A Complete Solution to the Harmonic Elimination Problem

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***Abstract*—The problem of eliminating harmonics in a switching converter is considered. That is, given a desired fundamental output voltage, the problem is to find the switching times (angles) that produce the fundamental while not generating specifically chosen harmonics. In contrast to the well known work of Patel and Hoft and others, here all possible solutions to the problem are found. This is done by first converting the transcendental equations that specify the harmonic elimination problem into an equivalent set of polynomial equations. Then, using the mathematical theory of resultants, all solutions to this equivalent problem can be found. In particular, it is shown that there are new solutions that have not been previously reported in the literature. The complete solutions for both unipolar and bipolar switching patterns to eliminate the fifth and seventh harmonics are given. Finally, the unipolar case is again considered where the fifth, seventh, 11th, and 13th harmonics are eliminated along with corroborative experimental results.**

***Index Terms*—Bipolar, harmonic elimination, switching converter, unipoplar.**

I. INTRODUCTION

THE PROBLEM of eliminating harmonics in switching converters has been the focus of research for many years. If the switching losses in an inverter are not a concern (i.e., switching on the order of a few kHz is acceptable), then the sine-triangle PWM method and its variants are very effective for controlling the inverter [1]. This is because the generated harmonics are beyond the bandwidth of the system being actuated and therefore these harmonics do not dissipate power. On the other hand, for systems where high switching efficiency is of utmost importance, it is desirable to keep the switching frequency much lower. In this case, another approach is to choose the switching times (angles) such that a desired fundamental output is generated and specifically chosen harmonics of the fundamental are suppressed [1]–[5]. This is referred to as *harmonic elimination* or *programmed harmonic elimination* as the switching angles are chosen (programmed) to eliminate specific harmonics.

In this work, it is shown how the complete solution (i.e., *all* possible solutions) to the problem considered in [2]–[5] is obtained. Specifically, in [2]–[4] the harmonic elimination problem was formulated as a set of transcendental equations that must be solved to determine the times (angles) in an electrical cycle for turning the switches on and off in a full bridge inverter so as to produce a desired fundamental amplitude while eliminating, for example, the fifth and seventh harmonics. These transcendental equations are then solved using *iterative numerical techniques* to compute the switching angles. (See Figs. 8–34 of [1] for a plot of these angles as a percent of the fundamental or Fig. 2.) Here a method is presented that not only obtains these solutions, but also another (different) set of the switching angles, and this other set of switching angles actually generates a smaller harmonic distortion due to the eleventh and thirteenth harmonics. The unipolar case is also considered (including the case where the fifth, seventh, eleventh, and thirteenth harmonics are eliminated) along with corroborative experimental results.

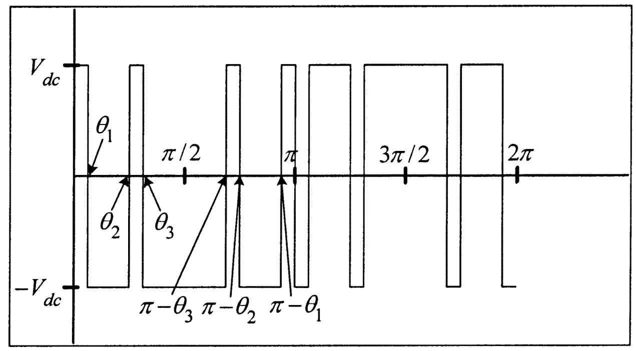

Fig. 1. Bipolar switching scheme.

The paper is organized as follows. In Section II, the solution method is illustrated for the bipolar case with the problem formulated as achieving the fundamental while not generating the fifth and seventh harmonics. In Section III, it is then shown how the method can be used in the case of a unipolar PWM switching scheme, again formulating the problem so as to achieve the fundamental while not generating the fifth and seventh harmonics. Section IV then formulates and solves the unipolar case using five switching angles in which the fundamental is achieved and the fifth, seventh, 11th, and 13th are not generated. Experimental results are presented in Section V, and a summary of the results is presented in Section VI.

II. BIPOLAR CASE

In this work, a standard H-bridge is used wherein choosing the switching angles θ_1, θ_2, θ_3 for the bipolar case results in an output waveform of the form shown in Fig. 1. (In this figure, the

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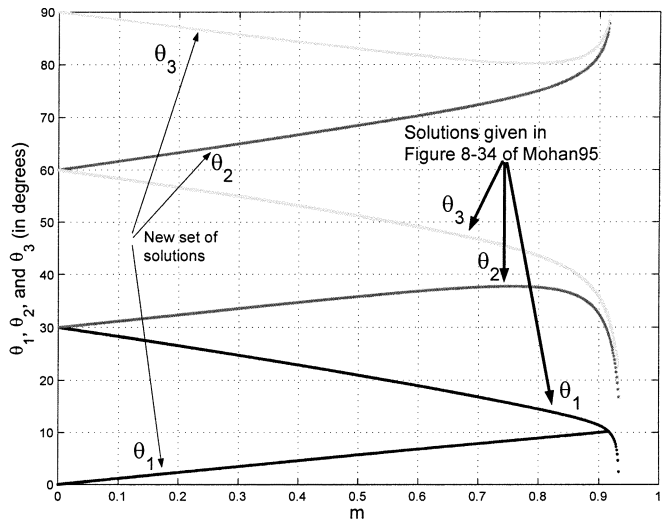

Fig. 2. Bipolar switching angles versus m.

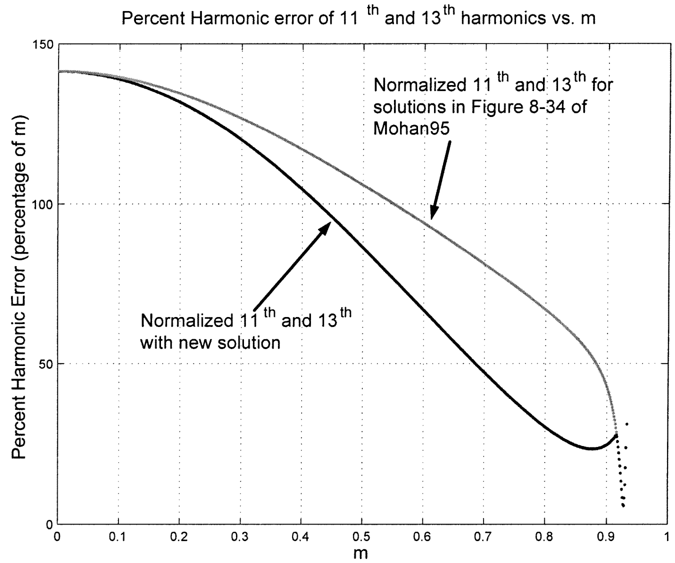

Fig. 3. Normalized error $\sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}$ for Bipolar PWM due to the 11th and 13th harmonics.

