote the Gamma function by $\Gamma(s)$. We have the following functional equation of the complete Riemann zeta function $\xi(s)$ (see, for example, Titchmarsh [15], (2.1.13))

$$\xi(s) = \xi(1-s), \qquad \xi(s) := s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.1}$$

In view of the Euler product, it is seen easily that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. It follows from the functional equation (1.1) and basic properties of the Gamma-function that $\zeta(s)$ vanishes in $\sigma < 0$ exactly at the so-called trivial zeros $s = -2m$, $m \in \mathbb{N}$. In 1859, Riemann stated that it seems likely that all nontrivial zeros lie on the so-called critical line $\sigma = 1/2$. This is the famous, yet unproved Riemann hypothesis. In 1896, Hadamard and de la Vallée-Poussin independently proved that $\zeta(1 + \mathrm{i}t) \neq 0$ for any $t \in \mathbb{R}$ (see Titchmarsh [15], page 45). Hence, we can also see that no zeros of $\zeta(s)$ lie on the line $\Re(s) = 0$ by (1.1). Therefore, the Riemann hypothesis is rewritten equivalently as

$$\textit{Riemann hypothesis} \quad \zeta(s) \neq 0 \qquad \text{for } 1/2 < \sigma < 1.$$

Put $Z_\sigma(t) := \zeta(\sigma - \mathrm{i}t)/\zeta(\sigma)$, $t \in \mathbb{R}$, then $Z_\sigma(t)$ is known to be a characteristic function when $\sigma > 1$ (see Khintchine [5] or Gnedenko and Kolmogorov [3], page 75). A distribution μ_σ on $\mathbb{R}$ is said to be a Riemann zeta distribution with parameter σ if it has $Z_\sigma(t)$ as its characteristic function. Recently, the Riemann zeta distribution is investigated by Lin and Hu [7], and Gut [4]. On the other hand, in Aoyama and Nakamura [1], Remark 1.13, it is showed that $Z_\sigma(t)$ is not a characteristic function for any $1/2 \leq \sigma \leq 1$. Afterwards, Nakamura [9] showed that $F_\sigma(t)$, where $F_\sigma(t) := f_\sigma(t)/f_\sigma(0)$ and $f_\sigma(t) := \zeta(\sigma - \mathrm{i}t)/(\sigma - \mathrm{i}t)$, is a characteristic function for any $0 < \sigma \neq 1$.

Note that there are some other papers connected to Riemann zeta function in probabilistic view. Biane Pitman and Yor [2] reviewed known results about $\xi(s)$ which are related to one-dimensional Brownian motion and to higher dimensional Bessel processes. Lagarias and Rains [6] treated $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and its generalizations and gave results connected to infinite divisibility.

1.2. Infinitely divisible and quasi-infinitely divisible distributions

A probability measure μ on $\mathbb{R}$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}$ such that $\mu = \mu_n^{n*}$, where μ_n^{n*} is the n-fold convolution of μ_n. For instance, normal, degenerate, Poisson and compound Poisson distributions are infinitely divisible.

Let $\widehat{\mu}(t)$ be the characteristic function of a probability measure μ on $\mathbb{R}$ and $\mathrm{ID}(\mathbb{R})$ be the class of all infinitely divisible distributions on $\mathbb{R}$. The following Lévy–Khintchine representation is well known (see Sato [14], Section 2). Put $D_b := \{x \in \mathbb{R}\colon -b \leq x \leq b\}$, where $b > 0$. If $\mu \in \mathrm{ID}(\mathbb{R})$, then one has

$$\widehat{\mu}(t) = \exp\left[-\frac{a}{2}t^2 + \mathrm{i}\lambda t + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{D_b}(x)\right)\nu(\mathrm{d}x)\right], \qquad t \in \mathbb{R}, \tag{1.2}$$

where $a \geq 0$, $\lambda \in \mathbb{R}$ and ν is a measure on $\mathbb{R}$ satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. Moreover, the representation of $\widehat{\mu}$ in (1.2) by a, ν, and λ is unique. If the Lévy measure ν in (1.2) satisfies $\int_{|x|<1} |x|\nu(\mathrm{d}x) < \infty$, then (1.2) can be written by

$$\widehat{\mu}(t) = \exp\left[-\frac{a}{2}t^2 + \mathrm{i}\lambda_0 t + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}tx} - 1\right)\nu(\mathrm{d}x)\right], \qquad \lambda_0 \in \mathbb{R}. \tag{1.3}$$

For example, the Lévy measure of $Z_\sigma(t) := \zeta(\sigma - \mathrm{i}t)/\zeta(\sigma)$ can be given as in the following (see Gnedenko and Kolmogorov [3], page 75). Let δ_x be the delta measure at x. Then we have

$$\log Z_\sigma(t) = \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1\right) N_\sigma(\mathrm{d}x), \qquad N_\sigma(\mathrm{d}x) := \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(\mathrm{d}x). \tag{1.4}$$

On the other hand, there are non-infinitely divisible distributions whose characteristic functions are the quotients of two infinitely divisible characteristic functions. That class is called class of *quasi-infinitely divisible distributions* and is defined as follows.

Quasi-infinitely divisible distribution. A distribution μ on $\mathbb{R}$ is called *quasi-infinitely divisible* if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure ν is a signed measure on $\mathbb{R}$ with total variation measure $|\nu|$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (|x|^2 \wedge 1)|\nu|(\mathrm{d}x) < \infty$.

We have to mention that the triplet (a, ν, λ) in this case is also unique if each component exists and that infinitely divisible distributions on $\mathbb{R}$ are quasi-infinitely divisible if and only if $a \geq 0$ and the negative part of ν in the Jordan decomposition equals zero. The measure ν is called *quasi*-Lévy measure and has appeared in some books and papers, for example, Gnedenko and Kolmogorov [3], page 81, Lindner and Sato [8], Niedbalska-Rajba [10], and others (see also Sato [13], Section 2.4).

