

A COMPLETELY MITOTIC NONRECURSIVE R.E. DEGREE

BY

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ABSTRACT. A nonrecursive r.e. degree d is constructed that has the property that every r.e. set of degree d is mitotic. The degree d has several other interesting properties including the property that any two r.e. sets of degree d are weak truth table equivalent.

The purpose of this paper is to prove the following:

Theorem. *There exists a nonrecursive r.e. degree d such that every r.e. set of degree d is mitotic.*

The notion of mitotic was defined by the author in [2]. A r.e. set is *mitotic* if it is the disjoint union of two r.e. sets both of the same degree. It was shown [2, Theorems 3 and 4] that there is a nonmitotic r.e. set of degree $0'$ and that for each nonrecursive r.e. set A there is a nonmitotic r.e. set recursive in A . From these facts one might suspect that every nonrecursive r.e. degree contains a nonmitotic r.e. set. However, as we shall show, such is not the case. The degree d that we construct has several other interesting properties. A r.e. set is *strongly mitotic* if it is the disjoint union of two recursively separable r.e. sets of the same degree. In [2, Theorem 5] it is shown that maximal sets can be mitotic yet it is quite easy to show that no maximal set can be strongly mitotic. Hence the notion of mitotic is truly weaker than the notion of strongly mitotic. The construction of the degree d actually yields the stronger result.

Corollary 1. *Every r.e. set of degree d is strongly mitotic.*

The following corollary is a consequence of the fact that every r.e. set which is mitotic is also autoreducible [2, Theorem 1].

Corollary 2. *Every r.e. set of degree d is autoreducible.*

The notion of weak truth table reducibility is defined in Rogers [3, p. 158]. As a by-product of the construction we obtain:

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Corollary 3. *Any two r.e. sets of degree d are weak truth table equivalent.*

Finally the degree d will have the property that $d'' = 0''$. This property and Corollaries 1 and 3 are strictly speaking not corollaries of the theorem but are corollaries of the construction. Hence we shall give their proofs after the proof of the theorem.

We now proceed with the proof.

I. The central ideas of the proof. Let W_0, W_1, \dots be a standard effective enumeration of the r.e. sets and let Φ_0, Φ_1, \dots be a standard effective enumeration of the partial recursive functionals. Define $c(i, X, n, s)$ to be the least number c greater than n such that if " $m \in X?$ " is a question asked in the computation of $\Phi_i^s(X; n)$ then $m \leq c$. Let b be a recursive function from N onto N^3 . We define the triple (θ_i, Ψ_i, Y_i) to be the triple $(\Phi_{i_0}, \Phi_{i_1}, W_{i_2})$ where $b(i) = (i_0, i_1, i_2)$. We will construct a r.e. set D in stages. The degree of D will be the d satisfying the theorem. If $\bar{D} \neq W_i$ then we say that *the nonrecursive condition of order i is satisfied*. If $\theta_i(D) = Y_i$ and $\Psi_i(Y_i) = D$ implies that there exist r.e. sets $Y_{i,0}$ and $Y_{i,1}$ such that $Y_{i,0} \cap Y_{i,1} = \emptyset$, $Y_{i,0} \cup Y_{i,1} = Y_i$, $D \leq_T Y_{i,0}$ and $D \leq_T Y_{i,1}$ then we say that *the mitotic condition of order i is satisfied*. Satisfying the nonrecursive condition of order i has priority $2i$, while satisfying the mitotic condition of order i has priority $2i + 1$.

We define the following auxiliary functions. Let $b(i) = (i_0, i_1, i_2)$.

1. $L(i, s) = \mu m [\theta_i^s(D^s; m) \neq Y_i^s(m) \text{ or } \Psi_i^s(Y_i^s; m) \neq D^s(m)]$,
2. $j(i, n, s) = \max\{c(i_0, D^{s'}, m, s') : m \leq n, s' \leq s\}$,
3. $k(i, n, s) = \max\{c(i_1, Y_i^{s'}, m, s') : m \leq n, s' \leq s\}$,
4. $l(i, n, s) =$ the greatest $l < L(i, s)$ such that $j(i, l, s) < n$ if there is such a number l . If there is no such l set $l(i, n, s) = -1$.

Remarks. (i) $l(i, n, s) < n \leq k(i, n, s)$. (ii) If $m \leq n$ then $j(i, m, s) \leq j(i, n, s)$, $k(i, m, s) < k(i, n, s)$, and $l(i, m, s) \leq l(i, n, s)$.

The proof is based on the following lemma.

Lemma. *Let i, x, y, s, s', s'' be given such that $x \leq y, s \leq s' < s''$. If*

- (i) $L(i, s) > k(i, y, s)$,
 - (ii) $L(i, s') > k(i, y, s)$ and $D^{s'}(m) = D^s(m)$ for all $m \leq j(i, k(i, y, s), s)$,
 - (iii) $x \in D^{s''} - D^{s'}$, $L(i, s'') > x$, $D^{s''}(m) = D^s(m)$ for all $m < x$,
- then there exists $z \in Y_i^{s''} - Y_i^{s'}$ such that $l(i, x, s) < z \leq k(i, y, s)$.*

Proof. Since $D^{s''}(m) = D^s(m)$ for all $m < x$ then $\theta_i^{s''}(D^{s''}; m) = \theta_i^s(D^s; m)$ for $m \leq l(i, x, s)$. If there exists $z \in Y_i^{s''} - Y_i^{s'}$ such that $z \leq l(i, x, s)$ then cer-

tainly $Y_i^{s''}(z) \neq \theta_i^{s''}(D^{s''}; z)$. Hence $L(i, s'') \leq x$ contrary to the hypothesis. On the other hand suppose there exists no $z \leq k(i, y, s)$ such that $z \in Y_i^{s''} - Y_i^{s'}$. Since $D^{s'}(m) = D^s(m)$ for all $m \leq j(i, k(i, y, s), s)$ then $\theta_i^{s'}(D^{s'}; m) = \theta_i^s(D^s; m)$ for all $m \leq k(i, y, s)$. Since $L(i, s') > k(i, y, s)$ then we must have $Y_i^{s'}(m) = Y_i^s(m)$ for all $m \leq k(i, y, s)$. Hence we have that there is no $z \leq k(i, y, s)$ such that $z \in Y_i^{s''} - Y_i^s$ which implies that $\Psi_i^{s''}(Y_i^{s''}; m) = \Psi_i^s(Y_i^s; m)$ for all $m \leq y$. Since $x \in D^{s''} - D^{s'}$ then $D^{s''}(x) \neq \Psi_i^{s''}(Y_i^{s''}; x)$. Hence $L(i, s'') \leq x$ contrary to the hypothesis.

Figure 1 illustrates the lemma.

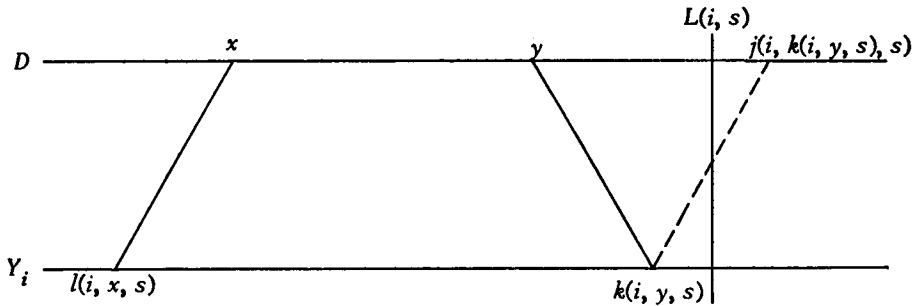


Figure 1

Notice that $\lim_s L(i, s) = \infty$ iff $\theta_i(D) = Y_i$ and $\Psi_i(Y_i) = D$. The basic lemma tells us how to indirectly control Y_i by controlling D . If $\theta_i(D) = Y_i$ and $\Psi_i(Y_i) = D$ then by the lemma we can force a number into Y_i in a specific interval if we put a number into D .

One mitotic condition. It is helpful to explain how to satisfy one mitotic condition, say the first mitotic condition, while satisfying all the nonrecursive conditions. For this purpose we shall only need a special case of the lemma, the case when $x = y$. In order to make D nonrecursive we may want to put relatively small numbers into D . Each time we do so we would like to force two numbers into Y_0 one for each of the two sets $Y_{0,0}$ and $Y_{0,1}$. Of course, we need only do this if it looks like $L(0, s)$ is becoming large, that is, if it looks like it may be the case that $\theta_0(D) = Y_0$ and $\Psi_0(Y_0) = D$. Assume $L(0, s)$ is large and we want to put n into D to satisfy some nonrecursive condition. By the lemma we can put n into D and force some number into Y_0 in a specific interval. Waiting in the wings is another number m which we will put into D to force another number into an interval disjoint from the first. Figure 2 helps illustrate this procedure.

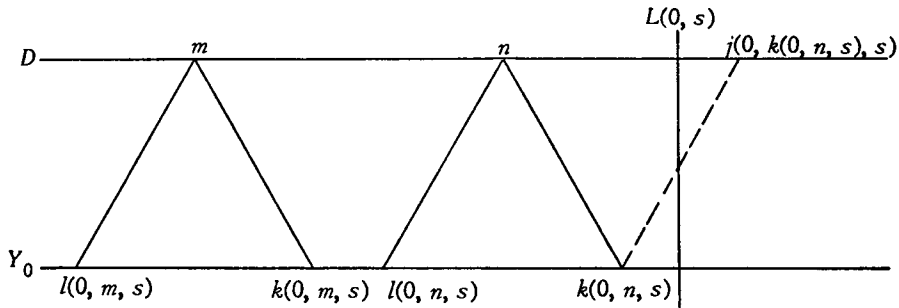


Figure 2

Choose m and n such that $L(0, s) > k(0, n, s)$ and $k(0, m, s) \leq l(0, n, s)$. Suppose n is ready to satisfy a nonrecursive condition at stage $s_0 + 1 > s$. We assume that no number $\leq j(0, k(0, m, s), s)$ was put into D after stage s and before stage $s_0 + 1$. We put n into D at stage $s_0 + 1$ and wait for a stage $s_1 > s_0 + 1$ such that $L(0, s_1) > n$. At stage $s_1 + 1$ we put m into D . Let s_2 be the least stage $> s_1 + 1$ such that $L(0, s_2) > m$. By the lemma we can conclude that there are two numbers y_0 and y_1 such that $y_0, y_1 \in Y_0^{s_2} - Y_0^s$, $l(0, m, s) < y_0 \leq k(0, m, s)$ and $l(0, n, s) < y_1 \leq k(0, n, s)$. We may now put y_0 into $Y_{0,0}$ and y_1 into $Y_{0,1}$. This procedure depends on the existence of stages s_1 and s_2 . However if one of s_1 and s_2 does not exist then the mitotic condition is satisfied vacuously since $\lim_s L(i, s) \neq \infty$. The preceding explains how to get two different numbers into Y_0 but only locally.

We now give a global picture of how to construct D just satisfying one mitotic condition. Candidates are chosen to follow nonrecursive conditions. The candidate of order i follows $\bar{D} \neq W_i$. Candidates are of two types, members of N (type 0) and members of $N \times N$ (type 1). Actually only the second coordinate of a member of $N \times N$ is used to satisfy a nonrecursive condition but it is easier to think of the pair as the candidate. In general an unsatisfied nonrecursive condition starts off with a type 0 candidate and then tries to attain a type 1 candidate. As D is constructed so are sets E, A, R_0 , and R_1 , and functions d and r . We shall always have $A^s \subseteq E^s \subseteq N \times N$, $R_0^s \cap R_1^s = \emptyset$ and $r(s) = \max(R_1^s \cup \{-1\})$. Figure 3 illustrates a stage s in the construction.