angle θ_1 corresponds to the time $(\theta_1/2\pi)T$, etc and 2π corresponds to the fundamental period T.) The Fourier series expansion of this output voltage waveform is

$$\begin{aligned} V(\omega t) = -\frac{4V_{dc}}{\pi} &\left\{ \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin(n\omega t)}{n} \right. \\ &\left. \times \left(1 - 2\cos(n\theta_1) + 2\cos(n\theta_2) - 2\cos(n\theta_3)\right) \right\}. \end{aligned} \tag{1}$$

Given a desired fundamental voltage V_1, the problem here is to determine the switching angles θ_1, θ_2, θ_3 so that

$$\begin{aligned} 1 - 2\cos(\theta_1) + 2\cos(\theta_2) - 2\cos(\theta_3) &= -m \\ 1 - 2\cos(5\theta_1) + 2\cos(5\theta_2) - 2\cos(5\theta_3) &= 0 \\ 1 - 2\cos(7\theta_1) + 2\cos(7\theta_2) - 2\cos(7\theta_3) &= 0 \end{aligned} \tag{2}$$

where $m \triangleq V_1/(4V_{dc}/\pi)$. This is a system of 3 transcendental equations in the unknowns θ_1, θ_2, θ_3. One approach to solving this set of nonlinear transcendental (2) is to use an iterative technique such as the Newton-Raphson method [3], [4]. Such a method results in the solution in Figs. 8–34 in [1] (or Fig. 2). Here, a methodology for finding *all* the solutions to (2) is presented, and our method not only gives the solutions reported in [1], [3], [4], but also a new set of solutions which are found to generate a lower harmonic distortion due to the 11th and 13th harmonics (see Fig. 3).

To use the method, the conditions (2) are first converted to an equivalent polynomial system. Specifically, one defines $x_1 = \cos(\theta_1), x_2 = \cos(\theta_2), x_3 = \cos(\theta_3)$ and uses the trigonometric identities

$$\begin{aligned} \cos(5\theta) &= 5\cos(\theta) - 20\cos^3(\theta) + 16\cos^5(\theta) \\ \cos(7\theta) &= -7\cos(\theta) + 56\cos^3(\theta) \\ &\quad - 112\cos^5(\theta) + 64\cos^7(\theta) \end{aligned} \tag{3}$$

to transform the conditions (2) into the equivalent conditions

$$\begin{aligned} p_1(x) &\triangleq 1 + m - 2x_1 + 2x_2 - 2x_3 = 0 \\ p_5(x) &\triangleq 1 + 2\sum_{i=1}^{3}(-1)^i \left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ p_7(x) &\triangleq 1 + 2\sum_{i=1}^{3}(-1)^i \\ &\quad \times \left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \end{aligned} \tag{4}$$

where $x = (x_1, x_2, x_3)$ and $m \triangleq V_1/(4V_{dc}/\pi)$. Equation (4) is a set of three *polynomial* equations in the three unknowns x_1, x_2, x_3. Further, the solutions must satisfy $0 \le x_3 < x_2 < x_1 \le 1$. Such a transformation to polynomial equations was also used in [5] where the polynomials were then solved using iterative numberical techniques. In contrast, it is shown here how the polynomial equations can be solved directly for all solutions.

A. Elimination Using Resultants

In order to explain how one computes the zero sets of polynomial systems, a brief discussion of the procedure of solving such systems is now given. A systematic procedure to do this is known as *elimination theory* and uses the notion of *resultants* [6]–[9]. Briefly, one considers $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. Then, for example, letting $a(x_1, x_2)$ and $b(x_1, x_2)$ have degrees 3 and 2, respectively in x_2, they may be written in the form

$$\begin{aligned} a(x_1, x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1, x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1). \end{aligned} \tag{5}$$

The $n \times n$ *Sylvester* matrix, where $n = \deg_{x_2}\{a(x_1,x_2)\} + \deg_{x_2}\{b(x_1,x_2)\} = 3 + 2 = 5$, is defined by

$$S_{a,b}(x_1) = \begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & 0 & b_2(x_1) \end{bmatrix}. \tag{6}$$

The *resultant* polynomial is then defined by

$$r(x_1) = \mathrm{Res}\left(a(x_1,x_2), b(x_1,x_2), x_2\right) \triangleq \det S_{a,b}(x_1) \tag{7}$$

and is the result of solving $a(x_1,x_2) = 0$ and $b(x_1,x_2) = 0$ simultaneously for x_1, i.e., eliminating x_2. See the Appendix for a brief explanation of this fact.

B. Solving the Bipolar Equations

Following the procedure just outlined [10], the resultant methodology is used to solve for *all* possible switching angles. That is, $x_3 = m - (x_1 + x_2)$ is used to eliminate x_3 from p_5 and p_7 in (4) to get the two polynomials equations $p_5(x_1,x_2) = 0$, $p_7(x_1,x_2) = 0$ in two unknowns which must be solved simultaneously. This is reduced to one polynomial in one unknown by computing the resultant polynomial $r_{p_5,p_7}(x_1)$ of the polynomial pair $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ (see [7] and [8] for background on resultants) to get

$$r_{p_5,p_7}(x_1) = 16\,777\,216 m^2 (1 + m - 2x_1)^4 r_{bi}^2(x_1) \tag{8}$$

where $r_{bi}(x_1)$ is a polynomial of 9th degree (see the Appendix). As the parameter m is incremented in steps of 0.01, the roots of $r_{bi}(x_1)$ are found and used to back solve for x_2 and x_1. The set of all three tuples $(x_{3\ell}, x_{2\ell}, x_{1\ell})$ which satisfy $0 \le x_{3\ell} < x_{2\ell} < x_{1\ell} \le 1$ then give

$$\{(\theta_{1l}, \theta_{2l}, \theta_{3l})\} = \left\{\left(\cos^{-1}(x_{1l}), \cos^{-1}(x_{2l}), \cos^{-1}(x_{3l})\right)\right\} \tag{9}$$

as the set of all possible solutions to (2) for the particular value of m. This computation was done as m was incremented between 0 and 1 resulting in the switching angles versus m as given in Fig. 2. As the figure shows, only at high values of $m (> 0.91)$ do the two sets of solutions merge into one.

To compare the two sets of solutions, the normalized magnitude of their 11th and 13th harmonics (i.e., $\sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}$ where a_k is the k^{th} harmonic) is plotted in Fig. 3. As this figure shows, the new set of solutions generates less harmonic distortion due to the 11th and 13th harmonics.

