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A compound class of Poisson and lifetime distributions

Said Alkarni and Aiman Oraby

Quantitative Analysis Dept, King Saud University, Saudi Arabia
E-mail: salkarni@ksu.edu.sa, Orabi_issr@yahoo.com

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Abstract: A new lifetime class with decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime distribution, where the compounding procedure follows same way that was previously carried out by Adamidis and Loukas(1998). A general form of probability, distribution, survival and hazard rate functions as well as its properties will be presented for such a class. This new class of distributions generalizes several distributions which have been introduced and studied in the literature.

Keywords: Lifetime distributions, decreasing failure, Poisson distribution

1. Introduction

In many practical applications such as biological science, physics, engineering and manufacture, the available data can be interpreted as lifetimes and it is important to predict future observations. In general, a population is expected to exhibit decreasing failure rate (DFR) when its behavior over time is characterized by 'work-hardening' (in engineering terms) or 'immunity' (in biological terms); sometimes the broader term 'infant mortality' is used to denote the DFR phenomenon (Adamidis and Loukas, 1998). The distributions with DFR are discussed in the works of Lomax (1954), Proschan (1963), Barlow et al. (1963), Barlow and Marshall (1964, 1965), Marshall and Proschan (1965), Cozzolino (1968), Dahiya and Gurland (1972), McNolty et al. (1980), Saunders and Myhre (1983), Nassar (1988), Gleser (1989), Gurland and Sethuraman (1994) and Adamidis and Loukas (1998). The exponential-Poisson (EP) distribution proposed by Kus (2007), and genearalized by Hemmati et al. (2011) using Wiebull distribution and the exponential-logarithmic distribution discussed by Tahmasbi and Rezaei (2008). A two-parameter distribution family with decreasing failure rate arising by mixing power-series distribution has been introduced by Chahkandi and Ganjali (2009). A Weibull power series class of distributions with Poisson presented by Morais and Barreto-Souza (2011).

This paper is organized as follow. In section 2, we define the class of Poisson lifetime distributions. In section 3, we present the density, survival and hazard functions and give some of their properties. In section 4, we discuss the estimation of parameters. The entropy for the class is presented in section 5.

2. The class

Let $\underline{\mathrm{T}} = (\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_Z)$ be independent identically distributed (iid) random variables with probability density function given by (1) and Z is a zero truncated Poisson variable with probability function represented by (2)

$$f_T(x;\underline{\theta}),\ \underline{\theta} = (\theta_1, \dots, \theta_k) \text{ for }, k \geq 1, x, \underline{\theta} \in \mathbb{R}^+ \quad (1)$$

$$f_Z(z;\lambda) = \frac{e^{-\lambda}}{1-e^{-\lambda}} z^{\lambda}\Gamma^{-1}(z+1), z \in \mathbb{N}, \lambda \in \mathbb{R}^+ \quad (2)$$

where$\Gamma(.)$ is the Gamma function with Z and T are independent. Let us define $X = \min(T_1, T_2, \dots, T_Z)$. Then, the probability function of X is

$$f_X(x;\underline{\theta},\lambda) = \frac{\lambda f_T(x;\underline{\theta})\, e^{-\lambda F_T(x;\underline{\theta})}}{1-e^{-\lambda}} \quad (3)$$

and the distribution function of X is

$$F_X(x;\lambda,\underline{\theta}) = \frac{1-e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}\,. \quad (4)$$

The proof of the results in (3) and (4) are presented in the following theorem.

Theorem 2.1

Suppose $\underline{T} = (Y_1, Y_2, \dots, Y_Z)$ with $f_{Y_i}(y,\underline{\theta}) = f_Y(y,\underline{\theta}),\ \underline{\theta} = (\theta_1, \dots, \theta_k)$ for $, k \geq 1, x, \underline{\theta} \in \mathbb{R}^+$ and Z is a zero truncated Poisson variable with probability function$f_Z(z;\lambda) = \frac{e^{-\lambda}}{1-e^{-\lambda}} z^{\lambda}\Gamma^{-1}(z+1), Z \in \mathbb{N}, \lambda \in \mathbb{R}^+$ where $\Gamma(.)$ is the Gamma function where Z and T are independent. If $X = \min(Y_1, Y_2, \dots, Y_Z)$, then the probability function of X is

$$f_X(x;\lambda,\underline{\theta}) = \frac{\lambda f_Y(x;\,\underline{\theta})\, e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}$$

and the distribution function of X is

$$F_X(x;\lambda,\underline{\theta}) = \frac{1-e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}.$$

