A complex hyperbolic Riley slice

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Abstract

We study subgroups of $\mathrm{PU}(2,1)$ generated by two unipotent maps A and B whose product AB is also unipotent. We show that the set of conjugacy classes of such groups is two dimensional and we provide coordinates on it. By considering their action on complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, we describe a two dimensional open disc of discrete groups of this type. In particular, our results give a proof of a conjecture of Schwartz for $(3,3,\infty)$-triangle groups. As a special case, we also consider a particular group on the boundary of this disc where the commutator $[A,B]$ is also unipotent. We show that the boundary of the quotient orbifold associated to the latter group is a spherical CR uniformisation of the Whitehead link complement.

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1 Introduction

The framework of this article is the study of the deformations of a discrete subgroup Γ of a Lie group H in a Lie group G containing H. This question has been addressed in many different contexts. A classical example is the one where Γ is a Fuchsian group, $H = \mathrm{PSL}(2,\mathbb{R})$ and $G = \mathrm{PSL}(2,\mathbb{C})$. When Γ is discrete, such groups are called quasi-Fuchsian. We will be interested in the case where Γ is a discrete subgroup of $H = \mathrm{SO}(2,1)$ and $G = \mathrm{SU}(2,1)$ (or their natural projectivisations over $\mathbb{R}$ and $\mathbb{C}$ respectively). The geometrical motivation is very similar: In the classical case, $\mathrm{PSL}(2,\mathbb{C})$ is the orientation preserving isometry group of hyperbolic 3-space $\mathbf{H}^3$ and a Fuchsian group preserves a totally geodesic hyperbolic plane $\mathbf{H}^2$ in $\mathbf{H}^3$. In our case $\mathrm{SU}(2,1)$ is (a triple cover of) the holomorphic isometry group of complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ and subgroups of $\mathrm{SO}(2,1)$ preserve a totally geodesic Lagrangian plane isometric to $\mathbf{H}^2$. Such a discrete subgroup of $\mathrm{SO}(2,1)$ is called $\mathbb{R}$-Fuchsian.

A motivating example is the case where Γ is an $\mathbb{R}$-Fuchsian representation ρ_0 of $\pi_1(\Sigma)$, the fundamental group of Σ, a closed Riemann surface without boundary and genus $g \geq 2$. Then this point ρ_0 in the representation variety admits a neighbourhood of maximal dimension containing only discrete and faithful representations. See [19, 23, 27].

We note that there is a second way to embed the hyperbolic plane totally geodesically in $\mathbf{H}^2_{\mathbb{C}}$, namely as the intersection of $\mathbf{H}^2_{\mathbb{C}}$ with a complex line. A discrete subgroup preserving such a complex line is called $\mathbb{C}$-Fuchsian. In contrast to the $\mathbb{R}$-Fuchsian case, if Σ is closed surface as above, then all nearby deformations of a $\mathbb{C}$-Fuchsian representation of $\pi_1(\Sigma)$ also preserve a complex line. This is a special case of the rigidity theorems due to Toledo [36] and Goldman and Millson [16] (see also [14], for an exposition in the context of $\mathrm{PU}(2,1)$). This contrasting behaviour is a consequence of the fact that $\mathbb{C}$-Fuchsian representations are reducible, whereas $\mathbb{R}$-Fuchsian ones are not.

For surfaces with punctures then $\mathbb{C}$-Fuchsian as well as $\mathbb{R}$-Fuchsian representations can be deformed in a non-trivial way. Several explicit constructions of such deformations have been given by various authors (see for instance [10, 12, 20, 21, 37, 39]). The quotient of $\mathbf{H}^2_{\mathbb{C}}$ by an $\mathbb{R}$ or $\mathbb{C}$-Fuchsian punctured surface group is a disc bundle over the surface. As the surface is non-compact, this bundle is trivial. Its boundary is a circle bundle over the surface. Things become interesting if we deform the group in such a way that a loop on the surface is represented by a parabolic map: the topology of the manifold at infinity can change. A hyperbolic manifold arising in this way was first constructed by Schwartz (see Theorem 1.1 below). Such three-manifolds appearing on the boundary of quotient of $\mathbf{H}^2_{\mathbb{C}}$ are naturally equipped with a *spherical CR structure*, which is the analogue of the flat conformal structure in the real hyperbolic case. These structures are examples of (X, G)-structure, where $X = S^3 = \partial\mathbf{H}^2_{\mathbb{C}}$ and $G = \mathrm{PU}(2,1)$. To any such structure on a three manifold M are associated a holonomy representation $\rho : \pi_1(M) \longrightarrow \mathrm{PU}(2,1)$ and a developing map $D = \widetilde{M} \longrightarrow X$. This motivates the study of representations of fundamental groups of hyperbolic three manifolds in $\mathrm{PU}(2,1)$ and PGL(3,$\mathbb{C}$) initiated by Falbel in [8], and continued in [11, 9] (see also [22]). Among PU(2,1)-representations, *uniformisations* (see Definition 1.3 in [5]) are of special interest. There, the manifold at infinity is the quotient of the discontinuity region by the group action. Schwartz's example provides a uniformisation of the Whitehead link complement, and Deraux and Falbel described a uniformisation of the complement of the figure eight knot in [6]. In [4], Deraux proved that this uniformisation was flexible.

The main goals of this paper are first to investigate the question of discreteness in the $\mathrm{PU}(2,1)$-representation variety of the fundamental group of the 3-punctured sphere where the holonomy around each boundary component is unipotent, and secondly to produce a spherical CR uniformisation of the complement of the Whitehead which is non conjugate to Schwartz's example.

Complex hyperbolic ideal triangle groups are an important motivation for this work. These groups were first studied by Goldman and Parker in [17] and are generated by three complex reflections (I_1, I_2, I_3) fixing pairwise asymptotic, distinct complex lines. They are natural $\mathrm{PU}(2,1)$-deformations of the classical ideal triangle group in the hyperbolic plane generated by three geodesic reflections fixing distinct, pairwise asymptotic geodesics. In a complex hyperbolic ideal triangle group, the maps $A = I_2I_1$, $B = I_1I_3$ and $AB = I_2I_3$ are unipotent parabolic. Furthermore, there are regular elliptic maps S and T (see Section 2.2) each of order three that cyclically permutes the fixed points of A, AB and B and the fixed points of A, B and BA respectively. It is not hard to show that $A = ST$ and $B = TS$; see Proposition 3.2. Therefore, the group $\langle A, B\rangle$ has index 2 in the ideal triangle group $\langle I_1, I_2, I_3\rangle$ and index 3 in the group $\langle S, T\rangle$ generated by two elements of order 3.

Complex hyperbolic ideal triangle groups are parametrised up to conjugation by a single real valued invariant, the *Cartan angular invariant*, denoted by $\mathbb{A}$, of the fixed points of A, B and AB; see Section 2.6 for precise definitions about the Cartan invariant. In particular $\mathbb{A} \in [-\pi/2, \pi/2]$, it vanishes if and only if the three points lie in a real plane, and is equal to $\pm\pi/2$ if and only if the three points lie in a complex line. Since we assume the three complex lines are distinct then $\mathbb{A} \in (-\pi/2, \pi/2)$. The main results we are interested in about complex hyperbolic triangle groups are summed up in the following theorem, saying that discreteness is controlled by a single element $(I_2I_3I_1)^2 = [A, B] = (ST^{-1})^3$.

Theorem 1.1 (Goldman, Parker [17], Schwartz [30, 31, 33])**.** *Let I_1, I_2, I_3 be complex involutions fixing distinct, pairwise asymptotic complex lines. Let $\mathbb{A}$ be the Cartan invariant of the fixed points of I_1I_2, I_2I_3 and I_3I_1 and let S be the regular elliptic map cyclically permuting these points.*

1. *The group $\langle I_1, I_2, I_3\rangle$ is a discrete and faithful representation of an (∞, ∞, ∞)-triangle group if and only if $I_1I_2I_3$ is non elliptic. This happens when* $|\mathbb{A}| \leq \arccos\sqrt{3/128}$.

2. *When $I_1I_2I_3$ is elliptic the group is not discrete. In this case* $\arccos\sqrt{3/128} < |\mathbb{A}| < \pi/2$.

3. *When $I_1I_2I_3$ is parabolic the quotient of $\mathbf{H}^2_{\mathbb{C}}$ by the group $\langle I_1I_2, S\rangle$ is a complex hyperbolic orbifold whose boundary is a spherical CR uniformisation of the Whitehead link complement. These groups have Cartan invariant $\mathbb{A} = \pm \arccos\sqrt{3/128}$,*

The question that initially motivated this work is to understand to what extend such behaviour can be described in other cases. For example, Will gave another family of discrete and faithful representations of the fundamental group of the 3-punctured sphere as a special case of [39]. There, he described a 1-parameter discrete and faithful $\mathrm{PU}(2,1)$-deformation of (the holonomy of) any finite volume hyperbolic structure on the surface $\Sigma_{g,p}$, where g is the genus and p the number of cusps. Roughly, his idea is to start from an $\mathbb{R}$-Fuchsian group and to bend along the edges of an ideal triangulation while keeping each ideal triangle in a Lagrangian plane. This construction provides embeddings of the Teichmüller space $\mathcal{T}_{g,p}$ into the $\mathrm{PU}(2,1)$-representation variety of $\pi_1(\Sigma_{g,p})$, of which images contain only discrete, faithful and parabolicity-preserving representations. Moreover the images of the boundary parabolic loops by the representations all are unipotent. In the case where $g = 0$ and $p = 3$, there is only one such hyperbolic structure, and we obtain a 1-parameter family of 3-punctured sphere groups. These groups are generated by two unipotent maps A and B such that AB is also unipotent (these three group elements correspond to peripheral curves of the 3-punctured sphere).

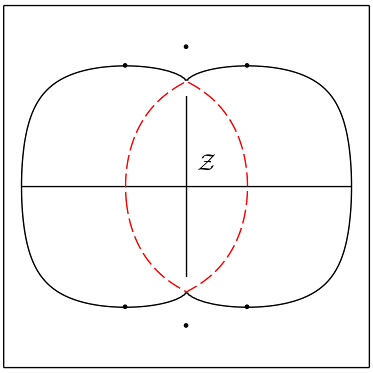

Figure 1: The parameter space. The exterior curve corresponds to classes of groups for which $[A, B]$ is parabolic. The portions of the horizontal and vertical axes correspond respectively to ideal triangle groups and bending groups, that are known to be discrete from previous work [17, 30, 31, 33, 39]. The isolated marked points correspond to groups studied by Deraux–Falbel and Falbel–Parker [6, 13]. The central dashed curve bounds the region $\mathcal{Z}$ where we prove discreteness.

Our focus will be on subgroups of $\mathrm{PU}(2,1)$ generated by two unipotent maps A and B whose product AB is also unipotent. Such a group always lies as an index 3 subgroup of $\langle S, T\rangle$ where S and T are regular elliptic elements of order three satisfying $ST = A$ and $TS = B$; see Proposition 3.2. We remark in Proposition 3.3 that the fundamental group of the Whitehead link complement always surjects onto this group $\langle S, T\rangle$. We show that, up to conjugation in $\mathrm{PU}(2,1)$, such groups are parametrised by the Cartan angular invariants of the fixed points of (A, AB, B) and the fixed points of (A, AB, BA), denoted by $(\alpha_1, \alpha_2) \in (-\pi/2, \pi/2)^2$; see Theorem 3.1. Note that Γ is $\mathbb{R}$-Fuchsian if and only if $(\alpha_1, \alpha_2) = (0, 0)$. Here is our main theorem; see Theorem 5.2 below:

Theorem 1.2. *Suppose that $\Gamma = \langle A, B\rangle$ is the group associated to parameters (α_1, α_2) satisfying $\mathcal{D}\big(4\cos^2(\alpha_1), 4\cos^2(\alpha_2)\big) > 0$, where $\mathcal{D}$ is the polynomial given by*

$$\mathcal{D}(x, y) = x^3y^3 - 9x^2y^2 - 27xy^2 + 81xy - 27x - 27.$$

Then Γ is discrete and isomorphic to the free group F_2. This region is $\mathcal{Z}$ in Figure 1.

The rough idea of the proof is to construct fundamental domains for these groups associated to their Ford domains. We thus have to consider a 2-parameters family of such polyhedra, and the polynomial $\mathcal{D}$ controls the combinatorial complexity within this family. In particular, Theorem 1.2 gives an explicit neighbourhood of the $\mathbb{R}$-Fuchsian group consisting of discrete and free representations. A consequence of Theorem 1.2 is that we can completely describe the groups with $\alpha_1 = 0$. These are Will's bending groups from [39] described above. It turns out, that they are also subgroups of $(3, 3, \infty)$-triangle groups. That is, there are three involutions I_1, I_2, I_3, each fixing a complex line, so that $S = I_2I_1$ and $T = I_1I_3$ have order 3 and $ST = A = I_2I_3$ is unipotent; see Proposition 3.6. Furthermore, writing $B = TS = I_1I_3I_2I_1$ we have $[A, B] = (ST^{-1})^3 = (I_2I_1I_3I_1)^3$. Our second main result is a statement analogous to Theorem 1.1 for $(3, 3, \infty)$-triangle groups, proving a special case of Conjecture 5.1 of Schwartz [32]. Compare with the proof of this conjecture for $(3, 3, n)$-triangle groups given by Parker, Wang and Xie in [28].

Theorem 1.3. *Let I_1, I_2 and I_3 be complex involutions fixing distinct complex lines and so that $S = I_2I_1$ and $T = I_1I_3$ have order three and $A = ST = I_2I_3$ is unipotent. Let $\mathbb{A}$ be the Cartan invariant of the fixed points of A, SAS^{-1} and $S^{-1}AS$.*

1. *The group $\langle I_1, I_2, I_3\rangle$ is a discrete and faithful representation of the $(3, 3, \infty)$-triangle group if and only if $I_2I_1I_3I_1$ is non-elliptic. This happens when $|\mathbb{A}| \le \arccos\sqrt{3/8}$.*

2. *When $I_2I_1I_3I_1$ is parabolic the quotient of $\mathbf{H}^2_{\mathbb{C}}$ by $\langle A, S\rangle$ is a complex hyperbolic orbifold whose boundary is a spherical CR uniformisation of the Whitehead link complement. These groups have Cartan invariant $\mathbb{A} = \pm\arccos\sqrt{3/8}$.*

The part of Theorem 1.3 where $[A, B] = (I_2I_1I_3I_1)^3$ is loxodromic follows by restricting Theorem 1.2 to the case where $\alpha_1 = 0$ (and writing $\alpha_2 = \mathbb{A}$), and also using Proposition 3.6 for the decomposition. In [39] Will used bending to prove these groups are discrete as long as $|\mathbb{A}| = |\alpha_2| \le \pi/4$. The gap between the vertical segment in Figure 1 and the curve where $[A, B]$ is parabolic illustrates the non-optimality of the result of [39].

The groups where $[A, B] = (I_2I_1I_3I_1)^3$ is parabolic are the focus of Section 6. The second part of Theorem 1.3 will be proved in Theorem 6.4. These groups were identified by Falbel, Koseleff and Rouillier in their census of $\mathrm{PGL}(3, \mathbb{C})$ representations of knot and link complement groups, see page 254 of [11].

As well as the ideal triangle groups and bending groups discussed above, there are some other previously studied discrete groups in this family. We give them in (α_1, α_2) coordinates and illustrate them in Figure 1.

1. The groups corresponding to $\alpha_1 = 0$ and $\alpha_2 = \pm\arctan\sqrt{7}$ have been studied in great detail by Deraux and Falbel who proved that they give a spherical CR uniformisation of the figure eight knot complement [6]. These groups mean that there is no statement for Theorem 1.3 analogous to the second part of Theorem 1.1: the group from [6] is contained in a discrete (non-faithful) $(3, 3, \infty)$ triangle groups where $I_2I_1I_3I_1$ is elliptic.

2. The groups where $\alpha_1 = \pm\pi/6$ and $\alpha_2 = \pm\pi/3$ are discrete, since they are subgroups of the Eisenstein-Picard lattice $\mathrm{PU}(2, 1; \mathbb{Z}[\omega])$, where ω is a cube root of unity. This lattice has been studied by Falbel and Parker in [13].

Finally, we explain the title and how work of Riley motivated our method of proof. Riley considered subgroups of $\mathrm{PSL}(2, \mathbb{C})$ generated by two parabolic maps A and B. There is a two (real) parameter family of such groups and he wanted to determine the parameters for which $\langle A, B\rangle$ is discrete and free. This parameter space has since been called *the Riley slice of Schottky space*. Riley also identified hyperbolic uniformisations of the figure eight knot complement as certain points in this space where the group is not free; our space also contains CR uniformisations of this group; see Figure 1. One of Riley's methods was to construct Ford domains (based at the parabolic fixed points of A). He wrote computer programmes to determine which parameters correspond to groups where the Ford domain has the same combinatorics. This work has been taken up more recently by Akiyoshi, Sakuma, Wada and Yamashita. In their book [1] they illustrate one of Riley's original computer pictures[1], Figure 0.2a, and their version of this picture, Figure 0.2b. The groups we consider in this paper correspond to the outermost region in these figures where the combinatorial structure is the simplest. This is the region $\mathcal{Z}$ defined by the polynomial $\mathcal{D}$ in Theorem 1.2. We believe that it should be possible to mimic Riley's approach and to construct regions in our parameter space where the Ford domain is more complicated. However, as with Riley's work, this may only be possible via computer experiments.

There is, conjecturally, one extremely significant difference between the classical Riley slice and our complex hyperbolic version. The boundary of the Riley slice is not a smooth curve and has a dense set of points where particular group elements are parabolic. (This is an extremely rich and beautiful topic and we refer the reader to [1] and the references it contains.) On the other hand, we believe that discreteness is completely controlled by the commutator $[A,B]$, or equivalently ST^{-1}. If this is true, then the boundary of our set is piecewise smooth; it is given by the simple closed curve in Figure 1.

This article is organised as follows. In Section 2 we gather together the necessary background facts on complex hyperbolic space and its isometries. In Section 3, we describe coordinates on the space of (conjugacy classes) of group generated by two unipotent isometries with unipotent product. Section 4 is devoted to the description of the isometric spheres that bound our fundamental domains. We state and apply the Poincaré polyhedron theorem in Section 5. In Section 6, we focus on the specific case where the commutator becomes parabolic, and prove that the corresponding manifold at infinity is homeomorphic to the complement of the Whitehead link. In Section 7, we give the the technical proofs which we have omitted for readability in the earlier sections.

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2 Preliminary material

Throughout we will work in the complex hyperbolic plane using a projective model and will therefore pass from projective objects to lifts of them. Our convention is that the same letter will be used to denote a point in $\mathbb{C}P^2$ and a lift of it to $\mathbb{C}^3$ with a bold font for the lift. As an example, each time p is a point of $\mathbf{H}^2_{\mathbb{C}}$, $\mathbf{p}$ will be a lift of p to $\mathbb{C}^3$.

2.1 The complex hyperbolic plane

Let H be the following matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The Hermitian product on $\mathbb{C}^3$ associated to H is given by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* H \mathbf{x}$. The corresponding Hermitian form has signature $(2,1)$, and we denote by V_- (respectively V_0 and V_+) the associated negative

[1]JRP has one of Riley's printouts of this picture dated 26th March 1979

(respectively null and positive) cones in $\mathbb{C}^3$.

Definition 1. The *complex hyperbolic plane* $\mathbf{H}^2_{\mathbb{C}}$ is the image of V_- in $\mathbb{C}P^2$ by projectivisation and its boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is the image of V_0 in $\mathbb{C}P^2$. The complex hyperbolic plane is endowed with the *Bergman metric*

$$ds^2 = \frac{-4}{\langle \mathbf{z},\mathbf{z}\rangle^2}\det\begin{pmatrix} \langle \mathbf{z},\mathbf{z}\rangle & \langle d\mathbf{z},\mathbf{z}\rangle \\ \langle \mathbf{z},d\mathbf{z}\rangle & \langle d\mathbf{z},d\mathbf{z}\rangle \end{pmatrix}.$$

The Bergman metric is equivalent to the *Bergman distance function* ρ defined by

$$\cosh^2\left(\frac{\rho(m,n)}{2}\right) = \frac{\langle \mathbf{m},\mathbf{n}\rangle\langle \mathbf{n},\mathbf{m}\rangle}{\langle \mathbf{m},\mathbf{m}\rangle\langle \mathbf{n},\mathbf{n}\rangle},$$

where $\mathbf{m}$ and $\mathbf{n}$ are lifts of m and n to $\mathbb{C}^3$.

Let $\mathbf{z} = [z_1,\, z_2,\, z_3]^T$ be a (column) vector in $\mathbb{C}^3 - \{\mathbf{0}\}$. Then $\mathbf{z} \in V_-$ (respectively V_0) if and only if $2\mathrm{Re}(z_1\overline{z}_3) + |z_2|^2 < 0$ (respectively $= 0$). Vectors in V_0 with $z_3 = 0$ must have $z_2 = 0$ as well. Such a vector is unique up to scalar multiplication. We call such its projectivisation the *point at infinity* $q_\infty \in \partial\mathbf{H}^2_{\mathbb{C}}$. If $z_3 \neq 0$ then we can use inhomogeneous coordinates with $z_3 = 1$. Writing $\langle \mathbf{z},\mathbf{z}\rangle = -2u$ we give $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$ *horospherical coordinates* $(z,t,u) \in \mathbb{C}\times\mathbb{R}\times\mathbb{R}_{\geq 0}$ defined as follows. A point $q \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ with horospherical coordinates (z,t,u) is represented by the following vector, which we call its *standard lift*.

$$\mathbf{q} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix} \text{ if } q \neq q_\infty, \quad \mathbf{q}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ if } q = q_\infty. \tag{1}$$

Points of $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$ have $u = 0$ and we will abbreviate $(z,t,0)$ to $[z,t]$.

Horospherical coordinates give a model of complex hyperbolic space analogous to the upper half plane model of the hyperbolic plane. The *Cygan metric* d_{Cyg} on $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$ plays the role of the Euclidean metric on the upper half plane. It is defined by the distance function:

$$d_{\mathrm{Cyg}}(p,q) = \big|\langle \mathbf{p},\mathbf{q}\rangle\big|^{1/2} = \Big||z-w|^2 + i\big(t - s + \mathrm{Im}(z\overline{w})\big)\Big|^{1/2} \tag{2}$$

where p and q have horospherical coordinates $[z,t]$ and $[w,s]$. We may extend this metric to points p and q in $\mathbf{H}^2_{\mathbb{C}}$ with horospherical coordinates (z,t,u) and (w,s,v) by writing

$$d_{\mathrm{Cyg}}(p,q) = \Big||z-w|^2 + |u-v| + i\big(t - s + \mathrm{Im}(z\overline{w})\big)\Big|^{1/2}$$

If (at least) one of p and q lies in $\partial\mathbf{H}^2_{\mathbb{C}}$ then we still have the formula $d_{\mathrm{Cyg}}(p,q) = \big|\langle \mathbf{p},\mathbf{q}\rangle\big|^{1/2}$.