III. Unipolar Case

The Fourier expansion of the unipolar waveform given in Fig. 4 is

$$V(\omega t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{dc}}{n\pi} \times \left(\cos(n\theta_1) - \cos(n\theta_2) + \cos(n\theta_3)\right)\sin(n\omega t). \tag{10}$$

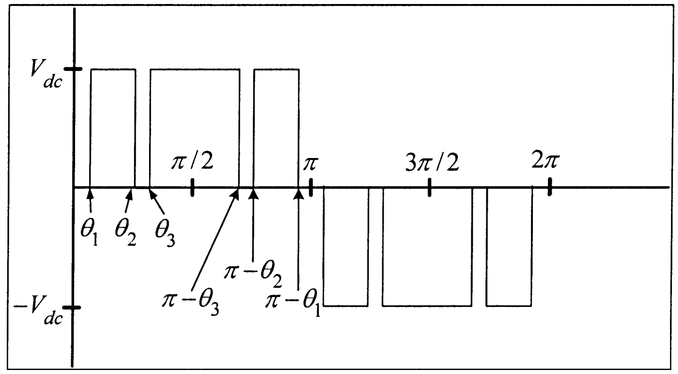

Fig. 4. Unipolar PWM switching scheme.

The problem is to determine the switching angles $\theta_1, \theta_2, \theta_3$ such that $(m \triangleq V_1/(4V_{dc}/\pi))$

$$\begin{aligned} \cos(\theta_1) - \cos(\theta_2) + \cos(\theta_3) &= m \\ \cos(5\theta_1) - \cos(5\theta_2) + \cos(5\theta_3) &= 0 \\ \cos(7\theta_1) - \cos(7\theta_2) + \cos(7\theta_3) &= 0. \end{aligned} \tag{11}$$

Converting (11) to polynomial equations

$$\begin{aligned} p_1(x) &\triangleq x_1 - x_2 + x_3 - m = 0 \\ p_5(x) &\triangleq \sum_{i=1}^{3} (-1)^{i-1}\left(5x_i - 20x_i^3 + 16x_i^5\right) = 0 \\ p_7(x) &\triangleq \sum_{i=1}^{3} (-1)^{i-1} \\ &\quad \times \left(-7x_i + 56x_i^3 - 112x_i^5 + 64x_i^7\right) = 0 \end{aligned} \tag{12}$$

as in the bipolar example, the resultant methodology as presented in [10] was again used to solve for *all* possible switching angles. That is, $x_3 = m - (x_1 + x_2)$ is used to eliminate x_3 from p_5 and p_7 in (12) to get the pair of polynomial equations $p_5(x_1,x_2) = 0$, $p_7(x_1,x_2) = 0$ that must be solved simultaneously. As in the bipolar case, this is done by computing resultant polynomial $r_{p_5,p_7}(x_1)$ of the pair $\{p_5(x_1,x_2), p_7(x_1,x_2)\}$ to get

$$r_{p_5,p_7}(x_1) = 16\,777\,216 m^4 (m - x_1)^4 r_{uni}^2(x_1) \tag{13}$$

where $r_{uni}(x_1)$ is a polynomial of ninth degree (see the Appendix).

As the parameter m is incremented in steps of 0.01, the roots of $r_{uni}(x_1)$ are found and used to back solve for x_2 and x_1. The set of all three tuples $(x_{3\ell}, x_{2\ell}, x_{1\ell})$ which satisfy $0 \le x_{3\ell} < x_{2\ell} < x_{1\ell} \le 1$ then give

$$\{(\theta_{1l}, \theta_{2l}, \theta_{3l})\} = \left\{\left(\cos^{-1}(x_{1l}), \cos^{-1}(x_{2l}), \cos^{-1}(x_{3l})\right)\right\} \tag{14}$$

as the set of all possible solutions to (11) for the particular value of m. The parameter m is then varied between 0 and 1, and these switching angles are plotted versus m in Fig. 5. Fig. 6 is a plot of magnitude of the distortion (i.e., $\sqrt{(a_{11}/a_1)^2 + (a_{13}/a_1)^2}$) due to the 11th and 13th harmonics. As seen in the figure, there are two sets of solutions for $m \in [0.5, 0.91]$ and that the two sets of solutions produce approximately the same distortion.

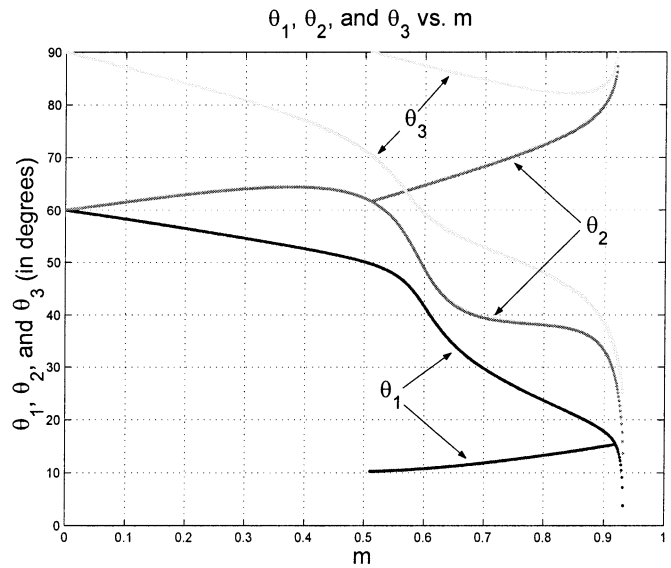

Fig. 5. Unipolar switching angles versus m.

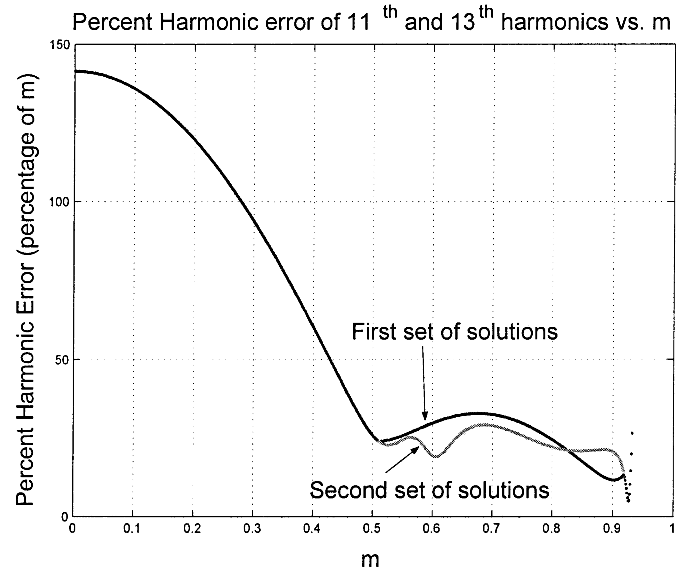

Fig. 6. Normalized error $\sqrt{(a_{11}/a_1)^2+(a_{13}/a_1)^2}$ for Unipolar PWM due to the 11th and 13th harmonics.