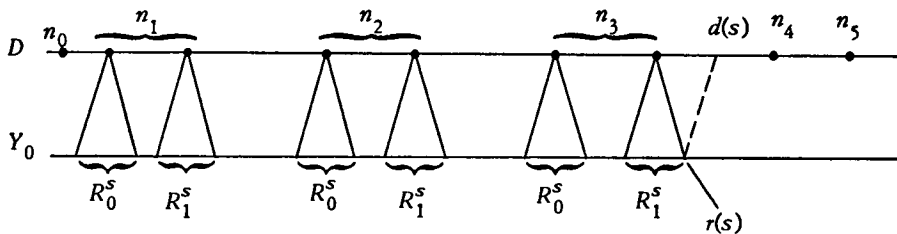


Figure 3

The point n_i is the candidate of order i at stage s and $A^s = E^s = \{n_1, n_2, n_3\}$. Notice that the first nonrecursive condition has only a type 0 candidate and will never attain a type 1 candidate. This is because the first nonrecursive condition is of higher priority than the first mitotic condition. If and when n_0 is put into D then the entire picture is "erased", that is, we set $A^s = E^s = \emptyset$, $R_0^s = R_1^s = \emptyset$, $d(s) = -1$, and cancel all candidates. Several things can happen at stage $s + 1$.

1. *A candidate is chosen.* No candidate for $\bar{D} \neq W_6$ exists, so one of type 0 is chosen at stage $s + 1$ which is larger than $d(s)$ and larger than any candidate of higher priority.

2. *A candidate is improved.* The condition $\bar{D} \neq W_4$ has a candidate of type 0 and can gain a candidate of type 1. There exists $p, q \in \bar{D}^s$ such that $r(s) < l(0, p, s)$, $k(0, p, s) < l(0, q, s)$, and $L(0, s) > k(0, q, s)$. The new candidate of order 4 becomes $n_4 = (p, q)$ and candidates of higher order are cancelled. We then have $A^{s+1} = E^{s+1} = \{n_1, n_2, n_3, n_4\}$, $R_0^{s+1} = R_0^s \cup [l(0, p, s), k(0, p, s)]$, $R_1^{s+1} = R_1^s \cup [l(0, q, s), k(0, q, s)]$ and $d(s + 1) = j(0, k(0, q, s), s)$. Figure 4 illustrates the situation at stage $s + 1$.

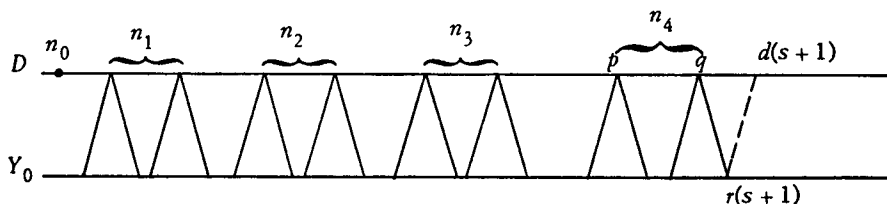


Figure 4

3. *A candidate is implemented.* We look at three cases. First suppose $n_0 \in W_0^{s+1}$. As was mentioned earlier we put n_0 into D at stage $s + 1$ and "erase" everything. The second case is when a type 0 candidate other than n_0 is implemented, for instance suppose $n_4 \in W_4^{s+1}$. Just put n_4 into D and relax for we have satisfied a nonrecursive condition without any need to worry about mitoticity. The third case is when we implement a type 1 candidate. Let $n_2 = (p, q)$ and suppose $q \in W_2^{s+1}$. Put q into D . We now set up a potential condition of order 2 which consists of the point (p, q) . The potential condition will be satisfied at stage s' if $L(0, s') > q$. We have $E^{s+1} = E^s$ but $A^{s+1} = \{n_1\}$. Further $d(s + 1) = d(s)$, $R_0^{s+1} = R_0^s$, and $R_1^{s+1} = R_1^s$. Notice that the point n_3 is no longer any good because a number less than it has entered D . The set A^s is the set of points that are still good. Cancel all candidates and potential conditions of order > 2 . Figure 5 illustrates the situation at stage $s + 1$.

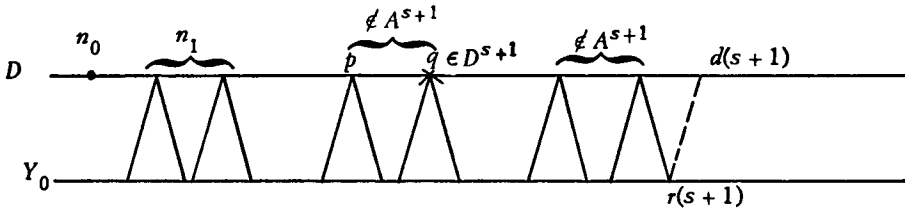


Figure 5

4. A potential condition is satisfied. Let us consider a stage $s' \geq s + 1$ (stage $s + 1$ as pictured in Figure 5). At stage $s + 1$ the potential condition (p, q) was instituted. Suppose $L(0, s') > q$, that is, the potential condition (p, q) is satisfied at stage s' . At stage $s' + 1$ put p into D . (There is no need now to set up another potential condition because we do not need to put a smaller than p into D .) Cancel all candidates of order > 2 .

The four situations we have just described are organized as follows. At stage $s + 1$ find the least i such that one of the following holds:

- A. The potential condition of order i is satisfied at stage s .
- B. The candidate of order i can be implemented at stage s and $D^s \cap W_i^s = \emptyset$.
- C. The candidate of order i can be improved at stage s .
- D. A candidate of order i can be chosen at stage s and $D^s \cap W_i^s = \emptyset$.

Now at stage $s + 1$ do the appropriate task described above in 1–4.

It should be clear that all the nonrecursive conditions are satisfied. Now suppose $\theta(D) = Y_0$ and $\Psi(Y_0) = D$. Since $\lim_s L(0, s) = \infty$ then we must have constructed infinitely many type 1 points. Hence we must have $\lim_s d(s) = \infty$. Notice there is at most one stage t such that $E^t \not\subseteq E^{t+1}$ (n_0 enters D at stage $t + 1$). Let t_0 be the least stage such that $E^s \subseteq E^{s+1}$ for all $s \geq t_0$. Define

$$R = \bigcup_{s \geq t_0} R_0^s, \quad Y_{0,0}^s = Y_0^s \cap R, \quad Y_{0,1}^s = Y_0^s \cap \bar{R},$$

$$Y_{0,0} = \bigcup_s Y_{0,0}^s \quad \text{and} \quad Y_{0,1} = \bigcup_s Y_{0,1}^s.$$

Now, R is recursive since its intervals are enumerated in increasing order.

Hence $Y_{0,0}$ and $Y_{0,1}$ are r.e. Clearly $Y_0 = Y_{0,0} \cup Y_{0,1}$ and $Y_{0,0} \cap Y_{0,1} = \emptyset$. We show now how to reduce D to $Y_{0,0}$. A similar argument will show $D \leq_T Y_{0,1}$.

Let x be a fixed argument. We can assume inductively that we know $D(y)$ for all $y < x$. Let $t_1 + 1$ be the least stage $\geq t_0$ such that $d(t_1 + 1) \geq x$. Let t_2 be the least stage $\geq t_1 + 1$ such that

- (i) $D(y) = D^{t_2}(y)$ for all $y < x$,
- (ii) $Y_{0,0}(y) = Y_{0,0}^{t_2}(y)$ for all $y \leq r(t_1 + 1)$,
- (iii) $L(0, t_2) > r(t_1 + 1)$,
- (iv) no potential condition exists at stage t_2 .

Now, $x \in D$ iff $x \in D^{t_2}$.

To argue this suppose $x \in D - D^{t_2}$. First of all x must be the first coordinate of a type 1 point created at stage $t_1 + 1$. If x were the second coordinate then at some time after x was put into D a number smaller than x would be put into D , otherwise, a potential condition would be in existence forever. At stage $t_1 + 1$ we have the situation described in Figure 6.

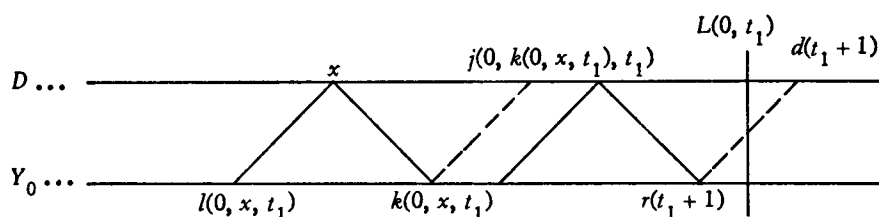


Figure 6

We certainly have $L(0, t_1) > k(0, x, t_1)$. At stage t_2 we must have $L(0, t_2) > k(0, x, t_1)$ and $D^{t_1}(y) = D^{t_2}(y)$ for all $y \leq j(0, k(0, x, t_1), t_1)$. There must be a stage $t_3 > t_2$ such that $x \in D^{t_3}$ and $L(0, t_3) > x$. By the basic lemma there exists a $z \in Y_0^{t_3} - Y_0^{t_2}$ such that $l(0, x, t_1) < z \leq k(0, x, t_1)$. Hence $z \in Y_{0,0} - Y_{0,0}^{t_2}$ and $z \leq r(t_1 + 1)$ which is impossible.

There is an important fact about the construction which should be noted. *The mitotic condition is never satisfied once and for all at any finite stage.* We must continually work on satisfying the mitotic condition. This kind of thing happens in the standard maximal set construction. If one considers a maximal condition as satisfied if $\bar{M} \cap W_i$ or $\bar{M} \cap \bar{W}_i$ is finite then these conditions are not satisfied at any finite stage but must continually be worked on. A useful tool in such constructions is the concept of "e-state" introduced by Yates [5]. Although we shall not mention "e-state" explicitly in our construction the idea is implicit in our notion of "type". The difficulties that arise in the construction may be best illustrated by indicating how to construct D satisfying just two of the mitotic conditions, say the first two.

Two mitotic conditions. In satisfying two mitotic conditions there are four types of points to be considered. Figure 7 illustrates these types of points.

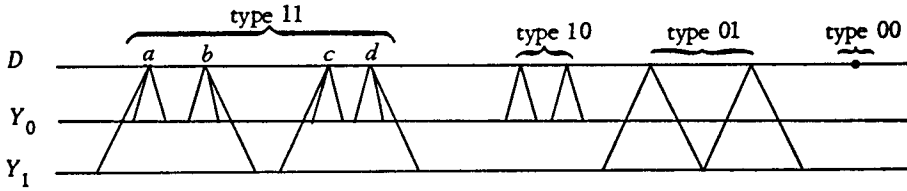


Figure 7

Notice that type 11 points are built up from type 10 points. The basic strategy in constructing D can be summarized as follows:

Satisfy $\bar{D} \neq W_i$ with an α point ($\alpha = 11, 10, 01, 00$) where α is greatest lexicographically.

