A COMPLETE THEORY FOR JOINTLY CONTINUOUS NONLINEAR SEMIGROUPS ON A COMPLETE SEPARABLE METRIC SPACE

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Abstract. For a jointly continuous semigroup of transformations on a complete separable metric space X an induced semigroup of linear transformations on an appropriate space of measures is defined. A complete characterization of generators of such semigroups is given and it is shown how to construct a jointly continuous semigroup on X from a generator taken from this characterized collection.

1. Linear Extension of Nonlinear Semigroups

For many years there has been sought a proper generalization of linear one-paramater semigroup theory to nonlinear cases. A satisfactory theory for a class of nonlinear semigroups would

- Assign to each semigroup in the class a generator.
- Characterize the set of all such generators.
- Show how to recover a semigroup from its generator.

This note deals with a complete theory of this nature for the class $SG(X)$ of nonlinear semigroups where X is a complete separable metric space and $SG(X)$ denotes the collection of all functions T with domain $[0, \infty)$ and range a collection of transformations from X to X so that

- $T(0) = J$, the identity transformation on X.
- $T(t)T(s) = T(t + s),\ t, s \geq 0$.
- If $g : [0, \infty) \times X \to X$ is defined by $g(t, x) = T(t)x,\ t \geq 0, x \in X$, then g is continuous.

Such a T is called a jointly continuous semigroup on X. A generator for each $T \in SG(X)$ will be a generator of a linear extension of T to a semigroup of measures on X. We now describe the needed measures. First denote by $B(X)$ the collection of Borel subsets of X, by $M(X)$ the set of Borel measures on X and by $M^+(X)$ its subset consisting of those members of $M(X)$ with values in $[0, \infty)$. A member $\mu \in M^+(X)$ is called compact regular if

$$\mu(\Omega) = \sup\{\mu(\Omega') : \Omega' \text{ compact}, \Omega' \subset \Omega,\ \Omega \in B(X)\}$$

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and a member of $M(X)$ is called compact regular provided it is the difference between two compact regular members of $M^+(X)$. Denote by $MCR(X)$ the collection of all compact regular members of $M(X)$.

For $T \in SG(X)$ the function U from $[0, \infty)$ to $MCR(X)$ defined by

$$(U(t)\mu)(\Omega) = \mu\{T(t)^{-1}(\Omega)\}, t \geq 0,\ \Omega \in B(X), \mu \in MCR(X) \tag{1}$$

is called the principal linear extension of T (here identify a point x with its corresponding Dirac measure δ_x).

There may be a smaller linear extensions of T, notably in case T itself is linear, but the above extension is the object of our concern here.

The following gives a generator for the principal linear extension of T:

Definition 1. *If $T \in SG(X)$*

$$C = \{(\mu, \nu) \in MCR(X)^2 : \lim_{t \to 0+} \int_\Omega (fT(t) - f)/t \, d\mu = \int_\Omega f \, d\nu,\ f \in C(X)\}$$

is called the extended generator of T and $G(X)$ denotes the collection of all extended generators of members of $SG(X)$.

We intend to give a characterization of $G(X)$ and show how to recover (by means of an exponential formula) a semigroup T from its generator. We next give a notion of convergence in $MCR(X)$:

Definition 2. *A sequence $\{\mu_n\}_{n=1}^\infty$ in $MCR(X)$ is said to have γ -limit $\nu \in MCR(X)$ provided that*

$$\lim_{n \to \infty} \int_X f \, d\mu_n = \int_X f \, d\mu,\ f \in C(X).$$

Theorem 1. *Suppose $T \in SG(X)$, U is its principal linear extension and C is the extended generator of T. Then if $\lambda \geq 0$, $(I - \lambda C)^{-1}$ exists and has domain all of $MCR(X)$. Moreover, if $\mu \in MCR(X)$ and $\lambda \geq 0$,*

$$\gamma - \lim_{n \to \infty} (I - (\lambda/n)C)^{-n} \mu = U(t)\mu.$$

Note that if $x \in X$,

$$\gamma - \lim_{n \to \infty} (I - (\lambda/n)C)^{-n} \delta_x = \delta_{T(\lambda)x}$$

so in this way, $T(\lambda)$ is recovered from C for $\lambda \geq 0$.

This note is intended to give, in a form more useful for applications, an account of recent work by J.R. Dorroh and the present author on nonlinear semigroups [3], [4],[5]. Definitions and statement of results are self-contained here but proofs rely heavily on [4] and [5].

