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# A Completeness Theorem for the General Interpreted Modal Calculus $M C^{\nu}$ of A. Bressan. 

Alberto Zanardo (*)

Summary - A new semantics is introduced for Bressan's modal calculus $M C^{v}$ based on types of all finite levels. By this semantics we extend a completeness theorem of Zane Parks concerning the first order segment of $M C^{y}$ deprived of the description operator, to a completeness theorem for (possibly contingent) modal theories based on the full calculus $M 0^{v}$.

## 1. Introduction.

In [11] Zane Parks gives a completeness theorem for the first order part of Bressan's calculus $M C^{\nu}$ deprived of descriptions. In this paper we extend Z. Parks' theorem to $M C^{v}$ itself (and every theory based on it, i.e. every $M C^{v}$-theory) treating types and descriptions too. Unlike [11] this paper deals also with contingent (i.e. not modally closed) theories based on $M C^{p}$. This is achieved by identifying particular semantical structures which are sound for the definition of $\mathscr{T}$-validity, where $\mathscr{T}$ is an arbitrary $M C^{\nu}$-theory.

The calculus $M C^{v}$ is based on the modal language $M L^{\nu}$. In [1] a semantics for $M L^{\nu}$ is introduced: starting from $\nu$ sets $D_{1}$ to $D_{\nu}$ of (typed) individuals and a set $\Gamma$ of elementary possible cases (elsewere called worlds, or points), for every type $t$ it is defined the set $Q I_{t}$ of quasi intensions-briefly $Q I s$-of type $t$ on which variables and constants of this type can be valued.

More precisely, $Q I_{i}$, the set (of individual concepts) on which individual variables (of type $i$ ) run, is $\left(\Gamma \rightarrow D_{i}\right)(i=1, \ldots, v) ; Q I_{\left(t_{1}, \ldots, t_{n}\right)}$,
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on which relational variables (of type $\left(t_{1}, \ldots, t_{n}\right)$ ) run, is $\left(\Gamma \rightarrow \mathscr{P}\left(Q I_{t_{1}} \times\right.\right.$ $\left.\times \ldots \times Q I_{t_{n}}\right)$ ); and $Q I_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}$, on which functional variables (of type $\left(t_{1}, \ldots, t_{n}: t_{0}\right)$ ) run, is ( $Q I_{t_{1}} \times \ldots \times Q I_{t_{n}} \rightarrow Q I_{t_{0}}$ ) $\left.{ }^{1}\right)$.

Those interpretations for $M L^{v}$ which are based on a structure of quasi intensios as the above one will be called standard interpretations; in these interpretations every $Q I_{t}$ is uniquely determined by the choice of the sets $D_{1}$ to $D_{\nu}$ and $\Gamma$.

Following [7], our definition of $\mathscr{T}$-validity shall refer to a wider class of interpretations, the so-called general interpretations. In general interpretations the set of objects with a certain type is not uniquely determined by the sets of objects of lower type-level; for instance, if $O_{t}$ and $O_{t^{\prime}}$ are the sets of the objects of type $t$ and $t^{\prime}$ respectively, then $O_{\left(t: t^{\prime}\right)}$ will be an arbitrary subset of ( $O_{t} \rightarrow O_{t^{\prime}}$ ) (closed with respect to definable functions). In particular, the set of individual concepts of type $r$ will be an arbitrary subset of ( $\Gamma \rightarrow D_{r}$ ), like in [11].

The proof of the completeness theorem for $M C^{v}$-theories is an Henkin type proof; i.e., starting from a consistent set $K$ of formulas, a general interpretation defined by means of linguistic entities is constructed in which $K$ is satisfiable. Such a general interpretation is denumerable and hence a form of the Löwenheim-Skolem theorem holds.

Let us remark that the semantics introduced in this way is essentially non-extensional, i.e. the extension of a $Q I \xi$ in a possible case $\gamma$, in general, does not depend only on the extension (in $\gamma$ ) of the parts of $\xi$, but on the whole intension of them (for more details, see N. Belnap's foreword to [1], or [3]).

At the end of N. 9 some hints are briefly given for the proof of a completeness theorem for contingent theories based on $M C^{v}$, i.e. theories whose proper axioms are arbitrary (possibly not modally closed). Of course, in the interpretations of such theories a particular elementary case is privileged, the so-called real case. Contingent theories are very important in view of axiomatizations of physical theories. Indeed, only some of these admit modally closed axiomatization in $M C^{v}$; in astronomy, for instance, contingent axioms are needed-see [2].

Let us remark that in [1] NN. 52, 53 the (modally closed) calculus $M C_{e}^{v}$ is introduced by which contingent theories can be investigated. In view of this possibility, the completeness theorem for modally
${ }^{(1)}$ If $A_{0}$ to $A_{n}$ are sets, we denote their cartesian product by $A_{0} \times A_{1} \times$ $\times \ldots \times A_{n}$, the class of subset of $A_{0}$ by $\mathscr{P} A_{0}$, and the set of mappings of $A_{1} \times \ldots \times A_{n}$ into $A_{0}$ by $A_{1} \times \ldots \times A_{n} \rightarrow A_{0}$. Furthermore, we denote the empty set by $\emptyset$.
closed theories covers all possibilities. However we shall consider contingent theories indipendently because they seem interesting at least from a formal point of view.

## 2. The modal language $M L^{\nu}$.

The modal language $M L^{\nu}$ is based on a type system $\tau^{\nu}$ that contains $\nu$ individual types: $1, \ldots, v$. We denote $n$-tuples by 〈...〉 and define $\tau^{\nu}$ recursively by
(a) $\{1, \ldots, v\} \subset \tau^{\nu}$ and
(b) if $t_{1}, \ldots, t_{n} \in \tau^{\nu}$ and $t_{0} \in \tau^{\nu} \cup\{0\}$, then $\left\langle t_{1}, \ldots, t_{n}, t_{0}\right\rangle \in \tau^{\nu}$.

We call types the elements of $\tau^{\nu} \cup\{0\}\left(=\bar{\tau}^{\nu}\right)$. A type $t$ of the form $\left\langle t_{1}, \ldots, t_{n}, t_{0}\right\rangle$ with $t_{0}=0\left[t_{0} \neq 0\right]$ is called (and used as) a relation [function] type and, following Carnap, is denoted by $\left(t_{1}, \ldots, t_{n}\right)\left[\left(t_{1}, \ldots\right.\right.$, $\left.\left.t_{n}: t_{0}\right)\right]$.

The symbols of the modal language $M L^{v}$ are the variables $v_{t n}$ and the constants $c_{i n}$ (where $n \in Z^{+}(=\{1,2, \ldots\})$ and $\left.t \in \tau^{\nu}\right)$, «~» (not), " $\wedge$ " (and), " $\square$ " (necessarily), "=" (identity), reversed iota «1» for descriptions, the comma, and the parentheses.

The class $\mathscr{E}_{t}$ of the designators or wfes (well formed expressions) of type $t\left(\in \overline{\boldsymbol{\tau}}^{v}\right)$ for $M L^{\nu}$ is defined recursively by the following (formation) rules $\left(f_{1}\right)$ to $\left(f_{8}\right)$ where $n\left[t_{0}\right]$ runs over $Z^{+}\left[\bar{\tau}^{p}\right]$ and $t, t_{1}, \ldots, t_{n}$ run over $\tau^{\nu}$.
$\left(f_{1}\right) \quad v_{t n}, c_{t n} \in \mathscr{E}_{t} ;$
$\left(f_{2}\right)$ if $\Delta, \Delta^{\prime} \in \mathscr{E}_{t}$, then $\Delta=\Delta^{\prime} \in \mathscr{E}_{0} ;$
$\left(f_{r}\right)$ if $\Delta_{1} \in \mathscr{E}_{t_{1}}, \ldots, \Delta_{n} \in \mathscr{E}_{t_{n}}^{\prime}$ and $\Delta \in \mathscr{E}_{\left\langle t_{1}, \ldots, t_{n}, t_{0}\right\rangle}$, then $\left(\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right)\right) \in \mathscr{E}_{t_{0}} ;$
$\left(f_{4}\right) \quad$ if $\Delta \in \mathscr{E}_{0}$, then $\left(\left(v_{t n}\right) \Delta\right) \in \mathscr{E}_{t} ;$
$\left(f_{5-8}\right)$ if $\Delta, \Delta^{\prime} \in \mathscr{E}_{0}$, then $(\sim \Delta),\left(\Delta \wedge \Delta^{\prime}\right),\left(\left(v_{t n}\right) \Delta\right),(\square \Delta) \in \mathscr{E}_{0}$.
The connectives $\vee, \supset$, and $\equiv, \exists$, and $\diamond$ (it is possible) are understood to be introduced in the usual way; furthermore we use $\left(\forall x_{1}, \ldots, x_{n}\right) p$ and $\left(\exists x_{1}, \ldots, x_{n}\right) p$ as metalinguistic abbreviations of $\left(x_{1}\right) \ldots\left(x_{n}\right) p$ and $\sim\left(\forall x_{1}, \ldots, x_{n}\right) \sim p$ respectively. In order to drop parentheses we consider $(x x),(x), \sim, \wedge, \vee, \supset$, and $\equiv$ as having decreasing cohesive powers and we use also dots to devide expressions.