IV. Unipolar PWM With Five Switching Angles

In the bipolar scheme, the RMS voltage $1/(2\pi)\int_0^{2\pi}\sqrt{V^2(\omega t)^2}d\omega = V_{dc}$ is constant because $V(\omega t) = \pm V_{dc}$ and therefore the THD is constant and is only being shifted in the frequency spectrum. However, the unipolar PWM scheme can also produce zero voltage and therefore inherently has lower harmonic content than the bipolar scheme. Consequently, this scheme is now considered for the case where five switching angles are used. The Fourier expansion of a unipolar waveform with switching angles θ_1, θ_2, θ_3, θ_4, θ_5 leads to the conditions

$$\begin{aligned}
&\cos(\theta_1)-\cos(\theta_2)+\cos(\theta_3)-\cos(\theta_4)+\cos(\theta_5)=m\\
&\cos(5\theta_1)-\cos(5\theta_2)+\cos(5\theta_3)-\cos(5\theta_4)+\cos(5\theta_5)=0\\
&\cos(7\theta_1)-\cos(7\theta_2)+\cos(7\theta_3)-\cos(7\theta_4)+\cos(7\theta_5)=0\\
&\cos(11\theta_1)-\cos(11\theta_2)+\cos(11\theta_3)-\cos(11\theta_4)\\
&\quad+\cos(11\theta_5)=0\\
&\cos(13\theta_1)-\cos(13\theta_2)+\cos(13\theta_3)-\cos(13\theta_4)\\
&\quad+\cos(13\theta_5)=0.
\end{aligned} \tag{15}$$

Here, $m \triangleq V_1/(4V_{dc}/\pi)$ is the modulation index and the angles must satisfy $\theta_1 \le \theta_2 \le \theta_3 \le \theta_4 \le \theta_5$ (see Fig. 9 for a typical waveform). Let $\theta_i' = \theta_i$ if the coefficient of $\cos(n\theta_i)$ is $+1$ and $\theta_i' = \pi - \theta_i$ if it is $-1(\cos(n\theta_i') = -\cos(n\theta_i)$ for n odd) and letting $x_1 = \cos(\theta_1')$, $x_2 = \cos(\theta_2')$, $x_3 = \cos(\theta_3')$, $x_4 = \cos(\theta_4')$, $x_5 = \cos(\theta_5')$ the conditions become

$$\begin{aligned}
p_1(x) &\triangleq x_1+x_2+x_3+x_4+x_5-m=0\\
p_5(x) &\triangleq \sum_{i=1}^{5}\left(5x_i-20x_i^3+16x_i^5\right)=0\\
p_7(x) &\triangleq \sum_{i=1}^{5}\left(-7x_i+56x_i^3-112x_i^5+64x_i^7\right)=0\\
p_{11}(x) &\triangleq \sum_{i=1}^{5}\left(-11x_i+220x_i^3-1232x_i^5+2816x_i^7\right.\\
&\qquad\left.-2816x_i^9+1024x_i^{11}\right)=0
\end{aligned}$$

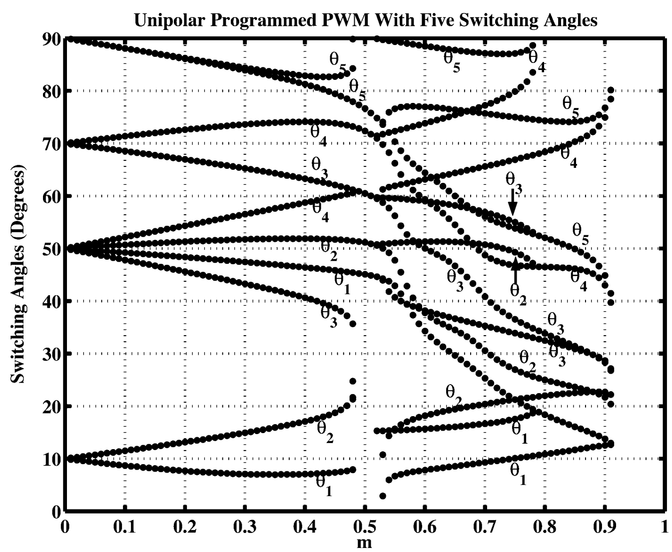

Fig. 7. Unipolar switching angles versus m with five switching angles.

$$\begin{aligned}
p_{13}(x) \triangleq \sum_{i=1}^{5}&\left(13x_i-364x_i^3+2912x_i^5-9984x_i^7+16\,640x_i^9\right.\\
&\left.-13\,312x_i^{11}+4096x_i^{13}\right)=0
\end{aligned} \tag{16}$$

where $0 \le x_5 \le -x_4 \le x_3 \le -x_2 \le x_1 \le 1$.

Remark: It is interesting to note that the set of polynomials in (16) are the same equations as that of a multilevel inverter with five dc sources and a fundamental frequency staircase output waveform [10]. The difference between the two solutions is in the region where the x_i must lie. In the multilevel case, the conditions are $0 \le x_5 \le x_4 \le x_3 \le x_2 \le x_1 \le 1$.

Following a procedure similar to that given in Sections II and III, one systematically solves these equations by elimination theory. This was done, and the *complete* set of switching angle solutions are plotted versus m in Fig. 7. Each set of solutions (θ_1, θ_2, θ_3, θ_4, θ_5) is labeled vertically in Fig. 7. Note that for $0 \le m \le 0.48$ there are two sets of solutions; for $0.40 \le m \le$

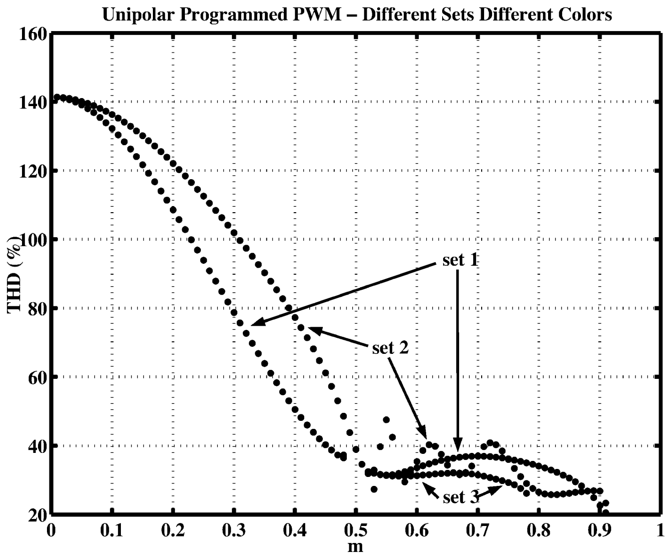

Fig. 8. THD versus m for each set of switching angles.