2.2 Isometries

Since the Bergman metric and distance function are both given solely in terms of the Hermitian form, any unitary matrix preserving this form is an isometry. Similarly, complex conjugation of points in $\mathbb{C}^3$ leaves both the metric and the distance function unchanged. Hence, complex conjugation is also an isometry.

Define $\mathrm{U}(2,1)$ to be the group of unitary matrices preserving the Hermitian form and $\mathrm{PU}(2,1)$ to be the projective unitary group obtained by identifying non-zero scalar multiples of matrices in $\mathrm{U}(2,1)$. We also consider $\mathrm{SU}(2,1)$ the subgroup of matrices in $\mathrm{U}(2,1)$ with determinant 1.

Proposition 2.1. *Every Bergman isometry of* $\mathbf{H}^2_{\mathbb{C}}$ *is either holomorphic or anti-holomorphic. The group of holomorphic isometries is* $\mathrm{PU}(2,1)$. *Every antiholomorphic isometry is complex conjugation followed by an element of* $\mathrm{PU}(2,1)$.

Elements of $\mathrm{SU}(2,1)$ fall into three types, according to the number and type of the fixed points of the corresponding isometry. Namely, an isometry is *loxodromic* (respectively *parabolic*) if it has exactly two fixed points (respectively exactly one fixed point) on $\partial\mathbf{H}^2_{\mathbb{C}}$. It is called *elliptic* when it has (at least) one fixed point inside $\mathbf{H}^2_{\mathbb{C}}$. An elliptic element $A \in \mathrm{SU}(2,1)$ is called *regular elliptic* whenever it has three distinct eigenvalues, and *special elliptic* if it has a repeated eigenvalue. The following criterion distinguishes the different isometry types.

Proposition 2.2 (Theorem 6.2.4 of Goldman [15])**.** *Let* $\mathcal{F}$ *be the polynomial given by* $\mathcal{F}(z) = |z|^4 - 8\mathrm{Re}\,(z^3) + 18|z|^2 - 27$, *and* A *be a non identity matrix in* $\mathrm{SU}(2,1)$. *Then*

1. *A is loxodromic if and only if $\mathcal{F}(\mathrm{tr}A) > 0$,*
2. *A is regular elliptic if and only if $\mathcal{F}(\mathrm{tr}A) < 0$,*
3. *if $\mathcal{F}(\mathrm{tr}A) = 0$, then A is either parabolic or special elliptic.*

We will be especially interested in elements of $\mathrm{SU}(2,1)$ with trace 0 and those with trace 3.

Lemma 2.3 (Section 7.1.3 of Goldman [15])**.**

1. *A matrix A in $\mathrm{SU}(2,1)$ is regular elliptic of order three if and only if its trace is equal to zero.*
2. *Let (p, q, r) be three pairwise distinct points in $\partial\mathbf{H}^2_{\mathbb{C}}$, not contained in a common complex line. Then there exists a unique order three regular elliptic isometry E so that $E(p) = q$ and $E(q) = r$.*

Suppose that $T \in \mathrm{SU}(2,1)$ has trace equal to 3. Then all T eigenvalues of T equal 1, that is T is *unipotent*. If T is diagonalisable then it must be the identity; if it is non-diagonalisable then it must fix a point of $\partial\mathbf{H}^2_{\mathbb{C}}$. Conjugating within $\mathrm{SU}(2,1)$ if necessary, we may assume that T fixes q_∞. This implies that T is upper triangular with each diagonal element equal to 1.

Lemma 2.4 (Section 4.2 of Goldman [15])**.** *Suppose that* $[w,s] \in \partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$. *Then there is a unique* $T_{[w,s]} \in \mathrm{SU}(2,1)$ *taking the point* $[0,0] \in \partial\mathbf{H}^2_{\mathbb{C}}$ *to* $[w,s]$. *As a matrix this map is:*

$$T_{[w,s]} = \begin{bmatrix} 1 & -\overline{w}\sqrt{2} & -|w|^2 + is \\ 0 & 1 & w\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Moreover, composition of such elements gives $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$ *the structure of the Heisenberg group*

$$[w,s] \cdot [z,t] = \big[w + z, s + t - 2\mathrm{Im}(z\overline{w})\big]$$

and $T_{[w,s]}$ *acts as left Heisenberg translation on* $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$.

The action of $T_{[w,s]}$ on horospherical coordinates is:

$$T_{[w,s]} : (z,t,u) \longmapsto \big(w + z, s + t - 2\mathrm{Im}(z\overline{w}), u\big).$$

An important observation is that this is an affine map, namely a translation and shear.

We can restate Lemma 2.4 in an invariant way. This result is actually true for any parabolic conjugacy class, as a special case of Proposition 3.1 in [25].

Corollary 2.5. *Let* p_1, p_2 *and* p_3 *be any three points in* $\partial\mathbf{H}^2_{\mathbb{C}}$. *Then there is a unique unipotent element of* $\mathrm{PU}(2,1)$ *fixing* p_1 *and taking* p_2 *to* p_3.

Proof. We can choose $A \in \mathrm{SU}(2,1)$ taking p_1 to q_∞ and p_2 to $[0,0]$. The result then follows from Lemma 2.4. ∎

2.3 Totally geodesic subspaces.

Maximal totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$ have real dimension 2, and they fall in two types. Complex lines are intersections with $\mathbf{H}^2_{\mathbb{C}}$ of projective lines in $\mathbb{C}P^2$. By Hermitian duality, any complex line L is *polar* to a point in $\mathbb{C}P^2$ that is outside the closure of $\mathbf{H}^2_{\mathbb{C}}$. Any lift of this point is called a *polar vector* to L. Any two distinct points p and q in the closure $\mathbf{H}^2_{\mathbb{C}}$ belong to a unique complex line, that is polar to the vector $\mathbf{p} \boxtimes \mathbf{q} = H\overline{\mathbf{p} \wedge \mathbf{q}}$ (see Section 2.2.7. in Chapter 2 of Goldman [15] for details).

The other type of maximal totally geodesic subspace is a Lagrangian plane. Lagrangian planes are PU(2,1) images of the set of real points $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$. In particular, real planes are fixed points sets of antiholomorphic isometric involutions (sometimes called *real symmetries*). The symmetry fixing $\mathbf{H}^2_{\mathbb{R}}$ is complex conjugation. In turn, the symmetry about any other Lagrangian plane $M \cdot \mathbf{H}^2_{\mathbb{R}}$, where $M \in \mathrm{SU}(2,1)$, is given by $\mathbf{z} \longmapsto M\overline{M^{-1}}\,\overline{\mathbf{z}} = M\overline{\big(M^{-1}\mathbf{z}\big)}$. Note that the matrix $N = M\overline{M^{-1}}$ satisfies $N\overline{N} = Id$: this reflects the fact that real symmetries are involutions. We refer the reader to Chapter 3 and 4 of Goldman [15].

2.4 Isometric spheres

Definition 2. For any $B \in \mathrm{SU}(2,1)$ that does not fix q_∞ the *isometric sphere* of B, denoted $\mathcal{I}(B)$, is defined to be

$$\mathcal{I}(B) = \Big\{ p \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}} \ : \ \big|\langle \mathbf{p}, \mathbf{q}_\infty\rangle\big| = \big|\langle \mathbf{p}, B^{-1}(\mathbf{q}_\infty)\rangle\big| = \big|\langle B(\mathbf{p}), \mathbf{q}_\infty\rangle\big| \Big\} \tag{4}$$

where $\mathbf{p}$ is the standard lift of $p \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ given in (1).

The *interior* of $\mathcal{I}(B)$ is the bounded component of the complement, namely,

$$\Big\{ p \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}} \ : \ \big|\langle \mathbf{p}, \mathbf{q}_\infty\rangle\big| > \big|\langle \mathbf{p}, B^{-1}(\mathbf{q}_\infty)\rangle\big| \Big\}.$$

The *exterior* of $\mathcal{I}(B)$ is the unbounded component of the complement.

Suppose B is written as a matrix as

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}. \tag{5}$$

Then $B^{-1}(\mathbf{q}_\infty) = \big[\overline{j},\, \overline{h},\, \overline{g}\big]^T$. Thus B fixes q_∞ if and only if $g = 0$. If B does not fix q_∞ (that is $g \neq 0$) the horospherical coordinates of $B^{-1}(q_\infty)$ are:

$$B^{-1}(q_\infty) = \Big[\overline{h}/\big(\overline{g}\sqrt{2}\big),\, \mathrm{Im}\big(\overline{j}/\overline{g}\big)\Big].$$

Lemma 2.6 (Section 5.4.5 of Goldman [15])**.** *Let $B \in \mathrm{PU}(2,1)$ be an isometry of $\mathbf{H}^2_{\mathbb{C}}$ not fixing q_∞.*

1. *The transformation B maps $\mathcal{I}(B)$ to $\mathcal{I}(B^{-1})$, and the interior of $\mathcal{I}(B)$ to the exterior of $\mathcal{I}(B^{-1})$.*
2. *For any $A \in \mathrm{PU}(2,1)$ fixing q_∞ and such that the corresponding eigenvalue has unit modulus, we have $\mathcal{I}(B) = \mathcal{I}(AB)$.*

Using the characterisation (2) of the Cygan metric in terms of the Hermitian form, the following lemma is obvious.

Lemma 2.7. *Suppose that $B \in \mathrm{SU}(2,1)$ written in the form (5) does not fix q_∞. Then the isometric sphere $\mathcal{I}(B)$ is the Cygan sphere in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ with centre $B^{-1}(q_\infty)$ and radius $r_A = 1/|g|^{1/2}$.*

The importance of isometric spheres is that they form the boundary of the *Ford polyhedron*. This is the limit of Dirichlet polyhedra as the centre point approaches $\partial\mathbf{H}^2_{\mathbb{C}}$; see Section 9.3 of Goldman [15]. The Ford polyhedron D for a discrete group Γ is the intersection of the (closures of the) exteriors of all isometric spheres for elements of Γ not fixing q_∞. That is:

$$D_\Gamma = \Big\{ p \in \mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}} \ : \ \big|\langle \mathbf{p}, \mathbf{q}_\infty\rangle\big| \geq \big|\langle \mathbf{p}, B^{-1}\mathbf{q}_\infty\rangle\big| \text{ for all } B \in \Gamma \text{ with } B(q_\infty) \neq q_\infty \Big\}.$$

Of course, just as for Dirichlet polyhedra, to construct the Ford polyhedron one must check infinitely many equalities. Therefore our method will be to guess the Ford polyhedron and check this using the Poincaré polyhedron theorem. When q_∞ is not a conical limit point, the Ford polyhedron is preserved by Γ_∞, the stabiliser of q_∞ in Γ. It is a fundamental polyhedron for the partition of Γ into Γ_∞-cosets. In order to obtain a fundamental domain for Γ, one must intersect the Ford domain with a fundamental domain for Γ_∞.

2.5 Cygan spheres and geographical coordinates.

We now give some geometrical results about Cygan spheres. They are, in particular, applicable to isometric spheres. The Cygan sphere $\mathcal{S}_{[0,0]}(r)$ of radius $r > 0$ with centre the origin $[0, 0]$ is the (real) hypersurface of $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$ described in horospherical coordinates by the equation

$$\mathcal{S}_{[0,0]}(r) = \Big\{ (z, t, u) \ : \ \big(|z|^2 + u\big)^2 + t^2 = r^4 \Big\}. \tag{6}$$

From (6) we immediately see that when written in horospherical coordinates the interior of $\mathcal{S}_{[0,0]}(r)$ is convex. The Cygan sphere $\mathcal{S}_{[w,s]}(r)$ of radius r with centre $[w, s]$ is the image of $\mathcal{S}_{[0,0]}(r)$ under the Heisenberg translation $T_{[w,s]}$. Since Heisenberg translations are affine maps in horospherical coordinates, we see that the interior of any Cygan sphere is convex. This immediately gives:

Proposition 2.8. *The intersection of two Cygan spheres is connected.*

Cygan spheres are examples of bisectors (otherwise called spinal hypersurfaces) and their intersection is an example of what Goldman calls an intersection of covertical bisectors. Thus Proposition 2.8 is a restatement of Theorem 9.2.6 of [15]. There is a natural system of coordinates on bisectors in terms of totally geodesic subspaces, see Section 5.1 of [15]. In particular for Cygan spheres, these are defined as follows:

Definition 3. Let $\mathcal{S}_{[0,0]}(r)$ be the Cygan sphere with centre the origin $[0, 0]$ and radius $r > 0$. The point $g(\alpha, \beta, w)$ of $\mathcal{S}_{[0,0]}(r)$ with *geographical coordinates* (α, β, w) is the point whose lift to $\mathbb{C}^3$ is:

$$\mathbf{g}(\alpha, \beta, w) = \begin{bmatrix} -r^2 e^{-i\alpha} \\ r w e^{i(-\alpha/2+\beta)} \\ 1 \end{bmatrix}, \tag{7}$$

where $\beta \in [0, \pi)$, $\alpha \in [-\pi/2, \pi/2]$ and $w \in [-\sqrt{2\cos(\alpha)}, \sqrt{2\cos(\alpha)}]$,

Let $\mathcal{S}_{[z,t]}(r)$ be the Cygan sphere with centre $[z, t]$ and radius r. Then geographical coordinates on $\mathcal{S}_{[z,t]}(r)$ are the images of geographical coordinates on $\mathcal{S}_{[0,0]}(r)$ under the Heisenberg translation $T_{[z,t]}$.

We will only be interested in geographical coordinates on $\mathcal{S}_{[0,0]}(1)$, the unit Cygan sphere centred at the origin. Note that for the point $g(\alpha, \beta, w)$ of this sphere, $\langle g(\alpha, \beta, u), g(\alpha, \beta, u)\rangle = w^2 - 2\cos(\alpha)$. Therefore the horospherical coordinates of $g(\alpha, \beta, w)$ are:

$$\Big(w e^{i(-\alpha/2+\beta)}/\sqrt{2}, \ \sin(\alpha), \ \cos(\alpha) - w^2/2 \Big)$$

In particular, the points of $\mathcal{S}_{[0,0]}(1)$ on $\partial\mathbf{H}^2_{\mathbb{C}}$ are those with $w = \pm\sqrt{2\cos(\alpha)}$.

The level sets of α and β are totally geodesic subspaces of $\mathbf{H}^2_{\mathbb{C}}$; see Example 5.1.8 of Goldman [15].

Proposition 2.9. *Let $\mathcal{S}_{[w,s]}(r)$ be a Cygan sphere with geographical coordinates (α, β, w).*

1. *For each $\alpha_0 \in (-\pi/2, \pi/2)$ the set of points $L_{\alpha_0} = \{g(\alpha,\beta,w) \in \mathcal{S}_{[w,s]}(r) \ : \ \alpha = \alpha_0\}$ is a complex line, called a* slice *of $\mathcal{S}_{[w,s]}(r)$.*
2. *For each $\beta_0 \in [0,\pi)$ the set of points $R_{\beta_0} = \{g(\alpha,\beta,w) \in \mathcal{S}_{[w,s]}(r) \ : \ \beta = \beta_0\}$ is a Lagrangian plane, called a* meridian *of $\mathcal{S}_{[w,s]}(r)$.*
3. *The set of points with $w = 0$ is the* spine *of $\mathcal{S}_{[w,s]}(r)$. It is a geodesic contained in every meridian.*

Remark 1. From (6), it is easy to see that projections of boundaries of Cygan spheres onto the z-factor are closed Euclidean discs in $\mathbb{C}$. This correspond to the vertical projection onto $\mathbb{C}$ in the Heisenberg group. This fact is often useful to prove that two Cygan spheres are disjoint.

2.6 Cartan's angular invariant.

Élie Cartan defined an invariant of triples of pairwise distinct points p_1, p_2, p_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$; see Section 7.1 of Goldman [15]. For any lifts $\mathbf{p}_j$ of p_j to $\mathbb{C}^3$, this invariant is defined by $\arg(-\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{p}_2\mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_1\rangle)$, where the argument is chosen to lie in $(-\pi, \pi]$. We state here some important properties of $\mathbb{A}$.

Proposition 2.10. *[Sections 7.1.1 and 7.1.2 of [15]]*

1. *$-\pi/2 \leq \mathbb{A}(p_1, p_2, p_3) \leq \pi/2$ for any triple of pairwise distinct points p_1, p_2, p_3.*
2. *$\mathbb{A}(p_1, p_2, p_3) = \pm\pi/2$ if and only if p_1, p_2, p_3 lie on the same complex line.*
3. *$\mathbb{A}(p_1, p_2, p_3) = 0$ if and only if p_1, p_2, p_3 lie on the same Lagrangian plane.*
4. *Two triples p_1, p_2, p_3 and q_1, q_2, q_3 have $\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(q_1, q_2, q_3)$ if and only if there exists $A \in \mathrm{SU}(2,1)$ so that $A(p_j) = q_j$ for $j = 1, 2, 3$.*
5. *Two triples p_1, p_2, p_3 and q_1, q_2, q_3 have $\mathbb{A}(p_1, p_2, p_3) = -\mathbb{A}(q_1, q_2, q_3)$ if and only if there exists an anti-holomorphic isometry A so that $A(p_j) = q_j$ for $j = 1, 2, 3$.*

3 The parameter space

3.1 Coordinates

Our space of interest is the following.

Definition 4. Let $\mathcal{U}$ be the set of $\mathrm{PU}(2,1)$-conjugacy classes of non-elementary pairs (A, B) such that A, B and AB are unipotent.

Here, by non-elementary, we mean that the two isometries A and B have no common fixed point in $\partial\mathbf{H}^2_{\mathbb{C}}$. In fact, a slightly stronger statement will follow from Theorem 3.1 below. Namely A and B do not preserve a common complex line and so the pair A, B have no common fixed point in $\mathbb{C}P^2$ (see Section 2.3). Another way to see this is that if A in $\mathrm{PU}(2,1)$ is unipotent and preserves a complex line, then its action on that complex line is via a unipotent element of $\mathrm{SL}(2,\mathbb{R})$ (that is parabolic with trace $+2$). It is well known that if A and B are unipotent elements of $\mathrm{SL}(2,\mathbb{R})$ whose product is also unipotent then A and B must share a fixed point (if A, B and AB are all parabolic with distinct fixed points, at least one of them should have trace -2).

Note that $BA = A^{-1}(AB)A = B(AB)B^{-1}$ and so if AB is unipotent then so is BA. If p_{AB} and p_{BA} in $\partial\mathbf{H}^2_{\mathbb{C}}$ are the fixed points of AB and BA then we have $A(p_{BA}) = p_{AB}$ and $B(p_{AB}) = p_{BA}$.

From Corollary 2.5 this means that A and B are uniquely determined by the fixed points of A, B, AB and BA. We describe a set of coordinates on $\mathcal{U}$ expressed in terms of the Cartan invariants of triples of these fixed points.

Theorem 3.1. *There is a bijection between $\mathcal{U}$ and the open square $(\alpha_1, \alpha_2) \in (-\pi/2, \pi/2)^2$, which is given by the map*

$$\Lambda : (A, B) \longmapsto (\mathbb{A}(p_A, p_{AB}, p_B), \mathbb{A}(p_A, p_{AB}, p_{BA})),$$

where p_A, p_B, p_{AB} and p_{BA} are the parabolic fixed points of the corresponding isometries.

This result can be see as a special case of the main result of [25]. For completeness, we include here a direct proof.

Proof. First, the two quantities $\alpha_1 = \mathbb{A}(p_A, p_{AB}, p_B)$ and $\alpha_2 = \mathbb{A}(p_A, p_{AB}, p_{BA})$ are invariant under $\mathrm{PU}(2,1)$-conjugation and thus the map Λ is well-defined. We must show that the image of Λ is $(-\pi/2, \pi/2)^2$. In other words, we must show $\alpha_1 \neq \pm\pi/2$ and $\alpha_2 \neq \pm\pi/2$.

Fix a choice of lifts $\mathbf{p}_A$, $\mathbf{p}_B$, $\mathbf{p}_{AB}$ and $\mathbf{p}_{BA}$ for the fixed points of A, B, AB and BA. Since the fixed points are assumed to be distinct, we see that the Hermitian product of each pair of these vectors does not vanish. The conditions $A(p_{BA}) = p_{AB}$ and $B(p_{AB}) = p_{BA}$ imply that there exist two non-zero complex numbers λ and μ satisfying

$$A\mathbf{p}_{BA} = \lambda \mathbf{p}_{AB} \quad \text{and} \quad B\mathbf{p}_{AB} = \mu \mathbf{p}_{BA}.$$

As AB is unipotent, its eigenvalue associated to $\mathbf{p}_{AB}$ is 1, and therefore $\lambda\mu = 1$. Moreover, using the fact that $\mathbf{p}_A$ and $\mathbf{p}_B$ are eigenvectors of A and B with eigenvalue 1, we have

$$\langle \mathbf{p}_{BA}, \mathbf{p}_A \rangle = \langle A\mathbf{p}_{BA}, A\mathbf{p}_A \rangle = \lambda \langle \mathbf{p}_{AB}, \mathbf{p}_A \rangle, \quad \langle \mathbf{p}_{AB}, \mathbf{p}_B \rangle = \langle B\mathbf{p}_{AB}, B\mathbf{p}_B \rangle = \mu \langle \mathbf{p}_{BA}, \mathbf{p}_B \rangle. \tag{8}$$

Using $\lambda\mu = 1$ and (8), it is not hard to show that $\mathbf{n}_1 = \lambda \mathbf{p}_{AB} - \mathbf{p}_{BA}$ is a polar vector for the complex line L_1 spanned by $\mathbf{p}_A$ and $\mathbf{p}_B$ (see Section 2.3). Moreover, $\langle \mathbf{p}_{AB}, \mathbf{n}_1 \rangle = -\langle \mathbf{p}_{AB}, \mathbf{p}_{BA} \rangle \neq 0$. Thus p_{AB} does not lie on L_1. That is, the three of points p_A, p_B, p_{AB} do not lie on the same complex line and so $\alpha_1 \neq \pm\pi/2$.

Likewise, again using $\lambda\mu = 1$ and (8) we find $\mathbf{n}_2 = \langle \mathbf{p}_B, \mathbf{p}_{AB} \rangle \mathbf{p}_A - \langle \mathbf{p}_A, \mathbf{p}_{AB} \rangle \mathbf{p}_B$ is a polar vector for L_2 and $\langle \mathbf{p}_A, \mathbf{n}_2 \rangle = -\langle \mathbf{p}_A, \mathbf{p}_{AB} \rangle \langle \mathbf{p}_A, \mathbf{p}_B \rangle \neq 0$. Hence p_A does not lie on L_2 and so $\alpha_2 \neq \pm\pi/2$. We remark that, by construction, we have $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle = 0$ and so in fact L_1 and L_2 are orthogonal.