1.3. Main results

In the present paper, we give a complete Riemann zeta distribution by the normalized complete Riemann zeta function

$$\Xi_\sigma(t) := \frac{\xi(\sigma - \mathrm{i}t)}{\xi(\sigma)},$$
$$\xi(\sigma - \mathrm{i}t) := (\sigma - \mathrm{i}t)(\sigma - 1 - \mathrm{i}t)\pi^{(\mathrm{i}t-\sigma)/2}\Gamma\left(\frac{\sigma - \mathrm{i}t}{2}\right)\zeta(\sigma - \mathrm{i}t),$$

for any $\sigma \in \mathbb{R}$. It should be mentioned that $\Xi_\sigma(t)$ is symmetric about the vertical axis $\sigma = 1/2$ by the functional equation (1.1). Therefore, we only have to consider the case $\sigma \geq 1/2$. In order to state the main results, we introduce the following pretended-infinitely divisible distribution.

Pretended-infinitely divisible distribution. A distribution μ on $\mathbb{R}$ is called *pretended-infinitely divisible* if it has a form of (1.2) with $a \in \mathbb{R}$ and the corresponding measure ν is a signed measure on $\mathbb{R}$ with $\nu(\{0\}) = 0$.

Namely, pretended-infinitely divisible distributions are infinitely divisible or quasi-infinitely divisible distributions without the condition $\int_{\mathbb{R}} (|x|^2 \wedge 1)|\nu|(\mathrm{d}x) < \infty$.

The main results in this paper are following four theorems.

Theorem 1.1. *The function* $\Xi_\sigma(t)$ *is a characteristic function for any* $\sigma \in \mathbb{R}$. *Moreover, the probability density function* $P_\sigma(y)$ *is given as follows*:

$$P_\sigma(y) := \begin{cases} \dfrac{2}{\xi(\sigma)} \displaystyle\sum_{n=1}^{\infty} f\left(n\mathrm{e}^{-y}\right)\mathrm{e}^{-\sigma y}, & y \le 0, \\ \dfrac{2}{\xi(\sigma)} \displaystyle\sum_{n=1}^{\infty} f\left(n\mathrm{e}^{y}\right)\mathrm{e}^{(1-\sigma)y}, & y > 0, \end{cases} \tag{1.5}$$

where $f(x) := 2\pi(2\pi x^4 - 3x^2)\mathrm{e}^{-\pi x^2}$.

Let $\mathcal{Z}$ and $\mathcal{Z}_+$ be the set of zeros of the Riemann zeta function which lie in the critical strip $\{s \in \mathbb{C}\colon\ 0 < \Re(s) < 1\}$, and the region $\{s \in \mathbb{C}\colon\ 0 < \Re(s) < 1, \Im(s) > 0\}$, respectively. If the Riemann hypothesis is true, then each $\rho \in \mathcal{Z}_+$ can be expressed by $\rho = 1/2 + \mathrm{i}\gamma$, where $\gamma > 0$.

Theorem 1.2. *The characteristic function* $\Xi_\sigma(t)$ *is a pretended-infinitely divisible characteristic function for any* $1/2 < \sigma < 1$ *if and only if the Riemann hypothesis is true. Furthermore, we have*

$$\begin{aligned} \Xi_\sigma(t) &= \exp\left[\int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1\right)\nu_\sigma(\mathrm{d}x)\right], \\ \nu_\sigma(\mathrm{d}x) &:= -\sum_{1/2+\mathrm{i}\gamma \in \mathcal{Z}_+} \frac{2\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1/2)x}}(\mathrm{d}x), \end{aligned} \tag{1.6}$$

under the Riemann hypothesis.

Let $\mathcal{Z}_+^R$ be the set of zeros of $\zeta(s)$ which lie on the half line $\{s \in \mathbb{C}\colon\ \Re(s) = 1/2, \Im(s) > 0\}$ and $\mathcal{Z}_+^N$ be the set of zeros of $\zeta(s)$ which lie in the region $\{s \in \mathbb{C}\colon\ 1/2 < \Re(s) < 1, \Im(s) > 0\}$. Note that $\mathcal{Z}_+^N = \varnothing$ if and only if the Riemann hypothesis is true. One has $\mathcal{Z} = \{\rho, 1-\rho\colon\ \rho \in \mathcal{Z}_+^R\} \cup \{\rho, 1-\rho, \overline{\rho}, 1-\overline{\rho}\colon\ \rho \in \mathcal{Z}_+^N\}$ from $\xi(s) = \xi(1-s)$ and $\xi(\overline{s}) = \overline{\xi(s)}$.

Theorem 1.3. *When* $\sigma \ge 1$, *we have*

$$\begin{aligned} \Xi_\sigma(t) &= \exp\left[\int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1\right)\nu_\sigma(\mathrm{d}x)\right], \\ \nu_\sigma(\mathrm{d}x) &:= -\sum_{1/2+\mathrm{i}\gamma \in \mathcal{Z}_+^R} \frac{2\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1/2)x}}(\mathrm{d}x) - \sum_{\beta+\mathrm{i}\gamma \in \mathcal{Z}_+^N} \left(\frac{2\cos(\gamma x)}{x\mathrm{e}^{(\sigma-\beta)x}} + \frac{2\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1+\beta)x}}\right)(\mathrm{d}x). \end{aligned} \tag{1.7}$$

Especially, $\Xi_\sigma(t)$ *is a pretended-infinitely divisible characteristic function when* $\sigma = 1$.