A wff (well formed formula) $p$ is said to be modally closed if it is contructed starting out from some wffs $\square p_{1}, \ldots, \square p_{n}$ by means of $\sim \wedge,\left(v_{t n}\right)$, and $\square$. The modal closure of $p$ is $p$ or $\square p$ according to whether $p$ is modally closed or not. The modal closure of the (extensional) closure ( $\left.\forall x_{1}, \ldots, x_{n}\right) p$ of $p$ is called the total closure of $p$.

If $a$ and $x$ are respectively a term and a variable of the same type, then we denote by $\Delta[x / a]$ the result of substituting occurrences of $a$ for free occurrences of $x$ in the wfe $\Delta$ after having performed changes of bound variables in it in order to obtain $a$ free for $x$ in (an «equivalent» of) $\Delta$.

Using the above convention we put

$$
\begin{equation*}
\left(\exists_{1} x\right) p \equiv_{D}(\exists x)(p \wedge(y)(p[x / y] \supset x=y)) \tag{2.1}
\end{equation*}
$$

Furthermore we assume that every expression used in what follows has a type, i.e. it is well formed. This will make several explanations unnecessary. For instance, if we speak of the term $\Delta\left(\Delta^{\prime}\right)$, where $\Delta \in \mathscr{E}_{\left(t^{\prime}: t\right)}$, this implies $\Delta^{\prime} \in \mathscr{E}_{t^{\prime}}$.

## 3. $M L^{\nu}$-interpretations.

In [1] N. 6 a semantical system is introduced for $M L^{v}: \nu+1$ nonempty sets $D_{1}, \ldots, D_{v}$, and $D_{v+1}=\Gamma$ are fixed (for $i=1, \ldots, v, D_{i}[\Gamma]$ is called the $i$-th individual domain [the set of elementary possible cases or $\Gamma$-cases]) and the class $Q I_{t}$ of the quasi-intensions-briefly QIs-of type $t\left(\in \bar{\tau}^{\nu}\right)$ based on $D_{1}$ to $D_{v+1}$ is defined recursively by the conditions (3.1-3) below ( $n \in \mathbb{Z}^{+} ; t_{0}, t_{1}, \ldots, t_{n} \in \tau^{\nu}$ );

$$
\begin{align*}
& Q I_{0}=\mathscr{P}(\Gamma), \quad Q I_{r}=\left(\Gamma \rightarrow D_{r}\right) \quad(r=1, \ldots, \nu)  \tag{3.1}\\
& Q I_{\left(t_{1}, \ldots, t_{n}\right)}=\mathscr{P}\left(Q I_{t_{1}} \times \ldots \times Q I_{t_{n}} \times \Gamma\right)\left(=\left(\Gamma \rightarrow \mathscr{P}\left(Q I_{t_{1}} \times \ldots \times Q I_{t_{n}}\right)\right)\right)  \tag{3.2}\\
& Q I_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}=\left(Q I_{t_{1}} \times \ldots \times Q I_{t_{n}} \rightarrow Q I_{t_{0}}\right) . \tag{3.3}
\end{align*}
$$

Furthermore a function $a^{\nu}$, of domain $\tau^{r}$, is considered such that $a_{t}^{v}={ }_{D} a^{v}(t) \in Q I_{t} ; a_{t}^{v}$ is called the non-existing object of type $t\left(\in \tau^{\nu}\right)$ because it serves to give a designatum to descriptions in the $\Gamma$-cases in which they do not fulfil their conditions of exact uniqueness. In [1] (p. 19) $a_{\left(t_{1}, \ldots, t_{n}\right)}^{v}$ is assumed to be the empty set, and in addition the counterdomain of $a_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}^{\nu}$ is assumed to be $\left\{a_{t_{0}}^{v}\right\}$. These (natural) conditions are conventional and we can omit them.

DeF. 3.1. In connection with the above sets $D_{1}$ to $D_{v+1}$ and function $a^{v}$ we assume $\mathscr{Q I}_{t}\left(t \in \bar{\tau}^{v}\right)$ to fulfil (3.1'-3') where (3.i') is what (3.i) becomes if we substitute $Q I_{t}$ with $\mathscr{Q}_{t}$ and the equality sign $"=»$ with the inclusion sign $« \subseteq \rrbracket$. If $a_{i}^{v} \in \mathscr{Q}_{t}$ for every $t \in \tau^{v}$, then we put

$$
\begin{equation*}
D=\left\{\mathscr{Q}_{t}: t \in \bar{\tau}^{v}\right\}, \quad \mathscr{D}=\left\langle D, a^{\nu}\right\rangle, \tag{3.4}
\end{equation*}
$$

and we say that $D$ is a QI-structure and $\mathscr{D}$ a QI-system.
An interpretation for $M L^{\nu}$ (briefly, an $M L^{\nu}$-interpretation) is then an ordered pair $\mathscr{M}=\langle\mathscr{D}, M\rangle$ where $\mathscr{D}$ is a $Q I$-system and $M$ is a valuation of the constants of $M L^{\nu}$, that is a function which, for all $t \in \tau^{\nu}$, assigns an element of $\mathscr{Q I}_{t}$ to every constant of type $t$. If $V$ is any valuation of the variables of $M L^{v}$ on the $M L^{v}$-interpretation $\mathscr{M}$ (briefly, $V$ is an $\mathscr{M}$-valuation), then the ordered pair $\mathscr{S}=\langle\mathscr{M}, V\rangle$ will be said an $M L^{v-s y s t e m . ~ A s ~ u s u a l, ~ i f ~} V$ and $V^{\prime}$ are two valuation such that $V(x)=V^{\prime}(x)$ for $x \notin\left\{x_{1}, \ldots, x_{n}\right\}$ and $V^{\prime}\left(x_{i}\right)=\xi_{i}$, then we denote $V^{\prime}$ by $V\binom{x_{1}, \ldots, x_{n}}{\xi_{1}, \ldots, \xi_{n}}$.

Every $Q I$-system, or $M L^{\nu}$-interpretation, or $M L^{\nu}$-system, based on a $Q I$-structure in which $\mathscr{Q} \mathscr{F}_{t}=Q I_{t}$ for all $t \in \bar{\tau}^{\nu}$, will be called standard $\left.{ }^{(2}\right)$.

Furthermore, in order to introduce a semantics for theories having some contingent axioms, we consider contingent $M L^{\nu}$-interpretations and systems; these are ordered pairs $\left\langle\mathscr{M}, \gamma_{R}\right\rangle$ and $\left\langle\mathscr{S}, \gamma_{R}\right\rangle$ respectively, where $\gamma_{R}$ is an arbitrarily fixed possible case, the so-called real case.

Def. 3.2. For $\gamma \in \Gamma$ and $\xi, \eta \in \mathscr{Q I}_{t}$ with $t \in \bar{\tau}^{y}$ we say that $\xi$ and $\eta$ are equivalent QIs of type $t$ in the case $\gamma$ (with respect to the QI-structure $D$ ), and we write

$$
\begin{equation*}
\xi={ }_{D, \gamma}^{t} \eta \quad\left(\text { or } \xi={ }_{\gamma}^{t} \eta\right), \tag{3.5}
\end{equation*}
$$

if one of the following conditions holds:
(a) $t \in\{1, \ldots, \nu\}$ or $t=\left(t_{1}, \ldots, t_{n}\right)$, and $\xi(\gamma)=\eta(\gamma)$;
(b) $t=\left(t_{1}, \ldots, t_{n}: t_{0}\right)$ and $\xi\left(\xi_{1}, \ldots, \xi_{n}\right)={ }_{\gamma}^{t_{0}} \eta\left(\xi_{1}, \ldots, \xi_{n}\right)$ for all $n$-tuples $\left\langle\xi_{1}, \ldots, \xi_{n}\right\rangle \in\left(Q \mathscr{I}_{t_{1}} \times \ldots \times Q \mathscr{I}_{t_{n}}\right) ;$
(c) $t=0$ and $\xi \cap\{\gamma\}=\eta \cap\{\gamma\}$.
$\left.{ }^{(2}\right)$ If $D\left(=\left\{Q I_{t}: t \in \bar{\tau}^{\nu}\right\}\right)$ and $D^{\prime}\left(=\left\{\mathscr{Q}_{i}: t \in \bar{\tau}^{\nu}\right\}\right)$ are two $Q I$-structures based on the same sets $D_{1}, \ldots, D_{\nu}$, and $\Gamma$, and $D$ is standard, then for $t \in\{0,1, \ldots, \nu\}, \mathscr{Q}_{t}$ is a subset of $Q I_{t}$ and, in general, $\mathscr{Q \mathscr { I } _ { i }}$ can be embedded in $Q I_{t}$. However, for the sake of simplicity, we shall write $\mathscr{Q} \mathscr{I}_{t} \subseteq Q I_{t}$ for all $t$.

The proof of Theor. 10.2 in [1], concerning standard $Q I$-structures, can be trivially adapted to demonstrate the following theorem in which $D$ is an arbitrary $Q I$-structure.