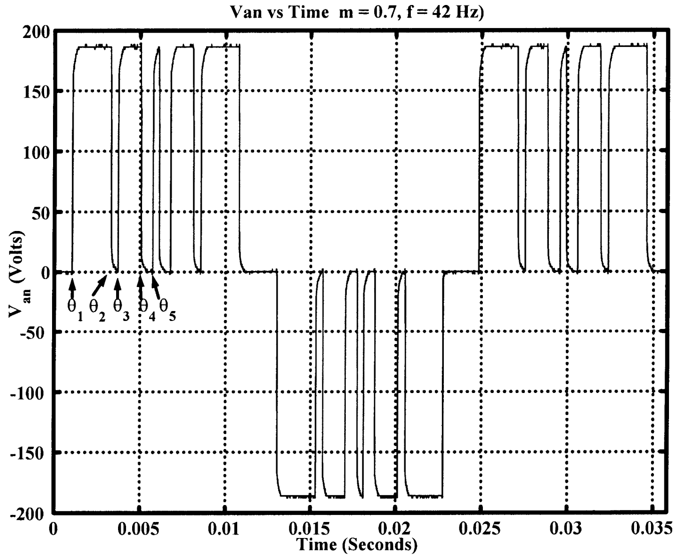

Fig. 9. Voltage waveform with $m = 0.7$ and $f = 42$ Hz.

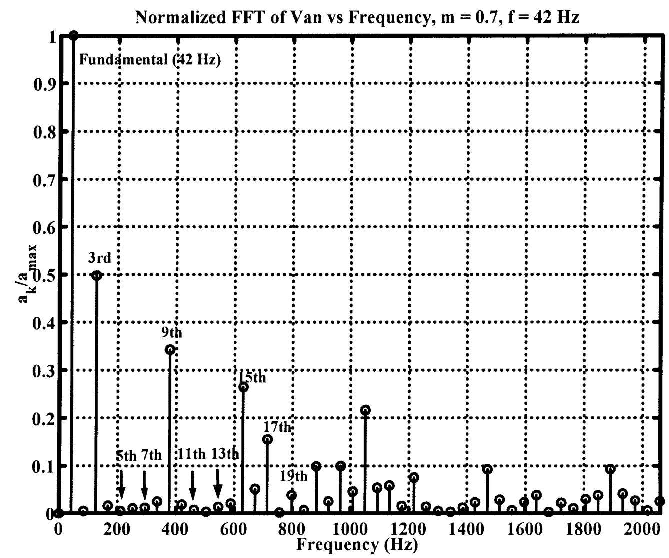

Fig. 10. FFT of the voltage waveform of Fig. 9 with $m = 0.7$ and $f = 42$ Hz.

0.53 there is only one solution set; for $0.53 \leq m \leq 0.78$ there are three sets of solutions; and finally, for $0.78 \leq m \leq 0.91$, there are again two sets of solutions.

The corresponding total harmonic distortion (THD) was computed out to the 31st according to

$$THD = \sqrt{\frac{V_5^2 + V_7^2 + V_{11}^2 + V_{13}^2 + V_{17}^2 + \cdots + V_{31}^2}{V_1^2}} \times 100 \tag{17}$$

and is plotted versus m in Fig. 8 for each of the solution sets shown in Fig. 7. As this figure shows, one can choose a particular solution for the switching angles such that the THD is 32% or less for $0.55 \leq m \leq 0.9$.

It is important to point out that if one had used an iterative method such as Newton-Raphson, then the third solution set that exists for $0.53 \leq m \leq 0.78$ would not have been found, and this is the solution set that results in the lowest THD for this range of modulation indices. The reason the Newton-Raphson method would not have found this solution set is simply due to the way it is implemented. One starts with an initial guess for the angles at $m = 0$. Then this solution is used as the initial guess for the solution when m is incremented by Δm to its next value and so on. At $m = 0$, the only possible solutions are $\theta_1 = 50°$, $\theta_2 = 50°$, $\theta_3 = 70°$, $\theta_4 = 70°$, $\theta_5 = 90°$ or $\theta_1 = 10°$, $\theta_2 = 10°$, $\theta_3 = 50°$, $\theta_4 = 50°$, $\theta_5 = 90°$. As Fig. 7 shows, if the first solution set is used as the starting point in the Newton-Raphson scheme for $m = 0$, then as m is incremented, one would obtain a set of solutions valid for $0 \leq m \leq 0.91$. If the second set of solutions is used as the starting point, then a set of solutions valid for $0 \leq m \leq 0.48$ would be obtained. Neither of these sets results in the minimum THD for $0.53 \leq m \leq 0.78$. *Consequently, the method proposed here that finds the complete solution set allows one to be sure that the solution with the lowest THD is used.* In the interesting work [13], a homotopy approach was used for the bipolar case only. Though it appears to be able to find all solutions in the bipolar case, it is not clear that it would be able to do so in the unipolar case (e.g., find the third set in Figs. 7 and 8).

V. Experimental Results

An inverter was used to perform experiments to validate the predicted results, that is, the elimination of the fifth, seventh, 11th, and 13th harmonics in the output of a three phase inverter. A real-time computing platform [11] was used to interface the logic signals from the computer to the gate driver board of the inverter. The switching algorithm is implemented as a lookup table in SIMULINK which is then converted to C code. The software provides icons to interface the SIMULINK model to the digital I/O board and converts the C code into executables. The computational time step size was 32 μm. The induction motor used in the experiments had the following name plate data:

$$\begin{aligned} \text{Rated hp} &= \frac{1}{3}\ \text{hp} \\ \text{Rated Current} &= 1.5\ \text{A} \\ \text{Rated Speed} &= 1725\ \text{rpm} \\ \text{Rated Voltage} &= 208\ \text{V(RMS line} - \text{to} - \text{line @ 60 Hz)}. \end{aligned}$$