Proof:

It is well known that the conditional density function can be defined as

$$f(x|z;\,\underline{\theta}) = z\, f_Y(x;\,\underline{\theta})[1 - F_Y(x;\,\underline{\theta})]^{z-1}$$

and hence the joint probability function of X and Z is

$$f_{X,Z}(x,z;\underline{\theta}) = f_Z(z;\lambda) f(x|z;\,\underline{\theta}) = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} f_Y(x;\,\underline{\theta}) \frac{\lambda^{z-1}[1-F_Y(x;\underline{\theta})]^{z-1}}{\Gamma z},$$

the marginal probability density function of X is

$$f_X(x;\underline{\theta},\lambda) = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} f_Y(x;\,\underline{\theta}) \sum_{z=1}^{\infty} \frac{[\lambda(1 - F_Y(x;\,\underline{\theta}))]^{z-1}}{\Gamma z} = \frac{\lambda f_Y(x;\,\underline{\theta})\, e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}$$

and

$$F_X(x;\underline{\theta},\lambda) = \int_0^x f_X(x;\underline{\theta},\lambda)\,dx = \int_0^x \frac{\lambda f_Y(x;\underline{\theta})\, e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}\,dx = \frac{1-e^{-\lambda F_Y(x;\underline{\theta})}}{1-e^{-\lambda}}.$$

We denote a random variable X following the Poisson lifetime distribution (PL) with parameters $\underline{\theta}$ and λ by $X\sim(\underline{\theta},\lambda)$. This new class of distributions generalizes several distributions which have been introduced and studied in the literature. For instance using the probability density and distribution function of exponential distribution in (3), we obtain the EP distribution (kus,2007) and using Wiebull probability density and distribution function gives WP distribution Hemmati et al. (2011).The model is obtained under the concept of population heterogeneity (through the process of compounding). An interpretation of the proposed model is as follows: a situation where failure (of a device for example) occurs due to the presence of an unknown number, Z, of initial defects of same kind (a number of semiconductors from a defective lot, for example). The Ts represent their lifetimes and each defect can be detected only after causing failure, in which case it is repaired perfectly (Adamidis and Loukas, 1998). Then the distributional assumptions given earlier lead to the PL distribution for modeling the time to the first failure X.

Table 1 shows the probability function and the distribution function for some lifetime distributions.

Table 1
Probability and distribution functions

	$f_X(x;\underline{\theta},\lambda)$	$F_X(x;\underline{\theta},\lambda)$
Exponential	$\frac{\lambda\beta}{1-e^{-\lambda}}e^{-\lambda-\beta x+\lambda\exp(-\beta x)}$	$\frac{e^{\lambda\exp(-\beta x)}-e^{\lambda}}{1-e^{-\lambda}}$
Weibull	$\frac{\lambda\beta\alpha}{1-e^{-\lambda}}(\alpha x)^{\beta-1}e^{-\lambda-(\alpha x)^{\beta}+\lambda\exp(\alpha x)^{\beta}}$	$\frac{e^{\lambda\exp(\alpha x)^{\beta}}-e^{\lambda}}{1-e^{-\lambda}}$
Rayleigh	$\frac{2\lambda\alpha^2}{1-e^{-\lambda}}xe^{-\lambda-\alpha^2x^2+\lambda\exp(\alpha^2x^2)}$	$\frac{e^{\lambda\exp(\alpha^2x^2)}-e^{\lambda}}{1-e^{-\lambda}}$
Pareto	$\frac{\lambda\alpha}{1-e^{-\lambda}}\frac{e^{-\lambda[1-\frac{1}{(1+x)^{\alpha}}]}}{(1+x)^{\alpha+1}}$	$\frac{1-e^{-\lambda[1-\frac{1}{(1+x)^{\alpha}}]}}{1-e^{-\lambda}}$

Some of the other lifetime distributions are excluded from this table since they do not have a nice forms.

3. Survival and hazard functions

Since the PL is not part of the exponential family, there is no simple form for moments see for instant (Kus, 2007) for the exponential case.

