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A COMPLEX VARIABLE INTEGRATION TECHNIQUE FOR THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

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Abstract. Starting from a complex variable formulation for the two-dimensional steady flow equations describing the motion of a viscous incompressible liquid, a method is developed which carries out three integrations of the fourth order system in parametric form containing three arbitrary real functions.

Introduction. It is a feature of nonlinear differential equations that even when an exact solution is available it is not always possible to express the dependent variables as explicit functions of the independent variables. Clearly this can be a disadvantage when the dependent variables represent unknown physical quantities and the independent variables may be space and time. However, in some cases the solution can be parametrized in terms of derivatives, as in the elementary example $x = pe^p$, $p = dy/dx$. Differentiation with respect to y, followed by integration with respect to p, leads to a second equation $y = (p^2 - p + 1)e^p + c$ from which the net gain is a parametrization of x and y in terms of the first derivative p containing an arbitrary constant c. In this example it is possible to eliminate p to determine a relation between x, y, and c, although this is not the case with the more general equation $x = f(p)$, which can be treated by the same technique. In general parametric representation represents a powerful method for displaying solutions of nonlinear differential equations especially when the solutions bifurcate as in the case of the Navier-Stokes equations.

The present paper attempts to extend this solution method to a certain class of partial differential equations and in particular the two-dimensional steady flow Navier-Stokes equations. Starting from a concise complex variable formulation for the flow equations first given in [1], and subsequently rediscovered by others, an integration technique is developed which exhibits solutions in implicit parametric form. The dependent variables in the complex system are the stream function ψ and an auxiliary function ϕ associated with the Bernoulli function, or total head of pressure. One major advantage in connection with the present analysis is that the complex flow equation is quasi-linear, autonomous, and contains only $\bar{z} = x - iy$ as independent variable.

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A second feature of the method is to extend the solution space to complex valued stream functions, which in turn leads to a complex integration of an equation of Riccati type. From this it is shown there is a special form of the complex stream function which is a solution of the original system. One further integration is possible and three integrations of the fourth system are expressed in implicit parametric form containing three arbitrary real functions.

The equations of motion. The equations of motion for the steady flow of an incompressible viscous liquid can be written in the form

$$-[\mathbf{q} \times \operatorname{curl} \underline{q}] = -\nabla B + \nu \nabla^2 \mathbf{q}, \tag{1}$$

$$\operatorname{div} \mathbf{q} = 0, \qquad B = p/\rho + \tfrac{1}{2}|\mathbf{q}|^2 \tag{2}$$

where $\underline{q}$ is the fluid velocity, p the pressure, ρ the density, ν the kinematic viscosity, and B the Bernoulli function, or total head of pressure. In this representation part of the nonlinear convection term has been absorbed into the Bernoulli function. For two-dimensional flow the fluid velocity can be prescribed in terms of a stream function $\psi(x, y)$ and

$$\mathbf{q} = u\hat{i} + v\hat{j} = \operatorname{curl}(-\psi\hat{k}) = -\psi_y\hat{i} + \psi_x\hat{j}. \tag{3}$$

The components of Eq. (1) are

$$-\psi_x \nabla^2 \psi = -B_x - \nu \frac{\partial}{\partial y} \nabla^2 \psi, \tag{4}$$

$$-\psi_y \nabla^2 \psi = -B_y + \nu \frac{\partial}{\partial x} \nabla^2 \psi, \tag{5}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{6}$$