Theorem 1.4. *When* $\sigma > 1$, *we have*

$$\Xi_\sigma(t) = \exp\left[\mathrm{i}t\lambda_\sigma + \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{D_{1/2}}(x)\right)\nu_\sigma(\mathrm{d}x)\right],$$

$$\lambda_\sigma := \frac{\mathrm{e}^{-\sigma/2}-1}{\sigma} + \frac{\mathrm{e}^{(1-\sigma)/2}-1}{\sigma-1} + \frac{\log \pi}{2} + \frac{1}{2}\int_0^1 \left(\frac{\mathrm{e}^{-\sigma x/2}}{1-\mathrm{e}^{-x}} - \frac{\mathrm{e}^{-x}}{x} \right) \mathrm{d}x - \frac{1}{2}\int_1^\infty \mathrm{e}^{-x} \frac{\mathrm{d}x}{x},$$

$$\nu_\sigma(\mathrm{d}x) := \frac{1(\mathrm{d}x)}{x\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-2x})} - \frac{1+\mathrm{e}^x}{x\mathrm{e}^{\sigma x}}(\mathrm{d}x) + \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r} \delta_{r \log p}(\mathrm{d}x).$$

Therefore, the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible when $\sigma > 1$.

We call the distribution defined by the characteristic function $\Xi_\sigma(t)$ the *completed Riemann zeta distribution*. It is well known that $\zeta(s)$ has zeros on $\Re(s) = 1/2$ (see Titchmarsh [15], Section 10). By the definition of pretended-infinitely divisible distribution and the fact that $\exp(z) \neq 0$ for any $z \in \mathbb{C}$, the characteristic function does not have zeros. Thus, $\Xi_\sigma(t)$ is not even a pretended-infinitely divisible characteristic function when $\sigma = 1/2$.

2. Proofs

2.1. Proof of Theorem 1.1

We quote the following fact from Patterson [12] (see also Biane Pitman and Yor [2], Section 2).

Lemma 2.1 (see Patterson [12], Section 2.10). *Let $f(x) := 2\pi(2\pi x^4 - 3x^2)\mathrm{e}^{-\pi x^2}$. Then we have*

$$\xi(s) = 2\int_1^\infty \sum_{n=1}^\infty f(nx)\big(x^{s-1/2} + x^{1/2-s}\big)x^{-1/2}\,\mathrm{d}x. \tag{2.1}$$

Note that the last integral is absolutely convergent for all values of s.

Proof of Theorem 1.1. By (2.1) and the change of variables $x = \mathrm{e}^{-y}$ and $x = \mathrm{e}^y$, we have

$$\begin{aligned}
\xi(\sigma - \mathrm{i}t) &= 2\int_1^\infty \sum_{n=1}^\infty f(nx)x^{\sigma-\mathrm{i}t-1}\,\mathrm{d}x + 2\int_1^\infty \sum_{n=1}^\infty f(nx)x^{\mathrm{i}t-\sigma}\,\mathrm{d}x \\
&= 2\int_0^{-\infty} \sum_{n=1}^\infty f\big(n\mathrm{e}^{-y}\big)\mathrm{e}^{(1+\mathrm{i}t-\sigma)y}\big(-\mathrm{e}^{-y}\big)\,\mathrm{d}y + 2\int_0^\infty \sum_{n=1}^\infty f\big(n\mathrm{e}^{y}\big)\mathrm{e}^{(\mathrm{i}t-\sigma)y}\big(\mathrm{e}^{y}\big)\,\mathrm{d}y \\
&= 2\int_{-\infty}^0 \mathrm{e}^{\mathrm{i}ty} \sum_{n=1}^\infty f\big(n\mathrm{e}^{-y}\big)\mathrm{e}^{-\sigma y}\,\mathrm{d}y + 2\int_0^\infty \mathrm{e}^{\mathrm{i}ty} \sum_{n=1}^\infty f\big(n\mathrm{e}^{y}\big)\mathrm{e}^{(1-\sigma)y}\,\mathrm{d}y.
\end{aligned}$$

Obviously, we have $f(x) = 2\pi(2\pi x^4 - 3x^2)\mathrm{e}^{-\pi x^2} > 0$ for any $x \geq 1$. Hence, one has $f(n\mathrm{e}^{-y}) > 0$ for any $y \leq 0$ and $n \in \mathbb{N}$, and $f(n\mathrm{e}^{y}) > 0$ for any $y > 0$ and $n \in \mathbb{N}$. Thus it holds that

$$\sum_{n=1}^{\infty} f\big(n\mathrm{e}^{-y}\big)\mathrm{e}^{-\sigma y} > 0, \qquad y \leq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} f\big(n\mathrm{e}^{y}\big)\mathrm{e}^{(1-\sigma)y} > 0, \qquad y > 0.$$

On the other hand, we have

$$\xi(\sigma) = 2\int_{-\infty}^{0} \sum_{n=1}^{\infty} f\big(n\mathrm{e}^{-y}\big)\mathrm{e}^{-\sigma y}\,\mathrm{d}y + 2\int_{0}^{\infty} \sum_{n=1}^{\infty} f\big(n\mathrm{e}^{y}\big)\mathrm{e}^{(1-\sigma)y}\,\mathrm{d}y > 0$$

from (2.1) and the argument above. Hence, $P_\sigma(y)$ defined by (1.5) is nonnegative. Therefore, we have $\Xi_\sigma(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ty} P_\sigma(y)\,\mathrm{d}y$, where $P_\sigma(y)$ is the probability density function. □

Remark 2.2. It should be emphasised that $\Xi_\sigma(t)$ is a characteristic function for any $\sigma \in \mathbb{R}$. On the other hand, $F_\sigma(t) := f_\sigma(t)/f_\sigma(0)$, where $f_\sigma(t) := \zeta(\sigma - \mathrm{i}t)/(\sigma - \mathrm{i}t)$, is not a characteristic function for $\sigma = 0, 1$ and $\sigma < -1/2$. This is proved as follows. When $\sigma = 1$, it is well known that $\zeta(1 + \mathrm{i}t) \neq 0$, $t \neq 0$, and $\zeta(s)$ has an only one pole at $s = 1$. Hence, we have

$$F_1(t) = \frac{1}{\zeta(1)} \frac{\zeta(1+\mathrm{i}t)}{1+\mathrm{i}t} = 0 \qquad \text{for any } t \neq 0,$$

which contradicts the uniform continuity of characteristic function $\widehat{\mu}(t)$ and $\widehat{\mu}(0) = 1$. A similar argument can be done when $\sigma = 0$ since $\zeta(s)/s$ has a simple pole at $s = 0$. By (1.1) and Stirling's formula, one has