We have already seen how 10, 01, and 00 points work. Type 11 points are constructed to enable us to satisfy the two mitotic conditions simultaneously. Let us examine the point $((a, b), (c, d))$ of Figure 7. Suppose d is ready to satisfy a nonrecursive condition at stage s_0 . Put d into D^{s_0+1} and wait for a stage s_1 when $L(0, s_1) > d$. Now put c into D^{s_1+1} and wait for a stage s_2 when $L(0, s_2) > c$. At this point housekeeping on the first mitotic condition is in order. Now wait further until $L(1, s_3) > c$. Now put b into D^{s_3+1} and wait until a stage s_4 when $L(0, s_4) > b$. Now put a into D^{s_4+1} and we are finished. A more sophisticated potential condition is used to keep track of how the point is progressing. It is important to notice that the second mitotic condition does not hold up the satisfaction of the first. While we are waiting for stage s_3 which concerns the second mitotic condition we are not in the middle of a point trying to satisfy the first mitotic condition. On the other hand we do allow the first mitotic condition to hold up satisfaction of the second. If the first mitotic condition holds up satisfaction of the second it does so only on a 11 point. We may now fall back on 01 points in order to satisfy the second mitotic condition.

Rather than describe a construction of D satisfying two mitotic conditions we shall just mention the difficulties that arise in satisfying two conditions and how we overcome them. It may be helpful to refer back to these explanations when reading the construction and proof.

(1) *The order of candidates.* In general our strategy is to try to satisfy a nonrecursive condition with the highest possible type point. To implement this we must always make sure that if $i < j$, p is the candidate of order i and q is the candidate of order j then p lies to the left of q . So if q is implemented then p is still good and can be implemented later.

(2) *The order of types of points.* Also to implement our strategy we must make sure that the lowest order candidates have the highest type points. How-

ever, 11 points are built up from 10 points hence we cannot say that 11 points must lie to the left of 10 points. These two types are *compatible*. On the other hand 10 points and 01 points are *incompatible* and 01 points must always lie to the right of 10 points. Let Figure 8 illustrate a situation at stage s .

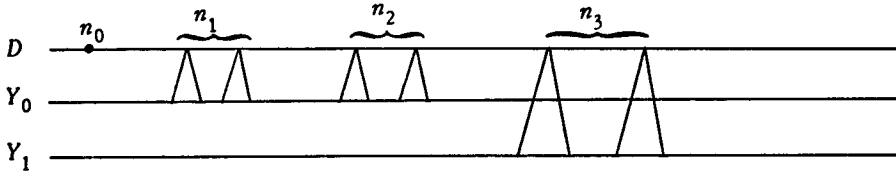


Figure 8

At stage $s + 1$ we may improve the candidate of order 3 to be a type 10 point. What do we do with the present candidate? Completely erase the point to get the situation illustrated in Figure 9.

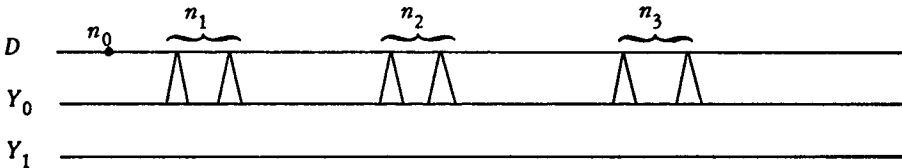


Figure 9

Although we may have forsaken the second mitotic condition it is only temporarily. If this situation keeps repeating we shall eventually be able to create 11 points which are better than 01 points.

(3) *An unbounded store of good points.* In order to create 11 points from 10 points we must be sure that we maintain an ever increasing store of good 10 points to build from. To do this we use the fact that there are infinitely many i such that $W_i = \emptyset$ (candidates of these orders are never implemented). We never use the first i good 10 points in constructing a type 11 candidate of order i .

(4) *A conflict between implementation and improvement.* Consider the situation at stage s illustrated in Figure 10.

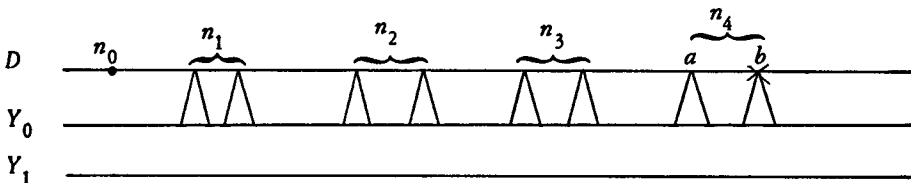


Figure 10

The number b is presently in D and we are waiting for $L(0, s)$ to exceed b in order to put a into D . However at stage $s + 1$ we could construct a type 11 candidate using n_2 and n_3 . Properly speaking this new candidate should be the candidate of order 2 replacing n_2 . This leaves the condition $\bar{D} \neq W_3$ without a candidate. However the condition $\bar{D} \neq W_4$ would still have a candidate. The order of candidates is ruined. To resolve this problem we do not form the type 11 candidate at this time. While we are waiting for $L(0, s)$ to exceed b we can think of the mitotic condition as temporarily satisfied for if $L(0, s)$ never exceeds b then it *is* satisfied. Hence there is no need at the moment for type 11 points to be constructed. While we are waiting for $L(0, s)$ to exceed b we shall say 10 and 11 are *frozen*.

(5) *A conflict between frozen types and improvement.* This conflict is best illustrated using three mitotic conditions. Suppose we have the following situation at stage s illustrated in Figure 11.

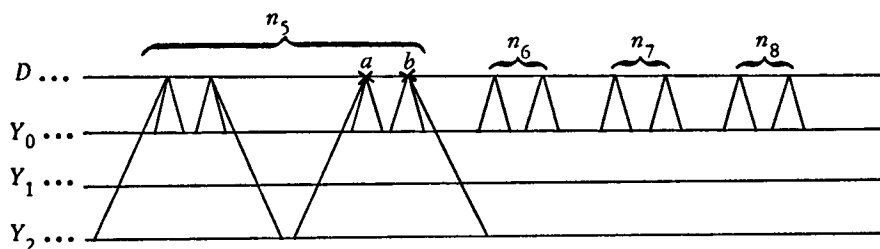


Figure 11

At stage s both a and b have been put into D and we are waiting. Now type 101 is frozen but *not* type 100. Suppose at stage $s + 1$ we could improve the candidate of order 6 to be a 110 point using n_6 and n_8 as subpoints. We go ahead and do just that to get the situation illustrated in Figure 12.

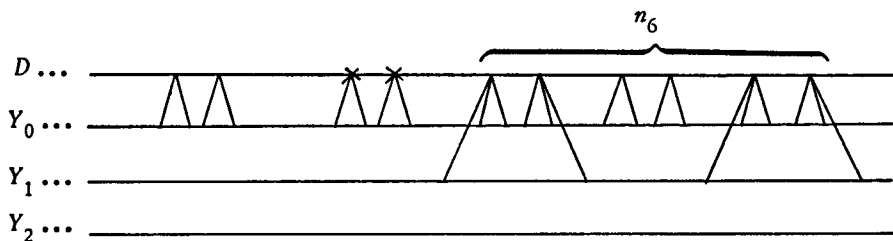


Figure 12

The candidate of order 5 goes out of existence. You may ask how can improving the candidate of order 6 have priority over continuing to implement the candidate

of order 5? When we put b into D we satisfied the condition $\bar{D} \neq W_5$, once and for all. The rest of the implementation of the candidate of order 5 is concerned with satisfying mitotic conditions. Thus we have not really violated the priorities at all but have adhered to them in constructing a 110 point over a 101 point.

The list (1)–(5) is not an exhaustive list of all the difficulties in the construction but only contains what we deem to be the most important. We now proceed with the formal proof.

II. Notation and definitions. Define $N^{[0]} = N$ and $N^{[k+1]} = N^{[k]} \times N^{[k]}$. We let $P = \bigcup \{N^{[k]} : k \in \mathbb{N}\}$ and call the members of P *points*. If $p \in N^{[k]}$ then we write $lhp = k$. Let Σ be the set of all finite sequences of 0's and 1's. We denote the empty sequence by λ , the length of a sequence σ by $lh\sigma$, and the concatenation of two sequences σ and τ by $\sigma * \tau$. For $n < lh\sigma$ the $(n + 1)$ st member of the sequence σ is denoted by σ_n . If $lh\sigma = lhr$ then $\sigma < \tau$ just in case $\sigma \neq \tau$ and if $n = \mu m$ ($\sigma_m \neq \tau_m$) then $\sigma_n = 0$. If $p \in P$ then $(p)_0$ and $(p)_1$ are the first and second coordinates of p respectively. If $lh\sigma \leq lhp$ then we define $(p)_\sigma$ inductively on the length of σ . We let $(p)_\lambda = p$ and $(p)_{\sigma * i} = ((p)_\sigma)_i$ where i is 0 or 1. For example, if $p = ((1, 2)(3, 4))$ then $(p)_0 = (1, 2)$, $(p)_1 = (3, 4)$, $(p)_{00} = 1$, $(p)_{01} = 2$, $(p)_{10} = 3$, and $(p)_{11} = 4$. We define $(p)_{\bar{0}} = (p)_\sigma$ where σ is the constant zero sequence of length equal to lhp . Likewise $(p)_{\bar{1}}$ is defined. If $p, q \in P$ then we write $p < q$ just in case $(p)_{\bar{1}} < (q)_{\bar{0}}$.

Let Γ be the set of all eventually zero infinite sequences of 0's and 1's. Members of Γ are used to classify points into types. When we were examining just two mitotic conditions a type was a sequence of length two but now that we are dealing with infinitely many mitotic conditions our types are infinite. If $\alpha \in \Gamma$ then α_n denotes the $(n + 1)$ st member of the sequence α . We denote the constant zero sequence by the symbol o , omicron. We consider two partial orders on Γ .

1. *The lexicographic order:* $\alpha \leq \beta$ if $\alpha = \beta$ or [$\alpha \neq \beta$ and if $n = \mu n'$ ($\alpha_{n'} \neq \beta_{n'}$) then $\alpha_n = 0$].

2. *The tree order:* $\alpha \subseteq \beta$ if there exists n such that, for all $m < n$, $\alpha_m = \beta_m$ and, for all $m \geq n$, $\alpha_m = 0$.

We define (level of α) $lev\alpha = \mu n$ ($\forall m \geq n, \alpha_m = 0$). Thus $lev\ o = 0$. Further, we define $\alpha * n$ to be the sequence β where $\beta_m = \alpha_m$ if $m \neq n$ and $\beta_n = 1$. Notice that if $n \geq lev\alpha$ then $lev\alpha * n = n + 1$ and $\alpha \subseteq \beta$ if and only if $\alpha_n = \beta_n$ for all $n < lev\alpha$. We define (extent of α) $ext\alpha = \text{Card}\{n : \alpha_n = 1\}$.

For each $\alpha \in \Gamma$ we will construct sets $E_\alpha, A_\alpha, R_{0,\alpha}$ and $R_{1,\alpha}$. These sets will be constructed in stages. We will always have $A_\alpha^s \subseteq E_\alpha^s \subseteq P$. The order on P always induces an order on E_α^s isomorphic to an initial segment of the natural numbers.

$$n(j, \alpha, s) = \begin{cases} \text{the } (j + 1)\text{st member of } A_\alpha^s & \text{if Card } A_\alpha^s \geq j + 1, \\ \text{the last member of } A_\alpha^s & \text{if } 0 < \text{Card } A_\alpha^s \leq j, \\ 0 & \text{if } A_\alpha^s = \emptyset, \end{cases}$$

$$m(j, \alpha, s) = \max \{ (n(j, \beta, s'))_1 : \beta \subseteq \alpha \text{ and } s' \leq s \},$$

$$r(\alpha, s) = \max (R_{1, \alpha}^s \cup \{ - 1 \}).$$

We shall also construct a function $\lambda \alpha sd(\alpha, s)$ with the property that if $\alpha \neq 0$ then $d(\alpha, s) > \max E_\alpha^s$. We define

$$d(s) = \max \{ d(\alpha, s) : \alpha \in \Gamma \}.$$

At various stages s we may appoint a point p to be the *candidate of order i* for some i . The point p will initially be a member of A_α^s for some α where $\text{lev } \alpha \leq i$. If p is appointed at stage s to be the candidate of order i and $p \in A_\alpha^s$ then we say p has *type α* . If the candidate of order i in existence at stage s is such that $(p)_1 \in D^s$ then the candidate is said to be *implemented* and a *potential condition* of order i must also be in existence. A potential condition associated with a candidate p of type α is of the form (p, α, σ, j) where $\sigma \in \Sigma$ and $j \in N$. We say that the *potential condition, (p, α, σ, j) , is satisfied at stage s* if $L(j, s) > (p)_\sigma$. For each i and s there is at most one candidate of order i and at most one potential condition of order i in existence at stage s . Once a candidate or potential condition is appointed it remains in existence until cancelled or changed. If an implemented candidate is cancelled at stage s then so also is the potential condition associated with it. It is important to notice that if a candidate p of type α is in existence at stage s that does not by definition imply that $p \in E_\alpha^s$. The type of a candidate refers to its initial status when appointed. However, we shall show in Claim 1 that if a candidate in existence at stage s has type α then it is indeed a member of E_α^s .