If $z = x + iy$, $\overline{z} = x - iy$, then

$$2\frac{\partial}{\partial \overline{z}} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \qquad 2\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \tag{7}$$

and Eqs. (4) and (5) can be combined in complex form to produce the single complex equation

$$-4\psi_{\overline{z}}\psi_{z\overline{z}} = -B_{\overline{z}} + 4\nu i \psi_{z\overline{z}\overline{z}}, \tag{8}$$

where $\nabla^2\psi \equiv 4\psi_{z\overline{z}}$. From Eq. (3) the velocity components can be combined to define a complex velocity given by

$$q = u + iv = 2i\psi_{\overline{z}}. \tag{9}$$

If a real function $\phi(x, y)$ is defined by $B = -\nu\nabla^2\phi = -4\nu\phi_{z\overline{z}}$, then (8) can be written as

$$\phi_{z\overline{z}\overline{z}} + i\psi_{z\overline{z}\overline{z}} + \nu^{-1}\psi_{\overline{z}}\psi_{z\overline{z}} = 0, \tag{10}$$

which implies

$$\phi_{\overline{z}\overline{z}} + i\psi_{\overline{z}\overline{z}} + (2\nu)^{-1}\psi_{\overline{z}}^2 = A''(\overline{z}). \tag{11}$$

The arbitrary analytic function $A(\overline{z})$ plays no role in the present analysis since it can be absorbed into the real function ϕ by replacing ϕ by $\phi - A(\overline{z}) - \overline{A}(z)$. In this case the Navier-Stokes equations can be expressed in the form (see [1]) by

$$\phi_{\overline{z}\overline{z}} + i\psi_{\overline{z}\overline{z}} + (2\nu)^{-1}\psi_{\overline{z}}^2 = 0. \tag{12}$$

The complex conjugate equation is

$$\phi_{zz} - i\psi_{zz} + (2\nu)^{-1}\psi_z^2 = 0. \tag{13}$$

Elimination of ϕ by differentiation yields the usual vorticity equation

$$\nu\nabla^4\psi = \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)}. \tag{14}$$

There are a few limiting cases worthy of note. First in the formal limit $\nu \to 0$, Eq. (13) reduces to

$$\nabla^2\psi = B_1(\psi), \tag{15}$$

where B_1 is an arbitrary real function and lines of constant vorticity coincide with the streamlines. The second case corresponds to the limit $\nu \to \infty$, in which case Eq. (12) becomes the creeping flow described by

$$\phi_{\overline{z}\overline{z}} + i\psi_{\overline{z}\overline{z}} = 0 \tag{16}$$

and the general solution is given by

$$\phi + i\psi = \overline{z}D_1(z) + E_1(z), \tag{17}$$

where $D_1(z)$ and $E_1(z)$ are analytic in the fluid region. The only known general solution of (14) is represented by the stream function

$$\psi = d(x^2 + y^2) + n(x, y), \qquad \nabla^2 n = 0, \ d \text{ constant}, \tag{18}$$

which is viscosity independent.

Method of solution. Define the operators by

$$L \equiv \phi_{\overline{z}\overline{z}} + i\psi_{\overline{z}\overline{z}} + (2\nu)^{-1}\psi_{\overline{z}}^2, \tag{19}$$

$$L_1 \equiv \phi_{\overline{z}\overline{z}} + iF_{\overline{z}\overline{z}} + (2\nu)^{-1}F_{\overline{z}}^2, \tag{20}$$

where ϕ, ψ are real functions of x, y, and F is a complex function of z, $\overline{z}$. The equation

$$L_1 - L \equiv i\exp(-(F + \psi)/2\nu i)\frac{\partial}{\partial\overline{z}}\{\exp((F + \psi)/2\nu i)(F_{\overline{z}} - \psi_{\overline{z}})\} = 0, \tag{21}$$

is of Riccati type and implies

$$\exp((F + \psi)/2\nu i)(F_{\overline{z}} - \psi_{\overline{z}}) = f(z), \tag{22}$$

where $f(z)$ is an arbitrary analytic function of z in the fluid region. Since L_1 is invariant under the transformation $F \to F + g(z)$, with $g(z)$ arbitrary, the arbitrary function $f(z)$ may be absorbed into F and Eq. (22) replaced by

$$\exp((F + \psi)/2\nu i)(F_{\overline{z}} - \psi_{\overline{z}}) = \exp((F + \psi)/2\nu i)\{F_\phi\phi_{\overline{z}} + (F_\psi - 1)\psi_{\overline{z}}\} = 1, \tag{23}$$

where F is now regarded as a function of ϕ, ψ.