$$\big|\zeta(s)\big| = \pi^{\sigma-1/2}\big(|t/2| + 2\big)^{-\sigma+1/2}\Big(1 + \mathrm{O}\big(\big(|t| + 2\big)^{-1}\big)\Big)\big|\zeta(1-s)\big|$$

for $\sigma < 0$. On the other hand, for any $\varepsilon > 0$ there are arbitrarily large t which satisfy $|\zeta(\sigma + \mathrm{i}t)| > (1-\varepsilon)\zeta(\sigma)$ when $\sigma > 1$ (see Titchmarsh [15], Theorem 8.4). Thus, we can find t which satisfies $|\zeta(s)| > \pi^{\sigma-1/2}|t/2|^{-\sigma+1/2}\zeta(1-\sigma)/2$. Hence, there exists $t \in \mathbb{R}$ such that $|F_\sigma(t)| > 1$ when $\sigma < -1/2$ by the factor $|t/2|^{-\sigma+1/2}$.

The absolute value of a characteristic function is not greater than 1 (see for instance Sato [14], Proposition 2.5). Hence, we have the following inequality by Theorem 1.1.

Corollary 2.3 (see Patterson [12], Section 2.11). *For any $t \in \mathbb{R}$ and $1/2 \leq \sigma$, we have*

$$\left|(\sigma + \mathrm{i}t)(\sigma - 1 + \mathrm{i}t)\pi^{-(\sigma+\mathrm{i}t)/2}\Gamma\left(\frac{\sigma + \mathrm{i}t}{2}\right)\zeta(\sigma + \mathrm{i}t)\right| \leq \sigma(\sigma - 1)\pi^{-\sigma/2}\Gamma\left(\frac{\sigma}{2}\right)\zeta(\sigma).$$

2.2. Proof of Theorem 1.2

Recall that $\mathcal{Z}$ is the set of zeros of the Riemann zeta function which lie in the critical strip $\{s \in \mathbb{C}\colon 0 < \Re(s) < 1\}$ (see Section 1.3). Observe that by the functional equation and $\overline{\zeta(s)} = \zeta(\overline{s})$

if $\rho \in \mathcal{Z}$ then $\overline{\rho}, 1-\rho, 1-\overline{\rho} \in \mathcal{Z}$. There are no real elements of $\mathcal{Z}$ since $\xi(\sigma) < 0$ and $0 < \Gamma(\sigma/2)$ when $0 < \sigma < 1$ (see Section 1.1 and the proof of Theorem 1.1). Now we quote the following fact from Patterson [12].

Lemma 2.4 (see Patterson [12], page 34). *Let $\mathcal{Z}_+ := \{\rho \in \mathcal{Z}\colon \Im(\rho) > 0\}$. Then $\sum_{\rho \in \mathcal{Z}_+} |\rho|^{-a}$ converges for all $a > 1$ and it holds that*

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \prod_{\rho \in \mathcal{Z}_+}\left(1-\frac{s}{\rho}\right)\left(1-\frac{s}{1-\rho}\right) \tag{2.2}$$

the product being absolutely convergent for all $s \in \mathbb{C}$.

Proof of Theorem 1.2. If $\Xi_\sigma(t)$ is a pretended-infinitely divisible characteristic function for any $1/2 < \sigma < 1$, then $\zeta(s) \neq 0$ for any $1/2 < \sigma < 1$ by $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, $\Gamma(s) \neq 0$ for any $1/2 < \sigma < 1$ and the representation (1.2).

Next suppose that the Riemann hypothesis is true. Then we have $\rho = 1/2 + \mathrm{i}\gamma$ and $1-\rho = 1/2 - \mathrm{i}\gamma$, where $\gamma > 0$ for $\rho \in \mathcal{Z}_+$. Note that the exponential distribution with parameter $a > 0$ is defined by $\mu(B) := a\int_{B\cap(0,\infty)} \mathrm{e}^{-ax}\,\mathrm{d}x$, where $B \in \mathfrak{B}(\mathbb{R})$. The characteristic function is given by $\widehat{\mu}(t) = a/(a-\mathrm{i}t)$ (see, for example, Sato [14], page 13). Moreover, it is well known that

$$\frac{a}{a-\mathrm{i}z} = \exp\left[\int_0^\infty \left(\mathrm{e}^{\mathrm{i}zx}-1\right)x^{-1}\mathrm{e}^{-ax}\,\mathrm{d}x\right], \qquad a > 0, z \in \mathbb{R} \tag{2.3}$$

(see, for instance, Sato [14], page 45). The formula above holds if a is replaced by α with $\Re(\alpha) > 0$. This is proved as follows. Put $\alpha = a + \mathrm{i}b$, $a > 0$ and $b \in \mathbb{R}$. Then one has

$$\begin{aligned}\frac{\alpha}{\alpha-\mathrm{i}z} &= \frac{a+\mathrm{i}b}{a}\frac{a}{a+\mathrm{i}b-\mathrm{i}z}\\ &= \exp\left[\int_0^\infty \left(\mathrm{e}^{\mathrm{i}(z-b)x}-1\right)x^{-1}\mathrm{e}^{-ax}\,\mathrm{d}x - \int_0^\infty \left(\mathrm{e}^{-\mathrm{i}bx}-1\right)x^{-1}\mathrm{e}^{-ax}\,\mathrm{d}x\right] \\ &= \exp\left[\int_0^\infty \left(\mathrm{e}^{\mathrm{i}zx}-1\right)x^{-1}\mathrm{e}^{-\alpha x}\,\mathrm{d}x\right], \qquad \Re(\alpha) > 0,\end{aligned} \tag{2.4}$$

