Research Article

A Comprehensive Family of Biunivalent Functions Defined by k-Fibonacci Numbers

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By using k-Fibonacci numbers, we present a comprehensive family of regular and biunivalent functions of the type $g(z) = z + \sum_{j=2}^{\infty} d_j z^j$ in the open unit disc $\mathfrak{D}$. We estimate the upper bounds on initial coefficients and also the functional of Fekete-Szegö for functions in this family. We also discuss few interesting observations and provide relevant connections of the result investigated.

1. Introduction and Notations

Let $\mathbb{C}$ be the set of all complex numbers and the disc $\{z \in \mathbb{C} : |z| < 1\}$ be symbolized by $\mathfrak{D}$. Let $\mathbb{N} = \mathbb{N}_0 \setminus \{0\} := \{1, 2, 3, \cdots\}$ and $\mathbb{R}$ be the collection of all real numbers. We denote the set of all normalized regular functions in $\mathfrak{D}$ that have the series of the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1}$$

by $\mathcal{A}$ and the symbol $\mathcal{S}$ stands for set of all functions of $\mathcal{A}$ that are univalent (or Schlicht) in $\mathfrak{D}$. As per the popular Koebe theorem (see [1]), every function $g \in \mathcal{S}$ has an inverse function given by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2 \omega^2 + \left(2d_2^2 - d_3\right)\omega^3 \\ - \left(5d_2^3 - 5d_2 d_3 + d_4\right)\omega^4 + \cdots, \tag{2}$$

such that $z = g^{-1}(g(z))$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, $z, \omega \in \mathfrak{D}$.

A function g of $\mathcal{A}$ is called biunivalent (or bi-Schlicht) in $\mathfrak{D}$ if both g and g^{-1} are univalent (or Schlicht) in $\mathfrak{D}$. Let Σ stands for the set of biunivalent (or bi-Schlicht) functions having the form (1). Historically, investigations of the family Σ begun five decades ago by Lewin [2] and Brannan and Clunie [3]. Later, Tan [4] found some coefficient estimates for biunivalent functions. In 1986 [5], Brannan and Taha introduced certain well-known subfamilies of Σ in $\mathfrak{D}$. Many interesting results related to initial bounds for some special families of Σ have appeared in [6–8].

In 2007, the concept of k-Fibonacci number sequence $\{F_{k,j}\}_{j=0}^{\infty}$, $k \in \mathbb{R}^+$ was examined by Falcón and Plaza [9] and is given by

$$F_{k,0} = 0,$$
$$F_{k,1} = 1, \tag{3}$$
$$F_{k,j+1} = kF_{k,j} + F_{k,j-1},$$

where $j \in \mathbb{N}$ and

$$F_{k,j} = \frac{(k - t_k)^j - t_k^j}{\sqrt{k^2 + 4}} \text{ with } t_k = \frac{k - \sqrt{k^2 + 4}}{2}. \qquad (4)$$

$F_{1,j} = F_j$ is the well-known Fibonacci number sequence. Özgür and Sokól in 2015 [10] proved that if

$$\tilde{p}_k(z) = \frac{1 + t_k^2 z^2}{1 - k t_k z - t_k^2 z^2}, \qquad (5)$$

then,

$$\tilde{p}_k(z) = 1 + (F_{k,0} + F_{k,2}) t_k z + (F_{k,1} + F_{k,3}) t_k^2 z^2 + \cdots$$
$$= 1 + k t_k z + (k^2 + 2) t_k^2 z^2 + \cdots, \qquad (6)$$

where t_k is as in (4) and $z \in \mathfrak{D}$. Further, if $\tilde{p}_k(z) = 1 + \Sigma \tilde{p}_{k,j} z^j$, then, we have

$$\tilde{p}_{k,j} = t_k^j (F_{k,j-1} + F_{k,j+1}), \quad j \in \mathbb{N}. \qquad (7)$$

Fibonacci polynomials, Pell-Lucas polynomials, Gegenbauer polynomials, Chebyshev polynomials, Horadam polynomials, Fermat-Lucas polynomials, and generalizations of them are potentially important in many branches such as architecture, physics, combinatorics, number theory, statistics, and engineering. Additional information is associated with these polynomials one can go through [11–13]. More details about the very popular functional of Fekete-Szegö for biunivalent functions based on k-Fibonacci numbers can be found in [14–20].