Consider the equation

$$AL + BL_1 = \frac{\partial}{\partial_{\bar{z}}}(C\phi_{\bar{z}} + D\psi_{\bar{z}}) + (K\phi_{\bar{z}} + Q\psi_{\bar{z}} + M)(C\phi_{\bar{z}} + D\psi_{\bar{z}} - 1) + (E\phi_{\bar{z}} + G\psi_{\bar{z}} + J)\{\exp((F+\psi)/2\nu i)[F_\phi\phi_{\bar{z}} + (F_\psi - 1)\psi_{\bar{z}}] - 1\}, \tag{2.4}$$

where A, B, C, D, E, G, J, K, M, Q are functions of ϕ, ψ. Written explicitly in terms of the derivatives of ϕ, ψ with respect to $\bar{z}$ Eq. (24) is of the form

$$A_1\phi_{\bar{z}\bar{z}} + A_2\psi_{\bar{z}\bar{z}} + A_3\phi_{\bar{z}}^2 + A_4\psi_{\bar{z}}^2 + A_5\phi_{\bar{z}}\psi_{\bar{z}} + A_6\phi_{\bar{z}} + A_7\psi_{\bar{z}} + A_8 = 0$$

and is an identity provided that $A_j = 0$, $j = 1, \dots, 8$. This yields the following set of eight equations:

$$C = A + B(1 + iF_\phi), \tag{25}$$

$$D = iA + iBF_\psi, \tag{26}$$

$$C_\phi + EF_\phi\exp((F+\psi)/2\nu i) + CK = B(iF_{\phi\phi} + (2\nu)^{-1}F_\phi^2), \tag{27}$$

$$C_\psi + D_\phi + E(F_\psi - 1)\exp((F+\psi)/2\nu i) + G\exp((F+\psi)/2\nu i)F_\phi + KD + CQ = 2B[iF_{\phi\psi} + (2\nu)^{-1}F_\phi F_\psi], \tag{28}$$

$$D_\psi + G\exp((F+\psi)/2\nu i)(F_\psi - 1) + QD = A/2\nu + B[iF_{\psi\psi} + (2\nu)^{-1}F_\psi^2], \tag{29}$$

$$-E + J\exp((F+\psi)/2\nu i)F_\phi + MC - K = 0, \tag{30}$$

$$-G + J\exp((F+\psi)/2\nu i)(F_\psi - 1) + MD - Q = 0, \tag{31}$$

$$J + M = 0. \tag{32}$$

If the derivatives of C, D are eliminated from Eqs. (25), (26), (27), (28), (29) the resulting subset of equations reduces to the system

$$A_\phi + B_\phi(1 + iF_\phi) + E\exp((F+\psi)/2\nu i)F_\phi + CK = BF_\phi^2/2\nu, \tag{33}$$

$$A_\psi + iA_\phi + B_\psi(1 + iF_\phi) + iB_\phi F_\psi + E\exp((F+\psi)/2\nu i)(F_\psi - 1) + G\exp((F+\psi)/2\nu i)F_\phi + KD + CQ = BF_\phi F_\psi/\nu, \tag{34}$$

$$iA_\psi + iB_\psi F_\psi + G\exp((F+\psi)/2\nu i)(F_\psi - 1) + QD = A/2\nu + BF_\psi^2/2\nu. \tag{35}$$

Also elimination of M from (30), (31), (32) gives the equations

$$E = J[\exp((F+\psi)/2\nu i)F_\phi - C] - K, \tag{36}$$

$$G = J[\exp((F+\psi)/2\nu i)(F_\psi - 1) - D] - Q. \tag{37}$$

Now addition of equations (33), (34), (35) results in one complex equation represented by

$$(1+i)(A_\phi + A_\psi) + (B_\phi + B_\psi)[1 + i(F_\phi + F_\psi)] + (E+G)\exp((F+\psi)/2\nu i)(F_\phi + F_\psi - 1) + (C+D)(K+Q) = A/2\nu + B(F_\phi + F_\psi)^2/2\nu. \tag{38}$$