by (2.3). Thus, it holds that

$$\begin{aligned}\left(1-\frac{\sigma-\mathrm{i}t}{\rho}\right)\left(1-\frac{\sigma}{\rho}\right)^{-1} &= \frac{1/2-\sigma+\mathrm{i}(\gamma+t)}{1/2+\mathrm{i}\gamma}\frac{1/2+\mathrm{i}\gamma}{1/2-\sigma+\mathrm{i}\gamma} = \frac{\sigma-1/2-\mathrm{i}\gamma-\mathrm{i}t}{\sigma-1/2-\mathrm{i}\gamma}\\ &= \exp\left[-\int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx}-1\right)\mathrm{e}^{(1/2-\sigma+\mathrm{i}\gamma)x}\frac{\mathrm{d}x}{x}\right],\end{aligned}$$

where $\sigma > 1/2$. It should be noted that we have $\sigma - \mathrm{i}t \neq \rho, 1-\rho$ when $\sigma > 1/2$ under the Riemann hypothesis. Therefore, one has

$$\begin{aligned}\varphi_\rho(t) &:= \left(1 - \frac{\sigma - \mathrm{i}t}{\rho}\right)\left(1 - \frac{\sigma}{\rho}\right)^{-1}\left(1 - \frac{\sigma - \mathrm{i}t}{1-\rho}\right)\left(1 - \frac{\sigma}{1-\rho}\right)^{-1} \\ &= \frac{\sigma - 1/2 - \mathrm{i}\gamma - \mathrm{i}t}{\sigma - 1/2 - \mathrm{i}\gamma}\frac{\sigma - 1/2 + \mathrm{i}\gamma - \mathrm{i}t}{\sigma - 1/2 + \mathrm{i}\gamma} \\ &= \exp\left[-2\int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\frac{\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1/2)x}}\,\mathrm{d}x\right].\end{aligned} \tag{2.5}$$

We remark that $x^{-1}\cos(\gamma x)\mathrm{e}^{(1/2-\sigma)x}(\mathrm{d}x)$ is not a measure but a signed measure since one has $-1 \le \cos(\gamma x) \le 1$ when $\gamma \in \mathbb{R}$. By (2.2) and the definition of $\Xi_\sigma(t)$, we have

$$\begin{aligned}\Xi_\sigma(t) &= \prod_{\gamma \in \mathcal{Z}_+} \frac{\sigma - 1/2 - \mathrm{i}\gamma + \mathrm{i}t}{\sigma - 1/2 - \mathrm{i}\gamma}\frac{\sigma - 1/2 + \mathrm{i}\gamma + \mathrm{i}t}{\sigma - 1/2 + \mathrm{i}\gamma} \\ &= \exp\left[-2\sum_{1/2+\mathrm{i}\gamma \in \mathcal{Z}_+}\int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\frac{\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1/2)x}}\,\mathrm{d}x\right].\end{aligned}$$

This equality implies (1.6). □

Remark 2.5. It should be mentioned that $\varphi_{1/2+\mathrm{i}\gamma}(t)$ defined by (2.5) is not a characteristic function for any $\sigma > 1/2$. It is proved by as follows. Obviously, one has

$$\left|\varphi_{1/2+\mathrm{i}\gamma}(t)\right|^2 = \frac{(\sigma - 1/2)^2 + \gamma^2 - t^2 + (2\sigma - 1)\mathrm{i}t}{(\sigma - 1/2)^2 + \gamma^2}.$$

If we take $t^2 = 2((\sigma - 1/2)^2 + \gamma^2)$, then $|\varphi_{1/2+\mathrm{i}\gamma}(t)|^2 > 1$.

2.3. Proof of Theorem 1.3

Recall that $\mathcal{Z}$, $\mathcal{Z}_+^R$ and $\mathcal{Z}_+^N$ is the set of zeros of $\zeta(s)$ which lie in $\{s \in \mathbb{C}\colon\ 0 < \Re(s) < 1\}$, $\{s \in \mathbb{C}\colon\ \Re(s) = 1/2, \Im(s) > 0\}$ and $\{s \in \mathbb{C}\colon\ 1/2 < \Re(s) < 1, \Im(s) > 0\}$, respectively. Then one has $\mathcal{Z} = \{\rho, 1-\rho\colon\ \rho \in \mathcal{Z}_+^R\} \cup \{\rho, 1-\rho, \overline{\rho}, 1-\overline{\rho}\colon\ \rho \in \mathcal{Z}_+^N\}$. We have the following by Lemma 2.4.

Lemma 2.6. *The sums* $\sum_{\rho \in \mathcal{Z}_+^R} |\rho|^{-a}$ *and* $\sum_{\rho \in \mathcal{Z}_+^N} |\rho|^{-a}$ *converge for all* $a > 1$ *and it holds that*

$$\begin{aligned}\xi(s) = &\prod_{1/2+\mathrm{i}\gamma \in \mathcal{Z}_+^R}\left(1 - \frac{s}{1/2 + \mathrm{i}\gamma}\right)\left(1 - \frac{s}{1/2 - \mathrm{i}\gamma}\right) \\ &\times \prod_{\rho \in \mathcal{Z}_+^N}\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right)\left(1 - \frac{s}{\overline{\rho}}\right)\left(1 - \frac{s}{1-\overline{\rho}}\right),\end{aligned} \tag{2.6}$$

the products being absolutely convergent for all $s \in \mathbb{C}$.