The recent research trends are the outcomes of the study of functions in Σ based on any one of the above-mentioned polynomials, which can be seen in the recent papers [21–28]. Generally, interest was shown to estimate the first two coefficient bounds and the functional of Fekete-Szegö for some subfamilies of Σ.

For functions g and f regular in $\mathfrak{D}$, g is said to subordinate f, if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$, and $g(z) = f(\psi(z)), z \in \mathfrak{D}$. This subordination is indicated as $g \prec f$. In particular, if $f \in \mathcal{S}$, then $g(z) \prec f(z) \Leftarrow g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the recent articles and the new trends on functions in Σ, we present a comprehensive family of Σ defined by using k-Fibonacci numbers as given by (3) with $F_{k,j}$ as in (4).

Throughout this paper, $g^{-1}(\omega) = f(\omega)$ is as in (2), $T_k = k - (k^2 + 2) t_k$, t_k is as in (4), and $\tilde{p}_k$ is as in (5).

Definition 1. A function $g \in \Sigma$ having the power series (1) is said to be in the family $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$, if

$$\frac{z\left(g'(z)\right)^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} \prec \tilde{p}_k(z), \quad z \in \mathfrak{D}$$
$$\frac{\omega\left(g'(\omega)\right)^\tau + \mu z^2 g''(\omega)}{\gamma g(\omega) + (1 - \gamma)\omega} \prec \tilde{p}_k(\omega), \quad \omega \in \mathfrak{D}, \qquad (8)$$

where $\tau \geq 1$, $0 \leq \gamma \leq 1$, and $\mu \geq 0$.

Remark 2. The function families $SR\mathfrak{S}_\Sigma^\tau(\gamma, 0, \tilde{p}_k)$ and $SR\mathfrak{S}_\Sigma^1(\gamma, \mu, \tilde{p}_k)$ were investigated by Frasin et al. [29].

It is interesting to note that (i) $\gamma = 1$, (ii) $\gamma = 0$, and (iii) $\mu = 1$ lead the family $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$ to various subfamilies, as illustrated in the following:

(1) $SR\mathfrak{S}_\Sigma^\tau(1, \mu, \tilde{p}_k) \equiv L_\Sigma^\tau(\mu, \tilde{p}_k)$ is the family of functions $g \in \Sigma$ satisfying

$$\frac{z\left(g'(z)\right)^\tau}{g(z)} + \mu\left(\frac{z^2 g''(z)}{g(z)}\right) \prec \tilde{p}_k(z) \text{ and } \frac{\omega\left(f'(\omega)\right)^\tau}{f(\omega)}$$
$$+ \mu\left(\frac{\omega^2 f''(\omega)}{f(\omega)}\right) \prec \tilde{p}_k(\omega), \quad z, \omega \in \mathfrak{D} \qquad (9)$$

(2) $SR\mathfrak{S}_\Sigma^\tau(0, \mu, \tilde{p}_k) \equiv K_\Sigma^\tau(\mu, \tilde{p}_k)$ is the set of functions $g \in \Sigma$ satisfying

$$\left(g'(z)^\tau + \mu z g''(z) \prec \tilde{p}_k(z) \text{ and } \left(f'(\omega)\right)^\tau$$
$$+ \mu \omega f''(\omega) \prec \tilde{p}_k(\omega), \quad z, \omega \in \mathfrak{D} \qquad (10)$$