There is a solution of Eq. (38) of the form

$$-(Q+K)=\Gamma_\phi+\Gamma_\psi, \qquad F_\phi+F_\psi-1=0, \tag{39}$$

where Γ is an arbitrary function of ϕ, ψ and the latter equation implies

$$F=\psi+N(\phi-\psi), \tag{40}$$

where $N(\phi-\psi)$ is an arbitrary function of $\phi-\psi$. Equation (38) now simplifies to the complex equation

$$(1+i)\{(A+B)_\phi+(A+B)_\psi\}=\frac{A+B}{2\nu}+(1+i)(A+B)(\Gamma_\phi+\Gamma_\psi), \tag{41}$$

for which the solution is

$$A+B=a(\phi-\psi)\exp(\psi/2\nu(1+i)+\Gamma), \tag{42}$$

where $a(\phi-\psi)\neq 0$ is an arbitrary complex function of $\phi-\psi$. Now from (25), (26) it follows that if

$$\begin{aligned} C\phi_{\bar z}+D\psi_{\bar z} &= A(\phi_{\bar z}+i\psi_{\bar z})+B(\phi_{\bar z}+iF_{\bar z}) \\ &= a(\phi-\psi)\exp(\psi/2\nu(1+i)+\Gamma)(\phi_{\bar z}+i\psi_{\bar z}) \\ &\quad + iB\exp(-(F+\psi)/2\nu i)=1, \end{aligned} \tag{43}$$

then from (22), (24) we have

$$AL+BL_1=0, \tag{44}$$

which in turn from (21), (22) implies that

$$L=L_1=0, \tag{45}$$

since $A+B\neq 0$. It is now sufficient to consider Eqs. (33), (35) which are given by

$$ES+CK=BF_\phi^2/2\nu-A_\phi-B_\phi(1+iF_\phi), \tag{46}$$

$$-GS+QD=A/2\nu+BF_\psi^2/2\nu-iA_\psi-iB_\psi F_\psi, \tag{47}$$

where $S=\exp((F+\psi)/2\nu i)N'(\phi-\psi)$. Elimination of E, G from (36), (37) results in the equations

$$S(S-C)J+(C-S)K=BF_\phi^2/2\nu-A_\phi-B_\phi(1+iF_\phi), \tag{48}$$

$$S(S+D)J+(S+D)Q=A/2\nu+BF_\psi^2/2\nu-iA_\psi-iB_\psi F_\psi. \tag{49}$$

Again elimination of Q from (49) and using (39) produces the equation

$$JS(S+D)-K(D+S)=A/2\nu+BF_\psi^2/2\nu-iA_\psi-iB_\psi F_\psi+(D+S)(\Gamma_\phi+\Gamma_\psi). \tag{50}$$

Further elimination of the function $(SJ-K)$ from (48) and (50) gives the equation

$$\begin{aligned} &(S+D)\{BF_\phi^2/2\nu-A_\phi-B_\phi(1+iF_\phi)\} \\ &\quad =(S-C)\{A/2\nu+BF_\psi^2/2\nu-iA_\psi-iB_\psi F_\psi+(S+D)(\Gamma_\phi+\Gamma_\psi)\}. \end{aligned} \tag{51}$$

This equation written explicitly in terms of A, B, N is expressed by

$$\begin{aligned}
&[\exp((N+2\psi)/2\nu i)N' + i(A+B-BN')][BN'^2/2\nu - A_\phi - B_\phi(1+iN')] \\
&\quad = [\exp((N+2\psi)/2\nu i)N' - A - B - iBN') \\
&\qquad \times \{A/2\nu + B(1-N')^2/2\nu - iA_\psi - iB_\psi(1-N') \\
&\qquad\quad + (\Gamma_\phi + \Gamma_\psi)[\exp((N+2\psi)/2\nu i)N' + i(A+B-BN')]\}. \qquad (52)
\end{aligned}$$