Using (3) and (4), survival function (also known reliability function)

$$s_X(x;\underline{\theta},\lambda) = 1 - F_X(x;\underline{\theta},\lambda) = \frac{e^{\lambda(1-F_T(x;\underline{\theta}))}-1}{e^{\lambda}-1} = \frac{e^{\lambda(s_T(x;\underline{\theta}))}-1}{e^{\lambda}-1}. \quad (5)$$

The hazard function (known as failure rate function)

$h_X(x;\underline{\theta},\lambda) = \frac{f_X(x;\underline{\theta},\lambda)}{s_X(x;\underline{\theta},\lambda)} = \frac{\lambda f_X(x;\underline{\theta})\, e^{-\lambda F_X(x;\underline{\theta})}}{e^{-\lambda F_X(x;\underline{\theta})}-e^{-\lambda}}$ and can be written in a simpler form

$$h_X(x;\underline{\theta},\lambda) = \frac{\lambda f_X(x;\underline{\theta})}{1-e^{-\lambda+F_X(x;\underline{\theta})}} = \frac{\lambda s_T(x;\underline{\theta}) h_T(x;\underline{\theta})}{1-e^{-\lambda s_T(x;\underline{\theta})}}. \quad (6)$$

Table 2 summarizes the survival functions and hazard rate functions for some distribution s of the class.

Table 2

Survival and hazard functions

	$s_X(x;\underline{\theta},\lambda)$	$h_X(x;\underline{\theta},\lambda)$
Exponential	$\frac{1-e^{\lambda \exp(\alpha x)}}{1-e^{-\lambda}}$	$\frac{\alpha\lambda(1-e^{\lambda})e^{-\lambda-\alpha x+\lambda \exp(\alpha x)}}{(1-e^{-\lambda})(1-e^{\lambda \exp(\alpha x)^{\beta}})}$
Weibull	$\frac{1-e^{\lambda \exp(\alpha x)^{\beta}}}{1-e^{-\lambda}}$	$\frac{\alpha\beta\lambda(1-e^{\lambda})(\alpha x)^{\beta-1}e^{-\lambda-(\alpha x)^{\beta}+\lambda \exp(\alpha x)^{\beta}}}{(1-e^{-\lambda})(1-e^{\lambda \exp(\alpha x)^{\beta}})}$
Rayleigh	$\frac{1-e^{\lambda \exp(\alpha^2 x^2)}}{1-e^{-\lambda}}$	$\frac{2\alpha^2\lambda(1-e^{\lambda})x\ e^{-\lambda-\alpha^2x^2+\lambda \exp(\alpha^2x^2)}}{(1-e^{-\lambda})(1-e^{\lambda \exp(\alpha^2x^2)})}$
Pareto	$\frac{e^{-\lambda[1-\frac{1}{(1+x)^{\alpha}}]}-e^{-\lambda}}{1-e^{-\lambda}}$	$\frac{\lambda\alpha}{(1+x)^{\alpha+1}(1-e^{-\lambda+(\left[1-\frac{1}{(1+x)^{\alpha}}\right]})}$

the hazard function is decreasing because the DFR property follows from the result of Barlow et al. (1963) on mixture.

4. Estimation

In what follows, we discuss the estimation of the parameters for PL distributions. Let $X_1, . . . ,X_n$ be a random sample with observed values $x_1, . . . ,x_n$ from a LP distributions with parameters $\underline{\theta}$ and λ . The log log–likelihood function based on the observed random sample size of $n, y_{obs} = (x_1, \ldots, x_n)$is given by

$$\ell(\underline{\theta};\ y_{obs})=n\log\lambda - n\log(1-e^{-\lambda}) + \sum_{i=1}^{n} \ln f_T(x_i;\ \underline{\theta}) - \lambda\sum_{i=1}^{n} F_T(x_i;\ \underline{\theta})$$

and subsequently the associated gradients are found to be

$$\frac{\partial\ell(\underline{\theta};\ y_{obs})}{\partial\lambda} = \frac{n}{\lambda} - \frac{n}{e^{\lambda}-1} - \sum_{i=1}^{n} F_T(x_i;\ \underline{\theta}),$$

$$\frac{\partial\ell(\underline{\theta};\ y_{obs})}{\partial\underline{\theta}} = \sum_{i=1}^{i=n} \frac{\frac{\partial f_T(x_i;\underline{\theta})}{\partial\underline{\theta}}}{f_T(x_i;\ \underline{\theta})} - \lambda\sum_{i=1}^{n} \frac{\partial}{\partial\underline{\theta}} F_T(x_i;\ \underline{\theta}).$$