(3) $SR\mathfrak{S}_\Sigma^\tau(\gamma, 1, \tilde{p}_k) \equiv M_\Sigma^\tau(\gamma, \tilde{p}_k)$ is the collection of functions $g \in \Sigma$ satisfying

$$\frac{z\left(g'(z)\right)^\tau + z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} \prec \tilde{p}_k(z), \quad z \in \mathfrak{D},$$
$$\frac{\omega\left(f'(\omega)\right)^\tau + \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} \prec \tilde{p}_k(\omega), \quad \omega \in \mathfrak{D} \qquad (11)$$

Remark 3. We note that (i) $L_\Sigma^\tau(1, \tilde{p}_k) \equiv M_\Sigma^\tau(1, \tilde{p}_k)$ and (ii) $K_\Sigma^\tau(1, \tilde{p}_k) \equiv M_\Sigma^\tau(0, \tilde{p}_k)$.

Remark 4.

   (i) When $\tau = 1$, the family $K_\Sigma^1(\mu, \tilde{p}_k)$ was introduced by Frasin et al. [30]

   (ii) The family $L_\Sigma^1(0, \tilde{p}_k) \equiv \mathcal{S}_\Sigma^*(\tilde{p}_k))$ was mentioned by Güney et al. [18], when $\mu = 0$ and $\tau = 1$

   (iii) For $\mu = 0$ and $k = 1$, the class $L_\Sigma^\tau(0, \tilde{p}_1) \equiv \mathcal{S}_\Sigma(\tilde{p}_1)$ was investigated by Magesh et al. [31]

We now state the following lemma, which we will be using in the proof of our theorem.

Lemma 5 (see [32]). *If $p \in P$, where P is the collection of regular functions p in $\mathfrak{D}$, satisfying $\mathfrak{R}(p(z)) > 0, z \in \mathfrak{D}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in \mathfrak{D}$, then $|p_i| \leq 2$, for each i.*

In the next section, we derive the estimates for $|d_2|, |d_3|$ and obtain the Fekete-Szegö [33] inequalities for functions in the class $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$.

2. Coefficient Bounds and Fekete-Szegö Functional

In this section, we offer to get the upper bounds on initial coefficients and find the functional of Fekete-Szegö for functions $\in SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$.

Theorem 6. *Let $\tau \geq 1, 0 \leq \gamma \leq 1$, and $\mu \geq 0$. If $g(z)$ of the form (1) in the family $SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$, then*

$$|d_2| \leq \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2 t_k + (2(\mu + \tau) - \gamma)^2 T_k|}}, \quad (12)$$

$$|d_3| \leq \frac{k|t_k|}{3(2\mu + \tau) - \gamma} + \frac{k^3 t_k^2}{|(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2 t_k + (2(\mu + \tau) - \gamma)^2 T_k|}, \quad (13)$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \dfrac{k\,|t_k|}{3(2\mu + \tau) - \gamma} & ; |1 - \delta| \leq J, \\[4mm] \dfrac{k^3 t_k^2 |1 - \delta|}{\left|(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2 t_k + (2(\mu + \tau) - \gamma)^2 T_k\right|} & ; |1 - \delta| \geq J, \end{cases} \quad (14)$$

where

$$J = \frac{1}{3(2\mu + \tau) - \gamma} \cdot \left[\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma) + (2(\mu + \tau) - \gamma)^2 \left|\frac{T_k}{k^2 t_k}\right|\right]. \quad (15)$$

where

$$m(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + u_1 z + u_2 z^2 + \cdots. \quad (18)$$

So it follows that

$$u(z) = \frac{m(z) - 1}{m(z) + 1} = \frac{u_1}{2} z + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \left(u_3 - u_1 u_2 + \frac{u_1^3}{4}\right)\frac{z^3}{2} + \cdots, \quad (19)$$