THeor. 3.1. Let $t \in \bar{\tau}^{\nu}$ and $\xi, \eta \in Q \mathscr{I}_{t}(\in D)$; then $\xi={ }_{D, \gamma} \eta$ for every $\gamma$ iff $\xi=\eta$ 。

## 4. Designation rules for $M L^{\nu}$ in connection with $M L^{\nu}$-systems,

For every $M L^{v}$-system $\mathscr{S}(=\langle\mathscr{M}, V\rangle)$ or contingent $M L^{v}$-system $\left\langle\mathscr{S}, \gamma_{R}\right\rangle$ we associate every $\Delta \in \mathscr{E}_{t}$ with an intensional designatum $\bar{Z}$ in $Q I_{t}$. This designatum is unique, as will appear from the nature of the rules. Hence we denote it by $\operatorname{des}_{\mathscr{C}}(\Delta)$ or $\operatorname{des}_{\mathscr{M} V}(\Delta)$. We define it recursively by the rules $\left(d_{1}\right)$ to ( $d_{9}$ ) below which are extensions to $M L^{v}$-systems of the rules $\left(\delta_{1}\right)$ to ( $\delta_{9}$ ) in [1], NN. 8, 11. For the sake of simplicity the equalities $\tilde{\Delta}_{i}=\operatorname{des}_{\mathscr{P}}\left(\Delta_{i}\right) \quad(i=0,1, \ldots, n)$, and $\tilde{R}=\operatorname{des}_{\mathscr{S}}(R)$, are assumed in the following table; furthermore, the afore-mentioned recursion consists of an induction on the number $\nu_{\Delta}$ of occurrences of 1 in $\Delta$ and, in connection with a given value of $\nu_{\Delta}$, of an induction on the length $l_{\Delta}$ of $\Delta$.

| Rule | If $\Delta$ is | Then $\tilde{\Delta}\left(=\operatorname{des}_{\mathscr{S}}(\Delta)\right)$ is |
| :---: | :---: | :---: |
| $\left(d_{1}\right)$ | $v_{t n}$ or $c_{t n}$ | $\nabla\left(v_{t n}\right)$ or $M\left(e_{t n}\right)$, respectively; |
| $\left(d_{2}\right)$ | $\Delta_{1}=\Delta_{2}\left(\Delta_{1}, \Delta_{2} \in \varepsilon_{t}\right)$ | $\left\{\gamma \in \Gamma: \widetilde{\triangle}_{1}={ }_{\gamma}^{t} \tilde{\Delta}_{2}\right\} ;$ |
| $\left(d_{3}\right)$ | $R\left(\Delta_{1}, \ldots, \Delta_{2}\right)\left(\in \mathcal{E}_{0}\right)$ | $\left\{\gamma \in \Gamma:\left\langle\tilde{\Delta}_{1}, \ldots, \tilde{\Delta}_{n}, \gamma\right\rangle \in \tilde{R}\right\} ;$ |
| ( $d_{4}$ ) | the term $\Delta_{0}\left(\Delta_{1}, \ldots, \Delta_{n}\right)$ | $\widetilde{\Delta}_{0}\left(\widetilde{U}_{1}, \ldots, \widetilde{\Delta}_{n}\right)$; |
| $\left(d_{5-6}\right)$ | $\sim \Delta_{1}$ or $\Delta_{1} \wedge \Delta_{2}$ | $\Gamma-\tilde{\triangle}_{1}$ or $\tilde{J}_{1} \cap \Lambda_{2}$, respectively; |
| $\left(d_{7}\right)$ | (x) $\Delta_{1}$ where $x$ is $v_{t n}$ | $\bigcap_{\xi \in \mathscr{Q} \mathscr{I}_{t}}^{\cap \operatorname{des} \mathscr{M} V^{\prime}\left(\Delta_{1}\right), \text { where } V^{\prime}=V\binom{x}{\xi} ; ~ ; ~ ; ~}$ |
| $\left(d_{8}\right)$ | $\square \Delta_{1}$ | $\Gamma$ if $\tilde{\Delta}_{1}=\Gamma, \emptyset$ otherwise; |
| $\left(d_{\theta}\right)$ | (2x) $\Lambda_{1}$ where $x$ is $v_{t n}$ | the element $\eta$ of $Q I_{t}$ (possibly not in $\mathscr{\mathscr { O }}{ }_{t}$ ) that fulfils conditions ( $\alpha$ ) and ( $\beta$ ) below. |

( $\alpha$ ) If $\gamma \in \operatorname{des}_{\mathscr{S}}\left(\sim\left(\exists_{1} x\right) A_{1}\right)$, then $\eta={ }_{D, \gamma}^{t} a_{t}^{\nu}$.
( $\beta$ ) If $\gamma \in \operatorname{des} \mathscr{P}\left(\left(\exists_{1} x\right) \Delta_{1}\right), \xi \in \mathscr{Q}_{t}$, and $\gamma \in \operatorname{des} \mathscr{M}_{V^{\prime}}\left(\Delta_{1}\right)$ for $V^{\prime}=V\left(\begin{array}{l}x \\ \xi=\eta, \gamma \\ \xi\end{array}\right)$, then $\eta={ }_{D, \gamma}^{t} \xi$.

Remark first that in $\left(d_{7}\right)$ the intersection is considered for $\xi \in \mathscr{Q I}_{t}$ and not for $\xi \in Q I_{t}$. Second, remark that $\operatorname{des}_{\mathscr{S}}(\Delta)$ may fail to be in $\mathscr{Q}_{t}$, which is unsatisfactory; however we are almost exclusively interested in general interpretations-see N. 6-in which, as we shall show, des $\mathscr{S}(\Delta)$ is always in $\mathscr{Q I}_{t}$ and hence the fact that $\operatorname{des}_{\mathscr{S}}(\Delta)$ does not necessarily belong to $\mathscr{Q}_{t}$ constitues no trouble for us.

For $\mathscr{S}$ standard, $\operatorname{des}_{\mathscr{S}}(\Delta)$ is the quasi intensional designatum of $\Delta$ according to NN. 8, 11 in [1]; in particular Theor. 11.1 in [1] holds. It is straightforward to check that this theorem can be extended to every $M L^{\nu}$-system:

THEOR. 4.1. For every choice of $\mathscr{M}$ and $V$, conditions $(\alpha)$ and $(\beta)$ are fulfilled by exactly one $\eta \in Q I_{t}$.

DeF. 4.1. Let $\left\langle\mathscr{M}, \gamma_{R}\right\rangle$ be a contingent $M L^{v}$-interpretation, $p$ a wff, and $K$ a class of wffs. We say that
(a) $p[K]$ is $\gamma$-satisfiable in $\left\langle\mathscr{M}, \gamma_{R}\right\rangle($ or $\mathscr{M})$ if, for some $\mathscr{M}$-valuation $V, \gamma \in \operatorname{des}_{\mathscr{M} V}(p)\left[\gamma \in \bigcap_{q \in \mathbb{K}} \operatorname{des}_{\mathscr{M} V}(q)\right] ;$
(b) $p$ is $\gamma$-true in $\left\langle\mathscr{M}, \gamma_{R}\right\rangle$ (or $\mathscr{M}$ ) if, for every $\mathscr{M}$-valuation $V$, $\gamma \in \operatorname{des}_{\mathscr{M}}(p) ;\left(b^{\prime}\right)\left\langle\mathscr{M}, \gamma_{R}\right\rangle$ (or $\mathscr{M}$ ) is a $\gamma$-model for $K$ if every formula in $K$ is $\gamma$-true in $\left\langle\mathscr{M}, \gamma_{R}\right\rangle$;
(c) $p$ is true in $\left\langle\mathscr{M}, \gamma_{R}\right\rangle[\mathscr{M}]$ if it is $\gamma_{R}$-true $[\gamma$-true for every $\gamma \in \Gamma]$; ( $\left.c^{\prime}\right)\left\langle\mathscr{M}, \gamma_{R}\right\rangle[\mathscr{M}]$ is a model for $K$ if every $p$ in $K$ is true in $\left\langle\mathscr{M}, \gamma_{R}\right\rangle[\mathscr{M}]$.

The following theorem is obviously true.
Theor. 4.2. If (1) $x$ is $v_{t n}, \Delta$ is a wfe, and a is a term of type $t$, (2) $V$ is an $\mathscr{M}$-valutation, and (3) $\xi=\operatorname{des}_{\mathscr{M} V}(a)$ and $V^{\prime}=V\binom{x}{\xi}$, then $\operatorname{des}_{\mathscr{M} V}(\Delta[x / a])=\operatorname{des}_{M_{V}}(\Delta)$.