We say that α is *frozen at stage s* if there exists a potential condition of the form (p, β, σ, i) such that if γ is the greatest member of Γ such that $\gamma \subseteq \beta$ and $\gamma \subseteq \alpha$ and $\sigma = \sigma' * \sigma''$ where $\text{lb } \sigma'' = \text{ext } \gamma$ then σ'' has a nonzero member. The notion of being frozen is explained in our list of difficulties (4) and (5). We should note that if α is frozen at stage s and $\alpha \subseteq \delta$ then δ is also frozen at stage s . Further if a potential condition of the form (p, β, σ, i) exists at stage s and if $\beta \subseteq \alpha$ then α is frozen at stage s . Basically α is frozen at stage s if we are presently implementing a candidate such that unless the potential condition associated with that candidate is satisfied at some time then only finitely many points of type α could ever be constructed because $\lim_s L(j, s) \neq \infty$ for some j such that $\alpha_j = 1$.

The candidate of order i can be improved at stage $s + 1$ if all the following hold:

- (1) an unimplemented candidate r of order i already exists at stage s ,
- (2) r has type α and there exists j, β, p, q such that
 - (a) $p \in A_{\beta}^s$ and $q \in A_{\beta}^s$,
 - (b) $p < q$,
 - (c) $\text{lev } \beta \leq j < i$,
 - (d) $\beta \subseteq \alpha$ and $\beta * j > \alpha$,
 - (e) $m(i, \beta, s) < p$,
 - (f) $\beta * j$ is not frozen at stage s ,
 - (g) $p >$ any unimplemented candidate of order $< i$ in existence at stage s ,
 - (h) $L(j, s) > k(j, (q)_{-1}^-, s)$,
 - (i) $r(\gamma, s) < l(j, (p)_{-0}^-, s)$ for all $\gamma \geq \beta * j$,
 - (j) $k(j, (p)_{-1}^-, s) < l(j, (q)_{-0}^-, s)$.

If indeed we do improve the candidate of order i to be the point (p, q) at stage $s + 1$ then we may explain the reasons for (a)–(j) by (a')–(j') below.

- (a') (p, q) will be a "good" point of type $\beta * j$.
- (b') If implemented the coordinates of (p, q) will enter D in descending order.
- (c') This insures that the candidate of order i can have only one of 2^i types.
- (d') $\beta \subseteq \alpha$ insures that there must have been a candidate of order i and type β before one of type $\beta * j$ was chosen while $\beta * j > \alpha$ insures that (p, q) is of higher type than r .

(e') This will insure that, for each $\delta \subseteq \beta$, A_{δ}^s will become unbounded if E_{δ}^s becomes unbounded (see difficulty (3)).

(f') (See difficulties (4) and (5) and the definition of frozen.)

(g') This is to insure the ordering of the candidates is correct. We only require that $p >$ any unimplemented candidates of order $< i$ not $>$ all candidates of order $< i$ for the following reason. An implemented candidate may be of type $< \beta * j$ and will be cancelled as soon as (p, q) is created. After all, the non-recursive condition associated with this low type candidate is already satisfied so there is no need to keep low type candidates around that will hold up the creation of high type candidates (difficulty (5) goes into this problem somewhat).

(h')–(j') We have gone into this extensively when we considered one mitotic condition.

The condition of order i requires attention at stage $s + 1$ if one of the following holds:

- A. The potential condition of order i is satisfied at stage s .
- B. $D^s \cap W_i^s = \emptyset$ and the candidate p of order i is such that $(p)_{-1}^- \in W_i^{s+1}$.
- C. The candidate of order i can be improved at stage $s + 1$.
- D. $D^s \cap W_i^s = \emptyset$ and there is no candidate of order i in existence at stage s .

III. The construction.

Stage 0.

$$D^0 = \emptyset, \quad d(\gamma, 0) = -1, \quad A_o^0 = E_o^0 = N,$$

$$A_\gamma^0 = E_\gamma^0 = \emptyset \quad \text{if } \gamma \neq o, \quad R_{0,\gamma}^0 = R_{1,\gamma}^0 = \emptyset.$$

Stage $s + 1$. Choose the condition of least order that requires attention at stage $s + 1$. If none go to the next stage. We adopt the case that corresponds to the first of the four possibilities that holds.

A. Let (p, α, σ, j) be the potential condition of order i satisfied at stage s . Let $n = 1 +$ the number of zeros at the end of σ .

Case 1. $j <$ the n th k such that $\alpha_k = 1$. In this case make no changes except that the potential condition of order i becomes (p, α, σ, k) where k is the least number $> j$ such that $\alpha_k = 1$.

Case 2. $j =$ the n th k such that $\alpha_k = 1$. Cancel all candidates and potential conditions of order $> i$. Let σ' be the immediate predecessor of σ in the $<$ ordering of Σ .

$$D^{s+1} = D^s \cup \{(p)_{\sigma'}\}, \quad d(\beta, s + 1) = \begin{cases} d(\beta, s) & \text{if } \beta \geq \alpha \text{ or } \beta \subseteq \alpha, \\ -1 & \text{otherwise,} \end{cases}$$

$$E_\beta^{s+1} = \begin{cases} E_\beta^s & \text{if } \beta \geq \alpha \text{ or } \beta \subseteq \alpha, \\ \emptyset & \text{otherwise,} \end{cases} \quad R_{j,\beta}^{s+1} = \begin{cases} R_{j,\beta}^s & \text{if } \beta \geq \alpha \text{ or } \beta \subseteq \alpha, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$A_\beta^{s+1} = \begin{cases} \{r \in A_\beta^s : r < p\} \cup \{r \in \bar{D}^{s+1} : r > d(s + 1)\} & \text{if } \beta = o, \\ \{r \in A_\beta^s : r < p\} & \text{if } \beta \neq o \text{ and } (\beta \subseteq \alpha \text{ or } \beta \geq \alpha), \\ \emptyset & \text{otherwise.} \end{cases}$$

If σ' is not a constant zero sequence then the potential condition of order i becomes (p, α, σ', k) where k is the least number such that $\alpha_k = 1$. If σ' is a constant zero sequence then cancel the candidate of order i . Hence the potential condition of order i is also cancelled.

Case 3. Neither Case 1 nor 2 applies. Go to the next stage. (We shall show in Claim 2 that Case 1 or Case 2 must always apply so that Case 3 exists only for the completeness of the construction.)

B. Let p be the candidate of order i and suppose p has type α . Cancel all

candidates and potential conditions of order $> i$.

$$D^{s+1} = D^s \cup \{(p)_{\bar{1}}\}, \quad d(\beta, s+1) = \begin{cases} d(\beta, s) & \text{if } (\beta \geq \alpha \text{ or } \beta \subseteq \alpha) \text{ and } \text{lev } \beta \leq i, \\ -1 & \text{otherwise,} \end{cases}$$

$$E_{\beta}^{s+1} = \begin{cases} E_{\beta}^s & \text{if } (\beta \geq \alpha \text{ or } \beta \subseteq \alpha) \text{ and } \text{lev } \beta \leq i, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$R_{j, \beta}^{s+1} = \begin{cases} R_{j, \beta}^s & \text{if } (\beta \geq \alpha \text{ or } \beta \subseteq \alpha) \text{ and } \text{lev } \beta \leq i, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$A_{\beta}^{s+1} = \begin{cases} \{r \in A_{\beta}^s: r < p\} \cup \{r \in \bar{D}^{s+1}: r > d(s+1)\} & \text{if } \beta = 0, \\ \{r \in A_{\beta}^s: r < p\} & \text{if } \beta \neq 0, \text{lev } \leq i \text{ and } (\beta \geq \alpha \text{ or } \beta \subseteq \alpha), \\ \emptyset & \text{otherwise.} \end{cases}$$

If $\alpha = 0$ do nothing more. If $\alpha \neq 0$ then appoint (p, α, σ, k) to be the potential condition of order i where k is the least number such that $\alpha_k = 1$ and σ is the constant one sequence of length lhp .

C. Let r be the candidate of order i that can be improved at stage $s+1$. Cancel all candidates and potential conditions of order $\geq i$. Choose the least j, β, p , and q in that order satisfying the definition of improvement.

$$D^{s+1} = D^s,$$

$$d(\gamma, s+1) = \begin{cases} d(\gamma, s) & \text{if } \gamma > \beta * j \text{ or } \gamma \subset \beta * j, \\ j(i, k(i, (q)_{\bar{1}}), s), s) & \text{if } \gamma = \beta * j, \\ -1 & \text{otherwise,} \end{cases}$$

$$E_{\gamma}^{s+1} = \begin{cases} E_{\gamma}^s & \text{if } \gamma > \beta * j \text{ or } \gamma \subset \beta * j, \\ E_{\gamma}^s \cup \{(p, q)\} & \text{if } \gamma = \beta * j, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$R_{k, \gamma}^{s+1} = \begin{cases} R_{k, \gamma}^s & \text{if } \gamma > \beta * j \text{ or } \gamma \subset \beta * j, \\ R_{k, \gamma}^s \cup [k(j, (p)_{\bar{0}}, s), k(j, (p)_{\bar{1}}, s)] & \text{if } \gamma = \beta * j \text{ and } k = 0, \\ R_{k, \gamma}^s \cup [k(j, (q)_{\bar{0}}, s), k(j, (q)_{\bar{1}}, s)] & \text{if } \gamma = \beta * j \text{ and } k = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$A_{\gamma}^{s+1} = \begin{cases} \{a \in A_{\gamma}^s: a < p\} \cup \{a \in \bar{D}^{s+1}: a > d(s+1)\} & \text{if } \gamma = 0, \\ \{a \in A^s: a < p\} & \text{if } \gamma > \beta * j \text{ or } (\gamma \subset \beta * j \text{ and } \gamma \neq 0), \\ A_{\gamma}^s \cup \{(p, q)\} & \text{if } \gamma = \beta * j, \\ \emptyset & \text{otherwise.} \end{cases}$$

If there exists a potential condition of order i' of the form (a, γ, σ, k) where $i' < i$ and $\gamma < \beta * j$ then cancel the candidate of order i' and hence the potential condition as well. (Note: we cannot have $\gamma \subseteq \beta * j$ otherwise $\beta * j$ would be frozen.) Appoint the point (p, q) to be the candidate of order i .

D. Let p be the least number of \bar{A}_0^s such that $p > d(s)$, $p >$ any candidate of order $< i$ and $p \geq m(i, o, s)$. Appoint p to be the candidate of order i , cancel all candidates of order $> i$ and make no other changes at stage $s + 1$ except to increase the arguments of functions to $s + 1$.