An inverse map to Λ can be defined as follows. Fix (α_1, α_2) in $(-\pi/2, \pi/2)^2$ and define

$$x_1 = \sqrt{2\cos(\alpha_1)} \quad \text{and} \quad x_2 = \sqrt{2\cos(\alpha_2)}, \quad \text{for } \alpha_i \in (-\pi/2, \pi/2), \text{ so } x_1, x_2 \in \mathbb{R}_+^*. \tag{9}$$

Now consider the following elements of $\mathrm{SU}(2,1)$:

$$A = \begin{bmatrix} 1 & -x_1 x_2^2 & -x_1^2 x_2^2 e^{-i\alpha_2} \\ 0 & 1 & x_1 x_2^2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ x_1 x_2^2 e^{-i\alpha_1} & 1 & 0 \\ -x_1^2 x_2^2 e^{i\alpha_2} & -x_1 x_2^2 e^{i\alpha_1} & 1 \end{bmatrix}, \tag{10}$$

Clearly, A and B are unipotent, and since $\mathrm{tr}(AB) = 3$, AB is also unipotent. The four fixed points can be lifted to the vectors

$$\mathbf{p}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_{AB} = \begin{bmatrix} -e^{i\alpha_1} \\ x_1 e^{i\alpha_2} \\ 1 \end{bmatrix}, \quad \mathbf{p}_{BA} = \begin{bmatrix} -e^{i\alpha_1} \\ -x_1 e^{-i\alpha_2} \\ 1 \end{bmatrix}. \tag{11}$$

They satisfy $\mathbb{A}(p_A, p_{AB}, p_B) = \alpha_1$ and $\mathbb{A}(p_A, p_{AB}, p_{BA}) = \alpha_2$.

Note that as either α_1 or α_2 tends to $\pm/\pi/2$ (that is x_1 or x_2 respectively tends to 0) so A and B both tend to the identity matrix. □

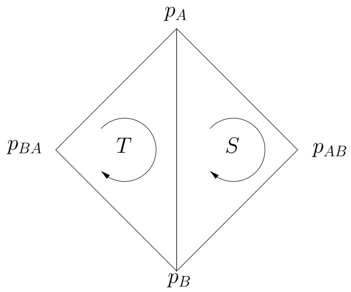


Figure 2: Action of S and T on the tetrahedron $(p_A, p_B, p_{AB}, p_{BA})$.

From now on, we will identify any conjugacy class of pair in $\mathcal{U}$ with its representative given by (10). We will repeatedly use the notation $x_i = \sqrt{2\cos(\alpha_i)}$ from (9) and, when necessary, we will freely combine x_i with trigonometric notation. It should be noted that the unipotent isometry A given by (10) is equal to $T_{[\ell_A, t_A]}$, where

$$\ell_A = x_1 x_2^2/\sqrt{2} = 2\cos(\alpha_1)\cos^2(\alpha_2) \quad \text{and} \quad t_A = x_1^2 x_2^2 \sin(\alpha_2) = 4\cos(\alpha_1)\cos(\alpha_2)\sin(\alpha_2). \tag{12}$$

3.2 Products of order 3 elliptics.

Definition 5. Let S and T be the order three regular elliptic maps cyclically permuting (p_A, p_{AB}, p_B) and (p_A, p_B, p_{BA}). Since $\alpha_1 \neq \pm\pi/2$ these triples of points do not lie on a complex line. Hence by Lemma 2.3 S and T are uniquely determined.

If p_A, p_B, p_{AB} and p_{BA} are given by the vectors in (11), then S and T are given by the following elements of SU(2, 1):

$$S = e^{-i\alpha_1/3}\begin{bmatrix} e^{i\alpha_1} & x_1 e^{i\alpha_1 - i\alpha_2} & -1 \\ -x_1 e^{i\alpha_2} & -e^{i\alpha_1} & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad T = e^{i\alpha_1/3}\begin{bmatrix} 0 & 0 & -1 \\ 0 & -e^{-i\alpha_1} & -x_1 e^{-i\alpha_1 - i\alpha_2} \\ -1 & x_1 e^{i\alpha_2} & e^{-i\alpha_1} \end{bmatrix}, \tag{13}$$

where as usual $x_i = \sqrt{2\cos(\alpha_i)}$; see (9).

Proposition 3.2. *For any pair $(A, B) \in \mathcal{U}$, let S and T be the isometries defined in Definition 5. Then $A = ST$ and $B = TS$.*

Proof. Using the action of S and T on the parabolic fixed points, we have $ST(p_A) = S(p_B) = p_A$ and $ST(p_{BA}) = S(p_A) = p_{AB}$. This means ST is a unipotent map fixing p_A and sending p_{BA} to p_{AB}. Hence $ST = A$. Similarly, $TS = B$. □

A more geometric proof of the existence of order three elliptic isometries decomposing pairs of parabolics as above can be found in a slightly more general context in [25].

One consequence of the existence of this decomposition as product of order three elliptic is that any group in $\mathcal{U}$ is the image of the fundamental group of the Whitehead link complement by a morphism to PU(2,1). This follows directly from the following.

Proposition 3.3. *The free product $\mathbb{Z}_3 * \mathbb{Z}_3$ is a quotient of the fundamental group of the Whitehead link complement.*

Proof. The fundamental group of the Whitehead link complement is presented by $\pi = \langle u, v | \mathrm{rel(u,v)} \rangle$, where

$$\mathrm{rel}(u, v) = [u, v] \cdot [u, v^{-1}] \cdot [u^{-1}, v^{-1}] \cdot [u^{-1}, v] \tag{14}$$

Making the substitution $u = st$ and $v = tst$, the relation becomes $\mathrm{rel}(st, tst) = [st, s^{-1}t^{-3}s^{-2}]$. This relation is trivial whenever $s^3 = t^3 = 1$. Therefore π surjects onto $\mathbb{Z}_3 * \mathbb{Z}_3$. □

3.3 Symmetries of the moduli space

The parameters (α_1, α_2) determine Γ up to $\mathrm{PU}(2,1)$ conjugation. We now show that there is an antiholomorphic conjugation that changes the sign of both α_1 and α_2.

Proposition 3.4. *There is an antiholomorphic involution ι with the properties:*

1. *ι interchanges p_A and p_B and interchanges p_{AB} and p_{BA};*
2. *ι conjugates S to T and A to B (and vice versa);*
3. *ι conjugates the group Γ with parameters (α_1, α_2) to the group with parameters $(-\alpha_1, -\alpha_2)$.*

Proof. The action on $\mathbb{C}^3$ of ι is:

$$\iota : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \overline{z}_3 \\ e^{-i\alpha_1}\overline{z}_2 \\ \overline{z}_1 \end{bmatrix}.$$

It is easy to see that ι^2 is the identity and that ι sends $\mathbf{p}_A$ to $\mathbf{p}_B$ and sends $\mathbf{p}_{AB}$ to $(-e^{-i\alpha_1})\mathbf{p}_{BA}$. Projectivising gives the first part.

Since A is the unique unipotent map fixing p_A and sending p_{BA} to p_{AB}, we see $\iota A \iota$ is the unique unipotent map fixing $\iota(p_A) = p_B$ and sending $\iota(p_{BA}) = p_{AB}$ to $\iota(p_{AB}) = p_{BA}$. Thus $\iota A \iota = B$ and so $\iota B \iota = A$. Applying Proposition 3.2 we see that $\iota S \iota = T$ and $\iota T \iota = S$, proving the second part.

The parameters associated to the group $\iota \Gamma \iota$ are $\mathbb{A}(\iota p_A, \iota p_{AB}, \iota p_B) = \mathbb{A}(p_B, p_{BA}, p_A) = -\alpha_1$ and $\mathbb{A}(\iota p_A, \iota p_{AB}, \iota p_{BA}) = \mathbb{A}(p_B, p_{BA}, p_{AB}) = -\alpha_2$. This completes the proof. □

There are other symmetries of the parameter space $\mathcal{U}$ that, in general, do not arise from conjugation by isometries.

Proposition 3.5. *Let $\phi_h : (\alpha_1, \alpha_2) \longmapsto (\alpha_1, -\alpha_2)$ and $\phi_v : (\alpha_1, \alpha_2) \longmapsto (-\alpha_1, \alpha_2)$ denote the symmetries about the horizontal and vertical axes of the (α_1, α_2)-square. Then $\phi_h \circ \phi_v$ induces the conjugation by ι given in Proposition 3.4. Moreover:*

1. *ϕ_h induces the change of generators $(S, T) \longmapsto (T^{-1}, S^{-1})$ and $(A, B) \longmapsto (A^{-1}, B^{-1})$.*
2. *ϕ_v induces the change of generators $(S, T) \longmapsto (S^{-1}, T^{-1})$ and $(A, B) \longmapsto (B^{-1}, A^{-1})$,*

Proof. Making the change ϕ_h to the points in (11) and then applying $\mathrm{diag}[1, -1, 1] \in \mathrm{PU}(2,1)$ fixes p_A and p_B and swaps p_{AB} and p_{BA}. Therefore it sends S to the map cyclically permuting (p_A, p_{BA}, p_B), which is T^{-1}. Similarly it sends T to S^{-1}.

It is clear that the change of generators $(S, T) \longmapsto (T^{-1}, S^{-1})$ sends $A = ST$ to $T^{-1}S^{-1} = A^{-1}$ and $B = TS$ to $S^{-1}T^{-1} = B^{-1}$.

The change of generators $(A, B) \longmapsto (A^{-1}, B^{-1})$ fixes p_A and p_B. Since it sends AB to $A^{-1}B^{-1} = (BA)^{-1}$ it sends p_{AB} to p_{BA} and similarly sends p_{BA} to p_{AB}. From this we can calculate the new Cartan invariants and we obtain the symmetry ϕ_h.

Hence all three conditions in the first part are equivalent. The second part then follows the first part and Proposition 3.4 by first applying ϕ_h and then conjugating by ι. □

The fixed point sets of these automorphisms are related to $\mathbb{R}$-decomposibility and $\mathbb{C}$-decomposibility of Γ.

Definition 6 (Compare Will [38]). A pair (S,T) of elements in $\mathrm{PU}(2,1)$ is $\mathbb{R}$-decomposible if there exists three antiholomorphic involutions $(\iota_1,\iota_2,\iota_3)$ such that $S=\iota_2\iota_1$ and $T=\iota_1\iota_3$.

A pair (S,T) of elements in $\mathrm{PU}(2,1)$ is $\mathbb{C}$-decomposible if there exists three involutions (I_1,I_2,I_3) in $\mathrm{PU}(2,1)$ such that $S=I_2I_1$ and $T=I_1I_3$.

The properties of $\mathbb{R}$ and $\mathbb{C}$-decomposibility have also been studied (in the special case of pairs of loxodromic isometries) from the point of view of traces in $\mathrm{SU}(2,1)$ in [38], and (in the general case) using cross-ratios in [29]. We could take either point of view here, but instead we choose to argue directly with fixed points.

Proposition 3.6. *Let (A,B) in $\mathcal{U}$, and (S,T) be the corresponding elliptic isometries.*

1. *If $\alpha_1=0$, then the pair (S,T) is $\mathbb{C}$-decomposible and the pair (A,B) is $\mathbb{R}$-decomposible. In particular, $\langle S,T\rangle$ has index 2 in a $(3,3,\infty)$-triangle group.*
2. *If $\alpha_2=0$, then the pair (S,T) is $\mathbb{R}$-decomposible and the pair (A,B) is $\mathbb{C}$-decomposible. In particular $\langle A,B\rangle$ has index two in a complex hyperbolic ideal triangle group.*

Proof. Consider the antiholomorphic involution $\iota_1:[z_1,\,z_2,\,z_3]\longmapsto\big[\overline{z}_1,\,-\overline{z}_2,\,\overline{z}_3\big]$. Applying ι_1 to the points in (11) with $\alpha_1=0$, we see that ι_1 fixes p_A and p_B and interchanges p_{AB} and p_{BA}. Therefore ι_1 conjugates A to A^{-1} and B to B^{-1}. Hence $A\iota_1A\iota_1$ and $\iota_1B\iota_1B$ are the identity. That is $\iota_2=A\iota_1$ and $\iota_3=\iota_1B$ are involutions. Hence (A,B) is $\mathbb{R}$-decomposible.

Again assuming $\alpha_1=0$, consider the holomorphic involution defined by $I_1=\iota_1\iota$ (where ι is the involution defined in Proposition 3.4). Then I_1 fixes p_{AB} and p_{BA} and interchanges p_A and p_B. Therefore, it conjugates S to S^{-1} and T to T^{-1}. This means $I_2=SI_1$ and $I_3=I_1T$ are involutions. Hence (S,T) is $\mathbb{C}$-decomposible.

Now consider the holomorphic involution $I_1':[z_1,\,z_2,\,z_3]\longmapsto[z_1,\,-z_2,\,z_3]$. This fixes p_A and p_B and when $\alpha_2=0$ it interchanges p_{AB} and p_{BA}. As above this means $I_2'=AI_1'$ and $I_3'=I_1'B$ are involutions and (A,B) is $\mathbb{C}$-decomposible. Finally, define $\iota_1'=I_1'\iota$. Arguing as above, again with $\alpha_2=0$, we see that $\iota_2'=S\iota_1'$ and $\iota_3'=\iota_1'T$ are involutions. Hence (S,T) is $\mathbb{R}$-decomposible. □

As indicated above, when $\alpha_1=0$ the group generated by (I_1,I_2,I_3) is a $(3,3,\infty)$ reflection triangle group. This group can be thought of as a limit as n tends to infinity of the $(3,3,n)$ triangle groups which have been studied by Parker, Wang and Xie in [28]. The special case $(3,3,4)$ has been studied by Falbel and Deraux in [6]. Both [6] and [28] constructed Dirichlet domains, and the Ford domain we construct can be seen as a limit of these. Moreover, $\mathbb{R}$-decomposibility of the pair (A,B) when $\alpha_1=0$ can be used to show that these groups correspond to the bending representations of the fundamental group of a 3-punctured sphere that have been studied in [39]. Ideal triangle groups have been studied in great detail in [17, 31, 30, 33, 34].

3.4 Isometry type of the commutator.

The isometry type of the commutator will play an important role in the rest of this paper. It is easily described using the order three elliptic maps given by Proposition 3.2.

Proposition 3.7. *The commutator $[A,B]$ has the same isometry type as ST^{-1}. More precisely, consider $\mathcal{G}(x_1^4,x_2^4)=\mathcal{G}\big(4\cos^2(\alpha_1),4\cos^2(\alpha_2)\big)$ where*

$$\mathcal{G}(x,y)=x^2y^4-4x^2y^3+18xy^2-27.$$

Then $[A, B]$ is loxodromic (respectively parabolic, elliptic) if and only if $\mathcal{G}(x_1^4, x_2^4)$ is positive (respectively zero, negative).

Proof. First, from $A = ST$, $B = TS$ and the fact that S and T have order 3, we see that

$$[A, B] = ABA^{-1}B^{-1} = STTST^{-1}S^{-1}S^{-1}T^{-1} = (ST^{-1})^3.$$

This implies that $[A, B]$ has the same isometry type as ST^{-1} unless ST^{-1} is elliptic of order three, in which case $[A, B]$ is the identity. This would mean that A and B commute, which can not be because their fixed point sets are disjoint.

Representatives of S and T in $\mathrm{SU}(2,1)$ are given in (13). A direct calculation using these matrices shows that $\mathrm{tr}(ST^{-1}) = x_1^2 x_2^4 e^{i\alpha_1/3}$. The function $\mathcal{G}(x_1^4, x_2^4)$ above is obtained by plugging this value in the function $\mathcal{F}$ given in Proposition 2.2. □

The null locus of $\mathcal{G}\big(4\cos^2(\alpha_1), 4\cos^2(\alpha_2)\big)$ in the square $(-\pi/2, \pi/2)^2$ is a curve, which we will refer to as the *parabolicity curve* and denote by $\mathcal{P}$. It is depicted on Figure 3. Similarly, the region where $\mathcal{G}$ is positive (thus $[A, B]$ loxodromic) will be denoted by $\mathcal{L}$. It is a topological disc, which is the connected component of the complement of the curve $\mathcal{P}$ that contains the origin. The region where $[A, B]$ is elliptic will be denoted by $\mathcal{E}$.

4 Isometric spheres and their intersections

4.1 Isometric spheres for S, S^{-1} and their A-translates.

In this section we give details of the isometric spheres that will contain the sides of our polyhedron D. The polyhedron D is our guess for the Ford polyhedron of Γ, subject to the combinatorial restriction discussed in Section 4.2.

We start with the isometric spheres $\mathcal{I}(S)$ and $\mathcal{I}(S^{-1})$ for S and its inverse. From the matrix for S given in (13), using Lemma 2.7 we see that $\mathcal{I}(S)$ and $\mathcal{I}(S^{-1})$ have radius $1/|-e^{-i\alpha_1/3}|^{1/2} = 1$ and centres $S^{-1}(q_\infty) = p_B$ and $S(q_\infty) = p_{AB}$ respectively; see (11). In particular, $\mathcal{I}(S)$ is the Cygan sphere $\mathcal{S}_{[0,0]}(1)$ of radius 1 centred at the origin; see (6). In our computations we will use geographical coordinates in $\mathcal{I}(S)$ as defined in Definition 3. The polyhedron D will be the intersection of the exteriors of $\mathcal{I}(S^{\pm 1})$ and all their translates by powers of A. We now fix some notation:

Definition 7. For $k \in \mathbb{Z}$ let $\mathcal{I}_k^+$ be the isometric sphere $\mathcal{I}(A^k S A^{-k}) = A^k \mathcal{I}(S)$ and let $\mathcal{I}_k^-$ be the isometric sphere $\mathcal{I}(A^k S^{-1} A^{-k}) = A^k \mathcal{I}(S^{-1})$.

With this notation, we have:

Proposition 4.1. *For any integer $k \in \mathbb{Z}$, the isometric sphere $\mathcal{I}_k^+$ has radius 1 and centre the point with Heisenberg coordinates $[k\ell_A, kt_A]$, where ℓ_A and t_A are as in (12). Similarly, the isometric sphere $\mathcal{I}_k^-$ has radius 1 and centre the point with Heisenberg coordinates $[k\ell_A + \sqrt{\cos(\alpha_1)}e^{i\alpha_2}, -\sin(\alpha_1)]$.*

Proof. As A is unipotent and fixes q_∞, it is a Cygan isometry, and thus preserves the radius of isometric spheres. This gives the part about radius. Moreover, it follows directly from Proposition 10 that A^k acts on the boundary of $\mathbf{H}^2_{\mathbb{C}}$ by left Heisenberg multiplication by $[k\ell_A, kt_A]$. This gives the part about centres by a straightforward verification. □

The following proposition describes a symmetry of the family $\{\mathcal{I}_k^\pm \ : \ k \in \mathbb{Z}\}$ which will be useful in the study of intersections of the isometric spheres $\mathcal{I}_k^\pm$.

Proposition 4.2. *Let φ be the antiholomorphic isometry $S\iota = \iota T$, where ι is as in Proposition 3.4. Then $\varphi^2 = A$, and φ acts on the Heisenberg group as a screw motion preserving the affine line parametrised by*

$$\Delta_\varphi = \left\{\delta_\varphi(x) = \left[x + i\frac{\sqrt{\cos(\alpha_1)}\sin(\alpha_2)}{2}, x\sqrt{\cos(\alpha_1)}\sin(\alpha_2) - \frac{\sin(\alpha_1)}{2}\right] \;:\; x \in \mathbb{R}\right\}. \quad (15)$$

Moreover, φ acts on isometric spheres as $\varphi(\mathcal{I}_k^+) = \mathcal{I}_k^-$ and $\varphi(\mathcal{I}_k^-) = \mathcal{I}_{k+1}^+$ for all $k \in \mathbb{Z}$.

Proof. Using the fact that $T = \iota S \iota$ we see that $A = ST = S\iota S\iota = \varphi^2$. Moreover $\varphi(p_A) = S\iota(p_A) = S(p_B) = p_A$. Hence φ is a Cygan isometry. It follows by direct calculation that φ sends $\delta_\varphi(x)$ to $\delta_\varphi(x + \ell_A/2)$, and so preserves Δ_φ. Moreover,

$$\varphi(p_{BA}) = S\iota(p_{BA}) = S(p_{AB}) = p_B, \quad \varphi(p_B) = S\iota(p_B) = S(p_A) = p_{AB}.$$

Hence φ sends $\mathcal{I}_{-1}^-$ to $\mathcal{I}_0^+$ since it is a Cygan isometry mapping the centre of $\mathcal{I}_{-1}^-$ to the centre of $\mathcal{I}_0^+$. Similarly, φ sends $\mathcal{I}_0^+$ to $\mathcal{I}_0^-$. The action on other isometric spheres follows since $\varphi^2 = A$. □

4.2 A combinatorial restriction.

The following section is the crucial technical part of our work. As most of the proofs are computational, we will omit many of them here; they will be provided in Section 7. We are now going to restrict our attention to those parameters in the region $\mathcal{L}$ such that the three isometric spheres $\mathcal{I}_0^+ = \mathcal{I}(S)$, $\mathcal{I}_0^- = \mathcal{I}(S^{-1})$ and $\mathcal{I}_{-1}^- = \mathcal{I}(T)$ have no triple intersection. We will describe the region we are interested in by an inequality on α_1 and α_2. Prior to stating it, let us fix a little notation.

We denote by $\alpha_2^{\lim} = \arccos\left(\sqrt{3/8}\right)$. Then $\mathcal{G}\left(4\cos^2(0), 4\cos^2(\pm\alpha_2^{\lim})\right) = \mathcal{G}(4, 3/2) = 0$ and so the two points $(0, \pm\alpha_2^{\lim})$ are the two cusps of the curve $\mathcal{P}$ located on the vertical axis (see figure 3). Now, let $\mathcal{R}$ be the rectangle (depicted in Figure 3) defined by

$$\mathcal{R} = \left\{(\alpha_1, \alpha_2) \;:\; |\alpha_1| \leqslant \pi/6,\, |\alpha_2| \leqslant \alpha_2^{\lim}\right\}. \quad (16)$$

We remark that in Lemma 7.3 we will prove that when $(\alpha_1, \alpha_2) \in \mathcal{R}$, the commutator $[A, B]$ is non elliptic. This means that $\mathcal{R}$ is contained in in the closure of $\mathcal{L}$.

Definition 8. Let $\mathcal{Z}$ denote the subset of $\mathcal{R}$ where the triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_{-1}^- \cap \mathcal{I}_0^-$ is empty.

The following proposition characterises those points (α_1, α_2) that lie in $\mathcal{Z}$.