## 5. An axiom system for the modal calculus $M O^{\nu}$ based on $M L^{\nu}$.

The axiom schemes A5.1-17 below for $M C^{y}$ are written following more [4] than [1]. For them we assume that (1) $p$ and $q$ are wffs, (2) $\Delta$ is a term, and (3) $x, y, z, x_{1}$ to $x_{n}, F, G, f$, and $g$ are distinct
variables.
A5.1-3 AS12.1-3 in $[B]$ that are equivalent to tautologies.
A5.4 $(x)(p \supset q) \supset .(x) p \supset(x) q$.
A5.5
$\square(p \supset q) \supset . \square p \supset \square q$.
A5.6 $\quad p \supset(x) p$, where $x$ is not free in $p$.
A5.7 $p \supset \square p$, where $p$ is modally closed.
A5.8 (x) $p \supset p[x / \Delta]$.
A5.9 $\quad \square p \supset p\left({ }^{3}\right)$.
A5.10, $11 \quad x=x ; x=y \wedge y=z \supset x=z$.
A5.12 $\square x=y \supset \Delta[z / x]=\Delta[z / y]$.
A5.13 $\quad F=G \equiv\left(\forall x_{1}, \ldots, x_{n}\right) \cdot F\left(x_{1}, \ldots, x_{n}\right) \equiv G\left(x_{1}, \ldots, x_{n}\right)$.
A5.14 $\quad f=g \equiv\left(\forall x_{1}, \ldots, x_{n}\right) f\left(x_{1}, \ldots, x_{n}\right)=g\left(x_{1}, \ldots, x_{n}\right)$.
A5.15 $\quad(\exists F)\left(\forall x_{1}, \ldots, x_{n}\right) \cdot p \equiv F\left(x_{1}, \ldots, x_{n}\right)$.
A5.16
( $\exists f)\left(\forall x_{1}, \ldots, x_{n}\right) \Delta=f\left(x_{1}, \ldots, x_{n}\right)$.
A5.17
(a) $\left(\exists_{1} x\right) p \wedge p[x / y] \supset y=(n x) p$;
(b) $\sim\left(\exists_{1} x\right) p \supset(x x) p=a_{t}^{*}$, where $x$ is $v_{t n}$ and $a_{t}^{*}={ }_{D}\left(v_{t n}\right) v_{t n} \neq v_{t n}$.

In addition to A5.1-17, we assume:
(*) if $p$ is an axiom of $M C^{y}$, then $(x) p$ and $\square p$ are axioms of $M C^{p}$, for every variable $x$ of $M L^{\text {. }}$.
E. Omodeo has proved in [1.0] that descriptions can be eliminated from $M C^{\nu}$ according to the Russel method, only provided we replace $M C^{\nu}$ with the calculus $M C^{v}$ which has those among the axioms above that do not contain 1, and has an additional axiom (no such axioms
$\left.{ }^{(3}\right)$ A5.5, A5.7, and A5.9 tell us that the modal calculus $M C^{v}$ is based on the Lewis system S5-see [9]. The semantical counterpart of this features is in the designation rule ( $d_{8}$ ) in N4.
are wanted in extensional logic). The last axiom can be
$(z)(\exists y) \square[(\exists x) p \wedge(x)(p \supset x=y) \cdot \vee \cdot \sim(\exists 1 x) p \wedge y=z]$.
In [1] further axioms are considered for $M C^{\nu}$; there are, for instance, the axiom of choice, an axiom asserting the existence of a contingent attribute, and two conventional axioms which are the syntactical counterpart of the conventional conditions on the nonexisting object. However we prefer to consider the "minimal» version of $M C^{\nu}$ (i.e. that based on A5.1-17) and hence to enclose other versions, like that considered in [1], in the wider concept of $M C^{y_{-}}$ theory.

We say that the theory $\mathscr{T}$ is an $M C^{v}$-theory if $\mathscr{T}$ has the symbols of $M L^{\nu}$ except some (perhaps all) constants, and the axioms of $\mathscr{T}$ are those of $M C^{v}$-to be called logical axioms-and other wffs to be called proper axioms. An $M C^{\nu}$-theory $\mathscr{T}$ is said to be modally closed if such are its proper axioms; otherwise $\mathscr{T}$ is said to be contingent.

The only deduction rule in $M C^{\nu}$ (and $M C^{v}$-theories) is the Modus Ponens. The definitions of wffs deducible from $K$ in $\mathscr{T}\left(K \vdash_{\mathscr{T}}\right)$, and theorems of $\mathscr{T}\left(F_{\mathscr{T}}\right)$ are as usual; furthermore we will omit the subscript $\mathscr{T}$ in ${\digamma_{\mathscr{F}}}$ when no confusion can arise.

It is very easy to realize that, if $s_{1}, \ldots, s_{n}$ is any string of modal quantifiers (that is, $s_{i}$ is $\square$ or $\diamond$ ) and $\mathscr{T}\left[\mathscr{T}^{\prime}\right]$ is an arbitrary [a modally closed] $M C^{\nu}$-theory, then

$$
\begin{align*}
& \stackrel{F}{\mathscr{F}} s_{1}, \ldots, s_{n} p \equiv s_{n} p \quad \text { and } \quad K \vdash_{\mathscr{T}} p \Rightarrow\{(x) q: q \in K\} \digamma_{\mathscr{T}}(x) p ;  \tag{5.1}\\
& K \overleftarrow{F}_{\overline{\mathscr{F}}}, p \Rightarrow\{\square q: q \in K\}_{\overleftarrow{F}^{\prime}} \square p \quad \text { and } \quad \stackrel{\zeta}{\mathscr{T}}^{\prime}(x) \square p \equiv \square(x) p . \tag{5.2}
\end{align*}
$$

Some contingent assertions concerning the real world-such as «at the instant $t$ the earth has angular velocity $w$ "-constitute some postulate of e.g. astronomy. The easiest way of treating such postulates is to give them modally closed forms by use of the calculus $M C_{e}^{v}$-see NN. 52, 53 in [1]-which is also an $M C^{v}$-theory. On the basis of this remark first a completeness theorem will be proved in connection with modally closed theories; then it will be extended to contingent theories, for greater (admittely formal) generality.

A (contingent or not) $M L^{\nu}$-interpretation in which the axioms of the $M 0^{v}$-theory $\mathscr{T}$ are true is said to be a model of $\mathscr{T}$ (briefly, a $\mathscr{T}$-model). It is straightforward to check that the theorems of an $M C^{\boldsymbol{p}}$-theory $\mathscr{T}$ are true in every $\mathscr{T}$-model.

## 6. General $M L^{p}$-interpretations.

Def. 6.1. Assume that (1) $\mathscr{S}=\langle\mathscr{M}, V\rangle$ is an $M L^{v-s y s t e m, ~(2) ~} n \geqslant 0$, $x_{1}$ to $x_{n}$ are distinct variables of the respective types $t_{1}$ to $t_{n}$, and $X=$ $=\left\{x_{1}, \ldots, x_{n}\right\}$, (3) $\Delta \in \mathscr{E}_{t_{0}}$ and either
(4) $t_{0}=0$ and $\xi$ is the set of the $(n+1)$-tuples $\left\langle\xi_{1}, \ldots, \xi_{n}, \gamma\right\rangle$ such that $\xi_{i} \in \mathscr{Q}_{t_{i}}(i=1, \ldots, n), \gamma \in \Gamma$, and $\gamma \in \operatorname{des}_{M_{V^{\prime}}}(\Delta)$ vith $V^{\prime}=$ $V\binom{x_{1}, \ldots, x_{n}}{\xi_{1}, \ldots, \xi_{n}}$, so that $\xi \in Q I_{\left(t_{1}, \ldots, t_{n}\right)}$; or
(4') $t_{0} \in \tau^{\nu}$ and $\xi$ is the element of $Q I_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}$ such that, for every choice of $\xi_{i} \in \mathscr{Q I}_{t_{i}}(i=1, \ldots, n), \xi\left(\xi_{1}, \ldots, \xi_{n}\right)=\operatorname{des}_{M^{\prime}}(\Delta)$ where $V^{\prime}$ $=V\binom{x_{1}, \ldots, x_{n}}{\xi_{1}, \ldots, \xi_{n}}$.

Then $\xi$ is said to be the $Q I$ denoted by $\Delta$ with respect to $X$ and $\mathscr{S}$, and this fact is expressed by $\xi=d(\Delta, X, \mathscr{S})=d(\Delta, X, \mathscr{M}, V)$.

Def. 6.2. The $Q I \xi$ is said to be definable with respect to the $M L^{p_{-}}$ interpretation $\mathscr{M}$ if there is a wfe $\Delta$ of $M L^{v}$, a finite set $X$ of variables, and an $\mathscr{M}$-valuation $V$ for which $\xi=d(\Delta, X, \mathscr{M}, V)$.

Def. 6.3. (a) The $M L^{\nu}$-interpretation $\mathscr{M}$ is said to be general if, for $t \in \bar{\tau}^{v}$, every $Q I$ of type $t$, definable with respect to $\mathscr{M}$, is in $\mathscr{Q}_{i}$;
(b) the $M L^{v}$-interpretation $\mathscr{M}$ is said to be weakly general if it becomes general by adding $\mathscr{Q}_{0}$ with the set $D_{0}=\left\{d(p, \emptyset, \mathscr{M}, V): p \in \mathscr{E}_{0}\right.$ and $V$ is an $\mathscr{M}$-valuation $\}\left({ }^{4}\right)$.

In the sequel we shall say that the $M L^{v}$-system $\mathscr{S}(=\langle\mathscr{M}, V\rangle)$ is general or weakly general if $\mathscr{M}$ is general or weakly general respectively.