This ends the construction. The set D is r.e. since all the functions and predicates used in the construction are recursive.

We say that stage $s + 1$ is an A-stage, B-stage, C-stage or D-stage pertaining to i corresponding to the case that applies at stage $s + 1$. We sometimes say that the candidate of order i is the point considered at stage $s + 1$.

Remarks on the construction.

1. If $\delta \neq \gamma$ then $E_\delta^s \cap E_\gamma^s = \emptyset$.
2. If $x \in E_\delta^s$ then $\text{ext } \delta = lbx$.
3. If $x < y$, $x \in E_\delta^s$, and $y \in E_\gamma^s$ then $\delta \geq \gamma$ or $\delta \subset \gamma$.
4. If $x \neq y$, $x \in A_\delta^s$, and $y \in A_\gamma^s$ then $x < y$ or $y < x$.
5. If $x \in A_\delta^s$ then $(x)_\sigma \in \bar{D}^s$ for all σ such that $lbx = lb\sigma$.
6. If $x \in E_\delta^s$ and $lb\sigma = lb\tau = lb\rho$ then $\sigma < \tau$ if and only if $(x)_\sigma < (x)_\tau$.
7. $r(\delta, s) \leq d(\delta, s)$, and $d(\delta, s) = -1$ iff $r(\delta, s) = -1$.
8. If $x \in E_\delta^s$ and $\delta \neq o$ then $(x)_1^- \leq r(\delta, s)$.
9. $R_{0,\delta}^s \cap R_{1,\delta}^s = \emptyset$.

10. If there exists a candidate of order i and type δ in existence at stage s then for each $\gamma \subset \delta$ there must have been a candidate of order i and type γ in existence at an earlier stage.

The proof of 1 through 10 is by induction on s . The remarks are clearly true for $s = 0$. Assume the remarks are true for s . It is not difficult to check that the remarks are true if stage $s + 1$ is an A-stage, a B-stage, or a D-stage. Let stage $s + 1$ be a C-stage. Suppose stage $s + 1$ pertains to i and j, β, p , and q are chosen. In most cases we need only consider the point (p, q) .

1. Let $\delta \neq \gamma$. If $\delta \neq \beta * j$ and $\gamma \neq \beta * j$ then $E_\delta^s \cap E_\gamma^s = \emptyset$ by the induction hypothesis. Let $\delta \neq \beta * j$. If $(p, q) \in E_\delta^{s+1}$ then $\delta > \beta * j$ or $\delta \subset \beta * j$ and $(p, q) \in E_\delta^s$. By the induction hypothesis $lb(p, q) = \text{ext } \delta$. Thus $\delta \not\subset \beta * j$. If $\delta > \beta * j$ then $(p, q) < r(\delta, s) < l(j, (p)_0^-, s) < p$ by part (i) of the definition of improvement. This is impossible.

2. Since $j \geq \text{lev } \beta$ then $\text{ext } \beta * j = \text{ext } \beta + 1$. Now, p and q are of type β thus $lb p = lb q = \text{ext } \beta$. Hence $\text{ext } \beta * j = lb(p, q)$.

3. First let $(p, q) < r$ and let r be of type γ . If $\beta * j < \gamma$ then by part (i) of

the definition of improvement $r < (p, q)$ which is impossible. Thus $\gamma \leq \beta * j$. On the other hand, let $r < (p, q)$ and let r be of type δ . If $\delta < \beta * j$ and $\delta \not\subseteq \beta * j$ then $E_{\delta}^{s+1} = \emptyset$.

4. Let $r \neq (p, q)$ and let $r \in A_{\delta}^{s+1}$. If $\delta = 0$ then $r < (p, q)$ or $r > d(\beta * j, s + 1) \geq ((p, q))_1^-$. If $\delta \neq 0$ and $\delta \neq \beta * j$ then by the construction $r \in A_{\delta}^s$ and $r < (p, q)$. If $\delta = \beta * j$ then by part (i) of the definition of improvement $r < (p, q)$.

5. Both p and q are members of A_{β}^s and $(p, q) \in A_{\beta * j}^{s+1}$. By the induction hypothesis $((p, q))_{\sigma} \in \bar{D}^s$ for all σ such that $lb(p, q) = lb\sigma$.

6. By part (b) of the definition of improvement $p < q$, hence the conclusion holds for (p, q) .

7. We must have $r(\beta * j, s + 1) = k(j, (q)_1^-, s) \leq j(i, k(j, (q)_1^-, s), s) = d(\beta * j, s + 1)$.

8. We have $(q)_1^- \leq k(j, (q)_1^-, s) = r(\beta * j, s + 1)$.

9. This is quite clear from (i) and (j) in the definition of improvement.

10. All we need show is that if $\gamma \subset \beta * j$ then there existed a candidate of order i and type γ at an earlier stage. At stage s the candidate of order i is of type α and $\beta \subseteq \alpha$ by part (d) of the definition of improvement. Hence $\gamma \subseteq \alpha$ and by the induction hypothesis there was a candidate of type γ at an earlier stage.

IV. Candidates and potential conditions. In this section we prove three technical claims.

Claim 1. (i) If $i < j$ and p and q are candidates of orders i and j respectively both in existence at stage s then $p < q$. (ii) If p is the candidate of order i and type α in existence at stage s then $p \in E_{\alpha}^s$. If p is unimplemented then $p \in A_{\alpha}^s$ and if p is implemented then there exists a potential condition of the form (p, α, σ, k) in existence at stage s .

The proof is by induction on s . Since no candidates are in existence at stage 0 then the claim holds for $s = 0$. Assume the claim is true for s . We first prove part (i) of the claim for $s + 1$. Since no new candidates are appointed in an A-stage or B-stage then we need only consider $s + 1$ to be a C-stage or D-stage. Let i, j, p , and q be given and let stage $s + 1$ pertain to k . First suppose $s + 1$ is a C-stage. We need only consider the case when q is the new candidate of order k . By the induction hypothesis $p \in E_{\alpha}^s$ for some α and since q was just chosen we have $q \in E_{\beta}^{s+1}$ for some β . If p is unimplemented then by part (g) of the definition of improvement $p < q$. If p is implemented then by the induction hypothesis there exists a potential condition of the form (p, α, τ, l) in existence at stage s . Now, p is still the candidate of order i at stage $s + 1$ hence $\alpha \geq \beta$.

For if $\alpha < \beta$ then by the construction p would have been cancelled as the candidate of order i at stage $s + 1$. By part (i) of the definition of improvement $p < q$. Now, suppose $s + 1$ is a D-stage. We need only consider the case when $k = i$ and p is appointed as the candidate of order i at stage $s + 1$. Stage $s + 1$ cannot be the first stage when a candidate of order i is appointed since a candidate of order j exists at stage s . Let $u + 1 < s + 1$ be the last stage when a candidate of order i was cancelled. Any candidate of order j was also cancelled at stage $u + 1$ so that q must have been appointed as a candidate after stage $u + 1$ and before stage $s + 1$. This is impossible.

We now proceed to prove part (ii) of the claim. Let p be the candidate of order i and type α in existence at stage $s + 1$. Assume stage $s + 1$ pertains to j . There are four cases to consider corresponding to the four types of stages possible.

Case A. If stage $s + 1$ is in Case 1 or Case 3 there is nothing more to show. If stage $s + 1$ is in Case 2 then $i \leq j$ otherwise p is cancelled at stage $s + 1$. By part (i) of the claim $p \leq q$ where q is the candidate of order j in existence at stage s . By the induction hypothesis $p \in E_\alpha^s$ and by Remark 3 and the construction $E_\alpha^s = E_\alpha^{s+1}$. Further, if p is unimplemented then $p \in A_\alpha^{s+1}$ since $p < q$. If p is implemented then the potential condition associated with p is either slightly changed at stage $s + 1$ or persists unchanged since p is not cancelled at stage $s + 1$.

Case B. Again $i \leq j$ otherwise p is cancelled. We have $p \leq q$ where q is the candidate of order j and type β . Now $E_\alpha^{s+1} = E_\alpha^s$ if $\alpha \subseteq \beta$ or $\alpha \geq \beta$ and $\text{lev } \alpha \leq j$. It is not difficult to see that $\text{lev } \alpha \leq i$ since p is of type α and the level of the type of any point of order i must be $\leq i$. Again by Remark 3, $E_\alpha^{s+1} = E_\alpha^s$. We now argue as in Case A.

Case C. If $i = j$ then the new candidate of order i is a member of A_α^{s+1} by definition. If $i < j$ then we argue as in Cases A and B.

Case D. Trivial.

Claim 2. *If (p, α, σ, j) is the potential condition of order i in existence at stage s then*

- (i) p is the candidate of order i in existence at stage s and is of type α ,
- (ii) $lhp = lh\sigma = \text{ext } \alpha$,
- (iii) p is implemented, in fact, $(p)_r \in D^s$ for all $r \geq \sigma$ such that $lbr = lh\sigma$,
- (iv) $\alpha_j = 1$,
- (v) $j < i$,
- (vi) σ is not a constant zero sequence,
- (vii) if $n = 1 +$ the number of zeros at the end of σ then $j \leq n$ th k such that $\alpha_k = 1$.

The proof is quite straightforward by induction on s . Claim 2 verifies the

parenthetical note that Case 3 of an A-stage never occurs.

Claim 3. *Let r and r' be unimplemented candidates of orders i and j and types α and β respectively in existence at stage s . If $i < j$ and $\alpha \subseteq \beta$ then $\alpha_k = \beta_k$ for all $k < i$. In particular, if $\text{lev } \beta \leq i$ then $\alpha = \beta$.*

The proof is by induction on s . The claim is vacuously true for $s = 0$. Assume it is true for s . We need only consider a C-stage $s + 1$ pertaining to j . Let (p, q) be the new candidate of order j and type $\gamma * l$. Let r be the unimplemented candidate of order i and type α . Assume $\alpha \subseteq \gamma * l$.

If $\alpha = \gamma * l$ then we are finished. So we can assume that $\alpha \subsetneq \gamma$. We now show that $\alpha_k = \gamma_k$ for all $k < i$. If $\gamma = o$ then there is nothing more to show. On the other hand if $\gamma \neq o$ let $s' + 1 < s + 1$ be the stage when p came into existence. The point r was already in existence at stage $s' + 1$ since $r < p$. Stage $s' + 1$ pertains to some $i' > i$ otherwise r would be cancelled as the candidate of order i at stage $s' + 1$. Thus the candidate of order i' is p of type γ . By the induction hypothesis, $\alpha_k = \gamma_k$ for all $k < i$.

If $l \geq i$ then $\alpha_k = (\gamma * l)_k$ for all $k < i$. On the other hand suppose $l < i$. We now show that the quadruple l, γ, p, q also satisfied the definition of improvement of the candidate of order i at stage $s + 1$. The conditions (a), (b), (f), (h), (i), and (j) hold since they are independent of i . We check the other conditions. (c) $\text{lev } \gamma \leq l < i$ since $\text{lev } \gamma * l \leq i$. Since $\gamma = \alpha$ then certainly $\gamma \subseteq \alpha$ and $\gamma * l > \alpha$. (e) $m(i, \gamma, s) \leq m(j, \gamma, s) < p$. (g) Since $p <$ any unimplemented candidate of order $< j$ then $p >$ any unimplemented candidate of order $< i$. Thus the condition of order i requires attention at stage $s + 1$ contrary to the hypothesis.