Proposition 4.3. *A parameter $(\alpha_1, \alpha_2) \in \mathcal{R}$ is in $\mathcal{Z}$ if and only if it satisfies*

$$\mathcal{D}(x_1^4, x_2^4) = \mathcal{D}\left(4\cos^2(\alpha_1), 4\cos^2(\alpha_1)\right) > 0,$$

where $\mathcal{D}$ is the polynomial given by

$$\mathcal{D}(x, y) = x^3y^3 - 9x^2y^2 - 27xy^2 + 81xy - 27x - 27.$$

The region $\mathcal{Z}$ is depicted in Figure 3: it is the interior of the central region of the figure. In fact, $\mathcal{Z}$ is the region in all of $\mathcal{L}$ where $\mathcal{I}_0^+ \cap \mathcal{I}_{-1}^- \cap \mathcal{I}_0^-$ is empty, but as proving this is more involved, we restrict ourselves to the rectangle $\mathcal{R}$. This provides a priori bounds on the parameters α_1 and α_2 that will make our computations easier. We will prove Proposition 4.3 in Section 7.3. It relies on Proposition 4.4, describing the set of points where $\mathcal{D}(x_1^4, x_2^4) > 0$ and on Proposition 4.5, which gives geometric properties of the triple intersection. Proofs of Proposition 4.4 and Proposition 4.5 will be given in Section 7.2 and Section 7.1 respectively.

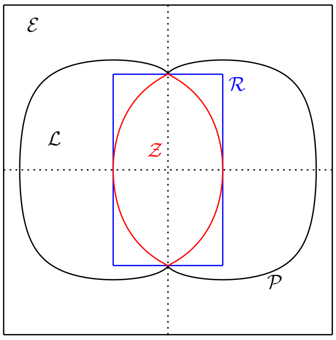


Figure 3: The parameter space, with the parabolicity curve $\mathcal{P}$ and the regions $\mathcal{E}$, $\mathcal{L}$. The region $\mathcal{Z}$ is the central region, which is contained in the rectangle $\mathcal{R}$.

Proposition 4.4. *The region $\mathcal{Z}$ is an open topological disc in $\mathcal{R}$, symmetric about the axes and intersecting them in the intervals $\{\alpha_2 = 0,\ -\pi/6 < \alpha_1 < \pi/6\}$ and $\{\alpha_1 = 0,\ -\alpha_2^{\lim} < \alpha_2 < \alpha_2^{\lim}\}$.*

Moreover, the intersection of the closure of $\mathcal{Z}$ with the parabolicity curve $\mathcal{P}$ consists of the two points $(0, \pm\alpha_2^{\lim})$.

Proposition 4.5. *1. The triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ is contained in the meridian $\mathfrak{m}$ of $\mathcal{I}_0^+$ defined in geographical coordinates by $\beta = (\pi - \alpha_1)/2$.*

2. If the triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ is non-empty, it contains a point in $\partial\mathbf{H}^2_{\mathbb{C}}$.

The second part of Proposition 4.5 is not true for general triples of bisectors. It will allow us to restrict ourselves to the boundary of $\mathbf{H}^2_{\mathbb{C}}$ to prove Proposition 4.3. Restricting ourselves to the region $\mathcal{Z}$ will considerably simplify the combinatorics of the family of isometric spheres $\{\mathcal{I}_k^\pm \ :\ k \in \mathbb{Z}\}$. The following fact will be crucial in our study.

Proposition 4.6. *Fix (α_1, α_2) a point in $\mathcal{Z}$. Then the isometric sphere $\mathcal{I}_0^+$ is contained in the exterior of the isometric spheres $\mathcal{I}_k^\pm$ for all k, except for $\mathcal{I}_1^+$, $\mathcal{I}_{-1}^+$, $\mathcal{I}_0^-$ and $\mathcal{I}_{-1}^-$.*

The proof of Proposition 4.6 will be detailed in Section 7.4). We can give more information about the intersections $\mathcal{I}_0^\pm$ with these four other isometric spheres.

Proposition 4.7. *If $(\alpha_1, \alpha_2) \in \mathcal{Z}$, then the intersection $\mathcal{I}_{-1}^- \cap \mathcal{I}_0^-$ is contained in the interior of $\mathcal{I}_0^+$.*

Proof. Since the point p_B is the centre of $\mathcal{I}_0^+$, it lies in its interior. Moreover, p_B lies on both $\mathcal{I}_{-1}^-$ and $\mathcal{I}_0^-$: indeed, $\langle \mathbf{p}_{AB}, \mathbf{p}_B \rangle = \langle \mathbf{p}_{BA}, \mathbf{p}_B \rangle = 1$. By Proposition 2.8, the intersection of the latter two isometric spheres is connected. This implies that $\mathcal{I}_{-1}^- \cap \mathcal{I}_0^-$ is contained in the interior of $\mathcal{I}_0^+$ for otherwise $\mathcal{I}_0^+ \cap \mathcal{I}_{-1}^- \cap \mathcal{I}_0^-$ would not be empty. □

Using Proposition 4.2, applying powers of φ to Propositions 4.6 and 4.7 gives the following results describing all pairwise intersections.

Corollary 4.8. *Fix $(\alpha_1, \alpha_2) \in \mathcal{Z}$ then for all $k \in \mathbb{Z}$:*

1. *$\mathcal{I}_k^+$ is contained in the exterior of all isometric spheres in $\{\mathcal{I}_k^\pm \ :\ k \in \mathbb{Z}\}$ except $\mathcal{I}_{k-1}^+$, $\mathcal{I}_{k-1}^-$, $\mathcal{I}_k^-$ and $\mathcal{I}_{k+1}^+$. Moreover, $\mathcal{I}_k^+ \cap \mathcal{I}_{k-1}^- \cap \mathcal{I}_k^- = \emptyset$ and $\mathcal{I}_k^+ \cap \mathcal{I}_{k-1}^+$ (respectively $\mathcal{I}_k^+ \cap \mathcal{I}_{k+1}^+$) is contained in the interior of $\mathcal{I}_{k-1}^-$ (respectively $\mathcal{I}_k^-$).*

2. *$\mathcal{I}_k^-$ is contained in the exterior of all isometric spheres in $\{\mathcal{I}_k^\pm \ :\ k \in \mathbb{Z}\}$ except $\mathcal{I}_{k-1}^-$, $\mathcal{I}_k^+$, $\mathcal{I}_{k+1}^+$, and $\mathcal{I}_{k+1}^-$. Moreover, $\mathcal{I}_k^- \cap \mathcal{I}_k^- \cap \mathcal{I}_{k+1}^- = \emptyset$ and $\mathcal{I}_k^- \cap \mathcal{I}_{k-1}^-$ (respectively $\mathcal{I}_k^- \cap \mathcal{I}_{k+1}^-$) is contained in the interior of $\mathcal{I}_k^+$ (respectively $\mathcal{I}_{k+1}^+$).*

5 Applying the Poincaré polyhedron theorem inside $\mathscr{Z}$.

5.1 The Poincaré polyhedron theorem

For the proof of our main result we need to use the Poincaré polyhedron theorem for coset decompositions. The general principle of this result is described in Section 9.6 of [2] in the context of the Poincaré disc. A generalisation to the case of $\mathbf{H}^2_{\mathbb{C}}$ has already appeared in Mostow [24] and Deraux, Parker, Paupert [7]. In these cases it was assumed that the stabiliser of the polyhedron is finite. In our case the stabiliser is the infinite cyclic group generated by the unipotent parabolic map A. There are two main differences from the version given in [7]. First, we require that the polyhedron D has infinitely many facets, the stabiliser group Υ is also infinite, but there are only finitely many Υ-orbits of facets. Secondly, D has an ideal boundary in $\partial\mathbf{H}^2_{\mathbb{C}}$ (which contains more than just cusps). In fact, the version we need has many things in common with the version given by Parker, Wang and Xie [28]. A more general statement will appear in Parker's book [26]. In what follows we will adapt our statement of the Poincaré theorem to the case we have in mind.

The polyhedron and its cell structure Let D be an open polyhedron in $\mathbf{H}^2_{\mathbb{C}}$. We define the ideal boundary $\partial_\infty D$ of D to be the intersection of $\overline{D}$ with $\partial\mathbf{H}^2_{\mathbb{C}}$. This polyhedron has a natural cell structure which we suppose is locally finite inside $\mathbf{H}^2_{\mathbb{C}}$. We suppose that the facets of D of all dimensions are piecewise smooth submanifolds of $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Let $\mathcal{F}_k(D)$ be the collection of facets of codimension k having non-trivial intersection with $\mathbf{H}^2_{\mathbb{C}}$. We suppose that facets are closed subsets of $\overline{\mathbf{H}^2_{\mathbb{C}}}$. We write f° to denote the interior of a facet f, that is the collection of points of f that are not contained in $\partial\mathbf{H}^2_{\mathbb{C}}$ or any facet of a lower dimension (higher codimension). Elements of $\mathcal{F}_1(D)$ and $\mathcal{F}_2(D)$ are respectively called *sides* and *ridges* of D. Since D is a polyhedron, $\mathcal{F}_0(D)=\overline{D}$ and each ridge in $\mathcal{F}_2(D)$ lies in exactly two sides in $\mathcal{F}_1(D)$. Similarly, the intersection of facets of D with $\partial\mathbf{H}^2_{\mathbb{C}}$ gives rise to a polyhedral structure on a subset of $\partial_\infty D$. We let $\mathcal{IF}_k(D)$ denote the ideal facets of $\partial_\infty D$ of codimension k so that each facet in $\mathcal{IF}_k(D)$ is contained in some facet of $\mathcal{F}_\ell(D)$ with $\ell<k$. In particular, we will also need to consider *ideal vertices* in $\mathcal{IF}_4(D)$. These are points of either the endpoints of facets in $\mathcal{F}_3(D)$ or else they are points of $\partial\mathbf{H}^2_{\mathbb{C}}$ contained in (at least) two facets of D that do not intersect inside $\mathbf{H}^2_{\mathbb{C}}$. Note that, since we have defined ideal facets to be subsets of facets, it may be that $\partial\mathbf{H}^2_{\mathbb{C}}$ contains points of $\partial_\infty D$ not contained in any ideal facet. In the case we consider, there will be one such point, namely the point at ∞ fixed by A.

The side pairing. We suppose that there is a *side pairing* $\sigma:\mathcal{F}_1(D)\longrightarrow \mathrm{PU}(2,1)$ satisfying the following conditions:

(1) For each side $s\in\mathcal{F}_1(D)$ with $\sigma(s)=S$ there is another side $s^-\in\mathcal{F}_1(D)$ so that S maps s homeomorphically onto s^- preserving the cell structure. Moreover, $\sigma(s^-)=S^{-1}$. Furthermore, if $s=s^-$ then $S=S^{-1}$ and S is an involution. In this case, we call $S^2=id$ a *reflection relation*.

(2) For each $s\in\mathcal{F}_1(D)$ with $\sigma(s)=S$ we have $\overline{D}\cap S^{-1}(\overline{D})=s$ and $D\cap S^{-1}(D)=\emptyset$.

(3) For each w in the interior s° of s there is an open neighbourhood $U(w)$ of w contained in $\overline{D}\cap S^{-1}(\overline{D})$.

In the example we consider D will be the Ford domain of a group. In particular, each side s will be contained in the isometric sphere $\mathcal{I}(S)$ of $S=\sigma(s)$. Indeed, $s=\mathcal{I}(S)\cap\overline{D}$. By construction we have $S:\mathcal{I}(S)\longmapsto\mathcal{I}(S^{-1})$ and in this case $s^-=\mathcal{I}(S^{-1})\cap\overline{D}$. The polyhedron D will be the (open) infinite sided polyhedron formed by the intersection of the exteriors of all the $\mathcal{I}(S)$ where $S=\sigma(s)$

and s varies over $\mathcal{F}_1(D)$. By construction, the sides of D are smooth hypersurfaces (with boundary) in $\mathbf{H}^2_{\mathbb{C}}$.

Suppose that D is invariant under a group Υ that is *compatible* with the side pairing map in the sense that for all $P \in \Upsilon$ and $s \in \mathcal{F}_1(D)$ we have $P(s) \in \mathcal{F}_1(D)$ and $\sigma(Ps) = P\sigma(s)P^{-1}$. We call the latter a *compatibility relation.* We suppose that there are finitely many Υ-orbits of facets in each $\mathcal{F}_k(D)$. In the example of a Ford domain Υ will be Γ_∞, the stabiliser of the point ∞ in the group Γ.

Ridges and cycle relations. Consider a ridge $r_1 \in \mathcal{F}_2(D)$. Then, r_1 is contained in precisely two sides of D, say s_0^- and s_1. Consider the ordered triple (r_1, s_0^-, s_1). The side pairing map $\sigma(s_1) = S_1$ sends s_1 to the side s_1^- preserving its cell structure. In particular, $S_1(r_1)$ is a ridge of s_1^-, say r_2. Let s_2 be the other side containing r_2. Then we obtain a new ordered triple (r_2, s_1^-, s_2). Now apply $\sigma(s_2) = S_2$ to r_2 and repeat. Because there are only finitely many Υ-orbits of ridges, we eventually find an m so that the ordered triple $(r_{m+1}, s_m^-, s_{m+1}) = (P^{-1}r_1, P^{-1}s_0^-, P^{-1}s_1)$ for some $P \in \Upsilon$ (we suppose this P is unique). We define a map $\rho : \mathcal{F}_2(D) \longrightarrow \mathrm{PU}(2,1)$ called the *cycle transformation* by $\rho(r_1) = P \circ S_m \circ \cdots \circ S_1$. (Note that for any ridge $r_1 = s_0^- \cap s_1$, the cycle transformation map $\rho(r_1) = R$ depends on a choice of one of the sides s_0^- and s_1. If we choose the other one then we send R to R^{-1}. This follows from the fact that then $\sigma(s_j^-) = \sigma(s_j)^{-1}$ and from the compatibility relations.) By construction, the cycle transformation $R = \rho(r_1)$ maps the ridge r_1 to itself setwise. However, R may not be the identity on r_1, nor on $\mathbf{H}^2_{\mathbb{C}}$. Nevertheless, we suppose that R has order n. The relation $R^n = id$ is called the *cycle relation* associated to r_1.

Writing the cycle transformation $\rho(r_1) = R$ in terms of P and the S_j, we let $\mathcal{C}(r_1)$ be the collection of suffix subwords of R^n. That is

$$\mathcal{C}(r_1) = \Big\{ S_j \circ \cdots \circ S_1 \circ R^k \ :\ 0 \le j \le m-1,\ 0 \le k \le n-1 \Big\}.$$

We say that *the cycle condition* is satisfied at r_1 provided:

(1)
$$r_1 = \bigcap_{C \in \mathcal{C}(r_1)} C^{-1}(\overline{D}).$$

(2) If $C_1, C_2 \in \mathcal{C}(r_1)$ with $C_1 \neq C_2$ then $C_1^{-1}(D) \cap C_2^{-1}(D) = \emptyset$.

(3) For each $w \in r_1^\circ$ there is an open neighbourhood $U(w)$ of w so that
$$U(w) \subset \bigcup_{C \in \mathcal{C}(r_1)} C^{-1}(\overline{D}).$$

Ideal vertices and consistent horoballs. Suppose that D has ideal vertices. In this case, we require that there is a system of *consistent horoballs* based at the ideal vertices and their images under the side paring maps. For each $\xi \in \mathcal{IF}_4(D)$ the consistent horoball H_ξ is a horoball based at ξ with the following property. Let $\xi \in \mathcal{IF}_4(D)$ and let $s \in \mathcal{F}_1(D)$ be a side with $\xi \in s$. Let $S = \sigma(s)$. Then S maps ξ to a point ξ^- in s^-. Note that ξ^- is not necessarily an ideal vertex. If it is not, we nevertheless define a consistent horoball H_{ξ^-} at ξ^-. In order for these horoballs to form a system of consistent horoballs we require that for each ideal vertex ξ and each side s with $\xi \in s$ the side pairing map $\sigma(s)$ should map the horoball H_ξ onto the horoball H_{ξ^-}. In particular, any cycle of side pairing maps sending ξ to itself must also send H_ξ to itself (setwise).

Statement of the Poincaré polyhedron theorem. We can now state the version of the Poincaré polyhedron theorem that we need (compare [24] or [7]).

Theorem 5.1. *Let D be a smoothly embedded polyhedron D in $\mathbf{H}^2_{\mathbb{C}}$ together with a side pairing $\sigma : \mathcal{F}_1(D) \longrightarrow \mathrm{PU}(2,1)$. Let $\Upsilon < \mathrm{PU}(2,1)$ be a group of automorphisms of D compatible with the side pairing and suppose that each $\mathcal{F}_k(D)$ contains finitely many Υ-orbits. Let Γ be the group generated by Υ and by the side pairing maps. Suppose that the cycle condition is satisfied for each ridge in $\mathcal{F}_2(D)$ and that there is a system of consistent horoballs at all the ideal vertices of D (if any). Then:*

(1) The images of D under the cosets of Υ in Γ tessellate $\mathbf{H}^2_{\mathbb{C}}$. That is $\mathbf{H}^2_{\mathbb{C}} \subset \bigcup_{A\in\Gamma} A(\overline{D})$ and $D \cap A(D) = \emptyset$ for all $A \in \Gamma - \Upsilon$.

(2) The group Γ is discrete and a fundamental domain for its action on $\mathbf{H}^2_{\mathbb{C}}$ is obtained from the intersection of D with a fundamental domain for Υ.

(3) A presentation for Γ (with respect to the generating set consisting of the generators of Υ and the side pairing maps) has the following set of relations: the relations in Υ, the compatibility relations between σ and Υ, the reflection relations and the cycle relations.

5.2 Application to our examples.

We are now going to apply Theorem 5.1 to the group generated by S and A. Explicit matrices for these transformations are provided in equations (10) and (13). Our aim is to prove:

Theorem 5.2. *Suppose that (α_1, α_2) is in $\mathcal{Z}$. That is, $\mathcal{D}\big(4\cos^2(\alpha_1), 4\cos^2(\alpha_1)\big) > 0$, where $\mathcal{D}(x,y)$ is the polynomial defined in Proposition 4.3. Then the group $\Gamma = \langle S, A\rangle$ associated to the parameters (α_1, α_2) is discrete and has the presentation*

$$\langle S,\, A \ :\ S^3 = (A^{-1}S)^3 = id\rangle. \tag{17}$$

We obtain the presentation $\langle S,\, T \ :\ S^3 = T^3 = id\rangle$ by changing generators to S and $T = A^{-1}S$.

Definition of the polyhedron and its cell structure. The infinite polyhedron we consider is the intersection of the exteriors of all the isometric spheres in $\{\mathcal{I}^{\pm}_k \ :\ k \in \mathbb{Z}\}$.

Definition 9. We call D the intersection of the exteriors of all isometric spheres $\mathcal{I}^+_k$ and $\mathcal{I}^-_k$ with centres $A^kS^{-1}(q_\infty)$ and $A^kS(q_\infty)$ respectively :

$$D = \Big\{ q \in \mathbf{H}^2_{\mathbb{C}} \ :\ d_{\mathrm{Cyg}}\big(q, A^kS^{\pm1}(q_\infty)\big) > 1 \text{ for all } k \in \mathbb{Z} \Big\}. \tag{18}$$

The set of sides of D is $\mathcal{F}_1(D) = \{s^+_k,\, s^-_k \ :\ k \in \mathbb{Z}\}$ where $s^+_k = \mathcal{I}^+_k \cap \overline{D}$ and $s^-_k = \mathcal{I}^-_k \cap \overline{D}$.

Using Corollary 4.8 we can completely describe s^+_k and s^-_k.

Proposition 5.3. *The side s^+_k (respectively s^-_k) is topologically a solid cylinder in $\mathbf{H}^2_{\mathbb{C}} \cup \partial\mathbf{H}^2_{\mathbb{C}}$. The intersection of ∂s^+_k (respectively s^-_k) with $\mathbf{H}^2_{\mathbb{C}}$ is the disjoint union of the topological discs $s^+_k \cap s^-_{k-1}$ and $s^+_k \cap s^-_k$ (respectively $s^-_k \cap s^+_k$ and $s^-_k \cap s^-_{k+1}$).*

Proof. Since s^+_k is contained in $\mathcal{I}^+_k$, its only possible intersections with other sides are contained in $\mathcal{I}^+_{k-1}$, $\mathcal{I}^-_{k-1}$, $\mathcal{I}^+_{k+1}$ and $\mathcal{I}^-_{k+1}$ by Corollary 4.8. Since $\mathcal{I}^+_k \cap \mathcal{I}^+_{k-1}$ and $\mathcal{I}^+_k \cap \mathcal{I}^+_{k+1}$ are contained in the interiors of other isometric spheres, the intersections $s^+_k \cap s^+_{k-1}$ and $s^+_k \cap s^+_{k+1}$ are empty. Also, $\mathcal{I}^+_k \cap \mathcal{I}^-_{k-1} \cap \mathcal{I}^-_k = \emptyset$ and so $s^+_k \cap s^-_{k-1}$ and $s^+_k \cap s^-_k$ are disjoint. Since isometric spheres are topological balls and their pairwise intersections are connected, the description of s^+_k follows. A similar argument describes s^-_k. □

The side pairing $\sigma : \mathcal{F}_1(D) \longrightarrow \mathrm{PU}(2,1)$ is defined by

$$\sigma(s_k^+) = A^k S A^{-k}, \qquad \sigma(s_k^-) = A^k S^{-1} A^{-k}. \tag{19}$$

Let $\Upsilon = \langle A \rangle$ be the infinite cyclic group generated by A. By construction the side pairing σ is compatible with Υ. Furthermore, using Proposition 5.3 the set of ridges is $\mathcal{F}_2(D) = \{r_k^+, r_k^- : k \in \mathbb{Z}\}$ where $r_k^+ = s_k^+ \cap s_k^-$ and $r_k^- = s_k^+ \cap s_{k-1}^-$. We can now verify that σ satisfies the first condition of being a side pairing.

Proposition 5.4. *The side pairing map $\sigma(s_k^+) = A^k S A^{-k}$ is a homeomorphism from s_k^+ to s_k^-. Moreover $\sigma(s_k^-)$ sends $r_k^+ = s_k^+ \cap s_k^-$ to itself and sends $r_k^- = s_k^+ \cap s_{k-1}^-$ to $r_{k+1}^- = s_k^- \cap s_{k+1}^+$.*

Proof. By applying powers of A we need only need to consider the case where $k = 0$. First, the ridge $r_0^+ = s_0^+ \cap s_0^- = \mathcal{I}(S) \cap \mathcal{I}(S^{-1})$ is defined by the triple equality

$$|\langle \mathbf{z}, \mathbf{q}_\infty \rangle| = |\langle \mathbf{z}, S^{-1}\mathbf{q}_\infty \rangle| = |\langle \mathbf{z}, S\mathbf{q}_\infty \rangle|. \tag{20}$$

The map S cyclically permutes $p_B = S^{-1}(q_\infty)$, $p_A = q_\infty$, $p_{AB} = S(q_\infty)$, and so maps r_0^+ to itself. Similarly, consider $r_0^- = s_0^+ \cap s_{-1}^-$. The side pairing map S sends $A^{-1}S(q_\infty)$, the centre of $\mathcal{I}_{-1}^-$, to

$$S(A^{-1}S)(q_\infty) = S(T^{-1}S^{-1})S(q_\infty) = ST^2(q_\infty) = (ST)S^{-1}(ST)(q_\infty) = AS^{-1}(q_\infty),$$

which is the centre of $\mathcal{I}_1^+$, where we have used $A^{-1} = T^{-1}S^{-1}$, $T^{-1} = T^2$ and $ST(q_\infty) = q_\infty$. Therefore $r_0^- = s_0^+ \cap s_{-1}^-$ is sent to $r_1^- = s_0^- \cap s_1^+$ as claimed. The rest of the result follows from our description of $s_k^\pm$ in Proposition 5.3. □

Local tessellation. We now prove local tessellation around the sides and ridges of D.

