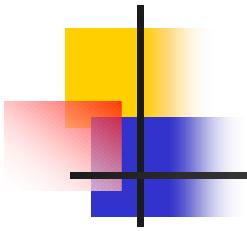
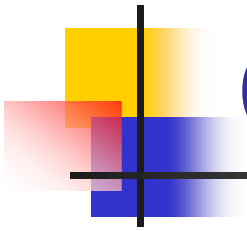
A Compressed Breadth-First Search for Satisfiability

DoRon B. Motter and Igor L. Markov
University of Michigan, Ann Arbor



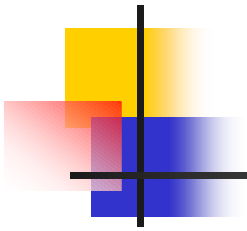
Motivation

- SAT is a fundamental problem in CS thry & apps
- "Efficient" SAT solvers abound (GRASP, Chaff)
- Many small instances are still difficult to solve
- We are pursuing novel algorithms for SAT facilitated by data structures with compression
  - Zero-suppressed Binary Decision Diagrams (ZDDs)
- Existing algorithms can be implemented w ZDDs
  - The DP procedure: Simon and Chatalic, *IJCAI 2000*
  - DLL: Aloul, Mneimneh and Sakallah, *DATE 2002*



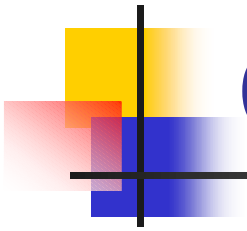
Outline

- Background
  - Partial truth assignments and implied clause classification
  - Representing collections of subsets with Zero-Suppressed BDDs (ZDDs)
- Cassatt: a simple example
- Cassatt: algorithm overview
  - Outer loop: process one variable at a time
  - Processing a given variable
  - Efficiency improvements using ZDDs
- Empirical results and conclusions



Partial Truth Assignments

- SAT instance: {V, C}
  - V: set of variables {a, b, … n}
  - C: set of clauses
    - Each clause is a set of literals over V
- Partial truth assignment to some $V \subseteq$ V
  - If it makes all literals in some clause false
    - call it invalid
  - Otherwise, call the assignment valid



Clause Classification

- With respect to a valid truth assignment, no clauses evaluate to false

⇒ Every clause must be either

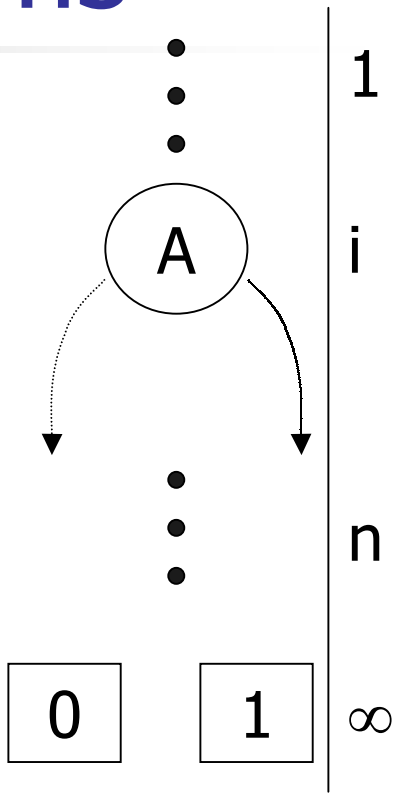
  - Unassigned
    - No literals in this clause are assigned
  - Satisfied
    - At least one literal in this clause is true
  - Open
    - At least one literal assigned, and all such literals are false

- {Open clauses} ⇔ partial truth assignment

⇒ Store sets of open clauses instead of assgnmts

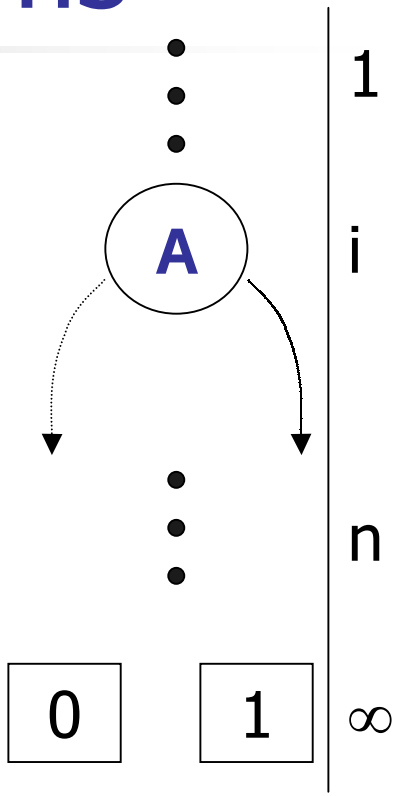
Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level number
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal



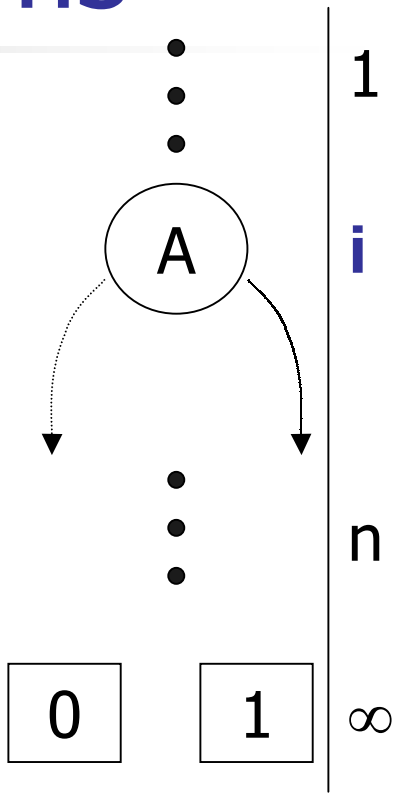
Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level number
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal



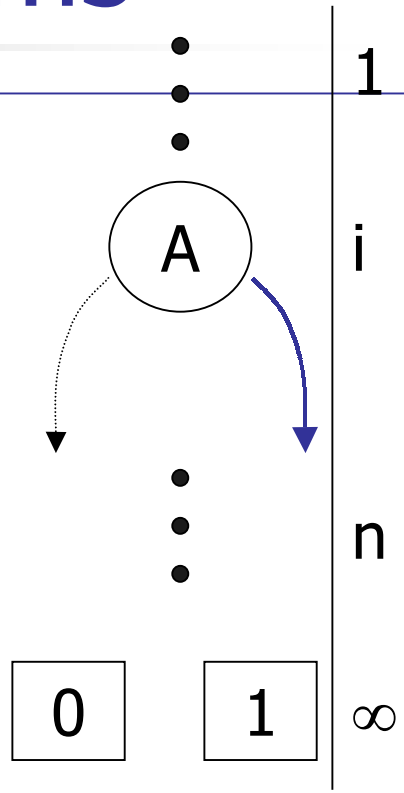
Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level number
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal



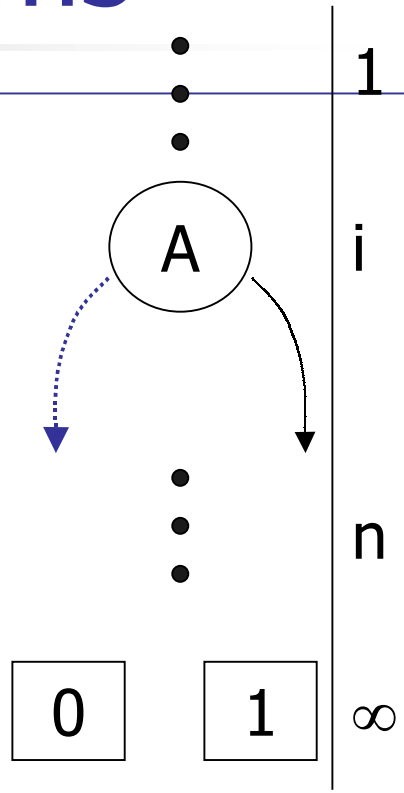
Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level number
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal



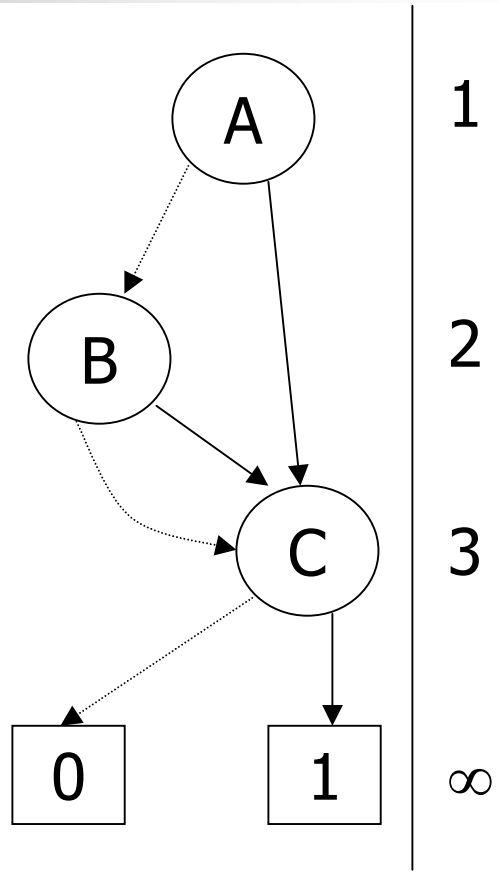
Binary Decision Diagrams

- BDD: A directed acyclic graph (DAG)
  - Unique source
  - Two sinks: the **0** and **1** nodes
- Each node has
  - Unique label
  - Level number
  - Two children at lower levels
    - T-Child and E-Child
- BDDs can represent Boolean functions
  - Evaluation is performed by a single DAG traversal