V. D is nonrecursive. We begin by making the following claim.

Claim 4. *For each i only finitely many stages pertain to i .*

Assume the claim is true for all $i' < i$. Let t be the least stage after which no condition of order $< i$ requires attention. If there exists a potential condition of order i at some stage s then for all $s' \geq s$, $D^{s'} \cap W_i^{s'} \neq \emptyset$. No candidate of order i is chosen or improved after stage s . Once a potential condition of order i is instituted no more than $i \cdot 2^t$ stages can pertain to i . Let us assume then that no potential condition of order i ever comes into existence. If there exists no candidate of order i in existence at stage $t + 1$ then we must have $D^t \cap W_i^t \neq \emptyset$. In this case the condition of order i never requires attention after stage t . We can assume that there exists a candidate of order i at stage $t + 1$. At all stages $\geq t + 1$ a candidate of order i is in existence. Let α be the lexicographically greatest member of Γ such that there exists a stage $s > t$ and a candidate of order i and type α in existence at stage s . There exists such an α since any

candidate of order i must have the level of its type $\leq i$ and there are only finitely many members of Γ with level $\leq i$.

Let $s_0 + 1 > t$ be the least stage pertaining to i such that a candidate of order i and type α is appointed at stage $s_0 + 1$. We now show that for all $s > s_0$ stage $s + 1$ does not pertain to i . Suppose the contrary. Let $s_1 + 1$ be the least stage $> s_0 + 1$ pertaining to i . The unimplemented candidate of order i in existence at stage $s_0 + 1$ is still in existence at stage $s_1 + 1$ since it cannot be cancelled by conditions of order $> i$. Stage $s_1 + 1$ must be a C-stage since no potential condition of order i ever exists. The new candidate of order i created at stage $s_1 + 1$ must have type $> \alpha$. This is impossible.

Claim 5. D is nonrecursive.

Let i be any number. If $D \cap W_i \neq \emptyset$ then we are finished. If $D \cap W_i = \emptyset$ then let t be a stage such that no stage $\geq t$ pertains to a condition of order $\leq i$. There must be a candidate p of order i in existence at stage t . Furthermore, for all $s \geq t$, p remains in existence. Now, $(p)_1 \notin W_i$ otherwise the condition of order i would require attention at some stage $\geq t$. By Remark 5, $(p)_1 \in \bar{D}^s$ for all $s \geq t$. Thus $(p)_1 \in \bar{D} \cap \bar{W}_i$. Hence $\bar{D} \neq W_i$.

VI. Every r.e. set Turing equivalent to D is mitotic. We make the following definitions.

$$E_\alpha = \bigcup_s \bigcap_{s' \geq s} E_\alpha^{s'}, \quad A_\alpha = \bigcup_s \bigcap_{s' \geq s} A_\alpha^{s'}$$

Claim 6. If E_α and E_β are both infinite then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

Suppose $\alpha \not\subseteq \beta$. If $\alpha = 0$ we are finished. If $\alpha \neq 0$ then for each s if a new point is added to E_β^s at stage $s + 1$ we have $E_\alpha^{s+1} = \emptyset$ unless $\alpha \subseteq \beta$. Since E_β is infinite then $E_\alpha^s = \emptyset$ for infinitely many s unless $\alpha \subseteq \beta$.

Claim 6 justifies the following definition. Let α be the infinite sequence of 0's and 1's such that $\alpha_i = 1$ if and only if there exists $a \in \Gamma$ such that $\alpha_i = 1$ and E_a is infinite. If $\alpha \in \Gamma$ then we can define $\alpha \subseteq \alpha$ just as we did earlier. It is not difficult to see that $\alpha \subseteq \alpha$ if and only if E_α is infinite. If E_α is infinite then $\alpha \subseteq \alpha$ by Claim 6. If $\alpha \subseteq \alpha$ then $\alpha \subseteq \beta$ where E_β is infinite. But the points of E_β are built up from the points of E_α . Thus E_α is infinite.

Claim 7. If E_α is infinite then A_α is infinite.

Since $E_0 = N$ we first prove the claim for $\alpha = 0$. Let i be any natural number. Let $t + 1$ be the greatest stage which pertains to a condition of order $\leq i + 1$ where $t + 1$ is not an A-stage in Case 1. Let p be the $(i + 1)$ st member

of A_o^{t+1} . We show by induction on s that for all $s \geq t + 1$ any candidate of order $> i + 1$ in existence at stage s is $> p$ and, further, p is the $(i + 1)$ st member of A_o^s . Since stage $t + 1$ is not an A-stage in Case 1 then all candidates and potential conditions of order $> i + 1$ are cancelled at stage $t + 1$. Let $s + 1 > t + 1$. If stage $s + 1$ is an A-stage in Case 1 then $A_o^s = A_o^{s+1}$. If stage $s + 1$ is an A-stage in Case 2 or a B-stage then it must pertain to some condition of order $j > i + 1$. By the induction hypothesis $p <$ the candidate of order j . Hence p remains the $(i + 1)$ st member of A_o^{s+1} . If stage $s + 1$ is a C-stage or a D-stage then the new candidate of order $> i + 1$ and is specifically chosen $\geq m(i + 1, o, s) > p$. The point p remains the $(i + 1)$ st member of A_o^{s+1} . We conclude that A_o contains at least $i + 1$ members.

Let $\alpha \neq o$ with level m and let n be any number. Assume A_α has at least n members. We show that A_α has at least $n + 1$ members. Let t be a stage such that for all $s \geq t, E_\alpha^s \neq \emptyset$ and A_α^s contains the first n members of A_α . Choose $i \geq \max\{m, n + 1\}$ such that $W_i = \emptyset$ and no stage $\leq t$ pertains to i . Such an i exists since there exists infinitely many i such that $W_i = \emptyset$. We now show that there exists a stage $s + 1 > t$ pertaining to i such that the candidate of order i is improved to type α . Since E_α is infinite there exists $i' > i$ and $s' + 1 > t$ such that the candidate of order i' is improved to type α at stage $s' + 1$. Let β be the type of the candidate of order i in existence at stage $s' + 1$. Such a candidate exists because $W_i = \emptyset$. By Claim 1 and Remark 3 $\beta \geq \alpha$ or $\beta \subseteq \alpha$. If $\beta > \alpha$ and $\alpha \not\subseteq \beta$ then $E_\alpha^s = \emptyset$ for some $s \geq t$. If $\alpha \subseteq \beta$ then by Remark 10 at some stage prior to stage $s' + 1$ but after stage t there must have been a candidate of order i and type α in existence. If $\beta \subseteq \alpha$ then $\beta = \alpha$ by Claim 3 since $\text{lev } \alpha \leq i$.

Now, let $s_0 + 1$ be the last stage such that the candidate of order i is improved to type α . No stage $> s_0 + 1$ pertains to a condition of order $< i$ except if the stage is an A-stage in Case 1. If not then at some stage $s_1 + 1 > s_0 + 1$ there is no candidate of order i in existence at stage $s_1 + 1$. We can argue as we did in the last paragraph that at some stage $> s_1 + 1$ the candidate of order i is improved to type α again. This is impossible. There are at least $n + 1$ members in $A_\alpha^{s_0+1}$. There are the n members of $A_\alpha^{s_0}$ plus the new candidate of order i in $A_\alpha^{s_0+1}$. Let p be the $(n + 1)$ st member of $A_\alpha^{s_0+1}$.

We show by induction on s if $s \geq s_0 + 1$ and q is a candidate of order $> i$ in existence at stage s then $q > p$, and further $p \in A_\alpha^s$. This is clear for $s = s_0 + 1$ since there are no candidates of order $> i$ in existence at stage $s_0 + 1$. If stage $s + 1$ is an A-stage in Case 1 then $A_\alpha^s = A_\alpha^{s+1}$. If stage $s + 1$ is any other type of stage then it pertains to some $j \geq i$. If stage $s + 1$ is an A-stage in Case 2 or B-stage then $j > i$ since $W_i = \emptyset$. By the induction hypothesis the candidate of order j in existence at stage s is $> p$ thus $p \in A_\alpha^{s+1}$. If stage $s + 1$ is a C-stage

then let β be the type of the new candidate of order j . If $\beta \leq \alpha$ then clearly the new point of order j is $> p$ and $p \in A_\alpha^{s+1}$. If $\alpha < \beta$ then $\alpha \subset \beta$ otherwise $E_\alpha^{s+1} = \emptyset$ which is impossible. If q is the new candidate of order j then $m(j, \alpha, s) < q$. Now, $j \geq i \geq n + 1$. Hence $q >$ the $(n + 1)$ st member of A_α^s if there are at least $n + 1$ members in A_α^s , which there are by the induction hypothesis. Thus $q > p$ and $p \in A_\alpha^{s+1}$. If stage $s + 1$ is a D-stage then $A_\alpha^s = A_\alpha^{s+1}$ and the candidate is chosen $> p$. We conclude that A_α contains at least $n + 1$ members.

Claim 8. *If $\theta_j(D) = Y_j$ and $\Psi_j(Y_j) = D$ then $\alpha_j = 1$.*

Let β be the greatest member of Γ such that $\text{lev } \beta \leq j + 1$ and $\beta \subseteq \alpha$. If $\beta_j = 1$ we are finished. Assume for reductio ad absurdum that $\beta_j = 0$. Clearly $E_{\beta * j}$ must be finite.

We begin by showing that there exist at most finitely many stages s such that a candidate of type $\geq \beta * j$ is appointed at stage s . Let t be a stage such that for all $s \geq t$, $E_\beta^s \neq \emptyset$. If a candidate of type $\gamma \geq \beta * j$ is appointed at a stage $s \geq t$ then $\gamma \supseteq \beta$ otherwise $E_\beta^s = \emptyset$. If p is appointed at stage s as a candidate of type γ where $\gamma \geq \beta * j$ and $\gamma \supseteq \beta$ then p is built up from members of $E_{\beta * k}^s$ where $\text{lev } \beta \leq k \leq j$. We can say that all types γ such that $\gamma \geq \beta * j$ and $\gamma \supseteq \beta$ are built up from finitely many types of the form $\beta * k$ where $\text{lev } \beta \leq k \leq j$. So it suffices to show that for all k such that $\text{lev } \beta \leq k \leq j$ there exist at most finitely many stages s such that a candidate of type $\beta * k$ is appointed at stage s . Suppose not. Let k be the least number $\geq \text{lev } \beta$ such that there exist infinitely many stages s such that a candidate of type $\beta * k$ is appointed at stage s . Let t_0 be a stage $\geq t$ such that no candidate of type γ is appointed at any stage $\geq t_0$ where $\gamma \supseteq \beta * l$ and $\text{lev } \beta \leq l < k$. Choose $t_1 \geq t_0$ such that if $s + 1 \geq t_1$ then at stage $s + 1$ no candidate of type γ is considered where $\gamma \supseteq \beta * l$ and $\text{lev } \beta \leq l < k$ and further $s + 1$ does not pertain to a number $\leq k + 1$. We now argue that if $s \geq t_1$ then $E_{\beta * k}^s \subseteq E_{\beta * k}^{s+1}$. Now, by the construction $E_{\beta * k}^{s+1} = \emptyset$ or $E_{\beta * k}^s \subseteq E_{\beta * k}^{s+1}$. If $E_{\beta * k}^{s+1} = \emptyset$ and $E_{\beta * k}^s \neq \emptyset$ then either stage $s + 1$ considers a candidate of type $\gamma > \beta * k$ (as in Parts A (Case 2), B, and C of the construction) or some candidate of order $\leq k + 1$ is implemented in part B of the construction. Since the second of these two possibilities is impossible then $\gamma \supseteq \beta$ or else $E_\beta^{s+1} = \emptyset$. Hence $\gamma \supseteq \beta$ and $\gamma > \beta * k$. This implies there exists an l such that $\text{lev } \beta \leq l < k$ such that $\gamma \supseteq \beta * l$ which is also impossible. Hence $E_{\beta * k}^s \subseteq E_{\beta * k}^{s+1}$ for all $s \geq t_1$. We conclude that $E_{\beta * k}$ is infinite since infinitely many candidates are appointed which are of type $\beta * k$. Now, $\text{lev } \beta * k \leq j + 1$ since $k \leq j$. This contradicts the assumption that β is the greatest member of Γ such that $\text{lev } \beta \leq j + 1$ and E_β is infinite.