Recalling rules $\left(d_{1}\right)$ to ( $d_{9}$ ) in N. 4 , we easily see that $\operatorname{des}_{\mathscr{M} V}(\Delta)$ $=d(\Delta, \emptyset, \mathscr{M}, V)$, and hence, if $\mathscr{M}$ is a general interpretation, then
${ }^{(4)}$ In [7] an interpretation is said to be general if its domain contains the designatum of every expression. This simple definition is equivalent (over the system in [7]) to ours; indeed, in the calculus investigated in [7], the $d\left(\Delta,\left\{x_{1}, \ldots, x_{n}\right\}, \mathscr{M}, V\right)$ is nothing else than $\operatorname{des}_{\mathscr{M} V}\left(\lambda x_{1}, \ldots, x_{n}\right) \Delta$. In [1] the operator $\lambda$ is defined by means of the operator 7 , thus the non-existing object may appear and hence the more elaborated definition 6.3 is needed.
the designatum of every expression is in the domain of $\mathscr{M}$; so that the function des $\mathscr{H}_{V}$ is satisfactory.

Theor. 6.1. (a) AA5.1-14 are true in every $M L^{\nu}$-interpretation; (b) $\mathscr{M}$ is a weakly general $M L^{v}$-interpretation iff $\mathscr{M}$ is an $M C^{v}$-model.

Proof. To prove part (a) is a matter of routine. Furthermore, we see that, if $\mathscr{M}$ is weakly general, then it is an $M C^{v}$-model; indeed the truth of AA5.15-17 follows directly from Defs. 6.1-3 and rule ( $d_{9}$ ) in N. 4. Now, in order to prove the other half of (b), first remark that, as is substantially shown in [1] (Theor. 40.1),

$$
\begin{align*}
& \vdash_{M C^{\nu}}(\exists F)\left(\forall x_{1}, \ldots, x_{n}\right) \square \cdot F\left(x_{1}, \ldots, x_{n}\right) \equiv p, \quad \text { and } \\
& \vdash_{M C^{p}}(\exists f)\left(\forall x_{1}, \ldots, x_{n}\right) \square f\left(x_{1}, \ldots, x_{n}\right)=\Delta\left(^{5}\right) \tag{6.1}
\end{align*}
$$

for every choice of the wff $p$, term $\Delta \in \mathscr{E}_{t_{0}}$, and variables $x_{1}, \ldots, x_{n}$ (of the respectives types $t_{1}, \ldots, t_{n}$ ), $F$, and $f$, with $F[f]$ not free in $p$ [ $\Delta$ ]. In addition $\mathscr{M}$ is an $M C^{v}$-model by an hypothesis; hence for every $p, x_{1}, \ldots, x_{n}$, and $F$ as above, and $\mathscr{M}$-valuation $V$, $\operatorname{des}_{\mathscr{M} V}(q)=\Gamma$, where $q$ is $(\exists F)\left(\forall x_{1}, \ldots, x_{n}\right) \square . F\left(x_{1}, \ldots, x_{n}\right) \equiv p$; but this is equivalent to the existence of a $\xi \in \mathscr{Q}_{\left(t_{1}, \ldots, t_{n}\right)}$ such that, for all $n$-tuples $\left\langle\xi_{1}, \ldots, \xi_{n}\right\rangle$ with $\xi_{i} \in \mathscr{\mathscr { L }} \mathscr{I}_{t_{i}}^{*},\left\langle\xi_{1}, \ldots, \xi_{n}, \gamma\right\rangle \in \xi$ iff $\gamma \in \operatorname{des}_{M^{\prime}}(p)$ with $V^{\prime}=V\binom{x_{1}, \ldots, x_{n}}{\xi_{1}, \ldots, \xi_{n}}$, hence $d\left(p,\left\{x_{1}, \ldots, x_{n}\right\}, \mathscr{M}, V\right)=\xi \in \mathscr{Q}_{\left(t_{1}, \ldots, t_{n}\right)}$. By similar reasonings (using (6.1) $)_{2}$ ) one can easily see that $d\left(\Delta,\left\{x_{1}, \ldots, x_{n}\right\}, \mathscr{M}, V\right) \in \mathscr{Q} \mathscr{F}_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}$ for every term $\Delta \in \mathscr{E}_{t_{0}}$, variables $x_{1}, \ldots, x_{n}$, and $\mathscr{M}$-valuation V. Q.E.D.

The following theorem refers, throught its assumption (2), to countable $Q I$-structures.

Theor. 6.2. Assume that (1) $\mathscr{M}$ is an $M L^{\nu}$-interpretation, (2) $V_{0}$ is an $\mathscr{M}$-valuation such that, for every $\xi \in \mathscr{Q}_{t}\left(t \in \bar{\tau}^{v}\right) \xi=\operatorname{des}_{\mathscr{M}_{V_{0}}}\left(\Delta^{\prime}\right)$ for some wfe $\Delta^{\prime}$, and (3) $\eta=d(\Delta, X, \mathscr{M}, V)$; then there exists a wfe $\Delta_{0}$ such that $\eta=d\left(\Delta_{0}, X, \mathscr{M}, V_{0}\right)$.

Proof. Let us first remark that $d(\Delta, X, \mathscr{M}, V)=d\left(\Delta, X, \mathscr{M}, V^{\prime}\right)$ if $V^{\prime}(x)=V(x)$ for every variable $x$ free in $\Delta$ and not belonging to $X$,
${ }^{(5)}$ Remark that in the proof of (6.1) in [1], A5.17 is effectively needed and hence it is not possible to strengten this half of part (b) by requiring $\mathscr{M}$ to be only a model of the part of $M C^{y}$ based on A5.1-16.
and hence we can suppose $V_{0}(x)$ arbitrary, for $x$ as above, and equal to $V(x)$ otherwise. Let $y_{1}, \ldots, y_{n}$ be the variables not in $X$ and free in $\Delta$, and let $\eta_{i}=V\left(y_{i}\right)(i=1, \ldots, n)$. By hypothesis (2) there exist $n$ wfes $a_{1}, \ldots, a_{n}$ such that $\eta_{i}=\operatorname{des}_{\mathscr{M}_{V_{0}}}\left(a_{i}\right)$. Let $\Delta_{0}=\Delta\left[y_{1} / a_{1}, \ldots, y_{n} / a_{n}\right]$. Then the equality $d(\Delta, X, \mathscr{M}, V)=d\left(\Delta_{0}, X, \mathscr{M}, V_{0}\right)$ is a straightforward consequence of Theor. 4.2.
Q.E.D.

## 7. Statement of the completeness theorem. Saturated sets.

If $p$ is a wff of the $M C^{v}$-theory $\mathscr{T}$, then $p$ is said to be $\mathscr{T}$-valid -briefly $\sqrt{\mathscr{T}} p$-iff $p$ is true in every general $\mathscr{T}$-model.

Now we can write our completeness theorem for modally closed $M C^{v}$-theories, which will be proved in the next sections.

Theor. 7.1. For every wff $p$ of the modally closed $M C^{\nu}$-theory $\mathscr{T}$, $\xi_{\mathscr{Y}} p$ iff $\stackrel{I}{\mathscr{T}} p$.

Let us first recall some standard definitions. A set $K$ of (closed or open) formulas is said to be $\mathscr{T}$-consistent if there is a formula which is not deducible from $K$ in $\mathscr{T}$. Of course, a maximal $\mathscr{T}$-consistent set $K$ contains every formula deducible from it in $\mathscr{T}$ and, in particular, every theorem of $\mathscr{T}$.

A language $\mathscr{L}^{\prime}$ is called an extension of the language $\mathscr{L}$ if it is obtained from $\mathscr{L}$ by adding a (possibly empty) set of new constants for each type (if $\mathscr{L}$ and $\mathscr{L}^{\prime}$ are based on a type system). An $\omega$-extension is an extension in which for each type the set of added constants is denumerable.

DeF. 7.1. Let the theory $\mathscr{T}$ be based on the language $\mathscr{L}$ and let $\mathscr{L}^{\prime}$ be an extension of $\mathscr{L}$. Then the set $H$ of wffs of $\mathscr{L}^{\prime}$ is said to be $\mathscr{T}-\mathscr{L}^{\prime}$ saturated provided conditions (i) and (ii) below hold:
(i) $H$ is maximal $\mathscr{T}^{\prime}$-consistent, where $\mathscr{T}^{\prime}$ is the extension of $\mathscr{T}$ obtained by adding $\mathscr{T}$ with the logical axioms involving all constants of $\mathscr{L}^{\prime}$,
(ii) if $(\exists x) p \in H$, then for some constant a of $\mathscr{L}^{\prime} p[x / a] \in H$.

Remark that, by the maximality of $H$, (ii) is equivalent to
(ii') if $p[x / a] \in H$ for all constants a of $\mathscr{L}^{\prime}$, then $(x) p \in H$.

Lemma 7.1. Assume that (1) $H$ is a $\mathscr{T}$ - $M L^{p}$-saturated set of wffs, (2) $\Gamma_{H}$ is defined by means of

$$
\begin{gather*}
\Gamma_{H}=\left\{\gamma: \gamma \text { is a } \mathscr{T}-M L^{p-s a t u r a t e d ~ s e t ~ o f ~ w f f s ~ a n d ~}\right.  \tag{7.1}\\
\{p: \square p \in H\} \subseteq \gamma\},
\end{gather*}
$$

(3) $\gamma_{1} \in \Gamma_{H}$, and (4) $\mathscr{T}$ is modally closed. Then $\square p \in \gamma_{1}$ iff, for all $\gamma_{2} \in \Gamma_{H}, p \in \gamma_{2}$.