It follows from (22) and (40) that if

$$L' \equiv \exp((N(\omega)+2\psi)/2\nu i)N'(\omega)\omega_{\overline{z}} - 1 = 0, \qquad \omega = \phi - \psi, \qquad (53)$$

then ϕ, ψ satisfy $L = 0$ since A, B, Γ are determined from Eqs. (42), (43), (52) and do not place any restriction on F, ϕ, ψ. The function $N(\omega)$ is an arbitrary function of ω. It is evident from (42), (52) that the function $a(\phi - \psi)$ can be absorbed into the function Γ, so without loss of generality $a(\phi - \psi) = 1$. Again from (43), (53) it follows $\phi_{\overline{z}}$, $\psi_{\overline{z}}$ can be expressed as

$$\begin{aligned}
(1+i)\phi_{\overline{z}} &= \exp(-\Gamma - \psi/2\nu(1+i))[1 - iB\exp(-(N+2\psi)/2\nu i)] \\
&\quad + \frac{i\exp(-(N+2\psi)/2\nu i)}{N'(\omega)} \\
&= A'(1+i), \qquad (54) \\
(1+i)\psi_{\overline{z}} &= \exp(-\Gamma - \psi/2\nu(1+i))[1 - iB\exp(-(N+2\psi)/2\nu i)] \\
&\quad - \frac{\exp(-(N+2\psi)/2\nu i)}{N'(\omega)} \\
&= (1+i)B', \qquad (55)
\end{aligned}$$

where A', B' are functions of ϕ, ψ defined by (54), (55). The integrability conditions $\phi_{z\overline{z}} = \phi_{\overline{z}z}$, $\psi_{z\overline{z}} = \psi_{\overline{z}z}$ require

$$A'_\phi\overline{A}' + A'_\psi\overline{B}' = \overline{A}'_\phi A' + \overline{A}'_\psi B', \qquad (56)$$

$$B'_\phi\overline{A}' + B'_\psi\overline{B}' = \overline{B}'_\phi A' + \overline{B}'_\psi B'. \qquad (57)$$

These equations together with (42), (52), (54), (55) define a self-consistent general solution of the Navier-Stokes equations. It is also remarked by essentially setting $f(z) = 1$ in (22) that Eqs. (54), (55), (56), (57) are independent of z, which would otherwise lead to inconsistency.

To achieve further progress in the integration process it is appropriate to return to (53) and eliminate ψ so that

$$4[h'(\omega)]^2\omega_z\omega_{\overline{z}} = 1, \qquad 4[h'(\omega)]^2 = \exp((N(\omega) - \overline{N}(\omega))/2\nu i)N'(\omega)\overline{N}'(\omega), \quad (58)$$

and it follows there exists a real function χ defined by

$$h'(\omega)\omega_x = \cos\chi, \qquad h'(\omega)\omega_y = \sin\chi. \qquad (59)$$

The consistency condition implies

$$\frac{\partial^2}{\partial x\partial y}[h(\omega)] = \cos\chi\chi_x = -\sin\chi\chi_y. \qquad (60)$$

The equation for χ is then of the form

$$\chi_x \cos\chi + \chi_y \sin\chi = 0, \tag{61}$$

which is a type of nonlinear wave equation for which the solution (see [2]) is

$$y - x\tan\chi = H(\tan\chi), \tag{62}$$

where $H(\tan\chi)$ is a real and arbitrary function of $\tan\chi$. Elimination of $\tan\chi$ from (59) gives the real equation

$$y - x(\omega_y/\omega_x) = H(\omega_y/\omega_x). \tag{63}$$

Again from Eq. (53) the ratio ω_y/ω_x can be expressed explicitly in terms of ω and ψ by