Let s_0 be a stage such that, for all $s \geq s_0$, $E_\beta^s \neq \emptyset$ and no candidate of

type $\geq \beta * j$ is appointed at stage s . Let i be the least number such that no candidate of order i has been chosen by stage s_0 , $j < i$ and $W_i = \emptyset$. Let $s_1 \geq s_0$ be a stage such that no stage $\geq s_1$ pertains to a condition of order $\leq i$. At stage s_1 there exists a candidate r of order i and type α for some α . Clearly r is the candidate of order i for all $s \geq s_1$. Since E_β is infinite there exists $i' > i$ such that at some stage $s \geq s_1$ a candidate of order i' and type β is appointed. Since $E_\alpha^s \neq \emptyset$ then $\alpha > \beta$ or $\alpha \subseteq \beta$.

If $\alpha > \beta$ then $\alpha \supset \beta$ otherwise $E_\beta^{s'} = \emptyset$, where $s' \geq s_0$ is the stage when r was chosen as the candidate of order i . If $\alpha \subseteq \beta$ then $\alpha = \beta$ by Claim 3. In any case $\beta \subseteq \alpha$. Since $s_1 \geq s_0$ then $\beta * j > \alpha$.

Since E_β is infinite so is A_β . Choose points p and q as follows.

- (i) $p \in A_\beta$ and $q \in A_\beta$,
- (ii) $p < q$,
- (iii) $p >$ any candidate ever appointed of order $< i$,
- (iv) $p > \max\{m(i, \beta, s) : s \in N\}$,
- (v) $\lim_s l(j, (p)_0^-, s) > \max\{r(\gamma, s) : \gamma \geq \beta * j \text{ and } s \in N\}$,
- (vi) $\lim_s k(j, (p)_1^-, s) < \lim_s l(j, (q)_0^-, s)$.

Such p and q exist because $\theta_j(D) = Y_j$ and $\Psi_j(Y_j) = D$ and there exist only finitely many stages when a candidate of type $\geq \beta * j$ is appointed.

Let $s_2 \geq s_1$ be a stage such that

- (1) p and q are in existence at all stages $\geq s_2$,
- (2) β is not frozen at stage s_2 ,
- (3) $L(j, s) > \lim_s k(j, (q)_1^-, s)$ for all $s \geq s_2$,
- (4) no stage $\geq s_2$ considers a candidate of type $\geq \beta * j$,
- (5) no candidate of type $\geq \beta * j$ is cancelled at any stage $\geq s_2$.

Since p and q are in A_β^s for all $s \geq s_2$ it must be the case that $l(j, (p)_0^-, s_2) = \lim_s l(j, (p)_0^-, s)$, $l(j, (q)_0^-, s_2) = \lim_s l(j, (q)_0^-, s)$ and $k(j, (p)_1^-, s_2) = \lim_s k(j, (p)_1^-, s)$. Clearly the candidate of order i can be improved at stage $s_2 + 1$

if we can show that $\beta * j$ itself is not frozen at stage s_2 . If $\beta * j$ is frozen at stage s_2 then there exists at stage s_2 a potential condition of the form (a, γ, σ, k) where if δ is the greatest member of Γ such that $\delta \subseteq \gamma$ and $\delta \subseteq \beta * j$, and $\sigma = \sigma' * \sigma''$ where $l b \sigma'' = \text{ext } \delta$ then σ'' has a nonzero member. Now, $\delta \not\subseteq \beta$ since otherwise β is frozen at stage s_2 . We conclude that $\delta = \beta * j$. Thus $\gamma \supseteq \beta * j$ and so $\gamma \geq \beta * j$. This potential condition is never cancelled for if it were then (5) above fails. If $k < j$ then the potential condition is eventually satisfied since $\beta_k = 1$ implies $\lambda s L(k, s)$ is unbounded. Thus $k \geq j$ otherwise (4) fails. Further $k \neq j$ since $\lambda s L(j, s)$ is unbounded by the hypothesis of the claim. Hence $k > j$. Let $n = 1 +$ the number of zeros at the end of σ . By Claim 2, $k \leq n$ th l

such that $\gamma_l = 1$. Now, $\sigma = \sigma' * \sigma''$ where $l\sigma'' = \text{ext } \beta * j$ and σ'' contains a non-zero member. Hence $n \leq \text{ext } \beta * j$. Since $k < n$ th l such that $\gamma_l = 1$ then k must be $\leq j$. We cannot have both $k > j$ and $k \leq j$ so we cannot have $\beta * j$ frozen at stage s_2 .

We have shown that the condition of order i requires attention at stage $s_2 + 1 \geq s_0$ which is impossible. Our assumption that $\beta_j = 0$ is false.

Claim 9. *If $\theta_j(D) = Y_j$ and $\Psi_j(Y_j) = D$ then there exists r.e. sets $Y_{j,0}$ and $Y_{j,1}$ such that $Y_{j,0} \cap Y_{j,1} = \emptyset$, $Y_{j,0} \cup Y_{j,1} = Y_j$, $D \leq_T Y_{j,0}$, and $D \leq_T Y_{j,1}$.*

Let $\alpha \subseteq \alpha$ be such that $\text{lev } \alpha = j + 1$. Such an α exists by Claim 8. Let t_0 be the last stage such that $E_\alpha^{t_0} = \emptyset$. For all $s \geq t_0$, $E_\alpha^s \subseteq E_\alpha^{s+1}$, $R_{0,\alpha}^s \subseteq R_{0,\alpha}^{s+1}$, $R_{1,\alpha}^s \subseteq R_{1,\alpha}^{s+1}$ and $d(\alpha, s) \leq d(\alpha, s + 1)$. Let $R = \bigcup \{R_{0,\alpha}^s : s \geq t_0\}$. The set R is recursive since the intervals that make up R are enumerated in increasing order. Define $Y_{j,0}^s = Y_j^s \cap R$ and $Y_{j,1}^s = Y_j^s \cap \bar{R}$. Now define $Y_{j,0} = \bigcup_s Y_{j,0}^s$ and $Y_{j,1} = \bigcup_s Y_{j,1}^s$. Clearly $Y_{j,0} \cap Y_{j,1} = \emptyset$ and $Y_{j,0} \cup Y_{j,1} = Y_j$.

We now show that $D \leq_T Y_{j,1}$. The proof that $D \leq_T Y_{j,0}$ is analogous. Let $M = \max\{d(\beta, s) : \beta > \alpha, \beta \not\geq \alpha \text{ and } s \in \mathbb{N}\}$. M exists otherwise $E_\alpha = \emptyset$. We now show how to compute $D(x)$ for $x > M$ inductively from $Y_{j,1}$.

Let $x > M$. Compute the least number $t_1 \geq t_0$ such that $d(\alpha, t_1 + 1) \geq x$. The stage t_1 exists because E_α is infinite. Let t_2 be the least stage $\geq t_1 + 1$ such that

- (i) $D(y) = D^{t_2}(y)$ for all $y < x$,
- (ii) $Y_{1,j}(y) = Y_{1,j}^{t_2}(y)$ for all $y \leq r(\alpha, t_1 + 1)$,
- (iii) $L(j, t_2) > r(\alpha, t_1 + 1)$,
- (iv) α is not frozen at stage t_2 .

We will show that $x \in D$ iff $x \in D^{t_2}$.

For reductio ad absurdum let us suppose that $x \in D - D^{t_2}$. Suppose $x \in D^{s_2+1} - D^{s_2}$. Let p be the member of E_α created at stage $t_1 + 1$.

Let p be the constant zero sequence of length $\text{ext } \alpha - 1$. We begin by making the following.

Subclaim. *There exists a candidate q of type $\beta \geq \alpha$ in existence at stage s_2 such that $x = (p)_0 * p$ and $p = (q)_\sigma$ for some σ of length $\text{ext } \beta - \text{ext } \alpha$.*

Assume that the subclaim has been proved and let β, q , and σ be as in the subclaim. Let r be the constant one sequence of length $\text{ext } \alpha - 1$. The number $(p)_{1*r}$ must enter D at some stage $s_0 + 1 < s_2 + 1$. At stage $s_0 + 1$ a potential condition of the form $(q, \beta, \sigma * 1 * r, l)$ comes into existence. Furthermore at every stage s such that $s_0 + 1 \leq s \leq s_2$ a potential condition of the form

$(q, \beta, \sigma * 1 * v, l)$ where $lvu = \text{ext } \alpha - 1$ is in existence otherwise $x = (p)_{0 * \rho}$ would never enter D at stage $s_2 + 1$. Hence α is frozen at each stage s such that $s_0 + 1 \leq s \leq s_2$. Since $t_2 \leq s_2$ then $t_2 \leq s_0$ otherwise α would be frozen at stage t_2 . Let $s_1 + 1$ be the stage such that $(p)_{1 * \rho} \in D^{s_1 + 1} - D^{s_1}$. Now, $s_0 < s_1 < s_2$. At stage $s_1 + 1$ a potential condition of the form $(q, \beta, \sigma * 1 * \rho, l)$ where l = the least number such that $\beta_l = 1$ comes into existence. There are exactly $\text{ext } \alpha - 1$ zeros at the end of $\sigma * 1 * \rho$. Hence the number n referred to in Case 1 of part A of the construction is actually equal to $\text{ext } \alpha$. Eventually a potential condition of the form $(q, \beta, \sigma * 1 * \rho, j)$ must come into existence since $j = \text{the}(\text{ext } \alpha)\text{th } k$ such that $\alpha_k = 1$. Let t_3 be the stage when this potential condition is satisfied and $(p)_{0 * r}$ actually enters D at stage $t_3 + 1$. We show now that $j, (p)_{1 * \rho}, (p)_{1 * r}, t_1, t_2, t_3$ satisfy the basic lemma. We certainly have $(p)_{1 * \rho} \leq (p)_{1 * r}$ and $t_1 \leq t_2 < t_3$. Now, the point $((p)_0, (p)_1)$ was constructed at stage $t_1 + 1$. Hence $L(j, t_1) > k(j, (p)_{1 * r}, t_1)$ and $d(\alpha, t_1 + 1) = j(j, k(j, (p)_{1 * r}, t_1), t_1)$. Since $(p)_{1 * r}$ enters D at stage $s_0 + 1$ and $s_0 \geq t_2$ then no number $\leq d(\alpha, t_1 + 1)$ may enter D at a stage s such that $t_1 \leq s \leq t_2$. Furthermore, no number $< (p)_{1 * \rho}$ may enter D at a stage s such that $t_1 \leq s \leq t_3$ otherwise $(p)_{0 * r}$ would never enter D at stage $t_3 + 1$. Now, $L(j, t_2) > r(\alpha, t_1 + 1) = k(j, (p)_{1 * r}, t_1)$ by the definition of t_2 and $L(j, t_3) > (p)_{1 * \rho}$ since the potential condition $(q, \beta, \sigma * 1 * \rho, j)$ is satisfied at stage t_3 . All the hypotheses of the basic lemma are satisfied. Hence there exists a $z \in Y_j^{t_3} - Y_j^{t_2}$ such that $l(j, (p)_{1 * \rho}, t_1) < z \leq k(j, (p)_{1 * r}, t_1)$. Now, $[l(j, (p)_{1 * \rho}, t_1), k(j, (p)_{1 * r}, t_1)] \subseteq \bar{R}$. Hence $z \in Y_{j,1}^{t_3} - Y_{j,1}^{t_2}$ and $z \leq r(\alpha, t_1 + 1)$. This contradicts part (ii) of the definition of t_2 . We conclude that our supposition that $x \in D - D^{t_2}$ is false.