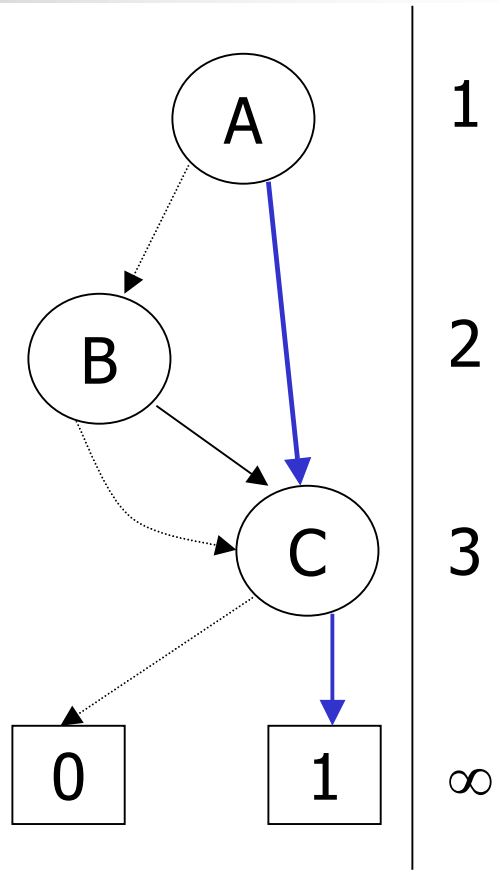
ZDD: Example

- Collection of subsets:
  - {1, 3}
  - {2, 3}
  - {3}



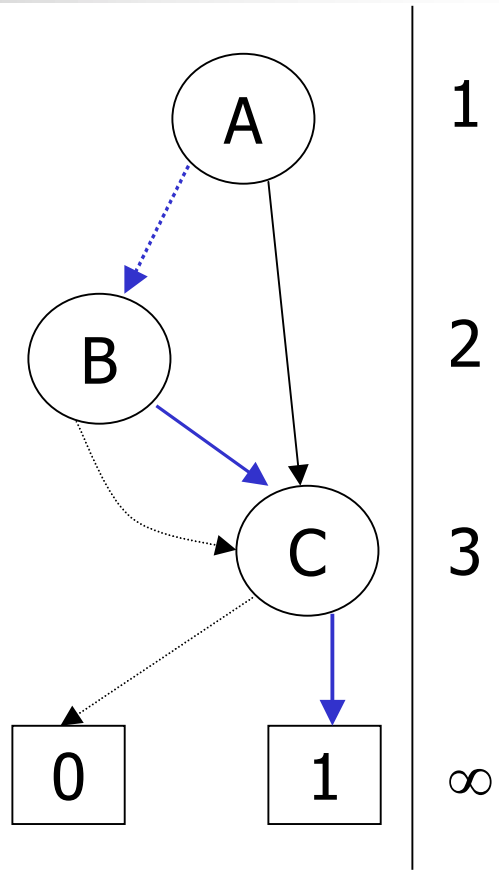
ZDD: Example

- Collection of subsets:
  - {1, 3}
  - {2, 3}
  - {3}



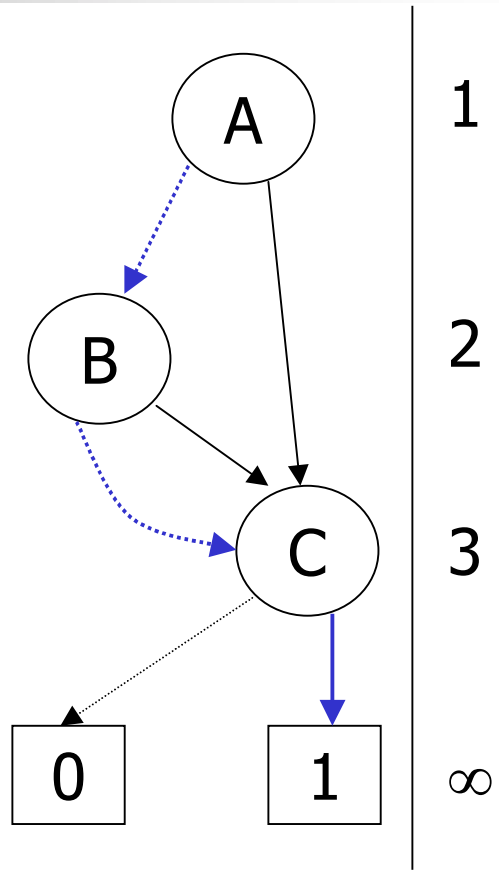
ZDD: Example

- Collection of subsets:
  - {1, 3}
  - {2, 3}
  - {3}



ZDD: Example

- Collection of subsets:
  - {1, 3}
  - {2, 3}
  - {3}



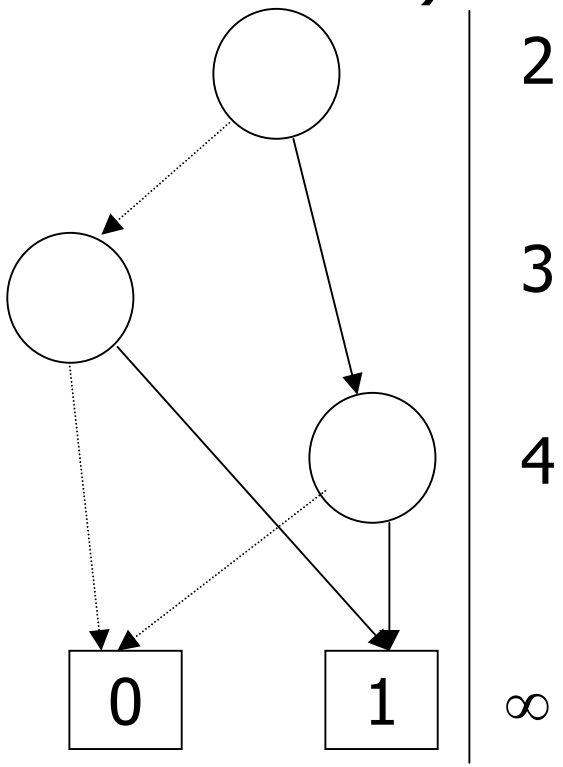
Zero-Supressed BDDs (ZDDs)

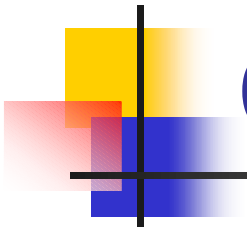
- Zero-supression rule
  - Eliminate nodes whose T-Child is **0**
  - No node with a given index $\Rightarrow$ assume a node whose T-child is **0**
- ZDDs can store a collection of subsets
  - Encoded by the collection's characteristic function
  - **0** is the empty collection $\varnothing$
  - **1** is the one-collection of the empty set $\{\varnothing\}$
- Zero-suppression rule enables compact representations of sparse or regular collections

Cassatt: Example

(b + c + ~d)(a+b)(~a + b + d)(a + ~b + ~c)

- a ← 1
  - activates clause 3 (satisfies 2, 4)
- a ← 0
  - activates clauses 2, 4 (sat 3)
- "Cut" clauses
  - 2, 3, 4





Cassatt: Example

(b + c + ~d)(a+b)(~a + b + d)(a + ~b + ~c)

- b ← 1
  - satisfies clauses 2, 3 (and 1)
- b ← 0
  - activates 1, satisfies 4, violates 2