- $s_k^\pm$. Since $\sigma(s_k^\pm) = A^k S^{\pm 1} A^{-1}$ sends the exterior of $\mathcal{I}_k^\pm$ to the interior of $\mathcal{I}_k^\mp$ we see that D and $A^k S^{\pm 1} A^{-k}(D)$ have disjoint interiors and cover a neighbourhood of each point in $s_k^\mp$. Together with Proposition 5.4 this means σ satisfies the three conditions of being a side pairing.

- r_0^+. Consider the case of $r_0^+ = s_0^+ \cap s_0^- = \mathcal{I}(S) \cap \mathcal{I}(S^{-1})$, which is given by (20). Observe that r_0^+ is mapped to itself by S. Using Proposition 5.4, we see that when constructing the cycle transformation for r_0^+ we have one ordered triple (r_0^+, s_0^-, s_0^+) and the cycle transformation $\rho(r_0^+) = S$. The cycle relation is $S^3 = id$ and $\mathcal{C}(r_0^+) = \{id,\, S,\, S^2\}$. Consider an open neighbourhood U_0^+ of r_0^+ but not intersecting any other ridge. The intersection of D with U_0^+ is the same as the intersection of U_0^+ with the Ford domain D_S for the order three group $\langle S \rangle$. Since S has order 3 this Ford domain is the intersection of the exteriors of $\mathcal{I}(S)$ and $\mathcal{I}(S^{-1})$. For z in D_S, $|\langle \mathbf{z}, \mathbf{q}_\infty \rangle|$ is the smallest of the three quantities in (20). Applying $S = \sigma(s_0^+)$ and $S^{-1} = \sigma(s_0^-)$ gives regions $S(D_S)$ and $S^{-1}(D_S)$ where one of the other two quantities is the smallest. Therefore $U_0^+ \cap S(U_0^+) \cap S(U_0^-)$ is an open neighbourhood of r_0^+ contained in $D \cup S(D) \cup S^{-1}(D)$. This proves the cycle condition at r_0^+.

- r_0^-. Now consider $r_0^- = s_0^+ \cap s_{-1}^-$. When constructing the cycle transformation for r_0^- we start with the ordered triple (r_0^-, s_{-1}^-, s_0^+). Applying $S = \sigma(s_0^+)$ to r_0^- gives the ordered triple (r_1^-, s_0^-, s_1^+), which is simply $(Ar_0^-, As_{-1}^-, As_0^+)$. Thus the cycle transformation of r_0^- is $\rho(r_0^-) = A^{-1}S = T^{-1}$, which has order 3. Therefore the cycle relation is $(A^{-1}S)^3 = id$, and $\mathcal{C}(r_0^-) = \{id,\, A^{-1}S,\, (A^{-1}S)^2\}$. Noting that $\mathcal{I}_0^+$ has centre $S^{-1}(q_\infty)S^{-1}A(q_\infty) = T(q_\infty)$ and $\mathcal{I}_{-1}^-$ has centre $A^{-1}S(q_\infty) = T^{-1}(q - \infty)$ we see $\mathcal{I}_0^+ = \mathcal{I}(T^{-1})$ and $\mathcal{I}_0^- = \mathcal{I}(T)$. Therefore a similar argument involving the Ford domain for $\langle T \rangle$ shows that the cycle condition is satisfied at r_0^-.

$r_k^{\pm}$. Using compatibility of the side pairings with $\Upsilon = \langle A \rangle$, we see that $\rho(r_k^+) = A^k S A^{-k}$ with cycle relation $(A^k S A^{-k})^3 = A^k S^3 A^{-k} = id$ and that the cycle condition is satisfied at r_k^+. Likewise, $\rho(r_k^-) = A^k(A^{-1}S)A^{-k} = A^{k-1}SA^{-k}$ with cycle relation $(A^{k-1}SA^{-k})^3 = A^k(A^{-1}S)A^{-k} = id$ and the cycle condition is satisfied at r_k^-.

This is sufficient to prove Theorem 5.2 by applying the Poincaré polyhedron theorem when D has no ideal vertices, that is to all groups Γ in the interior of $\mathcal{Z}$. In particular, Γ is generated by the generator A of Υ and the side pairing maps. Using the compatibility relations, there is only one side pairing map up to the action of Υ, namely S. There are no reflection relations, and (again up to the action of Υ) the only cycle relations are $S^3 = id$ and $(A^{-1}S)^3 = id$. Thus the Poincaré polyhedron theorem gives the presentation (17). This completes the proof of Theorem 5.2.

For groups on the boundary of $\mathcal{Z}$ the same result is also true: this follows from Chuckrow's theorem (see for instance Theorem 2.7 of [3]). We do not need to apply the Poincaré polyhedron theorem for these groups. However, to describe the manifold at infinity for the limit groups, we will need to know a fundamental domain, and we will have to go through a similar analysis in the next section.

6 The limit group.

In this section, we consider the group $\Gamma^{\lim}$, and unless otherwise stated, the parameters α_1 and α_2 will always be assumed to be equal to 0 and $\alpha_2^{\lim}$ respectively. We know already that $\Gamma^{\lim}$ is discrete and isomorphic to $\mathbb{Z}_3 * \mathbb{Z}_3$. Our goal is to prove that its manifold at infinity is homeomorphic to the complement of the Whitehead link. For these values of the parameters, the maps $S^{-1}T$ and ST^{-1} are unipotent parabolic (see the results of Section 3.4), and we denote by $V_{S^{-1}T}$ and $V_{ST^{-1}}$ respectively the sets of (parabolic) fixed points of conjugates of $S^{-1}T$ and ST^{-1} by powers of A.

1. As in the previous section, we apply the Poincaré polyhedron theorem, this time to the group $\Gamma^{\lim}$. We obtain an infinite A-invariant polyhedron, still denoted D, which is a fundamental domain for A-cosets. This polyhedron is slightly more complicated than the one in the previous section due to the appearance of ideal vertices that are the points in $V_{S^{-1}T}$ and $V_{ST^{-1}}$.

2. We analyse the combinatorics of the ideal boundary $\partial_\infty D$ of this polyhedron. More precisely, we will see that the quotient of $\partial_\infty D \setminus (\{p_A\} \cup V_{S^{-1}T} \cup V_{ST^{-1}})$ by the action of the group $\langle S, T \rangle$ is homeomorphic the the complement of the Whitehead link, which is Theorem 6.4.

6.1 Matrices and fixed points.

Before going any further, we provide specific expressions for the various objects we consider at the limit point. When $\alpha_1 = 0$ and $\alpha_2 = \alpha_2^{\lim}$, the map φ described in Proposition 4.2 is given in Heisenberg coordinates by

$$\varphi : [z, t] \longmapsto \left[\overline{z} + \sqrt{3/8} + i\sqrt{5/8}, -t + x\sqrt{5/2} + y\sqrt{3/2}\right]. \tag{21}$$

In particular its invariant line Δ_φ is parametrised by

$$\Delta_\varphi = \left\{\delta_\varphi(x) = \left[x + i\sqrt{5/32},\, x\sqrt{5/8}\right] \ :\ x \in \mathbb{R}\right\}. \tag{22}$$

The parabolic map $A = \varphi^2$ acts on Δ_φ as $A : \delta_\varphi(x) \longmapsto \delta_\varphi(x + \sqrt{3/2})$. As a matrix it is given by

$$A = \begin{bmatrix} 1 & -\sqrt{3} & -3/2 + i\sqrt{15}/2 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

We can decompose A into the product of regular elliptic maps S and T:

$$S = \begin{bmatrix} 1 & \sqrt{3}/2 - i\sqrt{5}/2 & -1 \\ -\sqrt{3}/2 - i\sqrt{5}/2 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -\sqrt{3}/2 + i\sqrt{5}/2 \\ -1 & \sqrt{3}/2 + i\sqrt{5}/2 & 1 \end{bmatrix}$$

These maps cyclically permute (p_A, p_{AB}, p_B) and (p_A, p_P, p_{PA}) where

$$\mathbf{p}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_{AB} = \begin{bmatrix} -1 \\ \sqrt{3}/2 + i\sqrt{5/2} \\ 1 \end{bmatrix}, \quad \mathbf{p}_{BA} = \begin{bmatrix} -1 \\ -\sqrt{3}/2 + i\sqrt{5}/2 \\ 1 \end{bmatrix}. \tag{24}$$

Using $\alpha_1 = 0$, we will occasionally use the facts from Proposition 3.6 that (S, T) is $\mathbb{C}$-decomposible and (A, B) is $\mathbb{R}$-decomposible.

As mentionned above, in the group $\Gamma^{\lim}$ the elements ST^{-1}, $S^{-1}T$, TST, STS and the commutator $[A, B] = (ST^{-1})^3$ are unipotent parabolic. For future reference, we provide here lifts of their fixed points, both as vectors in $\mathbb{C}^3$ and in terms of geographical coordinates $g(\alpha, \beta)$ (we omit the w coordinates: since we are on the boundary it is equal to $\sqrt{2\cos\alpha}$).

$$\begin{aligned}
\mathbf{p}_{ST^{-1}} &= \begin{bmatrix} -1/4 + i\sqrt{15}/4 \\ \sqrt{3}/4 + i\sqrt{5}/4 \\ 1 \end{bmatrix} = \mathbf{g}\left(\arccos(1/4), \pi/2\right), \\
\mathbf{p}_{S^{-1}T} &= \begin{bmatrix} -1/4 - i\sqrt{15}/4 \\ -\sqrt{3}/4 + i\sqrt{5}/4 \\ 1 \end{bmatrix} = \mathbf{g}\left(-\arccos(1/4), \pi/2\right), \\
\mathbf{p}_{TST} &= \begin{bmatrix} -1 \\ -3\sqrt{3}/4 + i\sqrt{5}/4 \\ 1 \end{bmatrix} = \mathbf{g}\left(0, -\arccos\left(\sqrt{27/32}\right)\right), \\
\mathbf{p}_{STS} &= \begin{bmatrix} -1 \\ 3\sqrt{3}/4 + i\sqrt{5}/4 \\ 1 \end{bmatrix} = \mathbf{g}\left(0, \arccos\left(\sqrt{27/32}\right)\right).
\end{aligned} \tag{25}$$

It follows from (21) that φ acts on these parabolic fixed points as follows:

$$\cdots p_{T^{-1}STST} \xrightarrow{\varphi} p_{TST} \xrightarrow{\varphi} p_{S^{-1}T} \xrightarrow{\varphi} p_{ST^{-1}} \xrightarrow{\varphi} p_{STS} \xrightarrow{\varphi} p_{STSTS^{-1}} \cdots \tag{26}$$

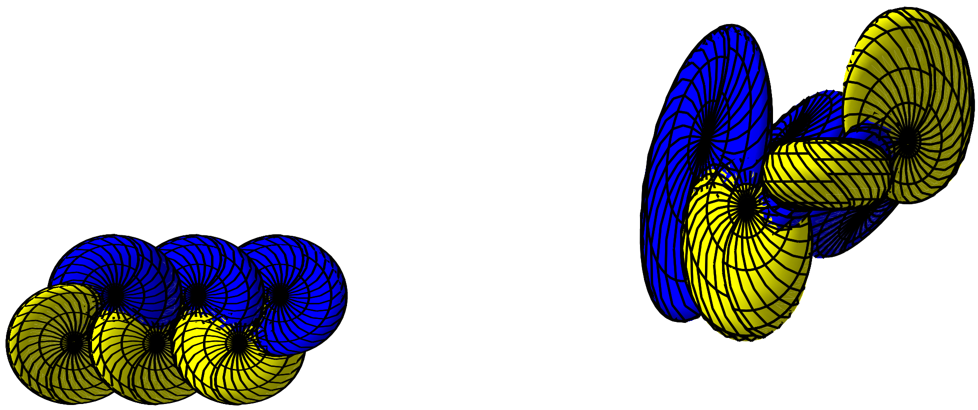
6.2 The Poincaré theorem for the limit group.

The limit group has extra parabolic elements. Therefore, in order to apply the Poincaré theorem, we must construct a system of consistent horoballs at these parabolic fixed points (see Section 5.1).

Lemma 6.1. *The isometric spheres $\mathcal{I}_1^+$ and $\mathcal{I}_{-1}^-$ are tangent at $p_{ST^{-1}}$. The isometric spheres $\mathcal{I}_{-1}^+$ and $\mathcal{I}_0^-$ are tangent at $p_{S^{-1}T}$.*

Proof. It is straightforward to verify that $|\langle \mathbf{p}_{ST^{-1}}, \mathbf{p}_{BA}\rangle| = |\langle \mathbf{p}_{ST^{-1}}, A(\mathbf{p}_B)\rangle| = 1$, and therefore $p_{ST^{-1}}$ belongs to both $\mathcal{I}_{-1}^-$ and $\mathcal{I}_1^+$. Projecting vertically (see Remark 1), we see that the projections of $\mathcal{I}_{-1}^-$ and $\mathcal{I}_1^+$ are tangent discs and as they are strictly convex, their intersection contains at most one point. This gives the result. The other tangency is along the same lines. □

A consequence of Lemma 6.1 is that the parabolic fixed points are tangency points of isometric spheres. The following lemma is proved in Section 7.1.



View from the positive t-axis   View from the negative y-axis

Figure 4: Various realistic views of the isometric spheres $\mathcal{I}_k^{\pm}$ for $-1 \leqslant k \leqslant 1$ for the limit group Γ^{lim}.

Lemma 6.2. *For the group Γ^{lim} the triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ contains exactly two points, namely the parabolic fixed points $p_{ST^{-1}}$ and $p_{S^{-1}T}$.*

Applying powers of φ, we see that these triple intersections are actually quadruple intersections of sides and triple intersections of ridges.

Corollary 6.3. *The parabolic fixed point $A^k(p_{ST^{-1}})$ lies on $\mathcal{I}_{k-1}^- \cap \mathcal{I}_k^+ \cap \mathcal{I}_k^- \cap \mathcal{I}_k^+$. In particular, it is the triple ridge intersection $r_k^- \cap r_k^+ \cap r_{k+1}^-$. Similarly, $A^k(p_{S^{-1}T})$ lies on $\mathcal{I}_{-1}^+ \cap \mathcal{I}_{-1}^- \cap \mathcal{I}_0^+ \cap \mathcal{I}_0^-$. In particular it is $r_{k-1}^+ \cap r_k^- \cap r_k^+$.*

To construct a system of consistent horoballs at the parabolic fixed points we must investigate the action of the side pairing maps on them. First, $p_{S^{-1}T} \in \mathcal{I}_{-1}^+ \cap \mathcal{I}_{-1}^- \cap \mathcal{I}_0^+ \cap \mathcal{I}_0^-$, we have

$$\begin{aligned}
\sigma(s_{-1}^+) = A^{-1}SA : p_{S^{-1}T} &\longmapsto p_{T^{-1}STST},\\
\sigma(s_{-1}^-) = A^{-1}S^{-1}A : p_{S^{-1}T} &\longmapsto p_{TST},\\
\sigma(s_0^+) = S : p_{S^{-1}T} &\longmapsto p_{ST^{-1}},\\
\sigma(s_0^-) = S^{-1} : p_{S^{-1}T} &\longmapsto p_{STS}.
\end{aligned}$$

Likewise $p_{ST^{-1}} \in \mathcal{I}_{-1}^- \cap \mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_1^+$. We have

$$\begin{aligned}
\sigma(s_{-1}^-) = A^{-1}S^{-1}A : p_{ST^{-1}} &\longmapsto A^{-2}(p_{ST^{-1}}),\\
\sigma(s_0^+) = S : p_{ST^{-1}} &\longmapsto p_{STS},\\
\sigma(s_0^-) = S^{-1} : p_{ST^{-1}} &\longmapsto p_{S^{-1}T},\\
\sigma(s_1^+) = ASA^{-1} : p_{ST^{-1}} &\longmapsto A^2(p_{ST^{-1}}).
\end{aligned}$$

We can combine these maps to show how the points $A^k(p_{ST^{-1}})$ and $A^k(p_{S^{-1}T})$ are related by the

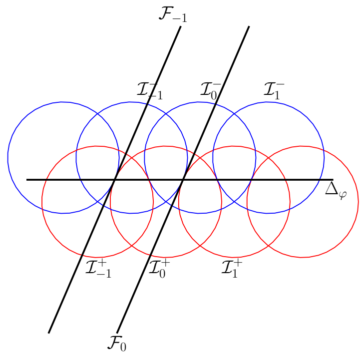
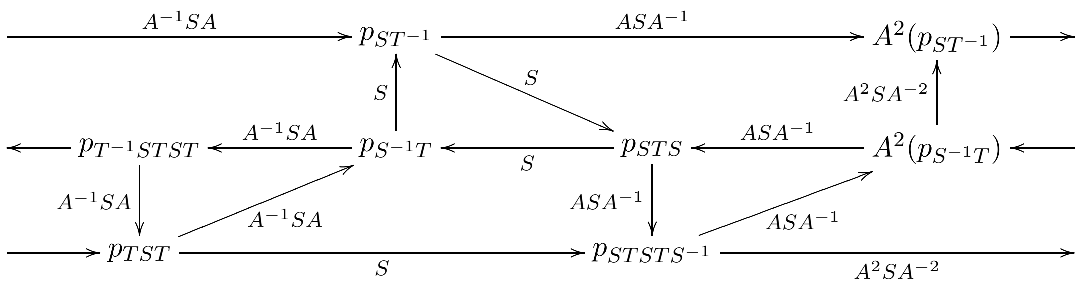


Figure 5: Vertical projection of the isometric spheres for the parameter values $\alpha_1 = 0$, $\alpha_2 = \alpha_2^{\text{lim}}$. Compare with Figure 4.

side pairing maps. This leads to an infinite graph, a section of which is:



(27)

From this it is clear that all the cycles in the graph (27) are generated by triangles and quadrilaterals. Up to powers of A, the triangles lead to the word S^3, which is the identity. Up to powers of A the quadrilaterals lead to words cyclically equivalent to the one coming from:

$$p_{S^{-1}T} \xrightarrow{S^{-1}} p_{STS} \xrightarrow{ASA^{-1}} p_{STSTS^{-1}} \xrightarrow{S^{-1}} p_{TST} \xrightarrow{A^{-1}SA} p_{S^{-1}T}$$

In other words, $p_{S^{-1}T}$ is fixed by $(A^{-1}SA)(S^{-1})(ASA^{-1})(S^{-1}) = (T^{-1}S)^3$. This is parabolic and so preserves all horoballs based at $p_{S^{-1}T}$.

Therefore, we can define a system of horoballs as follows. Let U_0^+ be a (sufficiently small) horoball based at $p_{S^{-1}T}$. Now define horoballs U_k^+ and U_k^- by applying the side pairing maps to U_0^+. Since every cycle in the graph (27) gives rise either to the identity map or to a parabolic map, this process is well defined and gives rise to a consistent system of horoballs. Therefore we can apply the Poincaré polyhedron theorem for the two limit groups. Using the same arguments as we did for groups in the interior of $\mathcal{Z}$, we see that Γ has the presentation (17).

6.3 The boundary of the limit orbifold.

Theorem 6.4. *The manifold at infinity of the group Γ^{lim} is homeomorphic to the Whitehead link complement.*

The ideal boundary of D is made up of those pieces of the isometric spheres $\mathcal{I}_k^\pm$ that are outside all other isometric spheres in $\{\mathcal{I}_k^\pm \ : \ k \in \mathbb{Z}\}$. Recall that the (ideal boundary of) the side $s_k^\pm$ is the part of $\partial\mathcal{I}_k^\pm$ which is outside (the ideal boundary of) all other isometric spheres. In this section, when we speak of sides and ridges in this section we implicitly mean their intersection with $\partial\mathbf{H}^2_{\mathbb{C}}$.

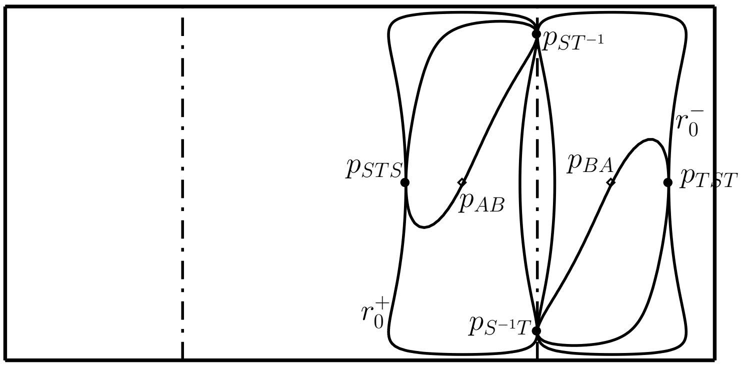


Figure 6: Intersections of the isometric spheres $\mathcal{I}_0^-$, $\mathcal{I}_{-1}^-$, $\mathcal{I}_1^+$ and $\mathcal{I}_{-1}^+$ with $\mathcal{I}_0^+$ in the boundary of $\mathbf{H}^2_{\mathbb{C}}$, viewed in geographical coordinates. Recall that $r_0^+ = \mathcal{I}_0^+ \cap \mathcal{I}_0^-$ and $r_0^- = \mathcal{I}_0^+ \cap \mathcal{I}_{-1}^-$. Here $\alpha \in [-\pi/2, \pi/2]$ is the vertical coordinates, and $\beta \in [-\pi, \pi]$ the horizontal one. The vertical dash-dotted segments $\beta = \pm\pi/2$ are the two halves of the boundary of the meridian $\mathfrak{m}$. The bigon between the two curves r_0^+ and r_0^- is $\mathcal{B}_0^+$ (see Proposition 6.5). Compare to Figure 2 of [6].

We will see that each isometric sphere in $\{\mathcal{I}_k^\pm \; : \; k \in \mathbb{Z}\}$ contributes a side $s_k^\pm$ made up of one quadrilateral, denoted by $\mathcal{Q}_k^\pm$ and one bigon $\mathcal{B}_k^\pm$. A very similar behaviour of isometric spheres has been observed by Deraux and Falbel in [6]. We begin by analysing the contribution of $\mathcal{I}_0^+$.