Proof. Let the function $g \in SR\mathfrak{S}_\Sigma^\tau(\gamma, \mu, \tilde{p}_k)$. Then, from Definition 1, we have

$$\frac{z\left(g'(z)\right)^\tau + \mu z^2 g''(z)}{\gamma g(z) + (1 - \gamma)z} \prec \tilde{p}_k(u(z)), \quad z \in \mathfrak{D}, \quad (16)$$

$$\frac{\omega\left(f'(\omega)\right)^\tau + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1 - \gamma)\omega} \prec \tilde{p}_k(v(\omega)), \quad \omega \in \mathfrak{D}. \quad (17)$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}_k$. Then, there exists a regular function u with $|u(z)| < 1$ in $\mathfrak{D}$ and $p(z) = \tilde{p}_k(u(z))$. Therefore, the function $m(z)$ is in the class P,

$$\tilde{p}_k(u(z)) = 1 + \tilde{p}_{k,1}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \cdots\right) + \tilde{p}_{k,2}\left(\frac{u_1 z}{2} + \left(u_2 - \frac{u_1^2}{2}\right)\frac{z^2}{2} + \cdots\right)^2 + \cdots$$

$$= 1 + \frac{\tilde{p}_{k,1} u_1 z}{2} + \left(\frac{1}{2}\left(u_2 - \frac{u_1^2}{2}\right)\tilde{p}_{k,1} + \frac{u_1^2}{4}\tilde{p}_{k,2}\right)z^2 + \cdots. \quad (20)$$

Similarly, it follows that

$$\tilde{p}_k(v(\omega)) = 1 + \frac{\tilde{p}_{k,1} v_1 \omega}{2} + \left(\frac{1}{2}\left(v_2 - \frac{v_1^2}{2}\right)\tilde{p}_{k,1} + \frac{v_1^2}{4}\tilde{p}_{k,2}\right)\omega^2 + \cdots, \tag{21}$$

where v is a regular function such that $|v(\omega)| < 1$ in $\mathfrak{D}$ such that $p(\omega) = \tilde{p}_k(v(\omega))$ and the function $l(\omega)$ is in the class P, where

$$l(\omega) = \frac{1 + v(\omega)}{1 - v(\omega)} = 1 + v_1 \omega + v_2 \omega^2 + \cdots. \tag{22}$$

By virtue of (14), (15), (18), and (19), we obtain

$$(2(\mu + \tau) - \gamma)\, d_2 = \frac{u_1 k\, t_k}{2}, \tag{23}$$

$$\begin{aligned}(3(2\mu + \tau) - \gamma)d_3 + \left(\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1)\right)d_2^2 \\ = \frac{1}{2}\left(u_2 - \frac{u_1^2}{2}\right)kt_k + \frac{u_1^2}{4}\left(k^2 + 2\right)t_k^2,\end{aligned} \tag{24}$$

$$-(2(\mu + \tau) - \gamma)d_2 = \frac{v_1 k\, t_k}{2}, \tag{25}$$

$$\begin{aligned}(3(2\mu + \tau) - \gamma)\left(2d_2^2 - d_3\right) + \left(\gamma^2 - 2\gamma(\mu + \tau) + 2\tau(\tau - 1)\right)d_2^2 \\ = \frac{1}{2}\left(v_2 - \frac{v_1^2}{2}\right)k\, t_k + \frac{v_1^2}{4}\left(k^2 + 2\right)t_k^2.\end{aligned} \tag{26}$$

From (21) and (23), we get

$$u_1 = -v_1, \tag{27}$$

and also,

$$2(2(\mu + \tau) - \gamma)^2 d_2^2 = \frac{\left(u_1^2 + v_1^2\right)k^2\, t_k^2}{4}. \tag{28}$$

If we add (26) and (24), then we obtain

$$\begin{aligned}2\left(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma)\right)d_2^2 = \frac{1}{2}\left(u_2 + v_2\right)k\, t_k \\ - \frac{1}{4}\left(k\, t_k - \left(k^2 + 2\right)t_k^2\right)\left(u_1^2 + v_1^2\right).\end{aligned} \tag{29}$$

Substituting the value of $\left(u_1^2 + v_1^2\right)$ from (26) in (27), we get

$$d_2^2 = \frac{k^3\, t_k^2(u_2 + v_2)}{4\left[\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2\, t_k + (2(\mu + \tau) - \gamma)^2 T_k\right]}, \tag{30}$$

which gets (10), on using Lemma 5.