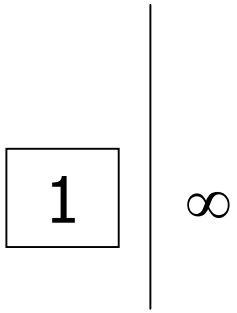
- "Cut" clauses
  - 1, 3, 4

Cassatt: Example

(b + c + ~d)(a+b)(~a + b + d)(a + ~b + ~c)

- c ← 1
  - violates 4, satisfies 1
- c ← 0
  - satisfies 4

- "Cut" clauses
  - 1, 3

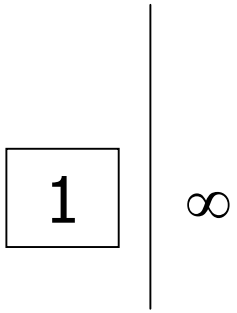


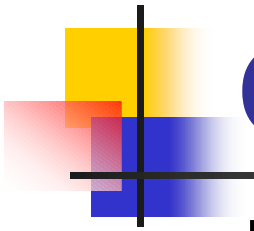
Cassatt: Example

(b + c + ~d)(a+b)(~a + b + d)(a + ~b + ~c)

- d ← 1
  - violates 1, satisfies 3
- d ← 0
  - violates 3, satisfies 1

- "Cut" clauses
  - ∅



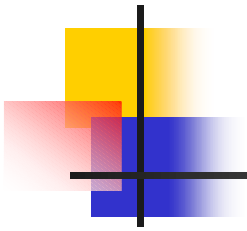


Cassatt: Algorithm Overview

- Maintain collection of subsets of open clauses
  - Analogous to maintaining all "promising" partial solutions of increasing depth
  - Enough information for BFS on the solution tree
- This collection of sets is called the front
  - Stored and manipulated in compressed form (ZDD)
  - Assumes a clause ordering (global indices)
    - Clause indices correspond to node levels in the ZDD
- Algorithm: expand one variable at a time
  - When all variables are processed two cases possible
    - The front is $\varnothing$ $\Rightarrow$ Unsatisfiable
    - The front is $\{\varnothing\}$ $\Rightarrow$ Satisfiable

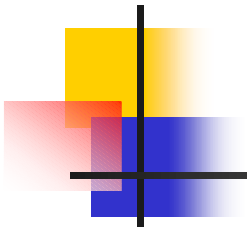
Cassatt: Algorithm Overview

```
Front ← 1     # assign {∅} to front
foreach v ∈ Vars
   Front2 ← Front
   Update(Front,  v ← 1)
   Update(Front2, v ← 0)
   Front ← Front ∪s Front2
if Front == 0 return Unsatisfiable
if Front == 1 return Satisfiable
```



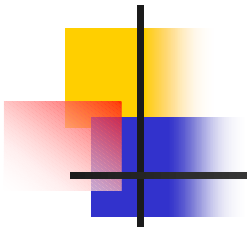
Processing a Single Variable

- Given:
  - Subset S of open clauses
  - Assignment of 0 or 1 to a single variable x
- Do they imply that some clauses must be violated?
  - I.e., does it correspond to a partial valid truth assignment ? (otherwise, can prune it)
- What subset S’ of clauses corresponds to the new truth assignment?
- In our BFS algorithm, we consider both 0 and 1



Detecting Violated Clauses

- Variables are processed in a static order
  - ⇒ Within each clause,
    some literal must be processed last
  - ⇒ The end literal of a clause is known beforehand
- For all literals in clause *C* to be false,
  it is *necessary and sufficient* that
  - Clause *C* must be open
  - The end literal of *C* must be assigned false



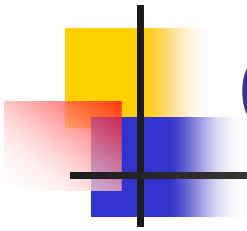
New Set of Open Clauses

- Given:
  - Subset S of open clauses
  - Assignment of 0 or 1 to a single variable x
  - The combination of the two is valid
- What subset S’ corresponds to the new truth assignment?