$$\frac{1}{\omega_x}L_2 = \frac{\omega_y}{\omega_x} - \frac{[\overline{N}'(\omega) - N'(\omega)\exp((N+\overline{N}+4\psi)/2\nu i)]}{i[\exp((N+\overline{N}+4\psi)/2\nu i)N'(\omega) + \overline{N}'(\omega)]} = 0, \tag{64}$$

so that Eqs. (53), (62) can be combined by elimination of ω_y/ω_x to give

$$\begin{aligned} L_3 = y + ix &\left[\frac{\overline{N}'(\omega) - N'(\omega)\exp((N+\overline{N}+4\psi)/2\nu i)}{iN'(\omega)\exp((N+\overline{N}+4\psi)/2\nu i) + \overline{N}'(\omega)i}\right] \\ &- H\left[\frac{\overline{N}'(\omega) - N'(\omega)\exp((N+\overline{N}+4\psi)/2\nu i)}{i\exp((N+\overline{N}+4\psi)/2\nu i)N'(\omega) + i\overline{N}'(\omega)}\right] = 0. \end{aligned} \tag{65}$$

Equations (64), (65), together with

$$L \equiv \omega_{\overline{z}\overline{z}} + (1+i)\psi_{\overline{z}\overline{z}} + (2\nu)^{-1}\psi_{\overline{z}}^2 = 0 \tag{66}$$

define a solution of the Navier-Stokes equations in which the complex function $N(\omega)$ and the real function H are arbitrary. Effectively three integrations of the fourth order system have been carried out and the resulting equations containing the arbitrary functions N and H can be displayed in parametric form by

$$L_3 = \frac{\partial L_3}{\partial y} \equiv L' = \frac{\partial L'}{\partial \overline{z}} = \frac{\partial L'}{\partial z} = \frac{\partial^2 L'}{\partial z \partial \overline{z}} - \frac{\partial^2 \overline{L}'}{\partial \overline{z}^2} = L = 0, \tag{67}$$

The system represented by (67) comprises 11 equations containing the six second-order derivatives ϕ_{zz}, $\phi_{\overline{z}\overline{z}}$, $\phi_{z\overline{z}}$, ψ_{zz}, $\psi_{\overline{z}\overline{z}}$, $\psi_{z\overline{z}}$, the four first-order derivatives ω_z, $\omega_{\overline{z}}$, ψ_z, $\psi_{\overline{z}}$ and ω, ψ. There are eight equations linear in the second-order derivatives and three linear in the first-order derivatives and it is a relatively routine but cumbersome procedure to eliminate these derivatives to produce two real equations containing ω, ψ and (x, y). This elimination process is best suited to a symbolic computer language program and will not be carried out here. In spite of this the underlying analytical structure of the solutions is revealed and it is hoped in a future publication to present the solutions in a more accessible form so that streamlines can be constructed for specific functions H and N. As an example to check the method of solution consider the simplest representation for $N(\omega)$ given by

$$N(\omega) = a\omega + k,$$

where a, k are complex constants. Equations (53), (66) are satisfied by

$$\phi = b\log z + \overline{b}\log\overline{z}, \qquad \psi = c\log z + \overline{c}\log\overline{z},$$

where the complex constants are determined from

$$\exp(k/2\nu i)a(\overline{b} - \overline{c}) = 1, \qquad a(b - c) + 2c = 0,$$
$$a(\overline{b} - \overline{c}) + 2\overline{c} = 0, \qquad -\overline{b} - i\overline{c} + (2\nu)^{-1}\overline{c}^2 = 0.$$

It is found that

$$\overline{c} = \nu i/2 - \nu, \qquad \overline{b} - (2\nu)^{-1}(\nu i/2 - \nu)^2 - i(\nu i/2 - \nu),$$

which verifies there is a solution for the stream function of the form

$$\psi = (\nu i/2 - \nu)\log z - (\nu i/2 + \nu)\log\overline{z}.$$

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