On using (25) in the subtraction of (24) from (26), we arrive at

$$d_3 = d_2^2 + \frac{k\, t_k(u_2 - v_2)}{4(3(2\mu + \tau) - \gamma)}. \tag{31}$$

Then, in view of Lemma 5 and equation (28), (29) reduces to (11).

From (28) and (29), for $\delta \in \mathbb{R}$, we can easily compute that

$$\begin{aligned}\left|d_3 - \delta d_2^2\right| = k\,|t_k|\bigg|\left(T(\delta) + \frac{1}{4(3(2\mu + \tau) - \gamma)}\right)u_2 \\ + \left(T(\delta) - \frac{1}{4(3(2\mu + \tau) - \gamma)}\right)v_2\bigg|,\end{aligned} \tag{32}$$

where

$$T(\delta) = \frac{(1 - \delta)k^2 t_k}{4\left[(\gamma^2 + (\tau - \gamma)(2\tau + 1) + 2\mu(3 - \gamma))k^2\, t_k + (2(\mu + \tau) - \gamma)^2 T_k\right]}. \tag{33}$$

In view of (4), we find that

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} \dfrac{k\,|t_k|}{(3(2\mu + \tau) - \gamma)} & ; 0 \leq |T(\delta)| \leq \dfrac{1}{4(3(2\mu + \tau) - \gamma)}, \\[4mm] 4k\,|t_k||T(\delta)| & ; |T(\delta)| \geq \dfrac{1}{4(3(2\mu + \tau) - \gamma)}, \end{cases} \tag{34}$$

which enable us to conclude (12) with J as in (13). Theorem 6 is proved. $\qquad\square$

Remark 7. By taking $\tau = 1$ in the above theorem, we obtain a result of Frasin et al. ([29], Corollary 3.4) and if we let $\mu = 0$ in the above theorem, we get another result of Frasin et al. ([29], Corollary 3.7).

Remark 8. Allowing $k = \gamma = 1$ and $\mu = 0$ in the above theorem, we have Theorem 2.3 of Magesh et al. [31].

Remark 9. Letting $\tau = \gamma = 1$ and $\mu = 0$ in the Theorem 6, we obtain two results of Güney et al. ([18], Corollary 10 and Corollary 23). Further, if we take $k = 1$, we get results of Güney et al. ([17], Corollary 1 and Corollary 4).

In Section 3, few interesting consequences and relevant observations of the main result are mentioned.

3. Outcome of the Main Result

By setting (i) $\gamma = 1$, (ii) $\gamma = 0$, and (iii) $\mu = 1$ in our main theorem, we obtain the following results, respectively.

Corollary 10. *If the function $g \in L_\Sigma^\tau(\mu, \tilde{p}_k)$, then*

$$|d_2| \leq \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\tau(2\tau - 1) + 4\mu)k^2 t_k + (2(\mu + \tau) - 1)^2 T_k|}},$$

$$|d_3| \leq \frac{k|t_k|}{3(2\mu + \tau) - 1} + \frac{k^3 t_k^2}{|(\tau(2\tau - 1) + 4\mu)k^2 t_k + (2(\mu + \tau) - 1)^2 T_{k.}|},$$

(35)

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \dfrac{k|t_k|}{3(2\mu + \tau) - 1} & ; |1 - \delta| \leq J_1, \\[4mm] \dfrac{k^3 t_k^2 |1 - \delta|}{|(\tau(2\tau - 1) + 4\mu)k^2 t_k + (2(\mu + \tau) - 1)^2 T_k|} & ; |1 - \delta| \geq J_1, \end{cases}$$

(36)

where

$$J_1 = \frac{1}{3(2\mu + \tau) - 1}\left(\tau(2\tau - 1) + 4\mu + (2(\mu + \tau) - 1)^2 \left|\frac{T_k}{k^2 t_k}\right|\right).$$

(37)

Remark 11.