Proof of Theorem 1.3. Put $\overline{s} = \sigma - \mathrm{i}t$. Then we have

$$\begin{aligned}
&\left(1-\frac{\overline{s}}{\rho}\right)\left(1-\frac{\sigma}{\rho}\right)^{-1}\left(1-\frac{\overline{s}}{1-\rho}\right)\left(1-\frac{\sigma}{1-\rho}\right)^{-1}\\
&\quad=\frac{\sigma-\beta-\mathrm{i}\gamma-\mathrm{i}t}{\sigma-\beta-\mathrm{i}\gamma}\frac{\sigma-1+\beta-\mathrm{i}\gamma-\mathrm{i}t}{\sigma-1+\beta-\mathrm{i}\gamma}\\
&\quad=\exp\left[-\int_0^\infty(\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{e}^{(\beta-\sigma+\mathrm{i}\gamma)x}\frac{\mathrm{d}x}{x}-\int_0^\infty(\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{e}^{(1-\beta-\sigma+\mathrm{i}\gamma)x}\frac{\mathrm{d}x}{x}\right]
\end{aligned}$$

from (2.4). By replacing ρ by $\overline{\rho}$, we obtain

$$\begin{aligned}
&\left(1-\frac{\overline{s}}{\overline{\rho}}\right)\left(1-\frac{\sigma}{\overline{\rho}}\right)^{-1}\left(1-\frac{\overline{s}}{1-\overline{\rho}}\right)\left(1-\frac{\sigma}{1-\overline{\rho}}\right)^{-1}\\
&\quad=\exp\left[-\int_0^\infty(\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{e}^{(\beta-\sigma-\mathrm{i}\gamma)x}\frac{\mathrm{d}x}{x}-\int_0^\infty(\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{e}^{(1-\beta-\sigma-\mathrm{i}\gamma)x}\frac{\mathrm{d}x}{x}\right].
\end{aligned}$$

We have to mention that one has $\beta - \sigma < 0$ and $1 - \beta - \sigma < 0$ since $\zeta(s) \neq 0$ for $\sigma \geq 1$ (see Remark 2.7 below). Hence, one has

$$\begin{aligned}
&\frac{(1-\overline{s}/\rho)(1-\overline{s}/\overline{\rho})(1-\overline{s}/(1-\rho))(1-\overline{s}/(1-\overline{\rho}))}{(1-\sigma/\rho)(1-\sigma/\overline{\rho})(1-\sigma/(1-\rho))(1-\sigma/(1-\overline{\rho}))}\\
&\quad=\exp\left[-2\int_0^\infty\left(\mathrm{e}^{\mathrm{i}tx}-1\right)\cos(\gamma x)\left(\mathrm{e}^{(\beta-\sigma)x}+\mathrm{e}^{(1-\beta-\sigma)x}\right)\frac{\mathrm{d}x}{x}\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Xi_\sigma(t)=\exp\Bigg[&-2\sum_{1/2+\mathrm{i}\gamma\in\mathcal{Z}_+^R}\int_0^\infty\left(\mathrm{e}^{\mathrm{i}tx}-1\right)\cos(\gamma x)\mathrm{e}^{(1/2-\sigma)x}\frac{\mathrm{d}x}{x}\\
&-2\sum_{\beta+\mathrm{i}\gamma\in\mathcal{Z}_+^N}\int_0^\infty\left(\mathrm{e}^{\mathrm{i}tx}-1\right)\cos(\gamma x)\left(\mathrm{e}^{(\beta-\sigma)x}+\mathrm{e}^{(1-\beta-\sigma)x}\right)\frac{\mathrm{d}x}{x}\Bigg]
\end{aligned}$$

by (2.6) and the definition of $\Xi_\sigma(t)$. □

Remark 2.7. By modifying the proof above, we can see that one has (1.7) for any $\sigma \geq \sigma_0 > 1/2$ if $\zeta(s)$ does not vanish for $\sigma \geq \sigma_0$.

2.4. Proof of Theorem 1.4

In order to prove Theorem 1.4, we first prove the following lemma which is an analogue of Nikeghbali and Yor [11], Lemma 2.9.

Lemma 2.8. *Let $G_\sigma(t) = \Gamma(\sigma - \mathrm{i}t)/\Gamma(\sigma)$ for $0 < \sigma$. Then $G_\sigma(t)$ is an infinitely divisible characteristic function for any $\sigma > 0$. Moreover, one has*

$$\begin{aligned}\log G_\sigma(t) &= \mathrm{i}t\lambda_\sigma^\# + \int_0^\infty \bigl(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1]}(x)\bigr)\nu_\sigma^\#(\mathrm{d}x),\\ \lambda_\sigma^\# &= C(\sigma) := \int_0^1 \left(\frac{\mathrm{e}^{-\sigma x}}{1-\mathrm{e}^{-x}} - \frac{\mathrm{e}^{-x}}{x}\right)\mathrm{d}x - \int_1^\infty \mathrm{e}^{-x}\frac{\mathrm{d}x}{x},\\ \nu_\sigma^\#(\mathrm{d}x) &:= \frac{1(\mathrm{d}x)}{x\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-x})}.\end{aligned}$$

Proof. By the integral representation of $\Gamma(s)$ and the change of variables $x = \mathrm{e}^{-y}$, we have

$$\begin{aligned}G_\sigma(t) &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \mathrm{e}^{-x}x^{\sigma-1-\mathrm{i}t}\,\mathrm{d}x = \frac{-1}{\Gamma(\sigma)}\int_\infty^{-\infty} \mathrm{e}^{-\mathrm{e}^{-y}}\mathrm{e}^{y(1-\sigma+\mathrm{i}t)y}\mathrm{e}^{-y}\,\mathrm{d}y\\ &= \frac{1}{\Gamma(\sigma)}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}ty}\exp\bigl(-\sigma y - \mathrm{e}^{-y}\bigr)\,\mathrm{d}y, \qquad \sigma > 0.\end{aligned}$$

Therefore, the probability density function is given by $\exp(-\sigma y - \mathrm{e}^{-y})/\Gamma(\sigma)$.