Proof. First we assume $(\alpha) \square p \in \gamma_{1}, p \notin \gamma_{2}\left(\in \Gamma_{H}\right)$. By assumption (1), (7.1), and Def. 7.1, $H, \gamma_{1}$, and $\gamma_{2}$ are maximal $\mathscr{T}$-consistent. Hence $\sim p \in \gamma_{2}$ so that $\square p \notin H$. Then $\sim \square p \in H$, hence $\square \sim \square p \in H$ and $\sim \square p \in \gamma_{1}$ which contrasts to $\square p \in \gamma_{1}$. We conclude that, if $\square p \in \gamma_{1}$, then ( $\beta$ ) $p \in \gamma_{2}$ for all $\gamma_{2} \in \Gamma_{H}$. We now conversely assume $(\beta)$ and $\square p \notin \gamma_{1}$. Let $K_{1}=\{q: \square q \in H\}$. $K_{1}$ satisfies condition (ii') of Def. 7.1; indeed, if $r[x / a] \in K_{1}$ for ever constant $a$ (of the same type of $x$ ), then $\square r[x / a] \in \boldsymbol{H}$ for every constant $a$ and, by the $\mathscr{T}$ - $M L^{\nu}$-saturation of $H,(x) \square r \in H$; but this is equivalent to $\square(x) r \in H$, and hence ( $x$ ) $r \in K_{1}$. The closure $K_{2}$ of $K_{1}$ through ${\digamma_{\mathscr{F}}}$ (i.e. $\left\{q: K_{1} \mathscr{F}_{\mathscr{F}} q\right\}$ ) satisfies condition (ii') of Def. 7.1 too; indeed, if $K_{1} \vdash_{\bar{T}} r[x / a]$ for every constant $a$, then $H \vdash_{\mathscr{F}} \square r[x / a]$ for every $a$ and $\square r[x / a] \in \boldsymbol{H}$ for every $a$, so that $(x) r \in K_{1}$ and $(x) r \in K_{2}$. Now, the closure $K_{3}$ of $K_{1} \cup\{\sim p\}$ through $F_{\mathscr{F}}$ can be shown to satisfy condition (ii') of Def. 7.1. Indeed, $K_{1} \cup\{\sim p\}{\digamma_{\mathscr{G}}} r[x / a]$ for every $a$, implies $K_{1} \vdash_{\mathscr{T}} \sim p \supset r[x / a]$ for every $a$, and $K_{1} \digamma_{\mathscr{F}}(y)(\sim p \supset r[x / y])$ for a suitable variable $y$ not free in $p$; hence, $K_{1} \stackrel{\xi}{\mathscr{T}}^{\sim} p \supset(y) r[x / y]$ and $K_{1} \cup\{\sim p\} \xi_{\mathscr{T}}(y) r[x / y]$, which is equivalent to $K_{1} \cup\{\sim p\} \vdash_{\mathscr{F}}(x) r . \quad K_{3}$ is also $\mathscr{T}$-consistent; otherwise $K_{1} \leftarrow_{\mathscr{F}} p, H \vdash_{\overline{\mathscr{F}}} \square p, \square \square p \in H$, and $\square p \in \gamma_{1}$, which contrasts to an hypothesis. Using the proof of Theor. 3 in [8], a $\mathscr{T}$ - $M L^{\nu}$-saturated extension of $K_{1} \cup\{\sim p\}$ can be constructed. Of course this contradictes the hypothesis $(\beta)$.
Q.E.D.

## 8. The Henkin construction on which the proof of the completeness theorem is based.

Let $\mathscr{T}$ be a modally closed $M C^{\nu}$-theory, and let $K$ be a $\mathscr{T}$-consistent set of formulas given arbitrarily. In this section we construct an $M L^{\nu}$-interpretation $\mathscr{M}_{0}$ in which the set $K$ will be proved to be sat-
isfiable-see N. 9 ; in N. 9 we also prove that $\mathscr{M}_{0}$ is a general $\mathscr{T}$-model, so that the completeness theorem will follow in a standard way.

The first step is to prove that the set $K$ has a $\mathscr{T}$ - $\mathscr{L}$-saturated extension $H$, for some extension $\mathscr{L}$ of $M L^{\nu}$. This can be easily achieved following, for instance, [9] (pp. 160, 161); furthermore, by replacing every constant $c_{t n}$ in $\mathscr{T}$ by $c_{t, 2 n}$, we can identify $\mathscr{L}$ with $M L^{v}\left({ }^{6}\right)$.

Now, in order to construct an interpretation $\mathscr{M}_{0}$, we first identify $D_{r}$ with $\mathscr{E}_{r}$-see N. $3-(r=1, \ldots, v)$ and $\Gamma$ with the set $\Gamma_{H}$ defined in (7.1), i.e. we identify individuals with individual expressions and possible cases with $\mathscr{T}$ - $M L^{v}$-saturated sets of formulas.

For $\gamma \in \Gamma$ we consider the equivalence relation $\approx_{\gamma}$ in $D_{1} \cup \ldots \cup D_{v}$ such that $\Delta_{1} \approx{ }_{\gamma} \Delta_{2}$ iff $\Delta_{1}=\Delta_{2} \in \gamma$. For $\Delta \in \mathscr{E}_{r}$ and $\gamma \in \Gamma$ let $Q_{\Delta}(\gamma)$ be a particular term $\Delta^{\prime}$ in $\mathscr{E}_{r}$ such that $\Delta^{\prime} \approx_{\gamma} \Delta$. Thus we have associated every $\Delta \in \mathscr{E}_{r}$ with a function $Q_{\Delta}$ to be dealt with as the $Q I$ of $\Delta$. We now give $Q_{\Delta}$ a meaning also for an arbitrary non-individual wfe $\Delta$ in $M L^{\nu}$, recursively, by means of the conditions (8.1-3) below.

$$
\begin{equation*}
\text { If } \Delta \in \mathscr{E}_{\left(t_{1}, \ldots, t_{n}: t_{0}\right)}, Q_{\Delta} \text { is the mapping of } \tag{8.2}
\end{equation*}
$$

$$
\begin{equation*}
\text { If } \Delta \in \mathscr{E}_{\left(t_{1}, \ldots, t_{n}\right)}, Q_{\Delta}=\left\{\left\langle Q_{\Delta_{1}}, \ldots, Q_{\Delta_{n}}, \gamma\right\rangle: \Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right) \in \gamma\right\} \tag{8.1}
\end{equation*}
$$

$$
\left\{Q_{A_{1}}: \Delta_{1} \in \mathscr{E}_{t_{1}}\right\} \times \ldots \times\left\{Q_{\Delta_{n}}: \Delta \in \mathscr{E}_{t_{n}}\right\} \text { into }\left\{Q_{\Delta_{0}}: \Delta_{0} \in \mathscr{E}_{t_{0}}\right\}
$$

$$
\text { for which } Q_{\Delta}\left(Q_{\Delta_{1}}, \ldots, Q_{A_{n}}\right)=Q_{\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right)}
$$

$$
\begin{equation*}
\text { If } \Delta \in \mathscr{E}_{0}, \text { then } Q_{\Delta}=\{\gamma: \Delta \in \gamma\} . \tag{8.3}
\end{equation*}
$$

Now the $M L^{\nu}$-interpretation $\mathscr{M}_{0}$ and the $M L^{\nu}$-system $\mathscr{S}_{0}\left(=\left\langle\mathscr{M}_{0}, V_{0}\right\rangle\right)$ can be defined by means of

$$
\begin{align*}
\mathscr{Q} \mathscr{I}_{t}= & \left\{Q_{\Delta}: \Delta \in \mathscr{E}_{t}\right\} \quad\left(t \in \bar{\tau}^{v}\right),  \tag{8.4}\\
& a_{t}^{v}=Q_{a_{i}} \quad\left(t \in \tau^{\nu}\right), \quad M_{0}\left(c_{t n}\right)=Q_{c t n}, \quad V_{0}\left(v_{t n}\right)=Q_{v_{t n}}
\end{align*}
$$

${ }^{(6)}$ In [7] the (correspondent of the) set $H$ is required only to be maximal $\mathscr{T}$-consistent and, in general, saturated sets are not considered. Our departure from [7] is necessary because of the different uses of the operator 1; in [7] (which follows [5]), 1 is a choice operator and, in particular, the formula $(\exists x) A(x) \supset A((2 x) A(x))$ is a valid formula. From the designation rule ( $d_{9}$ ) we see that the above formula may fail to be true if we refer to an $M L^{y}$ interpretation.

Theor. 8.1. For $t \in \tau^{v}$ and $\Delta, \Delta^{\prime} \in \mathscr{E}_{t}$ we have $\Delta=\Delta^{\prime} \in \gamma$ iff $Q_{\Delta}={ }_{\gamma}^{t} Q_{\Delta}$ -see (3.5).