Proposition 6.5. *The side s_0^+ of D has two connected components.*

1. *One of them is a quadrilateral, denoted $\mathcal{Q}_0^+$, whose vertices are points $p_{ST^{-1}}$, $p_{S^{-1}T}$, p_{STS} and p_{TST} (all of which are parabolic fixed points)*
2. *The other is a bigon, denoted $\mathcal{B}_0^+$, whose vertices are $p_{ST^{-1}}$ and $p_{S^{-1}T}$*

Proof. Since isometric spheres are strictly convex, the ideal boundaries of the ridges $r_0^+ = \mathcal{I}_0^+ \cap \mathcal{I}_0^-$ and $r_0^- = \mathcal{I}_0^+ \cap \mathcal{I}_{-1}^-$ are Jordan curves on $\mathcal{I}_0^+$. We still denote them by $r_0^\pm$. The interiors of these curves are respectively the connected components containing p_{AB} and p_{BA}. By Lemma 6.2 in Section 7.1, r_0^+ and r_0^- have two intersection points, namely $p_{S^{-1}T}$ and $p_{ST^{-1}}$, and that their interiors are disjoint. As a consequence the common exterior of the two curves has two connected components, and the points $p_{S^{-1}T}$ and $p_{ST^{-1}}$ lie on the boundary of both.

To finish the proof, consider the involution ι_1 defined in the proof of Proposition 3.6. (Note that since $\alpha_1 = 0$ this involution conjugates Γ^{lim} to itself.) In Heisenberg coordinates it is defined by $\iota_1 : [z, t] \longmapsto [-\overline{z}, -t]$ and is clearly a Cygan isometry. As in Proposition 3.6, ι_1 fixes p_A and p_B and it interchanges p_{AB} and p_{BA}. Thus it conjugates S to T^{-1} and so it interchanges $p_{ST^{-1}}$ and $p_{S^{-1}T}$ and it interchanges p_{STS} and p_{TST}. Moreover, since it is a Cygan isometry, ι_1 preserves $\mathcal{I}_0^+$ and interchanges $\mathcal{I}_{-1}^-$ and $\mathcal{I}_0^-$ and thus it also exchanges the two curves r_0^+ and r_0^-. Again, since it is a Cygan isometry, it maps interior to interior and exterior to exterior for both curves. As a consequence, the two connected components of the common exterior are either exchanged or both preserved.

Now consider the point with Heisenberg coordinates $[i, 0]$. It is fixed by ι_1, and belongs to the common exterior of both r_0^+ and r_0^- This implies that both connected components are preserved. Finally, since $p_{STS} \in \mathcal{I}_0^+ \cap \mathcal{I}_0^-$ and $p_{TST} \in \mathcal{I}_0^+ \cap \mathcal{I}_{-1}^-$ are exchanged by ι_1, these two points belong to the closure of the same connected component. As a consequence, one of the two connected components has $p_{ST^{-1}}$, $p_{S^{-1}T}$, p_{STS} and p_{TST} on its boundary. This is the quadrilateral. The other one has $p_{ST^{-1}}$ and $p_{S^{-1}T}$ on its boundary. This is the bigon. $\square$

We now apply powers of A to get a result about all the isometric sphere intersections in the ideal boundary of D. Define $\mathcal{Q}_0^- = \varphi(\mathcal{Q}_0^-)$ and $\mathcal{B}_0^- = \varphi(\mathcal{B}_0^-)$. Then applying powers of A we define quadrilaterals $\mathcal{Q}_k^\pm = A^k(\mathcal{Q}_0^\pm)$, and bigons $\mathcal{B}_k^\pm = A^k(\mathcal{B}_0^\pm)$. The action of the Heisenberg translation A and the glide reflection φ are:

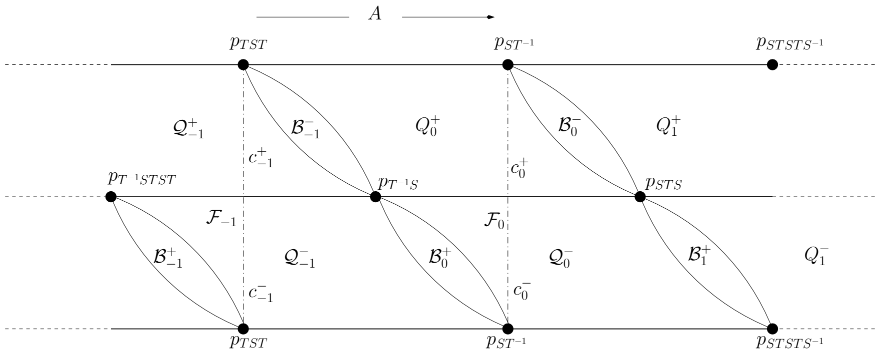


Figure 7: A combinatorial picture of ∂D. The top and bottom lines are identified.

Corollary 6.6. *For the group $\Gamma^{\lim}$, the (ideal boundary of) the side $s_k^\pm$ the quadrilateral $\mathcal{Q}_k^\pm$ and the bigon $\mathcal{B}_k^\pm$. The action of A and φ are as follows.*

(1) A maps $\mathcal{Q}_k^\pm$ to $\mathcal{Q}_{k+1}^\pm$, and $\mathcal{B}_k^\pm$ to $\mathcal{B}_{k+1}^\pm$.

(2) φ maps $\mathcal{Q}_k^+$ to $\mathcal{Q}_k^-$, $\mathcal{Q}_k^-$ to $\mathcal{Q}_{k+1}^+$, $\mathcal{B}_k^+$ to $\mathcal{B}_k^-$ and $\mathcal{B}_k^-$ to $\mathcal{B}_{k+1}^+$.

In order to understand the combinatorics of the sides of D, we describe the edges of the faces lying in $\mathcal{I}_0^+$. The three points $p_{S^{-1}T}$, $p_{ST^{-1}}$, p_{STS} lie on the ridge $r_0^+ = \mathcal{I}_0^+ \cap \mathcal{I}_0^-$. Likewise, the points $p_{ST^{-1}}$, $p_{S^{-1}T}$, p_{TST} lie in the ridge $r_0^- = \mathcal{I}_0^+ \cap \mathcal{I}_{-1}^-$. Indeed, these points divide (the ideal boundaries of) these ridges into three segments. We have listed the ideal vertices in positive cyclic order (see Figure 6). Using the graph (27), the action of the cycle transformations $\rho(s_0^+) = S$ and $\rho(r_0^-) = A^{-1}S = T^{-1}$ on these ideal vertices, and hence on the segments of the ridges, is:

$$p_{S^{-1}T} \xrightarrow{\quad S \quad} p_{ST^{-1}} \xrightarrow{\quad S \quad} p_{STS} \xrightarrow{\quad S \quad} p_{S^{-1}T},$$

$$p_{ST^{-1}} \xrightarrow{\quad A^{-1}S \quad} p_{S^{-1}T} \xrightarrow{\quad A^{-1}S \quad} p_{TST} \xrightarrow{\quad A^{-1}S \quad} p_{ST^{-1}}.$$

Furthermore, S maps p_{TST} to $p_{STSTS^{-1}}$.

The quadrilateral $\mathcal{Q}_0^+$ has two edges $[p_{S^{-1}T}, p_{TST}] \cup [p_{TST}, p_{ST^{-1}}]$ in the ridge r_0^- and two edges $[p_{ST^{-1}}, p_{STS}] \cup [p_{STS}, p_{S^{-1}T}]$ in the ridge r_0^+. It is sent by S to the quadrilateral $\mathcal{Q}_0^-$ with two edges $[p_{ST^{-1}}, p_{STSTS^{-1}}] \cup [p_{STSTS^{-1}}, p_{STS}]$ in r_1^- and two edges $[p_{STS}, p_{S^{-1}T}] \cup [p_{S^{-1}T}, p_{ST^{-1}}]$ in r_0^+. Similarly, the edges of the bigon $\mathcal{B}_0^+$ are the remaining segments in r_0^- and r_0^+, both with endpoints $p_{S^{-1}T}$ and $p_{ST^{-1}}$. It is sent by S to the bigon $\mathcal{B}_0^-$ with vertices $p_{ST^{-1}}$ and p_{STS}.

Applying powers of A gives the other quadrilaterals and bigons. As usual, the image under A^k can be found by adding k to each subscript and conjugating each side pairing map and ridge cycle by A^k. The combinatorics of D is summarised on Figure 7.

Lemma 6.7. *The line Δ_φ given in (22) is contained in the complement of D.*

Proof. As noted above, A acts on Δ_φ as a translation through $\sqrt{3/2}$. We claim that the segment of Δ_φ with parameter $x \in [-\sqrt{3/8}, \sqrt{3/8}]$ in contained in the interior of $\mathcal{I}_0^+$. Applying powers of A we see that each point of Δ_φ is contained in $\mathcal{I}_k^+$ for some k. Hence the line is in the complement of D.

Consider $\delta_\varphi(x) \in \Delta_\varphi$ with $x^2 \leq 3/8$. The Cygan distance between p_B and $\delta_\varphi(x)$ satisfies:

$$d_{\text{Cyg}}(p_B, \delta_\varphi(x))^4 = \left| -x^2 - 5/32 + ix\sqrt{5/8} \right|^2 = x^4 + 15x^2/16 + 25/1 - 24 \leq 529/1024.$$

Since $d_{\text{Cyg}}(p_B, \delta_\varphi(x)) < 1$ this means $\delta_\varphi(x)$ is in the interior of $\mathcal{I}_0^+$ as claimed. □

The following result, which will be proved in Section 7.5, is crucial for proving Theorem 6.4.

Proposition 6.8. *There exists a homeomorphism $\Psi : \mathbb{R}^3 \longrightarrow \partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$ mapping the exterior of $S^1 \times \mathbb{R}$, that is $\{(x, y, z) \;:\; x^2 + y^2 \geq 1\}$, homeomorphically onto D and so that $\Psi(x, y, z + 1) = A\Psi(x, y, z)$, that is Ψ is equivariant with respect to unit translation along the z axis and A.*

As a consequence of Proposition 6.8, D admits an A invariant 1-dimensional foliation, the leaves being the images of radial lines $\{(r\cos(\theta_0), r\sin(\theta_0), z_0) \;:\; r \geq 1\}$ that foliate the exterior of $S^1 \times \mathbb{R}$. Each of these leaves is a curve connecting a point of ∂D with q_∞. We can now prove Theorem 6.4.

Proof of Theorem 6.4. The union $\mathcal{Q}_0^+ \cup \mathcal{B}_0^+ \cup \mathcal{Q}_0^- \cup \mathcal{B}_0^-$ is a fundamental domain for the action of A on the boundary cylinder ∂D. As the foliation obtained above is A-invariant, the cone to the point q_∞ built over it via the foliation is a fundamental domain for the action of A over D, and thus, it is a fundamental domain for the action of Γ^{lim} on the region of discontinuity $\Omega(\Gamma^{\text{lim}})$.

This fundamental domain is the union of two pyramids $\mathcal{P}^+$ and $\mathcal{P}^-$, with respective bases $\mathcal{Q}_0^+ \cup \mathcal{B}_0^-$ and $\mathcal{Q}_0^- \cup \mathcal{B}_0^+$, and common vertex $q_\infty = p_{ST}$. The two pyramids share a common face, which is a triangle with vertices p_{STS}, $p_{T^{-1}S}$ and p_{ST}. Cutting and pasting, consider the union $\mathcal{P}^+ \cup S^{-1}(\mathcal{P}^-)$. It is again a fundamental domain for Γ^{lim}. The apex of $S^{-1}(\mathcal{P}^-)$ is $S^{-1}(q_\infty) = p_B = p_{TS}$. The image under S^{-1} of $\mathcal{Q}_0^-$ is $\mathcal{Q}_0^+$, and the bigon $\mathcal{B}_0^+$ is mapped by S^{-1} to another bigon connecting $p_{T^{-1}S}$ to p_{STS}. Since $\mathcal{B}_0^- = S(\mathcal{B}_0^+)$, this new bigon is the image of $\mathcal{B}_0^-$ under $S^{-2} = S$.

The resulting object is a is a polyhedron (a combinatorial picture is provided on Figure 8), whose faces are triangles and bigons. The faces of this octahedron are paired as follows.

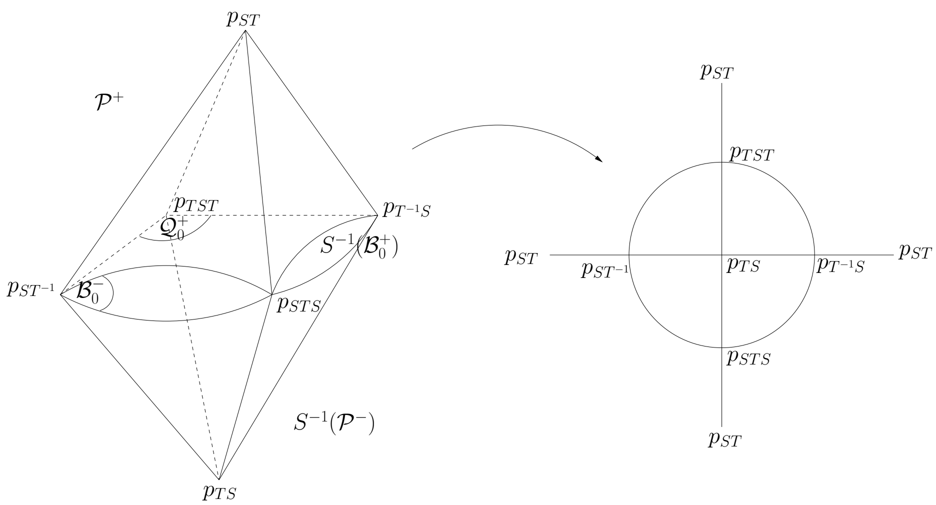


Figure 8: A combinatorial picture of the octahedron.

$$
\begin{aligned}
TS &: (p_{TS}, p_{T^{-1}S}, p_{STS}) \longmapsto (p_{TS}, p_{TST}, p_{TS^{-1}}),\\
ST &: (p_{ST}, p_{TST}, p_{T^{-1}S}) \longmapsto (p_{ST}, p_{ST^{-1}}, p_{STS}),\\
T &: (p_{ST}, p_{TST}, p_{ST^{-1}}) \longmapsto (p_{TS}, p_{T^{-1}S}, p_{TST}),\\
S &: (p_{TS}, p_{S^{-1}T}, p_{STS}) \longmapsto (p_{ST}, p_{STS}, p_{S^{-1}T}),\\
S &: (p_{ST^{-1}}, p_{STS}) \longmapsto (p_{STS}, p_{S^{-1}T})
\end{aligned}
$$

The last line is the bigon identification between $\mathcal{B}_0^-$ and $S^{-1}(\mathcal{B}_0^+)$. As the triangle $(p_{TS}, p_{S^{-1}T}, p_{STS})$ and the bigon $\mathcal{B}_0^-$ share a common edge and have the same face pairing they can be combined into a single triangle, as well as their images. Thus the last two lines may be combined into a single side with side pairing map S. We therefore obtain a true octahedron. The face identifications given above make the quotient manifold homeomorphic to the complement of the Whitehead link (compare for instance with Section 3.3 of [35]). □

7 Technicalities.

7.1 The triple intersections: proofs of Proposition 4.5 and Lemma 6.2.

In this section we first prove Proposition 4.5, which states that the triple intersection must contain a point of $\partial\mathbf{H}^2_{\mathbb{C}}$ and then we analyse the case of the limit group $\Gamma^{\lim}$, giving a proof of Lemma 6.2. First recall that the isometric spheres $\mathcal{I}_0^-$ and $\mathcal{I}_{-1}^-$ are the unit Heisenberg spheres with centres given respectively in geographical coordinates by (see 2.5)

$$
\begin{aligned}
\mathbf{p}_{AB} &= S(\infty) = g\big(-\alpha_1, -\alpha_1/2 + \alpha_2, \sqrt{2\cos(\alpha_1)}\big)\\
\mathbf{p}_{BA} &= A^{-1}S(\infty) = g\big(-\alpha_1, -\alpha_1/2 - \alpha_2 + \pi, \sqrt{2\cos(\alpha_1)}\big).
\end{aligned}
$$

Consider the two functions of points $q = g(\alpha, \beta, w) \in \mathcal{I}_0^+$ defined by

$$
\begin{aligned}
f^{[0]}_{\alpha_1,\alpha_2}(q) &= 2\cos^2(\alpha/2 - \alpha_1/2) + \cos(\alpha - \alpha_1) \qquad (28)\\
&\quad -4wx_1\cos(\alpha/2 - \alpha_1/2)\cos(\beta + \alpha_1/2 - \alpha_2) + w^2x_1^2,\\
f^{[-1]}_{\alpha_1,\alpha_2}(q) &= 2\cos^2(\alpha/2 - \alpha_1/2) + \cos(\alpha - \alpha_1)\\
&\quad +4wx_1\cos(\alpha/2 - \alpha_1/2)\cos(\beta + \alpha_1/2 + \alpha_2) + w^2x_1^2. \qquad (29)
\end{aligned}
$$

These functions characterise those points on $\mathcal{I}_0^+$ that belong to $\mathcal{I}_0^-$ and $\mathcal{I}_{-1}^-$.

Lemma 7.1. *A point q on $\mathcal{I}_0^+$ lies on $\mathcal{I}_0^-$ (respectively in its interior or exterior) if and only if it satisfies $f^{[0]}_{\alpha_1,\alpha_2}(q) = 0$ (respectively is negative or is positive). Similarly, a point q on $\mathcal{I}_0^+$ lies on $\mathcal{I}_{-1}^-$ (respectively in its interior or exterior) if and only if it satisfies $f^{[-1]}_{\alpha_1,\alpha_2}(q) = 0$ (respectively is negative or is positive).*

Proof. A point $q \in \mathcal{I}_0^+$ lies on $\mathcal{I}_0^-$ (respectively in its interior or exterior) if and only if its Cygan distance from the centre of $\mathcal{I}_0^-$, which is the point p_{AB}, equals 1 (respectively is less than 1 or greater than 1). Equivalently (see Section 2.4), the following quantity vanishes, is positive or negative respectively,

$$
\begin{aligned}
\big|\langle \mathbf{q}, \mathbf{p}_{AB}\rangle\big|^2 - 1 &= \Big|-e^{-i\alpha} + wx_1e^{-i\alpha/2 + i\beta - i\alpha_2} - e^{-i\alpha_1}\Big|^2 - 1\\
&= \Big|-2\cos(\alpha/2 - \alpha_1/2) + wx_1e^{i\beta + i\alpha_1/2 - i\alpha_2}\Big|^2 - 1\\
&= 4\cos^2(\alpha/2 - \alpha_1/2) - 1 - 4\cos(\alpha/2 - \alpha_1/2)wx_1\cos(\beta + \alpha_1/2 - \alpha/2) + w^2x_1^2\\
&= f^{[0]}_{\alpha_1,\alpha_2}(q).
\end{aligned}
$$

On the last line we used $2\cos^2(\alpha/2-\alpha_1/2)=1+\cos(\alpha-\alpha_1)$. This proves the first part of the Lemma and the second is obtained by a similar computation. □

Corollary 7.2. *For given (α_1,α_2), if the sum $f^{[0]}_{\alpha_1,\alpha_2}+f^{[-1]}_{\alpha_1,\alpha_2}$ is positive for all q, then the triple intersection $\mathcal{I}_0^+\cap\mathcal{I}_0^-\cap\mathcal{I}_{-1}^-$ is empty.*

See Figure 6. We can now prove Proposition 4.5.

Proof of Proposition 4.5. To prove the first part, note that a necessary condition for a point $q\in\mathcal{I}_0^+$ to be in the intersection $\mathcal{I}_0^-\cap\mathcal{I}_{-1}^-$ is that $f^{[0]}_{\alpha_1,\alpha_2}(q)-f^{[-1]}_{\alpha_1,\alpha_2}(q)=0$. This difference is:

$$\begin{aligned}
f^{[0]}_{\alpha_1,\alpha_2}(q)-f^{[-1]}_{\alpha_1,\alpha_2}(q) &= -4wx_1\cos(\alpha/2-\alpha_1/2)\big(\cos(\beta+\alpha_1/2-\alpha_2)+\cos(\beta+\alpha_1/2+\alpha_2)\big)\\
&= -8wx_1\cos(\alpha/2-\alpha_1/2)\cos(\beta+\alpha_1/2)\cos(\alpha_2).
\end{aligned}$$

Since α_1 and α_2 lie in $(-\pi/2,\pi/2)$ and $\alpha\in[-\pi/2,\pi/2]$, the only solutions are $\cos(\beta+\alpha_1/2)=0$ or $w=0$. Thus either $p=g(\alpha,\beta,w)$ lies on the meridian $\mathfrak{m}$, or on the spine of $\mathcal{I}_0^+$, and hence on every meridian, in particular on $\mathfrak{m}$ (compare with Proposition 2.9).

To prove the second part of Proposition 4.5, assume that the triple intersection contains a point $q=g\big(\alpha,(\pi/2-\alpha_1/2),w\big)$ inside $\mathbf{H}^2_{\mathbb{C}}$, that is such that $w^2<2\cos(\alpha)$, and

$$f^{[0]}_{\alpha_1,\alpha_2}(q)+f^{[-1]}_{\alpha_1,\alpha_2}(q)=0.$$

In view of Corollary 7.2, we only need to prove that there exists a point on $\partial\mathfrak{m}$ where the above sum is non-positive, and use the intermediate value theorem. To do so, let $\tilde\alpha$ be defined by the condition $2\cos(\tilde\alpha)=w^2$ and such that $\tilde\alpha$ and α_1 have opposite signs. Since $w^2<2\cos(\alpha)$, these conditions imply that $|\tilde\alpha|>|\alpha|$. We claim that the point $\tilde q=g\big(\tilde\alpha,(\pi-\alpha_1)/2,w\big)$ is satisfactory. Indeed, the conditions on $\tilde\alpha$ give

$$|\alpha-\alpha_1|\le|\alpha|+|\alpha_1|<|\tilde\alpha|+|\alpha_1|=|\tilde\alpha-\alpha_1|$$

where the last inequality follows from the fact that $\tilde\alpha$ and α_1 have opposite signs. Therefore

$$\cos(\tilde\alpha/2-\alpha_1/2)<\cos(\alpha/2-\alpha_1/2). \tag{30}$$

On the other hand, we have

$$\begin{aligned}
&f^{[0]}_{\alpha_1,\alpha_2}(q)+f^{[-1]}_{\alpha_1,\alpha_2}(q)\\
&\quad= 4\cos^2(\alpha/2-\alpha_1/2)+2\cos(\alpha-\alpha_1)-8wx_1\cos(\alpha/2-\alpha_1/2)\sin(\alpha_2)+2w^2x_1^2\\
&\quad= 8\cos^2(\alpha/2-\alpha_1/2)-2-8wx_1\cos(\alpha/2-\alpha_1/2)\sin(\alpha_2)+2w^2x_1^2.
\end{aligned}\tag{31}$$

We claim this is an increasing function of $\cos(\alpha/2-\alpha_1/2)$. In order to see this, observe that its derivative with respect to this variable is

$$16\cos(\alpha/2-\alpha_1/2)-8wx_1\sin(\alpha_2)>16\cos(\alpha/2-\alpha_1/2)-16\sqrt{\cos(\alpha)\cos(\alpha_1)}\ge 0,$$

where we used $x_1=\sqrt{2\cos(\alpha_1)}$, $w<\sqrt{2\cos(\alpha)}$ and $\sin(\alpha_2)\le 1$. Therefore,

$$\begin{aligned}
0 &= f^{[0]}_{\alpha_1,\alpha_2}(q)+f^{[-1]}_{\alpha_1,\alpha_2}(q)\\
&= 8\cos^2(\alpha/2-\alpha_1/2)-2-8wx_1\cos(\alpha/2-\alpha_1/2)\sin(\alpha_2)+2w^2x_1^2\\
&> 8\cos^2(\tilde\alpha/2-\alpha_1/2)-2-8wx_1\cos(\tilde\alpha/2-\alpha_1/2)\sin(\alpha_2)+2w^2x_1^2\\
&= f^{[0]}_{\alpha_1,\alpha_2}(\tilde q)+f^{[-1]}_{\alpha_1,\alpha_2}(\tilde q).
\end{aligned}$$

This proves our claim. □

We now prove Lemma 6.2 which completely describes the triple intersection at the limit point.