We now proceed to prove the subclaim. There certainly exists a candidate q of some type β in existence at stage s_2 such that $x = (q)_v$ for some v of length equal to $\text{ext } \beta$. We must have $\beta \leq \alpha$ or $\alpha \subseteq \beta$ since otherwise $E_\alpha^{s_2 + 1} = \emptyset$. If $\beta \leq \alpha$ then we can conclude that $\beta \subseteq \alpha$ otherwise $p < q$ which is impossible since $x \leq (p)_1^-$. Assume $\beta \subseteq \alpha$. We shall show that $\beta = \alpha$. Now, α is frozen from the stage when $(q)_1^-$ enters D until at least stage s_2 . Hence $(q)_1^-$ enters D after stage t_2 . Now, q is the candidate of order i for some i . We cannot have $i \leq j$ or else $E_\alpha^{s_2 + 1} = \emptyset$ where $s + 1 > t_2$ ($s + 1$ is the stage when $(q)_1^-$ enters D). Let $u_1 + 1$ be the stage when q was chosen as the candidate of order i . The point p was chosen at stage $t_1 + 1$ to be the candidate of order i' for some i' . We must have $u_1 \leq t_1$ or else $q > p$ which is impossible. Hence at stage $t_1 + 1$ both p and q are unimplemented candidates of orders i' and i respectively. Now $i \leq i'$ or else $q > p$. Now, $\text{lev } \alpha = j + 1 \leq i \leq i'$ hence by Claim 3 $\beta = \alpha$.

So far we have shown that $\alpha \subseteq \beta$. Since $x = (q)_v$ then $x = ((q)_\sigma)_\mu$ for some σ

of length $\text{ext } \beta - \text{ext } \alpha$ and μ of length $\text{ext } \alpha$. Since $t_1 + 1$ is the least stage such that $x \leq d(\alpha, t_1 + 1)$ and p was created at stage $t_1 + 1$ then $(q)_\sigma$ must be p . It remains to show that μ is the constant zero sequence of length $\text{ext } \alpha$. At stage $s_2 + 1$ a potential condition of order i of the form $(q, \beta, \sigma * \mu, k)$ is appointed where k is the least number such that $\beta_k = 1$. Suppose that μ is not the constant zero sequence of length $\text{ext } \alpha$. Then α is frozen at stage $s_2 + 1$. Further α must be frozen at every stage $s \geq s_2 + 1$ via a potential condition of order i of the form $(q, \beta, \sigma * \mu, l)$ for some l . This is shown by induction on s . There is no problem if $s + 1$ is an A-stage in Case 1 or a D-stage. There are three cases left to consider.

(1) $s + 1$ is an A-stage in Case 2. Let i' be the order of the potential condition satisfied at stage $s + 1$. If $i' \leq i$ then some number $< x$ must enter D at stage $s + 1$ which is impossible. If $i < i'$ then the potential condition of order i remains in force.

(2) $s + 1$ is a B-stage. Let i' be the order of the candidate implemented at stage $s + 1$. Again we must have $i < i'$ or else some number $< x$ enters D at stage $s + 1$. On the other hand if $i < i'$ then the potential condition remains in force.

(3) $s + 1$ is a C-stage. The new candidate r must be of order i' and of type γ where $\gamma \leq \alpha$ or $\gamma \supseteq \alpha$ since otherwise $E_\alpha^{s+1} = \emptyset$. Since α is frozen at stage s then we cannot have $\gamma \supseteq \alpha$. Since $\gamma \leq \alpha$ and $\alpha \subseteq \beta$ then $\gamma \leq \beta$. We of course cannot have $i = i'$. If $i < i'$ then $(q, \beta, \sigma * \mu, l)$ could only be cancelled if $\gamma > \beta$ which is not the case. Suppose $i' < i$. Let a be the candidate of order i' in existence at stage s just before r was chosen and suppose a has type δ . Now, since $i' < i$ then $a < q$ and $\delta \geq \beta$ or $\delta \subseteq \beta$. Since γ is to be an improvement over δ and $\gamma \leq \beta$ then $\delta \subseteq \beta$. Hence $\delta \subseteq \beta$. Let u be the stage when q was chosen as the candidate of order i . The point a must be the candidate of order i' in existence at stage u . By Claim 3 $\delta_k = \beta_k$ for all $k < i'$. Since $\gamma > \delta$ and $\text{lev } \gamma \leq i'$ then there exists $k' < i'$ such that $\gamma_k = \delta_k$ for all $k < k'$, $\gamma_{k'} = 1$ and $\delta_{k'} = 0$. Hence we must have $\gamma > \beta$ which is impossible. We conclude that μ is the constant zero sequence of length equal to $\text{ext } \alpha$.

This concludes the proof of the subclaim. The theorem is now proved where d is the degree of D .

VII. Proofs of the corollaries. Recall that we need to prove Corollaries 1 and 3 and the fact that $d'' = 0''$.

Proof of Corollary 1. The fact that every r.e. set of degree d is strongly mitotic is proved simply by noticing in the proof of Claim 9 that R is recursive, $Y_{j,0} \subseteq R$, and $Y_{j,1} \subseteq \bar{R}$.

Proof of Corollary 3. To show that every two r.e. sets of degree d are weak

truth table equivalent we show that every r.e. set of degree d is weak truth table equivalent to D . Let $\theta_j(D) = Y_j$ and $\Psi_j(Y_j) = D$. The reduction procedure used in Claim 9 to reduce D to $Y_{j,1}$ is a weak truth table reduction procedure since the number $d(t_1 + 1)$ can be found effectively from x . Hence $D \leq_w Y_{j,1}$ (" \leq_w " means "weak truth table in"). Trivially, $Y_{j,1} \leq_w Y_j$. Therefore $D \leq_w Y_j$. To show that $Y_j \leq_w D$ let t_0 be as in Claim 9 and let y be any number. Let u_1 be the least number $\geq t_0$ such that $r(\alpha, u_1 + 1) \geq y$. Let u_2 be the least stage $\geq u_1 + 1$ such that

- (i) $D(x) = D^{u_2}(x)$ for all $x \leq d(\alpha, u_1 + 1)$,
- (ii) $d(\alpha, u_2 + 1) > d(\alpha, u_2)$.

If we can show that $y \in Y_j$ iff $y \in Y_j^{u_2}$ then we have exhibited a reduction procedure that witnesses $Y_j \leq_w D$.

Let p be the candidate of type α created at stage $u_2 + 1$. We first show that no number $< (p)_0^-$ may enter D after stage u_2 . Suppose not, let $x < (p)_0^-$ be the least number that enters D after stage u_2 . We must have $x > d(\alpha, u_1 + 1)$ hence $x = (q)_\sigma$ for some σ and $q \in E_\alpha$. Let $u + 1$ be the stage when q was created. Clearly $u_2 + 1 \geq u + 1 > u_1$ and $d(\alpha, u + 1) > d(\alpha, u)$. Furthermore, we must have $D(z) = D^u(z)$ for all $z \leq (q)_0^-$ since x is the least number that enters D after stage u_2 and no number $< (q)_0^-$ may enter D between stage $u + 1$ and u_2 (or else x never would have been put into D). Since $q > d(\alpha, u_1 + 1)$ then $D(z) = D^u(z)$ for all $z \leq d(\alpha, u_1 + 1)$. We conclude that $u + 1 = u_2 + 1$ and $p = q$.

Since no number $< (p)_0^-$ may enter D after stage u_2 then using an argument like that in the basic lemma no number $\leq l(j, (p)_0^-, u_2)$ may enter Y_j after stage u_2 . For if $z \in Y_j - Y_j^{u_2}$ and $z \leq l(j, (p)_0^-, u_2)$ then $Y_j(z) \neq \theta_j^{u_2}(D^{u_2}; z)$. The computation on the right-hand side is protected since $D(w) = D^{u_2}(w)$ for all $w < (p)_0^-$. Now we specifically chose $r(\alpha, u_1 + 1) < l(j, (p)_0^-, u_2)$. Hence $Y_j(z) = Y_j^{u_2}(z)$ for all $z \leq r(\alpha, u_1 + 1)$ and in particular $Y_j(y) = Y_j^{u_2}(y)$ which was to be proved.

Proof that $d^u = 0^u$ [due to Paul F. Cohen]. First notice that if A is r.e. and $A \leq_T D$ then $A \leq_w D$. For if $A \leq_T D$ then $A \oplus D \equiv_T D$. Hence, by Corollary 3, $A \oplus D \leq_w D$. Trivially then $A \leq_w D$. From this we conclude that $\{e: W_e \leq_T d\} = \{e: W_e \leq_w D\}$. Now $\{e: W_e \leq_w D\} \in \Sigma_3$ since the set is defined by the predicate

$$\exists i \exists j \forall x \forall s \exists s'$$

$$[s' \geq s, \phi_i^{s'}(x) \text{ is defined,}$$

$$\Phi_j^{s'}(D^{s'}, x) \text{ is defined and equal to } W_e^{s'}(x) \text{ and } c(j, D^{s'}, x, s') \leq \phi_i^{s'}(x)]$$

Now, $\{e: W_e \leq_T d\} \in \Sigma_3$ and further $d < 0'$ since $0'$ contains a nonmitotic r.e.

set. By a theorem of Yales [6, Theorem 9] $d'' = 0''$. This proof parallels a proof of Jockusch [1, Corollary 8].

VIII. Conclusion and open questions. Let $P(W)$ be a property of r.e. sets satisfying the following conditions:

- (1) if $P(W)$ then W is nonrecursive,
- (2) there exists a W such that $P(W)$ and $W \in 0'$,
- (3) for all nonrecursive r.e. V there exists $W \leq_T V$ such that $P(W)$,
- (4) there exists a nonrecursive r.e. V such that for no $W \equiv_T V$ is $P(W)$ true.

The properties "W is nonmitotic" and "W is nonautoreducible" satisfies (1)–(4). From Sasso [5, Theorem 4] and from Corollary 3 we see that the property "there exists a r.e. V such that $W \equiv_T V$ and $W \not\equiv_w V$ " also satisfied (1)–(4). Are there any other interesting properties of r.e. sets that satisfy (1)–(4)? In particular, can a lattice theoretic property (lattice of r.e. sets) satisfy (1)–(4)?

There are several questions concerning the restrictions on the degrees of nonmitotic r.e. sets.

(a) Does the degree d in the present construction have $\text{jump} = 0'$? If not then can the construction be modified to produce such a degree?

(b) Does b r.e. and $b' = 0''$ or $b'' > 0''$ imply that b contains a nonmitotic r.e. set?

(c) Are the degrees of the nonmitotic r.e. sets dense in the r.e. degrees?

The following list contains conjectures that we feel may be solvable using modifications of the techniques of this paper.

(i) There exists a nonrecursive r.e. degree containing only one r.e. weak truth table degree and also having a nonmitotic r.e. set.

(ii) There exists a completely mitotic nonrecursive r.e. degree containing more than one r.e. weak truth table degree.

(iii) There exists a completely mitotic nonrecursive r.e. degree which is not completely strongly mitotic.

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