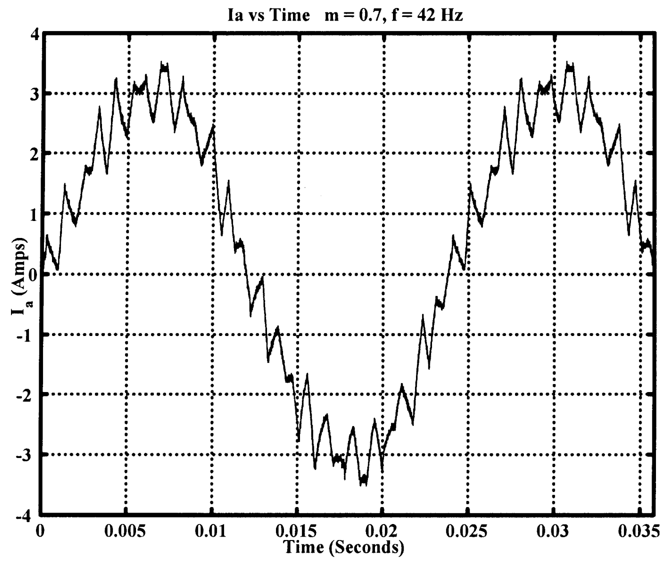

Fig. 11. Current waveform in phase a of the test (induction) motor with $m = 0.7$ and $f = 42$ Hz.

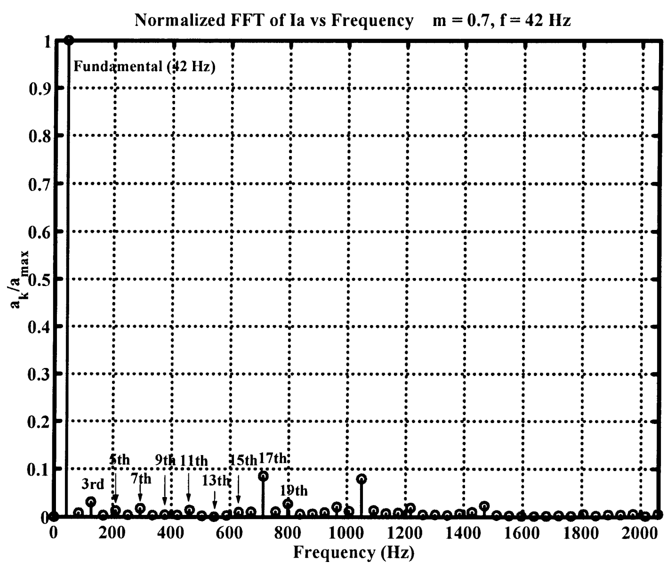

Fig. 12. FFT of the current in phase a of the test motor with $m = 0.7$ and $f = 42$ Hz.

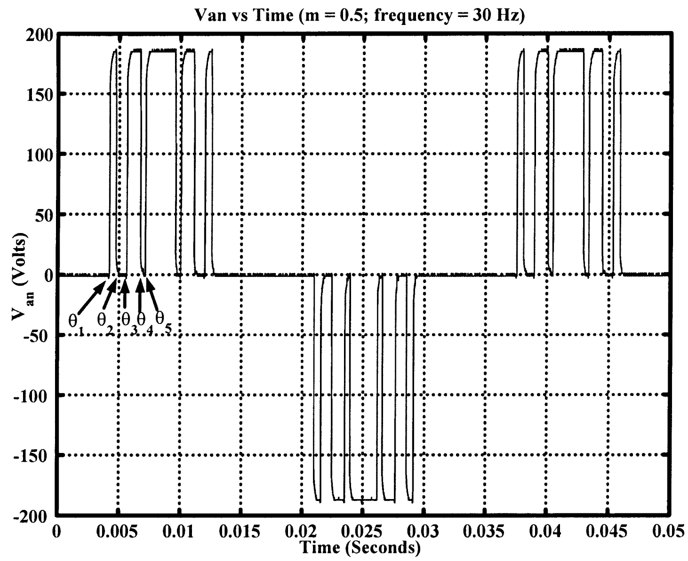

Fig. 13. Voltage waveform of phase a with $m = 0.5$ and $f = 30$ Hz.

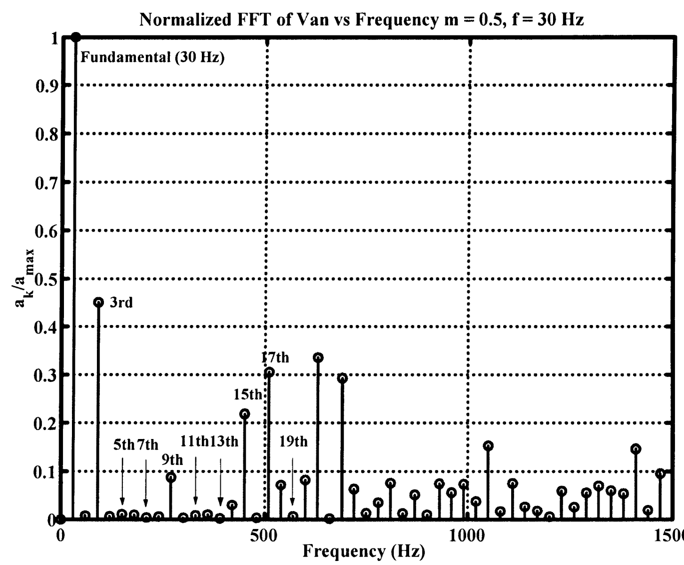

Fig. 14. FFT of the voltage waveform of Fig. 13 with $m = 0.5$ and $f = 30$ Hz.

Two sets of experiments were performed to compare with the computational results given in Figs. 7 and 8.

A. First Experimental Set

In this first experiment, the modulation index was set as $m = 0.7$ and the frequency $f = 42$ Hz. Fig. 9 shows the measured voltage waveform from phase a of the inverter output. A (normalized) fast fourier transform (FFT) of this waveform is plotted in Fig. 10. As predicted, the fifth, seventh, eleventh, and thirteenth harmonics are quite small consistent with their predicted value of zero. As can be seen from the harmonic specturm in Fig. 10, the lowest substantial nontriplen harmonics are the seventeenth and nineteenth. As the interest here is a three phase drive, the triplen harmonics in the phase voltages will cancel in the line-line voltages. Application of the voltages to the test motor resulted in a current waveform for phase a as given in Fig. 11 with its corresponding FFT plotted in Fig. 12. The total voltage THD computed using (17) was 29.7% based on the FFT data in Fig. 10 which compares well with the predicted value of 31.5% given in Fig. 8. The total current THD was found to be 12.6% using the FFT data in Fig. 12.