Next, we quote Malmstén's formula (see, for example, Whittaker and Watson [16], page 249)

$$\log\Gamma(s) = \int_0^\infty \left(\frac{\mathrm{e}^{-sx} - \mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} + (s-1)\mathrm{e}^{-x}\right)\frac{\mathrm{d}x}{x}, \qquad \sigma > 0.$$

Hence, it holds that

$$\begin{aligned}\log G_\sigma(t) &= \int_0^\infty \left(\frac{\mathrm{e}^{-(\sigma-\mathrm{i}t)x} - \mathrm{e}^{-\sigma x}}{1-\mathrm{e}^{-x}} - \mathrm{i}t\mathrm{e}^{-x}\right)\frac{\mathrm{d}x}{x}\\ &= \int_0^1 \left(\frac{\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx}{\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-x})} - \mathrm{i}t\mathrm{e}^{-x} + \frac{\mathrm{i}tx\mathrm{e}^{-\sigma x}}{1-\mathrm{e}^{-x}}\right)\frac{\mathrm{d}x}{x} + \int_1^\infty \left(\frac{\mathrm{e}^{\mathrm{i}tx} - 1}{\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-x})} - \mathrm{i}t\mathrm{e}^{-x}\right)\frac{\mathrm{d}x}{x}\\ &= \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1]}(x)}{x\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-x})}\,\mathrm{d}x + \mathrm{i}t\int_0^1 \left(\frac{\mathrm{e}^{-\sigma x}}{1-\mathrm{e}^{-x}} - \frac{\mathrm{e}^{-x}}{x}\right)\mathrm{d}x - \mathrm{i}t\int_1^\infty \mathrm{e}^{-x}\frac{\mathrm{d}x}{x}.\end{aligned}$$

Therefore, we obtain Lemma 2.8. □

For the reader's convenience, we give a proof of (1.4). By the Euler product of $\zeta(s)$ and the Taylor expansion of $\log(1-x)$, $|x| < 1$, one has

$$\begin{aligned}\log\frac{\zeta(\sigma - \mathrm{i}t)}{\zeta(\sigma)} &= \sum_p \log\frac{1-p^{-\sigma}}{1-p^{-\sigma+\mathrm{i}t}} = \sum_p\sum_{r=1}^\infty \frac{1}{r}p^{-r\sigma}\bigl(p^{r\mathrm{i}t} - 1\bigr)\\ &= \sum_p\sum_{r=1}^\infty \frac{1}{r}p^{-r\sigma}\bigl(\mathrm{e}^{r\mathrm{i}t\log p} - 1\bigr) = \int_{-\infty}^\infty \bigl(\mathrm{e}^{\mathrm{i}tx} - 1\bigr)\sum_p\sum_{r=1}^\infty \frac{1}{r}p^{r\sigma}\delta_{r\log p}(\mathrm{d}x).\end{aligned}$$

This equality implies (1.4).

Proof of Theorem 1.4. We have

$$\Xi_\sigma(t) = \pi^{\mathrm{i}t/2} G_{\sigma/2}(t/2) \frac{\sigma - \mathrm{i}t}{\sigma} \frac{\sigma - 1 - \mathrm{i}t}{\sigma - 1} \frac{\zeta(\sigma - \mathrm{i}t)}{\zeta(\sigma)}$$

by the definition of $\Xi_\sigma(t)$. It holds that

$$\begin{aligned}\log G_{\sigma/2}(t/2) &= \frac{\mathrm{i}t}{2} C(\sigma/2) + \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}(t/2)x} - 1 - \mathrm{i}(t/2)x 1_{[0,1]}(x)}{x\mathrm{e}^{\sigma x/2}(1 - \mathrm{e}^{-x})} \mathrm{d}x \\ &= \mathrm{i}t \frac{C(\sigma/2)}{2} + \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]}(x)}{x\mathrm{e}^{\sigma x}(1 - \mathrm{e}^{-2x})} \mathrm{d}x\end{aligned}$$

from Lemma 2.8. Obviously, one has $1/2 < r \log p$ for any integer r and prime number p since $\log 2 = 0.6931471806\ldots$. Hence by using (1.4), we have

$$\log \frac{\zeta(\sigma - \mathrm{i}t)}{\zeta(\sigma)} = \int_{-\infty}^\infty \big(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]}(x)\big) \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} \delta_{r \log p}(\mathrm{d}x).$$

When $\sigma > 1$, one has

$$\begin{aligned}\frac{\sigma - \mathrm{i}t}{\sigma} &= \exp\left[-\int_0^\infty \big(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]} + \mathrm{i}tx 1_{[0,1/2]}\big) \mathrm{e}^{-\sigma x} \frac{\mathrm{d}x}{x}\right] \\ &= \exp\left[-\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]}(x)}{x\mathrm{e}^{\sigma x}} \mathrm{d}x - \mathrm{i}t \frac{1 - \mathrm{e}^{-\sigma/2}}{\sigma}\right]\end{aligned}$$

by (2.3). Thus, it holds that

$$\begin{aligned}&\frac{\sigma - \mathrm{i}t}{\sigma} \frac{\sigma - 1 - \mathrm{i}t}{\sigma - 1} \\ &= \exp\left[-\int_0^\infty \big(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]}(x)\big) \frac{1 + \mathrm{e}^x}{x\mathrm{e}^{\sigma x}} \mathrm{d}x - \mathrm{i}t \left(\frac{1 - \mathrm{e}^{-\sigma/2}}{\sigma} + \frac{1 - \mathrm{e}^{-(\sigma-1)/2}}{\sigma - 1}\right)\right].\end{aligned}$$

If x is sufficiently large, then we have

$$\frac{1}{x\mathrm{e}^{\sigma x}(1 - \mathrm{e}^{-2x})} - \frac{1 + \mathrm{e}^x}{x\mathrm{e}^{\sigma x}} < 0.$$

Thus ν_σ in Theorem 1.4 is not a measure but a signed measure.