Proof. In the case $t \in\{1, \ldots, \nu\}$ the thesis is trivial.
Case $t=\left(t_{1}, \ldots, t_{n}\right)$. Since $\gamma$ is maximal consistent, by A5.13 $\Delta=$ $=\Delta^{\prime} \in \gamma$ iff $\left[\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right) \equiv \Delta^{\prime}\left(\Delta_{1}, \ldots, \Delta_{n}\right)\right] \in \gamma$ for all $\left\langle\Delta_{1}, \ldots, \Delta_{n}\right\rangle \in$ $\in \mathscr{E}_{t_{1}} \times \ldots \times \mathscr{E}_{t n}$. This holds iff for everyone of these $n$-tuples the conditions $\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right) \in \gamma$ and $\Delta^{\prime}\left(\Delta_{1}, \ldots, \Delta_{n}\right) \in \gamma$ are equivalent, and this holds iff $Q_{\Delta}={ }_{\gamma}^{t} Q_{\Delta^{\prime}}$-cf. (8.1).

Case $t=\left(t_{1}, \ldots, t_{n}: t_{0}\right)$. Let the thesis hold for $t=t_{0}$ as an inductive hypothesis. By A5.14 $\Delta=\Delta^{\prime} \in \gamma$ iff $\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right)=\Delta^{\prime}\left(\Delta_{1}, \ldots, \Delta_{n}\right) \in \gamma$ for every $n$-tuple $\left\langle\Delta_{1}, \ldots, \Delta_{n}\right\rangle \in \mathscr{E}_{t_{1}} \times \ldots \times \mathscr{E}_{i_{n}}$. By the inductive hypothesis this holds iff $Q_{\Delta\left(\Delta_{1}, \ldots, \Delta_{n}\right)}={ }_{\gamma}^{t_{i}} Q_{\Delta^{\prime}\left(\Lambda_{1}, \ldots, \Delta_{n}\right)}$ for all $n$-tuples above. By (8.2) this holds in turn iff $Q_{\Delta}={ }_{\gamma}^{t} Q_{A^{\prime}}$. Q.E.D.

Theors. 8.1 and 3.1 obviously imply the following
Theor. 8.2. For $t \in \tau^{\nu}$ and $\Delta, \Delta^{\prime} \in \mathscr{E}_{t}$, we have $\Delta=\Delta^{\prime} \in \gamma$ for every $\gamma \in \Gamma$ iff $Q_{\Delta}=Q_{\Delta^{\prime}}$.

Corollary 8.1. Every $\xi \in \mathscr{Q}_{t}\left(t \in \tau^{\nu}\right)$ has the form $Q_{c}$ for some constant $c$ of type $t$.

Proof. By (8.4), $\xi$ is $Q_{\Delta}$ for some term $\Delta$; furthermore, $(\exists x) \square$ ( $\Delta=x$ ) is provable in $M C^{\nu}$; hence by the $\mathscr{T}$ - $M L^{v}$-saturation of $H$, there is a constant $c$ such that $\square(\Delta=c) \in H$, that is, by (7.1), $\Delta=c \in \gamma$ for all $\gamma \in \Gamma$. The thesis follows now from Theor. 8.2. Q.E.D.

Theor. 8.3. For every wfe $\Delta$ of $M L^{\nu}, \operatorname{des}_{\mathscr{S}_{0}}(\Delta)=Q_{\Delta}$; and hence, by (8.3),

$$
\begin{equation*}
p \in \gamma \text { iff } \gamma \in \operatorname{des}_{\mathscr{S}_{0}}(p), \text { for every wff } p \tag{8.5}
\end{equation*}
$$

Proof. We use an induction on the number $\nu_{\Delta}$ of occurrences of 1 in $\Delta$, and for every $n(\geqslant 0)$ we treat the wfes $\Delta$ with $v_{\Delta}=n$ by induction on their lengths $l_{\Delta}$. For $l_{\Delta}=1, \Delta$ is $c_{t n}$ or $v_{t n}$, hence the thesis follows by $(8.4)_{4}$.

Case 1: $\Delta$ is the term $\Delta^{\prime}\left(\Delta_{1}, \ldots, \Delta_{n}\right)$. By the inductive hypothesis and (8.2), $\operatorname{des}_{\mathscr{S}_{0}}(\Delta)=\operatorname{des}_{\mathscr{S}_{0}}\left(\Delta^{\prime}\right)\left(\operatorname{des}_{\mathscr{S}_{0}}\left(\Delta_{1}\right), \ldots, \operatorname{des}_{\mathscr{S}_{0}}\left(\Delta_{n}\right)\right)=Q_{\Delta^{\prime}}\left(Q_{\Delta_{1}}, \ldots\right.$, $\left.Q_{\Delta_{n}}\right)=Q_{\Delta}$.

Case $2 a: \Delta$ is the wff $R\left(\Delta_{1}, \ldots, \Delta_{n}\right)$. Then by the designation rule $\left(d_{3}\right)$, the inductive hypothesis, and (8.1)-(8.3), $\gamma \in \operatorname{des}_{\mathscr{S}_{9}}(\Delta) \Leftrightarrow$
$\Leftrightarrow\left\langle\operatorname{des}_{\mathscr{S}_{0}}\left(\Delta_{1}\right), \ldots, \operatorname{des}_{\mathscr{S}_{0}}\left(\Delta_{n}\right), \gamma\right\rangle \in \operatorname{des}_{\mathscr{P}_{0}}(R) \Leftrightarrow\left\langle Q_{\Delta_{1}}, \ldots, Q_{\Delta_{n}}, \gamma\right\rangle \in Q_{R} \Leftrightarrow$ $\Leftrightarrow R\left(\Delta_{1}, \ldots, \Delta_{n}\right) \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case $2 b: \Delta$ is $\Delta_{1}=\Delta_{2}$ with $\Delta_{1}, \Delta_{2} \in \mathscr{E}_{t}$. Then by rule $\left(d_{2}\right)$, the inductive hypothesis, Theor. 8.1, and (8.3), $\gamma \in \operatorname{des}_{\mathscr{S}_{0}}(\Delta) \Leftrightarrow \operatorname{des}_{\mathscr{S}_{0}}\left(\Delta_{1}\right)={ }_{\gamma}^{l}$ $={ }_{\gamma}^{t} \operatorname{des}_{\mathscr{C}_{0}}\left(\Delta_{2}\right) \Leftrightarrow Q_{A_{1}}={ }_{\gamma}^{i} Q_{\Delta_{2}} \Leftrightarrow \Delta_{1}=\Delta_{2} \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case 2c: $\Delta$ is $\sim p$. Then $\operatorname{des}_{\mathscr{S}_{0}}(\Delta)=\Gamma-\operatorname{des}_{\mathscr{P}_{0}}(p)$ by rule $\left(d_{5}\right)$. Furthermore, since every $\gamma \in \Gamma$ is maximal consistent, by (8.3), $Q_{\Delta}=$ $=\Gamma-Q_{p}$. The thesis now follows by the inductive hypothesis.

Case $2 d: \Delta$ is $p \wedge q$. The proof is similar to the above one.
Case $2 e: \Delta$ is $\square p$. Then by rule ( $d_{8}$ ), the inductive hypothesis, Lemma 7.1, and (8.3), $\gamma \in \operatorname{des}_{\mathscr{S}_{0}}(\Delta) \Leftrightarrow \gamma_{1} \in \operatorname{des}_{\mathscr{S}_{0}}(p)$ for all $\gamma_{1}(\in \Gamma) \Leftrightarrow$ $\Leftrightarrow \gamma_{1} \in Q_{p}$ for all $\gamma_{1} \Leftrightarrow p \in \gamma_{1}$ for all $\gamma_{1} \Leftrightarrow \Delta \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case 2f: $\Delta$ is $(x) p$ where $x$ is $v_{t n}$. Then, by rule ( $\left.d_{7}\right), \gamma \in \operatorname{des}_{\mathscr{S}_{0}}(\Delta)$ iff for all $\xi \in \mathscr{Q}_{t}, \gamma \in \operatorname{des}_{\mathscr{M}_{0} V^{\prime}}(p)$ with $V^{\prime}=V\binom{x}{\xi}$. Since, by Corolary 8.1 , every $\xi \in \mathscr{Q}_{t}$ is a $Q_{b}$ (that is $\left.\operatorname{des}_{\mathscr{P}_{0}}(b)\right)$ for some constant $b$, and Theor. 4.2 holds, the last condition holds iff for every constant $b \in \mathscr{E}_{t}, \gamma \in \operatorname{des}_{\mathscr{S}_{0}}(p[x / b])$, which by the inductive hypothesis is equivalent to $\gamma \in Q_{p[x / b]}$ for all constants $b \in \mathscr{E}_{t}$ and hence, by (8.3), to $p[x / b] \in \gamma$ for the same constants. Since $\gamma$ is $\mathscr{T}-M L^{v}$-saturated, this holds iff ( $x$ ) $p \in \gamma$ and hence, by (8.3), iff $\gamma \in Q_{\Delta}$.

We conclude that the thesis holds for $\nu_{\Delta}=0$. Now fix $n>0$ and let the thesis hold for $v_{\Delta}<n$; and assume $v_{\Delta}=n$.