Proof of Lemma 6.2. From the first part of Proposition 4.5 we see that any point $q = g(\alpha, \beta, w)$ in $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ must lie on $\mathfrak{m}$, that is $\beta = (\pi - \alpha_1)/2$. For such points it is enough to show that $f^{[0]}_{0,\alpha_2^{\lim}}(q) + f^{[-1]}_{0,\alpha_2^{\lim}}(q) = 0$. Substituting $\alpha_1 = 0$ and $\sin(\alpha_2) = \sqrt{5/8}$, this becomes:

$$\begin{aligned} f^{[0]}_{0,\alpha_2^{\lim}}(q) + f^{[-1]}_{0,\alpha_2^{\lim}}(q) &= 4\cos^2(\alpha/2) + \cos(\alpha) - 4\sqrt{5}w\cos(\alpha/2) + 4w^2 \\ &= \Big(2\cos(\alpha/2) - \sqrt{5}w\Big)^2 + \big(2\cos(\alpha) - w^2\big). \end{aligned}$$

In order to vanish, both terms must be zero. Hence $w^2 = 2\cos(\alpha)$ and $2\cos(\alpha/2) = \sqrt{5}w = \sqrt{10\cos(\alpha)}$ (noting w cannot be negative since $\alpha \in [-\pi/2, \pi/2]$). This means $\alpha = \pm\arccos(1/4)$ and $w = \sqrt{2\cos(\alpha)} = 1/\sqrt{2}$. Therefore, the only points in $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ have geographical coordinates $g\big(\pm\arccos(1/4), \pi/2, 1/\sqrt{2}\big)$. Using (25), we see these points are $p_{ST^{-1}}$ and $p_{S^{-1}T}$. □

7.2 The region $\mathcal{Z}$ is an open disc in the region $\mathcal{L}$: Proof of Proposition 4.4.

Consider the group $\Gamma_{\alpha_1,\alpha_2}$ and, as usual, write $x_1^4 = 4\cos^2(\alpha_1)$ and $x_2^4 = 4\cos^2(\alpha_2)$. Recall, from Proposition 3.7, that (α_1, α_2) is in $\mathcal{L}$ (respectively $\mathcal{P}$) if $\mathcal{G}(x_1^4, x_2^4) > 0$ (respectively $= 0$) where:

$$\mathcal{G}(x, y) = x^2y^4 - 4x^2y^3 + 18xy^2 - 27. \tag{32}$$

Recall this means $[A, B]$ is loxodromic (respectively parabolic). Also (α_1, α_2) is in the rectangle $\mathcal{R}$ if and only if $(x_1^4, x_2^4) \in [3, 4] \times [3/2, 4]$. From Proposition 4.3, the point $(\alpha_1, \alpha_2) \in \mathcal{R}$ is in $\mathcal{Z}$ (respectively $\partial\mathcal{Z}$) if $\mathcal{D}(x_1^4, x_2^4) > 0$ (respectively $= 0$) where:

$$\mathcal{D}(x, y) = x^3y^3 - 9x^2y^2 - 27xy^2 + 81xy - 27x - 27. \tag{33}$$

Lemma 7.3. *Suppose $(\alpha_1, \alpha_2) \in \mathcal{R}$. Then $(\alpha_1, \alpha_2) \in \mathcal{L} \cup \mathcal{P}$. Moreover, $(\alpha_1, \alpha_2) \in \mathcal{P}$ if and only if $(\alpha_1, \alpha_2) = (0, \pm\alpha_2^{\lim})$.*

Proof. We first claim that the function $\mathcal{G}(x, y)$ has no critical points in $(0, \infty) \times (0, \infty)$. Indeed, the first partial derivatives of $\mathcal{G}(x, y)$ are

$$\mathcal{G}_x(x, y) = 2y^2(xy^2 - 4xy + 9), \quad \mathcal{G}_y(x, y) = 4xy(xy^2 - 3xy + 9).$$

These are not simultaneously zero for any positive values of x and y. As a consequence, the minimum of $\mathcal{G}$ on $[3, 4] \times [3/2, 4]$ is attained on the boundary of this rectangle. We then have:

$$\begin{aligned} \mathcal{G}(x, 3/2) &= \frac{27}{16}(4 - x)(5x - 4), & \mathcal{G}(x, 4) &= 9(32x - 3), \\ \mathcal{G}(3, y) &= 9(y - 1)\big(y^3 - 3y^2 + 3y + 3\big), & \mathcal{G}(4, y) &= (2y + 1)(2y - 3)^3. \end{aligned}$$

It is a simple exercise to check that under the assumptions that $3 \le x \le 4$ and $3/2 \le y \le 4$ all four of these terms are positive, except for when $(x, y) = (4, 3/2)$ in which case $\mathcal{G}(4, 3/2) = 0$. Then $(x_1^4, x_2^4) = (4, 3/2)$ if and only if $(\alpha_1, \alpha_2) = (0, \pm\alpha_2^{\lim})$; compare to Figure 3. □

Lemma 7.4. *The region $\mathcal{Z}$ is an open topological disc in $\mathcal{R}$ symmetric about the axes and intersecting them in the intervals $\{\alpha_2 = 0, -\pi/6 < \alpha_1 < \pi/6\}$ and $\{\alpha_1 = 0, -\alpha_2^{\lim} < \alpha_2 < \alpha_2^{\lim}\}$. Moreover, the only points of $\partial\mathcal{Z}$ that lie in the boundary of $\mathcal{R}$ are $(\alpha_1, \alpha_2) = (0, \pm\alpha_2^{\lim})$ and $(\alpha_1, \alpha_2) = (\pm\pi/6, 0)$.*

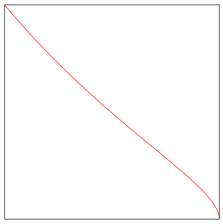


Figure 9: The null locus of $\mathcal{D}(x,y)$ in the rectangle $[3,4]\times[3/2,4]$.

Proof. First we examine the values of $\mathcal{D}(x,y)$ on the boundary of $[3,4]\times[3/2,4]$:

$$\begin{aligned} \mathcal{D}(x,3/2) &= \tfrac{27}{8}(x-4)(x^2-2x+2), & \mathcal{D}(x,4) &= (x-3)(3+8x)^2, \\ \mathcal{D}(3,y) &= 27(y-4)(y-1)^2, & \mathcal{D}(4,y) &= (16y-15)(2y-3)^2. \end{aligned} \tag{34}$$

We claim that, for any $y_0\in[3/2,4]$ the polynomial $\mathcal{D}(x,y_0)$ has exactly one root in $[3,4]$. Indeed, we have $\mathcal{D}(3,y_0)\le 0\le\mathcal{D}(4,y_0)$ and thus $\mathcal{D}(x,y_0)$ has at least one such root. The x-derivative of $\mathcal{D}$ is

$$\partial_x\mathcal{D}(x,y) = 3(x-3)y^2(xy+3y-6)+27(y-1)^3,$$

which is positive when $x\in[3,4]$ and $y\in[3/2,4]$. Thus $\mathcal{D}(x,y_0)$ is increasing, and the root is unique.

Similarly, we claim that, for any $x_0\in[3,4]$, the polynomial $\mathcal{D}(x_0,y)$ has a unique root in $[3/2,4]$. It is clear from (34) when $x_0=4$ (there the root is $y=3/2$). Now suppose $3\le x_0<4$. Arguing as before, we have $\mathcal{D}(x_0,3/2)<0\le\mathcal{D}(x_0,4)$. However, it is not true that $\mathcal{D}(x_0,y)$ is a monotone function of y. The partial derivative of $\mathcal{D}(x,y)$ with respect to y is

$$\partial_y\mathcal{D}(x,y) = 3x(x^2y^2-6xy-18y+27).$$

Therefore, for a fixed $x_0\in[3,4)$ we have $\partial_y\mathcal{D}(x_0,3/2)=27x_0^2(x_0-4)/4<0$. Since $\mathcal{D}(x_0,y)$ is a cubic with leading coefficient $x_0^3>0$, such that both $\mathcal{D}(x_0,3/2)$ and $\partial_y\mathcal{D}(x_0,3/2)$ are negative we see that $\mathcal{D}(x_0,y)$ has exactly one zero in $(3/2,\infty)$. Since $\mathcal{D}(x_0,4)\ge 0$ this zero must lies in $(3/2,4]$ as claimed.

Thus the zero-locus of $\mathcal{D}(x,y)$ in $[3,4]\times[3/2,4]$ is the graph of a continuous bijection connecting the two points $(3,4)$ and $(4,3/2)$. The polynomial $\mathcal{D}(x,y)$ is positive in the part of $[3,4]\times[3/2,4]$ above the zero-locus, that is containing the point $(x,y)=(4,4)$ (see Figure 9). Likewise, it is negative in the part below the zero locus, that is containing the point $(x,y)=(3,3/2)$. Changing coordinates to (α_1,α_2), we see that the zero locus of $\mathcal{D}\big(4\cos^2(\alpha_1),4\cos^2(\alpha_2)\big)$ in the rectangle $[0,\pi/6]\times[0,\alpha_2^{\lim}]$ is the graph of a continuous bijection connecting the points $(\alpha_1,\alpha_2)=(\pi/6,0)$ and $(0,\alpha_2^{\lim})$. Moreover, $\mathcal{D}$ is positive on the part below this curve, in particular on the interval $\alpha_1=0$ and $0\le\alpha_2<\alpha_2^{\lim}$ and the interval $\alpha_2=0$ and $0\le\alpha_1<\pi/6$. The region $\mathcal{Z}$ is the union of the four copies of this region by the symmetries about the horizontal and vertical coordinate axes. It is clearly a disc and contains the relevant parts of the axes. This completes the proof. □

Combining Lemmas 7.3 and 7.4 proves Proposition 4.4.

7.3 Condition for no triple intersections: Proof of Proposition 4.3.

In this section we find a condition on (α_1,α_2) that characterises the set $\mathcal{Z}$ where the triple intersection of isometric spheres $\mathcal{I}_0^+\cap\mathcal{I}_0^-\cap\mathcal{I}_{-1}^-$ is empty.

Lemma 7.5. *The triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ is empty if and only if $f_{\alpha_1,\alpha_2}(\alpha) > 0$ for all $\alpha \in [-\pi/2, \pi/2]$ where*

$$\begin{aligned} f_{\alpha_1,\alpha_2}(\alpha) &= 4\cos^2(\alpha/2-\alpha_1/2)+2\cos(\alpha-\alpha_1)+8\cos(\alpha)\cos(\alpha_1) \\ &\quad -16\sqrt{\cos(\alpha)\cos(\alpha_1)}\cos(\alpha/2-\alpha_1/2)\big|\sin(\alpha_2)\big|. \end{aligned} \tag{35}$$

Proof. By Corollary 7.2, it is enough to show that $f^{[0]}_{\alpha_1,\alpha_2} + f^{[-1]}_{\alpha_1,\alpha_2} > 0$. This sum is made explicit in (31). In view of the second part of Proposition (4.5), we can restrict our attention to showing that the triple intersection $\mathcal{I}_0^+ \cap \mathcal{I}_0^- \cap \mathcal{I}_{-1}^-$ contains no points of $\partial \mathbf{H}^2_{\mathbb{C}}$. That is, we may assume $w = \pm\sqrt{2\cos(\alpha)}$. Using the first part of Proposition 4.5 we restrict our attention to points m in the meridian $\mathfrak{m}$ where $\beta = (\pi - \alpha_1)/2$. The triple intersection is empty if and only if the sum $f^{[0]}_{\alpha_1,\alpha_2}(q) + f^{[-1]}_{\alpha_1,\alpha_2}(q)$ is positive for any value of α, where $q = g\big(\alpha, (\pi - \alpha_1/2), \pm\sqrt{2\cos(\alpha)}\big)$. When $w\sin(\alpha_2)$ is negative all terms in (31) are positive. Therefore we may suppose $w\sin(\alpha_2) = \sqrt{2\cos(\alpha_1)}\big|\sin(\alpha_2)\big| \geq 0$. Substituting these values in the expression for $f^{[0]}_{\alpha_1,\alpha_2}(q) + f^{[-1]}_{\alpha_1,\alpha_2}(q)$ given in (31) gives the function $f_{\alpha_1,\alpha_2}(\alpha)$ in (35). □

We want to convert (35) into a polynomial in a function of α. The numerical condition given in the statement of Proposition 4.3 will follow from the next lemma.

Lemma 7.6. *If $\alpha \in [-\pi/2, \pi/2]$ is a zero of f_{α_1,α_2} then $T_\alpha = \tan(\alpha/2) \in [-1, 1]$ is a root of the quartic polynomial $L_{\alpha_1,\alpha_2}(T)$, where*

$$\begin{aligned} L_{\alpha_1,\alpha_2}(T) &= T^4\left(2x_1^4x_2^4 - 4x_1^2x_2^4 + x_1^4 + 10x_1^2 + 1\right) - 8T^3\sin(\alpha_1)\left(x_1^2x_2^4 - x_1^2 - 1\right) \\ &\quad -2T^2\left(2x_1^4x_2^4 + 3x_1^4 - 9\right) + 8T\sin(\alpha_1)\left(x_1^2x_2^4 - x_1^2 + 1\right) \\ &\quad + \left(2x_1^4x_2^4 + 4x_1^2x_2^4 + x_1^4 - 10x_1^2 + 1\right) \end{aligned} \tag{36}$$

Proof. Squaring the two lines of (35) and using $\sqrt{2\cos(\alpha_1)}\big|\sin(\alpha_2)\big| \geq 0$, we see that the condition $f_{\alpha_1,\alpha_2}(\alpha) = 0$ is equivalent to

$$\Big(2\cos^2(\alpha/2-\alpha_1/2)+\cos(\alpha-\alpha_1)+4\cos(\alpha)\cos(\alpha_1)\Big)^2 = 64\cos(\alpha)\cos(\alpha_1)\cos^2(\alpha/2-\alpha_1/2)\sin^2(\alpha_2).$$

After rearranging and expanding, we obtain the following polynomial equation in $\cos(\alpha)$ and $\sin(\alpha)$.

$$\begin{aligned} 0 &= 4\left(8\cos^2(\alpha_1)\cos^2(\alpha_2)+2\cos^2(\alpha_1)-1\right)\cos^2(\alpha)+4\cos(\alpha_1)\big(8\cos^2(\alpha_2)-5\big)\cos(\alpha) \\ &\quad +8\cos(\alpha_1)\sin(\alpha_1)\left(4\cos^2(\alpha_2)-1\right)\cos(\alpha)\sin(\alpha)+4\sin(\alpha_1)\sin(\alpha)-4\cos^2(\alpha_1)+5. \end{aligned}$$

Substituting $\tan(\alpha/2) = T$, $2\cos(\alpha_1) = x_1^2$ and $2\cos(\alpha_2) = x_2^2$ into this equation gives $L_{\alpha_1,\alpha_2}(T)$. □

Before proving Proposition 4.3, we analyse the situation on the axes $\alpha_1 = 0$ and $\alpha_2 = 0$.

Lemma 7.7. *Let $L_{\alpha_1,\alpha_2}(T)$ be given by (36).*

1. *When $\alpha_2 = 0$ and $-\pi/6 < \alpha_1 < \pi/6$ then $L_{\alpha_1,0}(T)$ has two real double roots T_- and T_+ where $T_- < -1$ and $T_+ > 1$, and no other roots.*
2. *When $\alpha_1 = 0$ and $0 < \alpha_2 < \alpha_2^{\lim}$ or $-\alpha_2^{\lim} < \alpha_2 < 0$ the polynomial $L_{0,\alpha_2}(T)$ has no real roots.*

Proof. First, substituting $\alpha_2 = 0$ in (36) we find $L_{(\alpha_1,0)} = M_{\alpha_1}(T)^2$, where

$$M_{\alpha_1}(T) = T^2(3x_1^2 - 1) - 4T\sin(\alpha_1) - (3x_1^2 + 1).$$

The condition on α_1 guarantees that $3x_1^2 - 1 > 0$ and so as T tends to $\pm\infty$ so $M_{\alpha_1}(T)$ tends to $+\infty$. On the other hand,

$$M_{\alpha_1}(-1) = 4\sin(\alpha_1) - 2 < 0, \quad M_{\alpha_1}(1) = -4\sin(\alpha_1) - 2 < 0.$$

Therefore $M_{\alpha_1}(T)$ has two real roots $T_- < -1$ and $T_+ > 1$ as claimed. Since $M_{\alpha_1}(T)$ is quadratic, it cannot have any more roots. In particular, it is negative for $-1 \le T \le 1$.

Secondly, we substitute $\alpha_1 = 0$ in (36), giving:

$$L_{0,\alpha_2}(T) = \left(5T^2 - \frac{8x_2^4 + 3}{5}\right)^2 + \frac{32}{25}(2x_2^4 - 3)(4 - x_2^4).$$

When $\alpha_2 \in (-\alpha_2^{\lim}, \alpha_2^{\lim})$ and $\alpha_2 \neq 0$, we have $x_2^4 = 4\cos^2(\alpha_2) \in (3/2, 4)$. In particular, this means that $(2x_2^4 - 3)(4 - x_2^4) > 0$ and so $L_{0,\alpha_2}(T)$ has no real roots, proving the second part. □

We note that when $\alpha_1 = \alpha_2 = 0$ then $L_{0,0}(T)$ has double roots at $T = \pm\sqrt{7/5}$ and when $\alpha_1 = 0$ and $\alpha_2 = \pm\alpha_2^{\lim}$ then $L_{0,\pm\alpha_2^{\lim}}(T)$ has double roots at $T = \pm\sqrt{3/5}$.

Lemma 7.8. *If $(\alpha_1, \alpha_2) \in \mathcal{Z}$ then the polynomial $L_{\alpha_1,\alpha_2}(T)$ has no roots T in $[-1, 1]$.*

Proof. We analyse the number, type (real or non-real) and location of roots of the polynomial $L_{\alpha_1,\alpha_2}(T)$ when $(\alpha_1, \alpha_2) \in \mathcal{R}$. As $L_{\alpha_1\alpha_2}(T)$ has real coefficients, whenever it has only simple roots, its root set is of one of the following types:

(a) two pairs of complex conjugate non real simple roots,

(b) a pair of non-real complex conjugate simple roots and two simple real roots,

(c) four simple real roots.

But the set of roots of a polynomial is a continuous map (in bounded degree) for the Hausdorff distance on compact subsets of $\mathbb{C}$. In particular, the root set type of $L_{\alpha_1,\alpha_2}(T)$ is a continuous function of α_1 and α_2. This implies that it is not possible to pass from one of the above types to another without passing through a polynomial having a double root.

We compute the discriminant $\Delta_{\alpha_1,\alpha_2}$ of $L_{\alpha_1,\alpha_2}(T)$ (a computer may be useful to do so):

$$\Delta_{\alpha_1,\alpha_2} = 2^{16}x_1^4\big(x_1^4 + 1\big)^2\big(2x_1^2(2 - x_1^2)(4 - x_2^4) + (3x_1^2 - 1)^2\big)\big(4 - x_2^4\big)^2 \cdot \mathcal{D}\big(x_1^4, x_2^4\big) \tag{37}$$

where $\mathcal{D}(x, y)$ is as in Proposition 4.3, and $x_i = \sqrt{2\cos(\alpha_i)}$. The polynomial $L_{\alpha_1,\alpha_2}(T)$ has a multiple root in $\mathbb{C}$ if and only if $\Delta_{\alpha_1\alpha_2} = 0$. Let us examine the different factors.

- The first two factors x_1^4 and $(x_1^4 + 1)^2$ are positive when $(\alpha_1, \alpha_2) \in (-\pi/2, \pi/2)^2$.
- Note that $(2 - x_1^2)(4 - x_2^4) \ge 0$ and $(3x_1^2 - 1)^2 > 0$ when $\sqrt{3} \le x_1^2 \le 2$ and $x_2^4 \le 4$, and so the third factor is positive.

Thus, the only factors of $\Delta_{\alpha_1,\alpha_2}$ that can vanish on $\mathcal{R}$ are $(4 - x_2^4)^2 = 16\sin^4\alpha_2$ and $\mathcal{D}(x_1^4, x_2^4)$. In particular $L_{\alpha_1,\alpha_2}(T)$ has a multiple root in $\mathbb{C}$ if and only if one of these two factors vanishes. We saw in Proposition 4.4 that the subset of $\mathcal{R}$ where $\mathcal{D}(x_1^4, x_2^4) > 0$ is a topological disc $\mathcal{Z}$, symmetric about the α_1 and α_2 axes and intersecting them in the intervals $\{\alpha_2 = 0,\ -\pi/6 < \alpha_1 < \pi/6\}$ and $\{\alpha_1 = 0,\ -\alpha_2^{\lim} < \alpha_2 < \alpha_2^{\lim}\}$. Therefore, the rectangle $\mathcal{R}$ contains two open discs on which $\Delta_{\alpha_1,\alpha_2} > 0$, namely

$$\mathcal{Z}^+ = \{(\alpha_1, \alpha_2) \in \mathcal{Z} \ :\ \alpha_2 > 0\}, \quad \mathcal{Z}^- = \{(\alpha_1, \alpha_2) \in \mathcal{Z} \ :\ \alpha_2 < 0\}.$$

These two sets each contain an open interval of the α_2 axis. We saw in the second part of Lemma 7.7 that on both these intervals $L_{\alpha_1,\alpha_2}(T)$ has no real roots, that is its roots are of type (a). Therefore it has no real roots on all of $\mathcal{Z}^+$ and $\mathcal{Z}^-$.

The only points of $\mathcal{Z}$ yet to be considered are those in the interval $\{\alpha_2 = 0,\ -\pi/6 < \alpha_1 < \pi/6\}$. We saw in the first part of Lemma 7.7 that for such points $L_{\alpha_1,\alpha_2}(T)$ has no roots with $-1 \le T \le 1$. This completes the proof of Proposition 4.3. □

7.4 Pairwise intersection: Proof of Proposition 4.6.

Proposition 4.6 will follow from the next lemma.