B. Second Experimental Set

In the second experiment, the modulation index was set as $m = 0.5$ and the frequency $f = 30$ Hz. Fig. 13 shows the measured output voltage waveform from phase a of the inverter, and its corresponding FFT is plotted in Fig. 14. As predicted, the fifth, seventh, eleventh, and thirteenth harmonics are essentially zero consistent with their predicted value of zero. Application of the voltages to the test motor resulted in the current waveform given in Fig. 15, and the FFT of this waveform is presented in Fig. 16. The total voltage THD computed using (17) was 43.3% based on the FFT data in Fig. 14 which compares favorably with the predicted value of 39% given in Fig. 8. The total current THD was found to be 17.6% using the FFT data in Fig. 16.

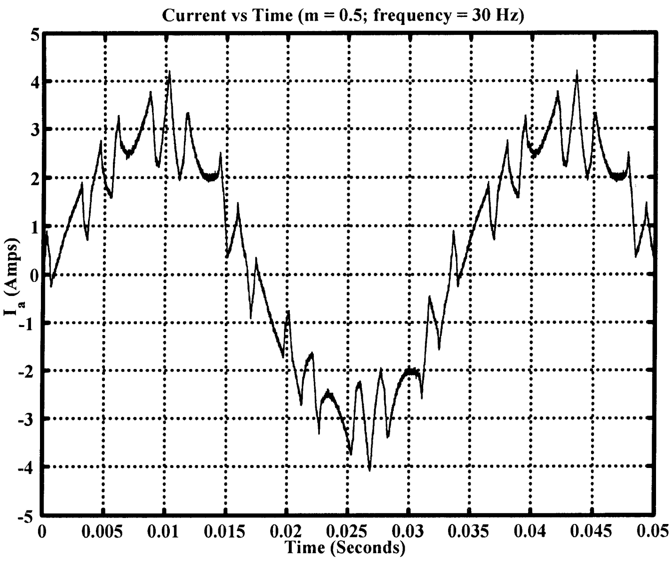

Fig. 15. Current waveform in phase a of the test (induction) motor with $m = 0.5$ and $f = 30$ Hz.

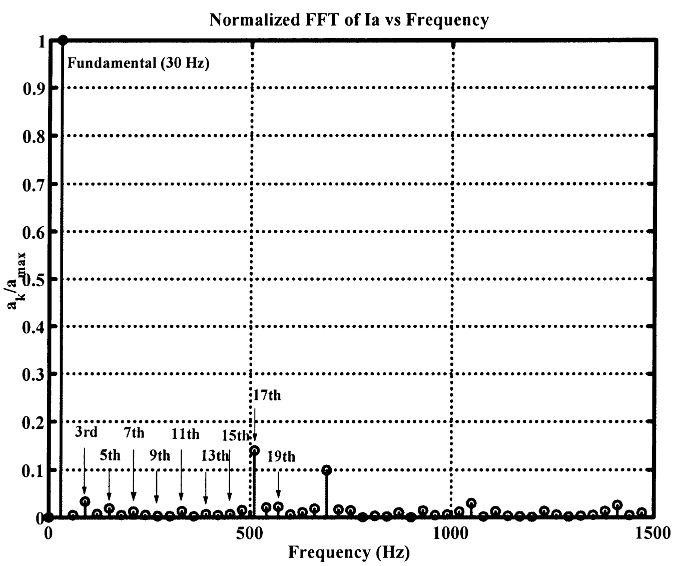

Fig. 16. FFT of the current in phase a of the test motor with $m = 0.5$ and $f = 30$ Hz.

VI. Conclusion

The complete solution to the harmonic elimination problem can be found using the theory of resultants from elimination theory. The solution is complete in the sense that any and all solutions were found. Experimental work was presented to corroborate the developed technique.

Appendix I

Resultants [7]–[9], [12]

Given two polynomials $a(x_1, x_2)$ and $b(x_1, x_2)$ how does one find their common zeros? That is, the values (x_{10}, x_{20}) such that

$$a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0.$$

Consider $a(x_1, x_2)$ and $b(x_1, x_2)$ as polynomials in x_2 whose coefficients are polynomials in x_1. There is always a polynomial $r(x_1)$ (called the *resultant polynomial*) such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1).$$

So if $a(x_{10}, x_{20}) = b(x_{10}, x_{20}) = 0$ then $r(x_{10}) = 0$, that is, if (x_{10}, x_{20}) is a common zero of the pair $\{a(x_1, x_2), b(x_1, x_2)\}$, then the first coordinate x_{10} is a zero of $r(x_1) = 0$. To see how one obtains $r(x_1)$, let

$$\begin{aligned} a(x_1, x_2) &= a_3(x_1)x_2^3 + a_2(x_1)x_2^2 + a_1(x_1)x_2 + a_0(x_1) \\ b(x_1, x_2) &= b_2(x_1)x_2^2 + b_1(x_1)x_2 + b_0(x_1). \end{aligned}$$

Next, see if polynomials of the form

$$\begin{aligned} \alpha(x_1, x_2) &= \alpha_1(x_1)x_2 + \alpha_0(x_1) \\ \beta(x_1, x_2) &= \beta_2(x_1)x_2^2 + \beta_1(x_1)x_2 + \beta_0(x_1) \end{aligned}$$

can be found such that

$$\alpha(x_1, x_2)a(x_1, x_2) + \beta(x_1, x_2)b(x_1, x_2) = r(x_1). \tag{18}$$

Equating powers of x_2, this equation may be rewritten in matrix form as

$$\begin{bmatrix} a_0(x_1) & 0 & b_0(x_1) & 0 & 0 \\ a_1(x_1) & a_0(x_1) & b_1(x_1) & b_0(x_1) & 0 \\ a_2(x_1) & a_1(x_1) & b_2(x_1) & b_1(x_1) & b_0(x_1) \\ a_3(x_1) & a_2(x_1) & 0 & b_2(x_1) & b_1(x_1) \\ 0 & a_3(x_1) & 0 & 0 & b_2(x_1) \end{bmatrix} \times \begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix on the left-hand side is called the *Sylvester* matrix and is denoted here by $S_{a,b}(x_1)$. The inverse of $S_{a,b}(x_1)$ has the form

$$S_{a,b}^{-1}(x_1) = \frac{1}{\det S_{a,b}(x_1)} \operatorname{adj}\left(S_{a,b}(x_1)\right)$$

where $\operatorname{adj}(S_{a,b}(x_1))$ is the adjoint matrix and is a 5×5 *polynomial* matrix in x_1. Solving for $\alpha_i(x_1)$, $\beta_i(x_1)$ gives

$$\begin{bmatrix} \alpha_0(x_1) \\ \alpha_1(x_1) \\ \beta_0(x_1) \\ \beta_1(x_1) \\ \beta_2(x_1) \end{bmatrix} = \frac{\operatorname{adj} S_{a,b}(x_1)}{\det S_{a,b}(x_1)} \begin{bmatrix} r(x_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Choosing $r(x_1) = \det S_{a,b}(x_1)$ guarantees that $\alpha_0(x_1)$, $\alpha_1(x_1)$, $\beta_0(x_1)$, $\beta_1(x_1)$, $\beta_2(x_1)$ are polynomials in x_1. That is, the *resultant polynomial* defined by $r(x_1) = \det S_{a,b}(x_1)$ is the polynomial required for (18).