Case 3: $\Delta$ has the form ( $x x$ ) $p$ where $x$ is $v_{t n}$. By the inductive hypothesis the thesis holds for $p$ and every wff $q$ that contains $p$ and has no occurrences of 2 outside $p$. Let $q$ be $\left(\exists_{1} x\right) p$. Remark that by Theor. 4.1, the transitivity of the relation $=_{\gamma}^{t}$, and rule ( $d_{9}$ ), it suffices to prove that $Q_{\Delta}$ is equivalent to $a_{t}^{v}$ in the cases $\gamma_{1} \notin \operatorname{des}_{\mathscr{S}_{0}}(q)$, and to some $Q I \xi$ such that $\gamma_{2} \in \operatorname{des}_{\mathscr{M}_{0} V^{\prime}}(p)$, for $V^{\prime}=V\binom{x}{\xi}$, in the cases $\gamma_{2} \in \operatorname{des}_{\mathscr{S}_{0}}(q)$.

Let $\gamma_{1} \in \operatorname{des}_{\mathscr{S}_{0}}(\sim q)$. By the inductive hypothesis, $\gamma_{1} \in Q_{\sim q}$, i.e. $\sim q \in$ $\in \gamma_{1}$; hence, by $\mathrm{A} 5.17(b),(x) p=a_{t}^{*} \in \gamma_{1}$. Then, by Theor. 8.1,

$$
\begin{equation*}
Q_{\Delta}={ }_{\gamma_{1}}^{t} Q_{a^{*}}\left(=a_{t}^{v}\right) \quad \text { for all } \quad \gamma_{1} \in \operatorname{des}_{\mathscr{S}_{0}}\left(\sim\left(\exists_{1} x\right) p\right) \tag{8.6}
\end{equation*}
$$

Now let $\gamma_{2} \in \operatorname{des}_{\mathscr{S}_{1}}(q)$. By the inductive hypothesis $\gamma_{2} \in Q_{q}$, i.e. $\left(\exists \exists_{1} x\right) p \in \gamma_{2}$. Then by A5.17 (a), (2.1), and (5.1) $)_{2}$,

$$
\begin{equation*}
(y)(p[x / y] \supset y=(x x) p) \in \gamma_{2} \quad \text { and } \quad(\exists x) p \in \gamma_{2} \tag{8.7}
\end{equation*}
$$

Since $\gamma_{2}$ is $\mathscr{T}-M L^{v}$-saturated, by $(8.7)_{2} p[x / a] \in \gamma_{2}$ for some constant $a$; furthermore by (8.7) $(p[x / a] \supset a=(x x) p) \in \gamma_{2}$ : Hence $(a=(x) p) \in \gamma_{2}$ which is equivalent to

$$
\begin{equation*}
Q_{\Delta}={ }_{\gamma_{2}}^{t} Q_{a} \quad \text { for some } a \text { and } \quad \gamma_{2} \in \operatorname{des}_{\mathscr{S}_{0}}\left(\left(\exists_{1} x\right) p\right) . \tag{8.8}
\end{equation*}
$$

By the inductive hypothesis, $\operatorname{des}_{\mathscr{P}_{0}}(p[x / a])=Q_{p[x / a]}$, so that by (8.3), $\gamma_{2} \in \operatorname{des}_{\mathscr{P}_{0}}(p[x / a])$ which by Theor. 4.2 is equivalent to

$$
\begin{equation*}
\gamma_{2} \in \operatorname{des}_{\mathscr{H}_{0} V^{\prime}}(p) \quad \text { for } \quad V^{\prime}=V\binom{x}{\operatorname{des}_{\mathscr{S}_{0}}(a)}=V\binom{x}{Q_{a}} . \tag{8.9}
\end{equation*}
$$

The thesis follows now by (8.6), (8.8), and (8.9). Q.E.D.

## 9. Accomplishment of the proof of the completeness theorems.

Let us return now to our completeness theorem. It remains to prove that $H$ (and hence $K$ ) is $\gamma$-satisfiable-see Def. 4.1-in $\mathscr{M}_{0}$ for some $\gamma \in \Gamma$. Recalling how the set $\Gamma_{H}$ of the elementary possible cases was constructed-i.e. (7.1)-we note that $H$ itself is an element, $\bar{\gamma}$, of $\Gamma_{H}$. Hence, by (8.5) applied to $\bar{\gamma}$, we have

$$
\begin{equation*}
p \in H \quad \text { iff } \quad \bar{\gamma} \in \operatorname{des}_{\boldsymbol{M}_{0} V_{0}}(p), \quad \text { for every wff } p, \tag{9.1}
\end{equation*}
$$

that is, $H$ is $\bar{\gamma}$-satisfiable in $\mathscr{M}_{0}$.
We now prove that $\mathscr{M}_{0}$ is a general $\mathscr{T}$-model. $\mathscr{M}_{0}$ is a $\mathscr{T}$-model; indeed, every $\gamma \in \Gamma_{H}$ (being maximal $\mathscr{T}$-consistent) contains the axioms of $\mathscr{T}$ and their extensional closure, and hence, by (8.5), des $\mathscr{M}_{0}(p)=\Gamma$ for every $\mathscr{M}_{0}$-valuation $V$ and every axiom $p$ of $\mathscr{T}$. Since $\mathscr{T}$ is an $M C^{v-t h e o r y, ~ b y ~ T h e o r . ~ 6.1, ~} \mathscr{M}_{0}$ is weakly general. Furthermore, let $\xi=d\left(p, \emptyset, \mathscr{M}_{0}, V\right)$ for some wff $p$ and $\mathscr{M}_{0}$-valuation $V$. Since, by (8.4) and Theor. 8.3, every $Q I$ (in the domain of $\mathscr{M}_{0}$ ) has the form $\operatorname{des}_{\mathscr{S}_{0}}(\Delta)$ for some $\Delta$, we may use Theor. 6.2 to derive $\xi=d\left(p^{\prime}, \emptyset\right.$, $\left.\mathscr{M}_{0}, V_{0}\right)$, for some wff $p^{\prime}$. But $d\left(p^{\prime}, \emptyset, \mathscr{M}_{0}, V_{0}\right)=\operatorname{des}_{\mathscr{L}_{0}}\left(p^{\prime}\right)$. Therefore by Theor. $8.3, \xi=Q_{p^{\prime}}$ and $\xi \in \mathscr{Q}_{0}$; that is, $\mathscr{M}_{0}$ is general. Thus Theor. 7.1 has been proved.

We can now briefly show how a completeness theorem for contingent $M C^{\nu}$-theories can be proved.

The proof of such a theorem can be wholly analogue to the one relative to modally closed $M C^{v}$-theories; that is, it consists in the construction (from a given contingent $M C^{\nu}$-theory $\mathscr{T}_{o}$ ) of a contingent $\mathscr{T}_{c}$-model $\left\langle\mathscr{M}, \gamma_{R}\right\rangle$ in which a given $\mathscr{T}_{o}$-consistent set $K_{0}$ of formulas is $\gamma_{R}$-satisfiable. However, in this case, we cannot use every result relative to modally closed theories; for instance, (5.2) does not hold when $\mathscr{T}$ is (properly) contingent and hence the proof of Lemma 7.1-in which (5.2) is applied-fails to be valid.

In any case, it is not necessary to repeat the whole proof of the completeness theorem; indeed, by an easy device, we can use the preceding proof for our present goals.

Let $\mathscr{T}_{\sigma}$ be a contingent $M C^{\nu}$-theory and let $K_{0}$ be a $\mathscr{T}_{0}$-consistent set of formulas. Let us denote by $C$ the set of contingent proper axioms of $\mathscr{T}_{\sigma}$ (that is, the proper axioms of $\mathscr{T}_{c}$ that are not modally closed) and by $\mathscr{T}$ the modally closed part of $\mathscr{T}_{\sigma}$ (that is the $M^{V_{-}}$ theory obtained from $\mathscr{T}_{c}$ by subtracting $C$ from the set of its axioms). Furthermore, let

$$
K=K_{0} \cup\{(. .) p: p \in C\}, \quad \begin{align*}
& \text { where }(. .) p \text { denotes the }  \tag{9.2}\\
& \\
& \text { extensional closure of } p
\end{align*}
$$

Obviously, $K$ is $\mathscr{T}_{\sigma}$-consistent.
We may now build up a $\mathscr{T}-M L^{\nu}$-saturated extension $H$ of $K$ and the general $M L^{\nu}$-system $\mathscr{S}_{0}$ just as in N. 8. Of course, Theor. 8.3 holds and, in particular, (9.1) holds too.

If we consider the contingent $M L^{p}$-interpretation $\mathscr{M}_{0}=\left\langle\mathscr{M}_{0}, \gamma_{R}\right\rangle$ where $\gamma_{R}$ is $H$, then, by (9.1), $H$ is $\gamma_{R}$-satisfiable in $\mathscr{M}_{g}$.

It remains to prove that $\mathscr{M}_{c}$ is a $\mathscr{T}_{c}$-model. We already know that $\mathscr{M}_{\sigma}$ is a $\mathscr{T}$-model, then let $p$ be a contingent axiom of $\mathscr{T}_{c}$; by (9.2) the extensional closure $p^{\prime}$ of $p$ belongs to $H$ and hence, by (9.1), $\gamma_{R} \in \operatorname{des}_{\mathscr{M}_{0} V_{0}}\left(p^{\prime}\right)$ and $\gamma_{R} \in \operatorname{des}_{\mathscr{M}_{0} V}(p)$ for all $\mathscr{M}_{0}$-valuation $V$. That is, $\mathscr{M}_{o}$ is a general $\mathscr{T}_{o}$-model.

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