2. Characterization of $G(X)$

Denote by $C(X)$ the Banach space, with sup norm, of all bounded continuous real-valued functions on X. A second topology for $C(X)$, the β topoology ([5],[8]) is related to the following notion of convergence:

Definition 3. *A sequence* $\{f_n\}_{n=1}^{\infty}$ *in* $C(X)$ *is* β *convergent to* $f \in C(X)$ *provided (1) there is* $M > 0$ *so that* $\|f_n\| \leq M,\ n = 1, 2, \ldots$ *and (2)* $\{f_n\}$ *converges uniformly to* f *on every compact subset of* X. *A collection* Z *of transformations from* $C(X)$ *to* $C(X)$ *is* β *equicontinuous if for each sequence* $\{f_n\}_{n=1}^{\infty}$, β *convergent to some* $f \in C(X)$, *(a) there is* $M > 0$ *so that* $\|Wf_n\| \leq M, n = 1, 2, \ldots,$ *and (b) if* Ω *is a compact subset of* X *and* $\epsilon > 0$, *there is* N *so that if* $n > N$,

$$|(Wf_n)(x) - (Wf)(x)| < \epsilon,\ x \in \Omega, W \in Z.$$

From [4] there is the following:

Definition 4. *Denote by* $LG(X)$ *the collection of all linear transformations* A *from a subset of* $C(X)$ *to* $C(X)$ *so that*
(i) A *is a derivation (if* $f, g \in D(A)$ *then* $fg \in D(A)$ *and* $A(fg) = fAg + gAf$*).*
(ii) $D(A)$ *is a* β *dense subset of* $C(X)$.
(iii) If $\lambda > 0$, $(I - \lambda A)^{-1}$ *exists, has domain all of* $C(X)$ *and is nonexpansive.*
(iv) If $\eta > 0$, *then*

$$\{(I - (\lambda/n)A)^{-n} : 0 \leq \lambda \leq \eta,\ n = 1, 2, \ldots\}$$

is β *equicontinuous.*

Theorem 2. $C \in G(X)$ *if and only if there is* $A \in LG(X)$ *so that*

$$C = \{(\mu, \nu) \in MCR(X)^2 : \int_X Af\ d\mu = \int_X f\ d\nu,\ f \in D(A)\}. \tag{2}$$

3. Proofs

The dual of the space $C(X)$, considered with the β toplogy, is denoted by $C(X)^{*\beta}$. A crucial fact relating the present note with [3],[4],[5] is the following from [8]:

Theorem 3. *The linear functional* g *on* $C(X)$ *is in* $C(X)^{*\beta}$ *if and only if there is* $\mu \in MCR(X)$ *such that*

$$g(f)\ =\ \int_X f\ d\mu,\ f \in C(X). \tag{3}$$

Denote by Q the transformation from $C(X)^{*\beta}$ to $MCR(X)$ so that (3) holds if and only if $\mu = Qg$. In [4], for $T \in SG(X)$, the linear representation S of T is defined as the function on $[0, \infty)$ so that

$$(S(t)f)(x) = f(T(t)x),\ x \in X,\ f \in C(X),\ t \geq 0 \tag{4}$$

and T was analyzed in terms of a generator A of S which satisfies Definition 4. In [5], given $T \in SG(X)$, a semigroup V on $C(X)^{*\beta}$ was defined so that

$$(V(t)h)(f) = h(fT(t)),\ h \in C(X)^{*\beta},\ f \in C(X),\ t \geq 0.$$

Such a semigroup is related very closely to (1).

Proof. [Theorem 1] Suppose $T \in SG(X)$ and denote by U, C the principal linear extension and extended generator respectively of T. Suppose also that $\lambda > 0$. We intend to show first that

$$(I - \lambda C)^{-1} \text{ exists and has domain all of } MCR(X).$$

Define $J_\lambda : C(X)^{*\beta} \to C(X)^{*\beta}$ so that if $h \in C(X)^{*\beta}$ then

$$(J_\lambda h)(f) = (1/\lambda) \int_0^\infty e^{-s/\lambda} h(fT(s))\, ds,\ f \in C(X).$$

Using [5] or analogous calculations in [4]

$$J_\lambda = (I - \lambda \overline{C})^{-1} \tag{5}$$

where

$$\overline{C} = \{(h,k) \in {C(X)^{*\beta}}^2 : kf = \lim_{t\to 0+} h(fT(t) - f)/t,\ f \in C(X)\}.$$