APPENDIX II
RESULTANT POLYNOMIALS $r_{bi}(x_1)$ AND $r_{uni}(x_1)$

$$
\begin{aligned}
r_{bi}(x_1) = {} & 6125m^2+6125m^3-12\,250m^4-14\,700m^5 \\
& + 4900m^6+10\,500m^7+1925m^8-2030m^9 \\
& - 910m^{10}+70m^{12}+15m^{13}+m^{14} \\
& - 24\,500mx_1-24\,500m^2x_1+110\,250m^3x_1 \\
& + 132\,300m^4x_1-88\,200m^5x_1-159\,600m^6x_1 \\
& - 25\,200m^7x_1+43\,400m^8x_1+20\,300m^9x_1 \\
& - 1820m^{11}x_1-420m^{12}x_1-30m^{13}x_1 \\
& - 73\,500mx_1^2-514\,500m^2x_1^2-514\,500m^3x_1^2 \\
& + 632\,100m^4x_1^2+1\,058\,400m^5x_1^2+126\,000m^6x_1^2 \\
& - 415\,800m^7x_1^2-205\,800m^8x_1^2+21\,840m^{10}x_1^2 \\
& + 5460m^{11}x_1^2+420m^{12}x_1^2+294\,000x_1^3 \\
& + 1\,274\,000mx_1^3+1\,225\,000m^2x_1^3-2\,156\,000m^3x_1^3 \\
& - 3\,822\,000m^4x_1^3-235\,200m^5x_1^3+2\,284\,800m^6x_1^3 \\
& + 1\,209\,600m^7x_1^3-11\,200m^8x_1^3-161\,280m^9x_1^3 \\
& - 43\,680m^{10}x_1^3-3640m^{11}x_1^3-882\,000x_1^4 \\
& - 1\,176\,000mx_1^4+4\,410\,000m^2x_1^4+8\,232\,000m^3x_1^4 \\
& - 235\,200m^4x_1^4-7\,761\,600m^5x_1^4-4\,401\,600m^6x_1^4 \\
& + 201\,600m^7x_1^4+823\,200m^8x_1^4+240\,240m^9x_1^4 \\
& + 21\,840m^{10}x_1^4-980\,000x_1^5-7\,056\,000mx_1^5 \\
& - 12\,152\,000m^2x_1^5+980\,000m^3x_1^5 \\
& + 16\,228\,800m^4x_1^5+9\,878\,400m^5x_1^5 \\
& - 1\,478\,400m^6x_1^5-3\,040\,800m^7x_1^5-952\,000m^8x_1^5 \\
& - 95\,200m^9x_1^5+4\,704\,000x_1^6+11\,760\,000mx_1^6 \\
& - 19\,992\,000m^3x_1^6-12\,936\,000m^4x_1^6 \\
& + 5\,644\,800m^5x_1^6+8\,064\,000m^6x_1^6 \\
& + 2\,721\,600m^7x_1^6+302\,400m^8x_1^6-1\,568\,000x_1^7 \\
& + 3\,136\,000mx_1^7+16\,464\,000m^2x_1^7 \\
& + 10\,192\,000m^3x_1^7-11\,289\,600m^4x_1^7 \\
& - 14\,425\,600m^5x_1^7-5\,376\,000m^6x_1^7 \\
& - 672\,000m^7x_1^7-4\,704\,000x_1^8-14\,112\,000mx_1^8 \\
& - 9\,408\,000m^2x_1^8+9\,408\,000m^3x_1^8 \\
& + 15\,052\,800m^4x_1^8+6\,585\,600m^5x_1^8 \\
& + 940\,800m^6x_1^8+3\,136\,000x_1^9+6\,272\,000mx_1^9 \\
& - 6\,272\,000m^3x_1^9-3\,763\,200m^4x_1^9-627\,200m^5x_1^9
\end{aligned}
$$

$$
\begin{aligned}
r_{uni}(x_1) = {} & 6125m-49\,000m^3+137\,200m^5-179\,200m^7 \\
& + 116\,480m^9-35\,840m^{11}+4096m^{13} \\
& - 12\,250x_1+220\,500m^2x_1-882\,000m^4x_1 \\
& + 1\,512\,000m^6x_1-1\,245\,440m^8x_1 \\
& + 465\,920m^{10}x_1-61\,440m^{12}x_1-367\,500mx_1^2 \\
& + 2\,352\,000m^3x_1^2-5\,644\,800m^5x_1^2 \\
& + 6\,048\,000m^7x_1^2-2\,795\,520m^9x_1^2 \\
& + 430\,080m^{11}x_1^2+269\,500x_1^3-3\,430\,000m^2x_1^3 \\
& + 12\,230\,400m^4x_1^3-17\,337\,600m^6x_1^3 \\
& + 10\,106\,880m^8x_1^3-1\,863\,680m^{10}x_1^3 \\
& + 2\,940\,000mx_1^4-16\,464\,000m^3x_1^4 \\
& + 31\,987\,200m^5x_1^4-24\,192\,000m^7x_1^4 \\
& + 5\,591\,040m^9x_1^4-1\,470\,000x_1^5+13\,720\,000m^2x_1^5 \\
& - 39\,513\,600m^4x_1^5+39\,782\,400m^6x_1^5 \\
& - 12\,185\,600m^8x_1^5-7\,056\,000mx_1^6 \\
& + 32\,928\,000m^3x_1^6-45\,158\,400m^5x_1^6 \\
& + 19\,353\,600m^7x_1^6+2\,744\,000x_1^7-17\,248\,000m^2x_1^7 \\
& + 35\,123\,200m^4x_1^7-21\,504\,000m^6x_1^7 \\
& + 4\,704\,000mx_1^8-18\,816\,000m^3x_1^8 \\
& + 15\,052\,800m^5x_1^8-1\,568\,000x_1^9+6\,272\,000m^2x_1^9 \\
& - 5\,017\,600m^4x_1^9
\end{aligned}
$$

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