(i) By taking $\mu = 0$ and $k = 1$ in the above corollary, we obtain Theorem 2.3 of Magesh et al. [31]

(ii) By allowing $\mu = 0$ and $\tau = 1$ in the above corollary, we get two results Güney et al. ([18], Corollary 10 and Corollary 23)

Corollary 12. *If the function $g \in K_\Sigma^\tau(\mu, \tilde{p}_k)$, then*

$$|d_2| \leq \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\tau(2\tau + 1) + 6\mu)k^2 t_k + 4(\mu + \tau)^2 T_k|}},$$

$$|d_3| \leq \frac{k^3 t_k^2}{|(\tau(2\tau + 1) + 6\mu)k^2 t_k + 4(\mu + \tau)^2 T_k|} + \frac{k|t_k|}{3(2\mu + \tau)},$$

(38)

and for some $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \dfrac{k|t_k|}{3(2\mu + \tau)} & ; |1 - \delta| \leq J_2, \\[4mm] \dfrac{k^3 t_k^2 |1 - \delta|}{|(\tau(2\tau + 1) + 6\mu)k^2 t_k + 4(\mu + \tau)^2 T_k|} & ; |1 - \delta| \geq J_2, \end{cases}$$

(39)

where

$$J_2 = \frac{1}{3(2\mu + \tau)}\left[\tau(2\tau + 1) + 6\mu + 4(\mu + \tau^2)\left|\frac{T_k}{k^2 t_k^2}\right|\right].$$

(40)

Remark 13. For $\tau = 1$, Corollary 12 reduces to a result of Frasin et al. ([30], Corollary 3.6). Further, allowing $k = 1$, we get Corollary 10 of Altnkaya [22].

Corollary 14. *If the function $g \in M_\Sigma^\tau(\gamma, \tilde{p}_k)$, then*

$$|d_2| \leq \frac{k\sqrt{k}\,|t_k|}{\sqrt{|((1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma)k^2 t_k + (2(1 + \tau) - \gamma)^2 T_k|}},$$

$$|d_3| \leq \frac{k|t_k|}{3(2 + \tau) - \gamma} + \frac{k^3 t_k^2}{|(4\mu + 1)k^2 t_k + (2\mu + 1)^2 T_k|},$$

(41)

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \dfrac{k|t_k|}{3(2 + \tau) - \gamma} & ; |1 - \delta| \leq J_3, \\[4mm] \dfrac{k^3 t_k^2 |1 - \delta|}{|((1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma)k^2 t_k + (2(1 + \tau) - \gamma)^2 T_k|} & ; |1 - \delta| \geq J_3, \end{cases}$$

(42)

where

$$J_3 = \frac{1}{3(2 + \tau) - \gamma}\left(((1 - \gamma)^2 + (\tau - \gamma)(2\tau + 1) + 5 - \gamma) + (2(1 + \tau) - \gamma)^2\left|\frac{T_k}{k^2 t_k}\right|\right).$$

(43)

4. Conclusion

A comprehensive family of biunivalent (or bi-Schlicht) functions is introduced by using k-Fibonacci numbers. Bounds of the first two coefficients $|d_2|$ and $|d_3|$ and the celebrated Fekete-Szegö functional have been fixed for this family. Through corollaries of our main result, we have highlighted many interesting new consequences.

A comprehensive family examined in this research paper could inspire further research related to other aspects such as a comprehensive family using q-derivative operator, a meromorphic biunivalent function family associated with Al-Oboudi differential operator, and a comprehensive family using integrodifferential operator.

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors confirm that there are no competing interests regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally in the preparation of this manuscript and have approved the final version of the manuscript.

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