Finally, we show $\int_{\mathbb{R}} (|x|^2 \wedge 1) |\nu_\sigma|(\mathrm{d}x) < \infty$ when $\sigma > 1$. By using $(1 - \mathrm{e}^{-2})x \le 1 - \mathrm{e}^{-2x}$ for $0 \le x < 1$ and $1 - \mathrm{e}^{-2} \le 1 - \mathrm{e}^{-2x}$ for $x \ge 1$, we have

$$\int_0^\infty \frac{(1 - \mathrm{e}^{-2})(|x|^2 \wedge 1)}{x\mathrm{e}^{\sigma x}(1 - \mathrm{e}^{-2x})} \mathrm{d}x \le \int_0^1 \frac{\mathrm{d}x}{\mathrm{e}^{\sigma x}} + \int_1^\infty \frac{\mathrm{d}x}{x\mathrm{e}^{\sigma x}} < \int_0^\infty \frac{\mathrm{d}x}{\mathrm{e}^{\sigma x}} < \infty.$$

Obviously, it holds that

$$\int_0^\infty \frac{(1+\mathrm{e}^x)(|x|^2\wedge 1)}{x\mathrm{e}^{\sigma x}}\,\mathrm{d}x < 2\int_0^\infty \frac{(|x|^2\wedge 1)\,\mathrm{d}x}{x\mathrm{e}^{(\sigma-1)x}} < 2\int_0^\infty \frac{\mathrm{d}x}{\mathrm{e}^{(\sigma-1)x}} < \infty.$$

From $\sum_p p^{-\sigma} < \sum_{n=2}^\infty n^{-\sigma} = \zeta(\sigma) - 1$, one has

$$\begin{aligned}
&\int_0^\infty \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma}\delta_{r\log p}(\mathrm{d}x)\\
&= \sum_p \sum_{r=1}^\infty \frac{1}{r} p^{-r\sigma} < \sum_p \sum_{r=1}^\infty p^{-r\sigma} < \sum_{n=1}^\infty n^{-\sigma} + \sum_p \sum_{r=2}^\infty p^{-r\sigma}\\
&= \zeta(\sigma) + \sum_p \frac{p^{-2\sigma}}{1-p^{-\sigma}} < \zeta(\sigma) + \sum_{n=2}^\infty \frac{n^{-2\sigma}}{1-2^{-\sigma}}\\
&< \zeta(\sigma) + \left(1-2^{-\sigma}\right)^{-1}\zeta(2\sigma) < \infty.
\end{aligned}$$

Therefore the characteristic function $\Xi_\sigma(t)$ is not infinitely divisible but quasi-infinitely divisible. □

Remark 2.9. Suppose $\sigma \neq 1$ and put

$$\Xi_\sigma^*(t) := \frac{\sigma-1}{\sigma-1-\mathrm{i}t}\,\Xi_\sigma(t).$$

Then $\Xi_\sigma^*(t)$ is a characteristic function for any $\sigma \neq 1$ by the fact that the product of a finite number of characteristic functions is also a characteristic function. By modifying the proof above, we have

$$\begin{aligned}
\Xi_\sigma^*(t) &= \exp\left[\mathrm{i}t\lambda_\sigma^* + \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx 1_{[0,1/2]}(x)\right)\nu_\sigma^*(\mathrm{d}x)\right],\\
\lambda_\sigma^* &:= \frac{1-\mathrm{e}^{-\sigma/2}}{\sigma} + \frac{\log\pi}{2} + \frac{1}{2}\int_0^1 \left(\frac{\mathrm{e}^{-\sigma x/2}}{1-\mathrm{e}^{-x}} - \frac{\mathrm{e}^{-x}}{x}\right)\mathrm{d}x - \frac{1}{2}\int_1^\infty \mathrm{e}^{-x}\frac{\mathrm{d}x}{x},\\
\nu_\sigma^*(\mathrm{d}x) &:= \frac{1(\mathrm{d}x)}{x\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-2x})} - \frac{1(\mathrm{d}x)}{x\mathrm{e}^{\sigma x}} + \sum_p \sum_{r=1}^\infty \frac{p^{-r\sigma}}{r}\delta_{r\log p}(\mathrm{d}x)
\end{aligned}$$

for $\sigma > 1$. Therefore the characteristic function $\Xi_\sigma^*(t)$ is infinitely divisible for any $\sigma > 1$ since one has

$$\frac{1}{x\mathrm{e}^{\sigma x}(1-\mathrm{e}^{-2x})} - \frac{1}{x\mathrm{e}^{\sigma x}} > 0, \qquad x > 0.$$

Moreover, we can see that every characteristic function $\Xi_\sigma^*(t)$ is a pretended-infinitely divisible characteristic function for each $1/2 < \sigma < 1$ if and only if the Riemann hypothesis is true by an argument similar to that in the proof of Theorem 1.2. In addition, it holds that

$$\Xi_\sigma^*(t) = \exp\left[\int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}tx} - 1\right)\nu_\sigma^*(\mathrm{d}x)\right],$$

$$\nu_\sigma^*(\mathrm{d}x) := \frac{1_{(-\infty,0)}(\mathrm{d}x)}{-x\mathrm{e}^{(\sigma-1)x}} - \sum_{1/2+\mathrm{i}\gamma\in\mathcal{Z}_+} \frac{2\cos(\gamma x)}{x\mathrm{e}^{(\sigma-1/2)x}} 1_{(0,\infty)}(\mathrm{d}x),$$

for $1/2 < \sigma < 1$, under the Riemann hypothesis. This is proved by (1.6) and

$$\frac{\sigma-1}{\sigma-1-\mathrm{i}t} = \frac{1-\sigma}{1-\sigma+\mathrm{i}t} = \exp\left[\int_0^{\infty} \left(\mathrm{e}^{-\mathrm{i}tx} - 1\right)\mathrm{e}^{(\sigma-1)x}\frac{\mathrm{d}x}{x}\right]$$
$$= \exp\left[\int_0^{-\infty} \frac{(\mathrm{e}^{\mathrm{i}tx} - 1)\,\mathrm{d}x}{x\mathrm{e}^{(\sigma-1)x}}\right] = \exp\left[-\int_{-\infty}^{0} \frac{(\mathrm{e}^{\mathrm{i}tx} - 1)\,\mathrm{d}x}{x\mathrm{e}^{(\sigma-1)x}}\right]$$

when $1/2 < \sigma < 1$.

It is well known that convolving a density with a normal density to make distributions more well-behaved. In this case the exponential distribution is the one that makes things nicer since when $\sigma > 1$, the complete Riemann zeta distribution defined by $\Xi_\sigma(t)$ and the distribution defined by $\Xi_\sigma^*(t)$ are quasi-infinitely divisible and infinitely divisible, respectively.

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