Inspection shows that

$$C = Q\overline{C}Q^{-1}.$$

Note that then

$$(I - \lambda C)^{-1} \ = \ Q(I - \lambda\overline{C})^{-1}Q^{-1}$$

and consequently for each positive integer n,

$$(I - (\lambda/n)C)^{-n} = Q(I - \lambda\overline{C})^{-n}Q^{-1}.$$

From [5] or analogous calculations in [4] one has that if $g \in C(X)^{*\beta}$ and $f \in C(X)$, then

$$\lim_{n\to\infty} ((I - (\lambda/n)\overline{C})^{-n} g)f = (V(\lambda)g)f$$

where V is as above and so

$$\gamma - \lim_{n\to\infty} (I - (\lambda/n)C)^{-n}\mu = U(\lambda)\mu.$$

The final part of the Theorem then follows. □

Proof. [Theorem 2] Suppose $C \in G(X)$ and denote by T a member of $SG(X)$ for which C is its extended generator. Using [4] T has the linear representation S given by (4). Following [4], denote the Lie generator of T by

$$A = \{(f,g) \in C(X)^2 : \lim_{t\to\infty} (f(T(x)) - f(x))/t = g(x),\ x \in X\}.$$

Suppose $\mu \in MCR(X)$ and $\nu = C\mu$. Then

$$\lim_{t\to 0+} \int_X (fT(t) - f)/t\, d\mu = \int_X f\, d\nu,\ f \in C(X)\} \tag{6}$$

and consequently

$$\int_X Af\, d\mu \ = \ \int_X f\, d\nu,\ f \in D(A) \tag{7}$$

since

$$\beta - \lim_{t\to 0} (fT(t) - f)/t = Af,\ f \in D(A)$$

and members of $C(X)^{*\beta}$ are β continuous. This shows that if $C \in G(X)$ and $\nu = C\mu$, then (7) holds.

Now suppose $A \in LG(X)$ and

$$\overline{C} = \{(\mu,\nu) \in MCR(X)^2 : \int_X Af\, d\mu = \int_X f\, d\nu,\ f \in D(A)\}.$$

Since $A \in LG(X)$, by Theorem 3.2 of [4] there is a unique $T \in SG(X)$ whose Lie generator is A. Denote by C the extended generator of T. Thus (7) holds since (6) holds for all $(\mu, \nu) \in C$. But (7) is both necessary and sufficient for $(\mu, \nu) \in C$. Thus $\overline{C} = C$ and the argument is finished. □

4. Indication of Applications

A main point of semigroup theory is the identification of time-dependent partial differential equations which have global solutions in positive time. If one starts with a semigroup T and defines a generator B (it helps for X to be a subset of a Banach space here) by differentiating T at 0:

$$Bx = \lim_{t \to 0+} (1/t)(T(t)x - x),\ x \in D(B), \tag{8}$$

then it often happens that various trajectories y of T satisfy

$$y'(t) = By(t),\ t \geq 0.$$

A traditional approach, always found in established linear theories, is to start with a class of semigroups, define a generator for each member of the class by (8) and then classify these generators, obtaining as a result a class of equations which can be solved by an exponential formula. There have been many interesting results in the theory of nonlinear semigroups (for example see [1],[2]) but is probably generally agreed that nonlinear theory has not approached linear theory in its development.

An example of Webb [9] shows an obstacle to a certain path of development: Suppose F is the function on $[-1, 1]$ so that $F(x) = x, x \in [0, 1]$ and $F(x) = 2x, x \in [-1, 0)$. Denote by T the semigroup on $C([-1, 1])$ so that

$$(T(t)f)(x) = F(t + F^{-1}(f(x))),\ x \in [-1, 1],\ t \geq 0,\ f \in C([-1, 1]).$$

It was shown that every element in the domain of the generator (in the sense of (8)) is either a nonegative valued function or else a negative valued function. The paucity of the domain of this generator discouraged one from hope of recovery of the semigroup from its generator by means of an exponential formula (or in any other way). However the development in the present note may be used to extend the semigroup of this example to an appropriate space of measures and to obtain a sufficiently based generator which allows recovery of the full semigroup for the generator. Use of the present development comes at a price since $MCR(X)$ is a much larger space than X. However we close this paper with a note of optimism that the structure of this work can be used in a concrete, even computational, way.

Some years ago this this author wrote a code for a fully nonlinear filter (for a small company - the work was never published). The main part of the code, apart from a section which accomodated Baysean updates, consisted in a numerical solution to Kolomogorov's forward equation (Fokker-Planck equation, see for example [6], p 168). This is a stochastic equation but without the 'noise' term, it is essentially a time-dependent equation for a semigroup U as above, with corresponding semigroup T giving the deterministic dynamics. Now the noise term taken by itself gives rise to a diffusion semigourp W. Numerically the splitting method actually used in the code is an implementation of a Trotter product (such a product would require a generalization of the one in [7], p 53):

$$(U(t/n)W(t/n))^n.$$

With careful programming and a good systematic selection of computational 'windows', measures can be evolved numerically.

The present note offers a new development, both practical and theoretical, for nonlinear semigroups.

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