New Set of Open Clauses

- Given:
  - Subset S of open clauses
  - Assignment of 0 or 1 to a single variable *x*
- In the table below, select
  - Row: current status of a clause C $\in$ S
  - Column: location of literal *l* in C (*l* corresp. to *x*)

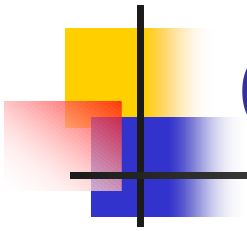
	Beginning	Middle	End	None
Satisfied	Impossible	No Action	No Action	No Action
Open	Impossible	**if**$(t(l)=0)$ $S' \leftarrow S' \bigcup C$	No Action	$S' \leftarrow S' \bigcup C$
Unassigned	**if**$(t(l)=0)$ $S' \leftarrow S' \bigcup C$	Impossible	Impossible	No Action



Gaining Efficiency Using ZDDs

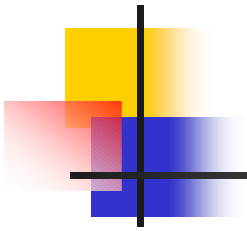
- Use ZDD to store the collection of all subsets of open clauses (front)
  - Achieves data compression (in some cases, with exponential compression ratio)
  - Improves memory requirements of BFS
- Use ZDD algorithms to consider all subsets in the ZDD at the same time
  - Implicit (symbolic) manipulation of compressed data

Gaining Efficiency Using ZDDs



- Given:
  - Assignment of 0 or 1 to a single variable *x*
- Consider its effect on all clauses
  - It violates some clauses
    - *x* corresponds to the end literal *l* of some clause *C*, and *l* is assigned false
  - It satisfies some clauses
    - *x* appears in *C*, and its literal *l* is assigned true
  - It activates some clauses
    - *x* corresponds to the beginning literal *l* for *C*, and *l* is assigned false

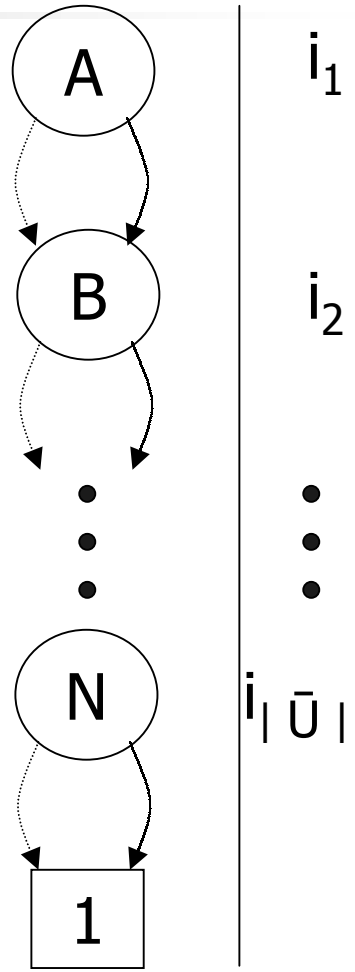
Newly Violated Clauses

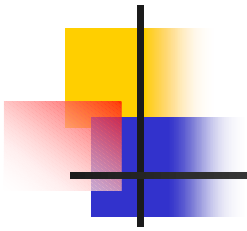


- Given:
  - Subset U of violated clauses
- Each set S in the ZDD containing $u \in U$ must be removed
  - This branch cannot yield satisfiability
- Efficient implementation in terms of ZDD ops
  - Form the ZDD containing all possible subsets of $\bar{U}$: the set-complement to U
  - Intersect this with the original front

Newly Violated Clauses

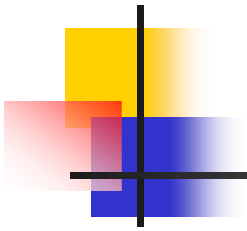
- Build the ZDD containing all subsets of $\bar{U}$
  - For each element in $\bar{U}$
    - add a don't care node at that level
  - Size is $O(\bar{U})$
  - Exponential compression in this simple case





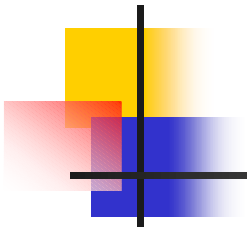
Newly Satisfied Clauses

- Given:
  - Set F of newly-satisfied clauses
- If $f \in F$ is in some subset of the front
  - It has now been satisfied
  - Any occurrence of f in the ZDD must be removed
- Implementation
  - The ZDD Existential Abstraction operation