Conditional upon the value of $\hat{\underline{\theta}}$, where $\hat{\lambda}$ and $\hat{\underline{\theta}}$ are the maximum likelihood estimates (MLEs) for the parameters λ and $\hat{\underline{\theta}}$ respectively. Based on the underline distribution, the maximum likelihood estimation of the parameters can be found analytically using an EM algorithm. Newton–Raphson algorithm is one of the standard methods to determine the MLEs of the parameters. To employ the algorithm, second derivatives of the log-likelihood are required for all iteration. EM algorithm is a very powerful tool in handling the incomplete data problem (Dempster et al., 1977; McLachlan and Krishnan, 1997). It is an iterative method by repeatedly replacing the missing data with estimated values and updating the parameter estimates. It is especially useful if the complete data set is easy to analyze. As pointed out by Little and Rubin (1983), the EM algorithm will converge reliably but rather slowly (as compared to the Newton–Raphson method) when the amount of information in the missing data is relatively large. Recently, EM algorithm has been used by several authors such as Adamidis and Loukas (1998), Adamidis (1999), Ng et al. (2002), Karlis (2003) and Adamidis et al. (2005).

To start the algorithm, hypothetical complete-data distribution is defined with density function

$$\mathrm{f}_{\mathrm{X,Z}}(x, \mathrm{z}; \underline{\theta}) = \frac{\lambda \mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}} f_T(x;\ \underline{\theta}) \frac{\lambda^{\mathrm{z}-1}\left[1-F_T(x;\ \underline{\theta})\right]^{\mathrm{z}-1}}{\Gamma \mathrm{z}}$$

with $x, \lambda, \underline{\theta} \in \mathbb{R}^{+}$ and $z \in \mathbb{N}$. Thus, it is straightforward to verify that the Estep of an EM cycle requires the computation of the conditional expectation of $\left(\mathrm{Z}|\mathrm{X}; \lambda^{(h)}, \underline{\theta}^{(h)}\right)$ where $(\lambda^{(h)}, \underline{\theta}^{(h)})$ are the current estimate of $(\lambda, \underline{\theta})$. The EM cycle is completed with M-step, which is complete data maximum likelihood over$(\lambda, \underline{\theta})$, with the missing Z's replaced by their conditional expectations $E\left(\mathrm{Z}|\mathrm{X}; \lambda, \underline{\theta}\right)$ (Adamidis and Loukas, 1998). Thus, an EM iteration

$$\lambda^{h+1} = n^{-1}(1-e^{-\lambda^{h+1}} \sum_{i=1}^{n} [1+\lambda^{h}[1-F_T(x;\ \underline{\theta}^{h})]]$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial \underline{\theta}^{h+1}} log f_T(x;\ \underline{\theta}^{h+1}) = -\lambda^{h} \sum_{i=1}^{n} [1-F_T(x;\ \underline{\theta}^{h})] \frac{\partial}{\partial \underline{\theta}^{h+1}} \log\, [1-F_T(x;\ \underline{\theta}^{h+1}).$$

It can be seen that only a one-dimensional search such as Newton–Raphson is required for M-step of an EM cycle. Ng et al. (2002) used the same method for estimation of the parameters of the Weibull distribution based on progressively type-II right censored sample.

5. Entropy for the class

IfX is a random variable having an absolutely continuous cumulative distribution function $F_X(x)$ and probability distribution function $f_X(x)$,then the basic uncertainty measure for distribution$F$ (called the entropy of F) is defined as

$$H_X(X) = -E\left(log f(X)\right). \tag{7}$$

Statistical entropy is a probabilistic measure of uncertainty or ignorance about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Since Shannon's (1948) pioneering work on the mathematical theory of communication, entropy (7) has been used as a major tool in information theory and in almost every branch of science and engineering. Numerous entropy and information indices, among them the Renyi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields. Substitute in (7), the entropy for the LP class is given by

$$H_X(X) = H_T(X)+\log(\frac{1-e^{-\lambda}}{\lambda}) + \lambda$$

Where $H_T(X)$ is the entropy of any lifetime distribution in the class. Note that as λ increases the $H_X(X)$ increases too which is very logical since the increase rate of accidents the entropy increases. The above treatment is different from the quantum entropy discussed in Ref. [27-31].

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