Lemma 7.9. *If $0 < x \le 4$ and $\mathcal{D}(x,y) \ge 0$ then $xy \ge 6$ with equality if and only if $(x,y) = (4,3/2)$.*

Proof. Substituting $y = 6/x$ in (33) and simplifying, we find $\mathcal{D}(x,6/x) = -27(x-4)(x-9)/x$. When $0 < x \le 4$ we see immediately that this is non-positive and equals zero if and only if $x = 4$. This means that $xy - 6$ has a constant sign on the region where $\mathcal{D}(x,y) > 0$. Checking at $(x,y) = (4,4)$ we see that it is positive. □

Proof of Proposition 4.6. To prove the disjointness of the given isometric spheres we calculate the Cygan distance between their centres. Since all the isometric spheres have radius 1, if we can show their centres are a Cygan distance at least 2 apart, then the spheres are disjoint. (Note that the Cygan distance is not a path metric, so it maybe the distance is less than 2 but the spheres are still disjoint. This will not be the case in our examples.)

The centre of I_k^+ is $A^k(p_B) = \big[kx_1x_2^2/\sqrt{2}, kx_1^2x_2^2\sin(\alpha_2)\big]$; see Proposition 4.1. We will show that $d_{\rm Cyg}\big(A^k(p_B), p_B\big)^4 > 16$ when $k^2 \ge 4$ and $(\alpha_1,\alpha_2) \in \mathcal{R}$, that is $(x_1^4, x_2^4) \in [3,4] \times [3/2,4]$:

$$\begin{aligned} d_{\rm Cyg}\big(A^k(p_B), p_B\big)^4 &= \frac{k^4x_1^4x_2^8 + k^2x_1^4x_2^4(4-x_2^4)}{4} \\ &\ge \frac{27k^4}{16}. \end{aligned}$$

This number is greater than 16 when $k \ge 2$ or $k \le -2$ as claimed.

Again using Proposition 4.1, the centre of $\mathcal{I}_k^-$ is $A^k(p_{AB}) = \big[(kx_1x_2^2 + x_1e^{i\alpha_2})/\sqrt{2}, -\sin(\alpha_1)\big]$. We suppose $(x_1^4, x_2^4) \in [3,4] \times [3/2,4]$ satisfies $x_1^4x_2^4 \ge 6$, which is valid for $(\alpha_1,\alpha_2) \in \mathcal{Z}$ by Lemma 7.9.

$$\begin{aligned} d_{\rm Cyg}\big(A^k(p_{AB}), p_B\big)^4 &= \frac{\big(k(k+1)x_1^2x_2^4 + x_1^2\big)^2 + 4 - x_1^4}{4} \\ &= 1 + \frac{k^2(k+1)^2x_1^4x_2^8 + 2k(k+1)x_1^4x_2^4}{4} \\ &\ge \left(\frac{3k(k+1)}{2} + 1\right)^2. \end{aligned}$$

This number is at least 16 when $k \ge 1$ or $k \le -2$ as claimed. Moreover, we have equality exactly when $k = 1$ or $k = -2$ and when $x_1^4x_2^4 = 6$ and $x_2^4 = 3/2$; that is when $(x_1^4, x_2^4) = (4, 3/2)$. □

7.5 $\partial_\infty D$ is a cylinder: Proof of Proposition 6.8.

To prove Proposition 6.8, we adopt the following strategy.

- **Step 1.** First, we intersect D with a fundamental domain D_A for the action of A on the Heisenberg group. The domain D_A is bounded by two parallel vertical planes F_{-1} and F_0 that are boundaries of *fans* in the sense of [18]. These two fans are such that $A(F_{-1}) = F_0$ (see Figure 5 for a view of the situation in vertical projection). We analyse the intersections of F_0 and F_{-1} with D, and show that they are topological circles, denoted by c_{-1} and c_0 with $A(c_{-1}) = c_0$.
- **Step 2.** Secondly, we consider the subset of the complement of D which is contained in D_A, and prove that it is a 3-dimensional ball that intersects F_{-1} and F_0 along topological discs (bounded by c_{-1} and c_0). This proves that $D \cap D_A$ is the complement a solid tube in D_A, which is unknotted using Lemma 6.7. Finally, we prove that, gluing together copies by powers of A of $D \cap D_A$, we indeed obtain the complement of a solid cylinder.

We construct a fundamental domain D_A for the cyclic group of Heisenberg translations $\langle A \rangle$. The domain D_A will be bounded by two fans, chosen to intersect as few bisectors as possible. The fan F_0 will pass through $p_{ST^{-1}}$ and will be tangent to both $\mathcal{I}_1^+$ and $\mathcal{I}_{-1}^-$; compare Figure 5. Similarly, $F_{-1} = A^{-1}(F_0)$ will pass through $A^{-1}(p_{ST^{-1}}) = p_{TST}$ and be tangent to both $\mathcal{I}_0^+$ and $\mathcal{I}_{-2}^-$. We first give F_0 and F_{-1} in terms of horospherical coordinates and then we give them in terms of their own geographical coordinates (see [18]). In horospherical coordinates they are:

$$F_0 = \Big\{[x+iy,t] \ : \ 3x\sqrt{3} - y\sqrt{5} = \sqrt{2}/2\Big\}, \tag{38}$$

$$F_{-1} = \Big\{[x+iy,t] \ : \ 3x\sqrt{3} - y\sqrt{5} = -4\sqrt{2}\Big\}. \tag{39}$$

This leads to the definition of D_A:

$$D_A = \Big\{[x+iy,t] \ : \ -4\sqrt{2} \le 3x\sqrt{3} - y\sqrt{5} \le \sqrt{2}/2\Big\}. \tag{40}$$

We choose geographical coordinates (ξ, η) on F_0: the lines where ξ is constant (respectively η is constant) are boundaries of complex lines (respectively Lagrangian planes). These coordinates correspond to the double foliation of fans by real planes and complex lines, which is described in Section 5.2 of [18]. The particular choice is made so that the origin is the midpoint of the centres of $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$. Doing so gives the fan F_0 as the set of points $f(\xi,\eta)$:

$$f(\xi,\eta) = \left\{ \left[\frac{\sqrt{5}\xi + \sqrt{3} + 3i\sqrt{3}\xi + i\sqrt{5}}{4\sqrt{2}},\ \eta - \xi/4 \right] \ : \ \xi,\, \eta \in \mathbb{R} \right\}.$$

The standard lift of $f(\xi,\eta)$ is given by

$$\mathbf{f}(\xi,\eta) = \begin{bmatrix} -\xi^2 - \sqrt{15}\xi/4 - 1/4 + i\eta - i\xi/4 \\ \sqrt{5}\xi/4 + \sqrt{3}/4 + 3i\sqrt{3}\xi/4 + i\sqrt{5}/4 \\ 1 \end{bmatrix}.$$

Using the convexity of Cygan spheres, we see that their intersection with F_0 (or F_{-1}) is one of: empty, a point or a topological circle. For the particular fans and isometric spheres of interest to us, the possible intersections are summarised in the following result:

Proposition 7.10. *The intersections of the fans F_{-1} and F_0 with the isometric spheres $\mathcal{I}_k^\pm$ are empty, except for those indicated in the following table.*

$\bigcap$	$\mathcal{I}_{-2}^-$	$\mathcal{I}_{-2}^+$	$\mathcal{I}_{-1}^-$	$\mathcal{I}_{-1}^+$	$\mathcal{I}_0^-$	$\mathcal{I}_0^+$	$\mathcal{I}_1^-$	$\mathcal{I}_1^+$
F_0	$\emptyset$	$\emptyset$	$\{p_{ST^{-1}}\}$	$\emptyset$	*a circle*	*a circle*	$\emptyset$	$\{p_{ST^{-1}}\}$
F_{-1}	$\{p_{TST}\}$	$\emptyset$	*a circle*	*a circle*	$\emptyset$	$\{p_{TST}\}$	$\emptyset$	$\emptyset$

Moreover, the point $p_{S^{-1}T}$ belongs to the interior of D_A. The parabolic fixed points $A^k(p_{ST^{-1}})$ lie outside D_A for all $k \geq 1$ and $k \leq -1$; parabolic fixed points $A^k(p_{S^{-1}T})$ lie outside D_A for all $k \neq 0$.

A direct consequence of this proposition is that the only point in the closure of the quadrilateral $\mathcal{Q}_{-1}^-$ and the bigon $\mathcal{B}_{-1}^-$ that lie on F_0 is their vertex p_{TST}.

Proof. The part about intersections of fans and isometric spheres is proved easily by projecting vertically onto $\mathbb{C}$, as in the proof of Proposition 4.6 (see Figure 5). Note that as isometric spheres are strictly convex, their intersections with a plane is either empty or a point or a topological circle. The part about the parabolic fixed points is a direct verification using (38) as well as (25). □

We need to be slightly more precise about the intersection of F_0 with $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$.

Proposition 7.11. *The intersection of F_0 with $\mathcal{I}_0^+ \cup \mathcal{I}_0^-$ (and thus with ∂D) is a topological circle c_0, which is the union of two topological segments c_0^+ and c_0^-, where the segment $c_0^\pm$ is the part of $F_0 \cap \mathcal{I}_0^\pm$ that is outside $\mathcal{I}_0^\mp$. The two segments c_0^+ and c_0^- have the same endpoints: one of them is $p_{ST^{-1}}$, and we will denote the other by q_0. Moreover, the point q_0 lies on the segment $[p_{STS}, p_{S^{-1}T}]$ of $\mathcal{I}_0^+ \cap \mathcal{I}_0^-$.*

The point q_0 appears in Figures 10, 11 and 12

Proof. The point $f(\xi,\eta)$ of the fan F_0 lies of $\mathcal{I}_0^+$ whenever $1 = \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_B\rangle\right|$ and on $\mathcal{I}_0^-$ whenever $1 = \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_{AB}\rangle\right|$. We first find all points on $F_0 \cap \mathcal{I}_0^+ \cap \mathcal{I}_0^-$. These correspond to simultaneous solutions to:

$$1 = \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_B\rangle\right| = \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_{AB}\rangle\right| \tag{41}$$

Computing these products and rearranging, we obtain

$$\begin{aligned}\left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_B\rangle\right|^2 &= (\xi^2+1/4)^2 + \xi^2 + \eta^2 + \xi\Big(\sqrt{15}\xi^2 + \sqrt{15}/4 - \eta\Big)/2,\\ \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_{AB}\rangle\right|^2 &= (\xi^2+1/4)^2 + \xi^2 + \eta^2 - \xi\Big(\sqrt{15}\xi^2 + \sqrt{15}/4 - \eta\Big)/2.\end{aligned}$$

Substracting, we see that solutions to (41) must either have $\xi = 0$ or $\eta = \sqrt{15}(\xi^2 + 1/4)$. Substituting these solutions into $1 = \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_B\rangle\right|^2$ we see first that $\xi = 0$ implies $1 = \eta^2 + 1/16$; and secondly that $\eta = \sqrt{15}(\xi^2 + 1/4)$ implies

$$1 = (\xi^2+1/4)^2 + \xi^2 + 15(\xi^2+1/4)^2 = (4\xi^2+1)^2 + \xi^2.$$

Clearly the only solution to this equation is $\xi = 0$. So both cases lead to the solutions $(\xi,\eta) = (0, \pm\sqrt{15}/4)$. Thus the only points satisfying (41), that is the points in $F_0 \cap \mathcal{I}_0^+ \cap \mathcal{I}_0^-$, are

$$f(0,\sqrt{15}/4) = \left[\frac{\sqrt{3}+i\sqrt{5}}{4\sqrt{2}},\ \frac{\sqrt{15}}{4}\right] \text{ and } f(0,-\sqrt{15}/4) = \left[\frac{\sqrt{3}+i\sqrt{5}}{4\sqrt{2}},\ \frac{-\sqrt{15}}{4}\right].$$

Note that the first of these points is $p_{ST^{-1}}$. We call the other point q_0.

These two points divide $F_0 \cap \mathcal{I}_0^+$ and $F_0 \cap \mathcal{I}_0^-$ into two arcs. It remains to decide which of these arcs is outside the other isometric sphere. Clearly $\left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_B\rangle\right| > \left|\langle \mathbf{f}(\xi,\eta), \mathbf{p}_{AB}\rangle\right|$ if and only if $\xi\big(\sqrt{15}\xi^2 + \sqrt{15}/4 - \eta\big) > 0$. Close to $\eta = -\sqrt{15}/4$ we see this quantity changes sign only when ξ does. This means that if $f(\xi,\eta) \in \mathcal{I}_0^-$ with $\xi > 0$ then $f(\xi,\eta)$ is in the exterior of $\mathcal{I}_0^+$. Similarly, if $f(\xi,\eta) \in \mathcal{I}_0^+$ with $\xi < 0$ then $f(\xi,\eta)$ is in the exterior of $\mathcal{I}_0^-$. In other words, c_0^+ is the segment of $F_0 \cap \mathcal{I}_0^+$ where $\xi < 0$ and c_0^- is the segment of $F_0 \cap \mathcal{I}_0^-$ where $\xi > 0$.

Finally, consider the involution $I_2 = SI_1$ in $\mathrm{PU}(2,1)$ from the proof of Proposition 3.6. (Note that since $\alpha_1 = 0$ this involution conjugates $\Gamma^{\lim}$ to itself.) The involution I_2 preserves F_0, acting on

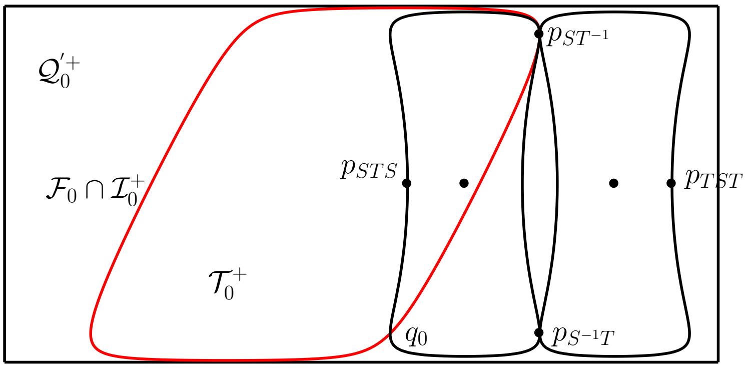


Figure 10: The intersection of F_0 with I_0^+ drawn on $\mathcal{I}_0^+$, in geographical coordinates.

it by $f(\xi,\eta)$ to $f(-\xi,\eta)$, and hence interchanging the components of its complement. In Heisenberg coordinates I_2 is given by

$$I_2 : \Big[x+iy,\, t\Big] \longleftrightarrow \Big[-x-iy+\sqrt{3/8}+i\sqrt{5/8},\, t-\sqrt{5/2}\,x+\sqrt{3/2}\,y\Big]. \tag{42}$$

As I_2 is elliptic and fixes the point q_∞, it is a Cygan isometry (see Section 2.4). Since it interchanges p_B and p_{AB}, it also interchanges $\mathcal{I}_0^+$ and $\mathcal{I}_0^-$. Hence their intersection is preserved setwise. The involution I_2 also interchanges $p_{S^{-1}T}$ and p_{STS} contained in $\mathcal{I}_0^+ \cap \mathcal{I}_0^-$ (but not on F_0). Therefore, these two points lie in different components of the complement of F_0. Hence there must be a point of F_0 on the segment $[p_{S^{-1}T}, p_{STS}]$. This point cannot be $p_{ST^{-1}}$, and so must be q_0 (see Figure 10). □

Let D^c denote closure of the complement of D in $\partial\mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$.

Proposition 7.12. *The closure of the intersection $D^c \cap D_A$ is a solid tube homeomorphic to a 3-ball.*

Proof. We describe the combinatorial cell structure of $D^c \cap D_A$; see Figure 12. Using Proposition 7.11, it is clear D^c intersects F_0 in a topological disc whose boundary circle is made up of two edges, $c_0^\pm$ and two vertices $p_{ST^{-1}}$ and q_0. Combinatorially, this is a bigon. Applying A^{-1} we see D^c intersects F_{-1} in a bigon with boundary made up of edges $c_{-1}^\pm$ and two vertices p_{TST} and q_{-1}.

Moreover, Proposition 7.11 immediately implies that c_0 cuts $\mathcal{Q}_0^\pm$ into a quadrilateral and a triangle, which we denote by $\mathcal{Q}'^\pm_0$ and $\mathcal{T}_0^\pm$. Since D_A contains $p_{S^{-1}T}$ and p_{TST} we see that D_A contains $\mathcal{Q}'^+_0$ and $\mathcal{T}_0^-$. These have vertex sets $\{p_{ST^{-1}},\, p_{TST},\, p_{S^{-1}T},\, q_0\}$ and $\{p_{S^{-1}T},\, p_{ST^{-1}},\, q_0\}$ respectively. Applying A^{-1} we see that c_{-1} cuts $\mathcal{Q}_{-1}^\pm$ into a quadrilateral an a triangle. Of these the quadrilateral $\mathcal{Q}'^-_{-1}$ and the triangle $\mathcal{T}_{-1}^+$ lie in D_A. Finally, the bigons $\mathcal{B}_0^+$ and $\mathcal{B}_{-1}^-$ also lie in D_A.

In summary, the boundary of $D^c \cap D_A$ has a combinatorial cell structure with five vertices $\{p_{ST^{-1}},\, p_{S^{-1}T},\, p_{TST},\, q_0,\, q_{-1}\}$ and eight faces.

$$\{\mathcal{Q}'^+_0,\, \mathcal{Q}'^-_{-1},\, \mathcal{T}_0^-,\, \mathcal{T}_{-1}^+,\, \mathcal{B}_0^+,\, \mathcal{B}_{-1}^-,\, F_0 \cap D^c,\, F_{-1} \cap D^c\}.$$

These are respectively two quadrilaterals, two triangles and four bigons. Therefore, in total the cell structure has $(2\times 4 + 2\times 3 + 4\times 2)/2 = 11$ edges. Therefore the Euler characteristic of $\partial(D^c \cap D_A)$ is

$$\chi\big(\partial(D^c \cap D_A)\big) = 5 - 11 + 8 = 2.$$

Hence $\partial(D^c \cap D_A)$ is indeed a sphere. This means $D^c \cap D_A$ is a ball as claimed. □

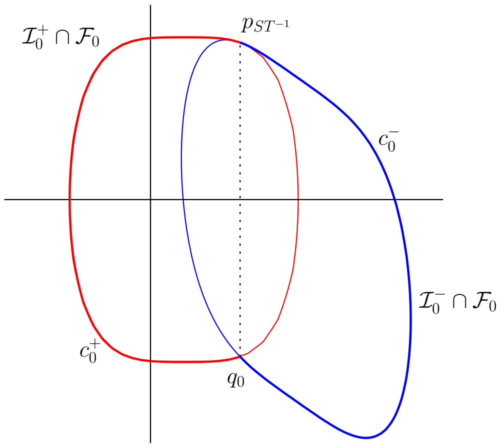


Figure 11: The intersection of F_0 wit $\mathcal{I}_0^+ \cap \mathcal{I}_0^-$. The disc $\mathcal{D}_0$ is the interior of $c_0 = c_0^+ \cap c_0^-$. The two segments c_0^+ and c_0^- are the thicker parts of $F_0 \cap \mathcal{I}_0^+$ and $F_0 \cap \mathcal{I}_0^-$.

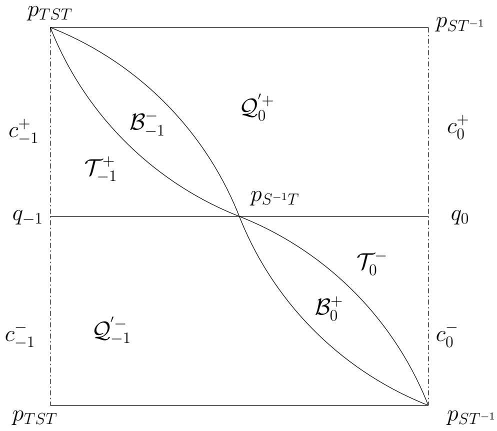


Figure 12: A combinatorial picture of the intersection of ∂D with D_A . The top and bottom lines are identified. The curve c_0 corresponds to the right hand side of the figure.

Remark 2. The combinatorial structure described on Figure 12 is quite simple. However, the geometric realisation of this structure is much more intricate. As an example, there are fans F parallel to F_0 and F_{-1} whose intersection with D^c is disconnected. This means that the foliation described right after Proposition 6.8 that is used in the proof of Theorem 6.4 is actually quite "distorted".

We are know ready to prove Proposition 6.8.

Proposition 7.13. *There is a homeomorphism $\Psi_A : \mathbb{R}^2 \times [0,1] \longrightarrow D_A$ that satisfies $\Psi_A(x,y,1) = A\Psi_A(x,y,0)$ and so that Ψ_A restricts to a homeomorphism from the exterior of $S^1 \times [0,1]$, that is $\big\{(x,y,z)\ :\ x^2+y^2 \geq 1,\, 0 \leq z \leq 1\big\}$, to $D \cap D_A$.*

Proof. We have shown Proposition 7.12 that $D^c \cap D_A$ is a solid tube homeomorphic to a 3-ball and (using Proposition 7.11) that D^c intersects ∂D_A in two discs, one in F_0 bounded by c_0 and the other in F_{-1} bounded by c_{-1}. This means we can construct a homeomorphism Ψ_A^c from the solid cylinder $\big\{(x,y,z)\ :\ x^2+y^2 \leq 1,\, 0 \leq z \leq 1\big\}$ to $D^c \cap D_A$ so that the restriction of Ψ_A^c to $S^1 \times [0,1]$ is a homeomorphism to $\partial D \cap D_A$, with $\Psi_A^c : S^1 \times \{0\} \longmapsto c_{-1}$ and $\Psi_A^c : S^1 \times \{1\} \longmapsto c_0$. Adjusting Ψ_A^c if necessary, we can assume that $\Psi_A^c(x,y,1) = A\Psi_A^c(x,y,0)$.

Furthermore, in Lemma 6.7, we showed that D^c contains the invariant line Δ_φ of φ. This means that the cylinder $D^c \cap D_A$ is a thickening of $\Delta_\varphi \cap D_A$ and so, in particular, it cannot be knotted. Hence we can extend Ψ_A^c to homeomorphism $\Psi_A : \mathbb{R}^2 \times [0,1] \longrightarrow D_A$ satisfying $\Psi_A(x,y,1) = A\Psi_A(x,y,0)$. In particular, Ψ maps $\big\{(x,y,z)\ :\ x^2+y^2 \geq 1,\, 0 \leq z \leq 1\big\}$ homeomorphically to $D \cap D_A$ as claimed. □

Finally, we prove Proposition 6.8 by extending $\Psi_A : \mathbb{R}^2 \times [0,1] \longrightarrow D_A$ equivariantly to a homeomorphism $\Psi : \mathbb{R}^3 \longmapsto \partial \mathbf{H}^2_{\mathbb{C}} - \{q_\infty\}$. That is, if $(x,y,z+k) \in \mathbb{R}^3$ where $k \in \mathbb{Z}$ and $z \in [0,1]$, we define $\Psi(x,y,z+k) = A^k(x,y,z)$. Since $\Psi(x,y,1) = A\Psi(x,y,0)$ there is no ambiguity at the boundary.

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