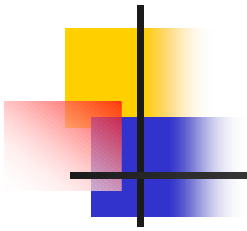
Newly Activated Clauses

- Given:
  - Set A of activated clauses
- Each $a \in A$ must be added to every set in the front
- Implementation:
  - The ZDD Cartesian Product operation



Pseudocode

```
Front ← 1     # assign {∅} to front
foreach v ∈ Vars
   Front2 ← Front
   Update(Front,  v ← 1)
   Update(Front2, v ← 0)
   Front ← Front ∪s Front2
if Front == 0 return Unsatisfiable
if Front == 1 return Satisfiable
```



Pseudocode

`Update(ZDD Z, v ← value)`

Find the set U of violated clauses

$Z \leftarrow Z \cap \mathbf{2}^{\sim U}$

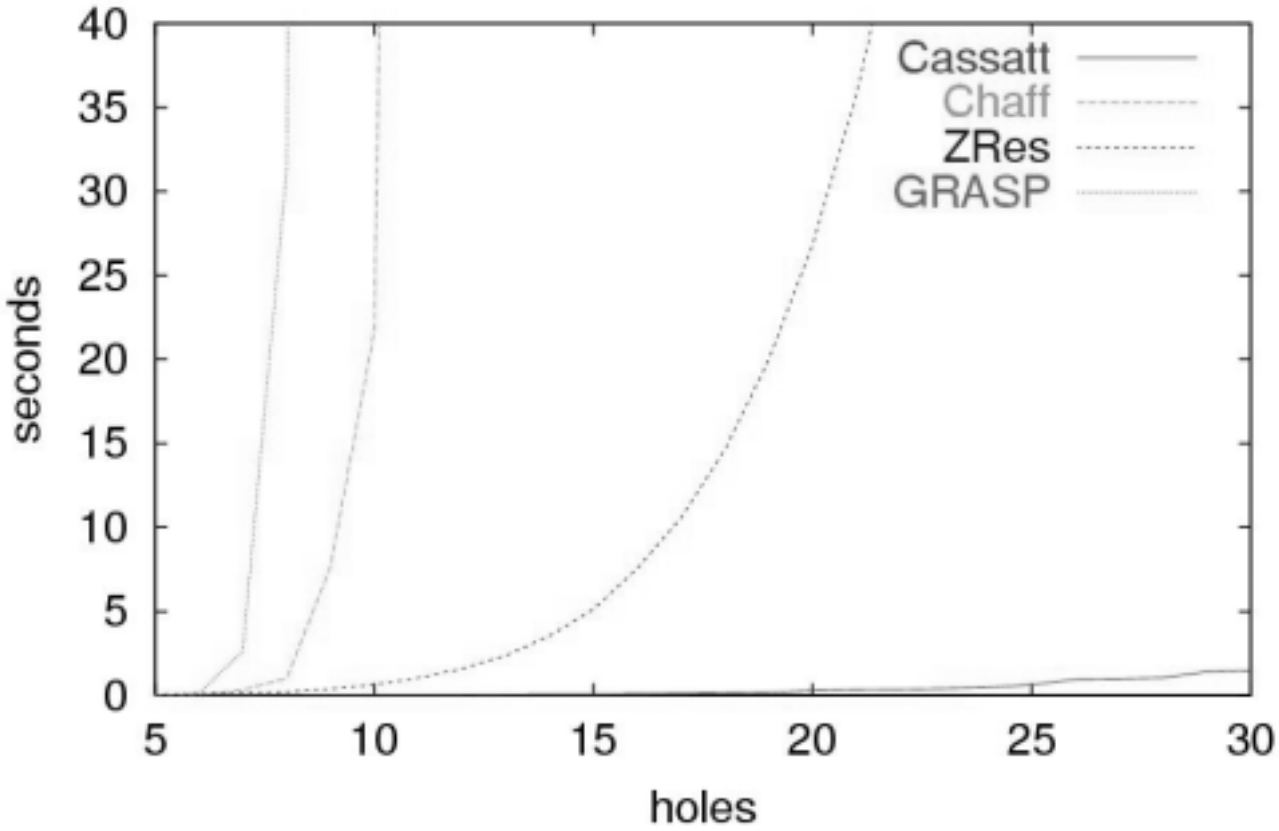
Find the set F of satisfied clauses

`Z ← ExistentialAbstract(Z, F)`

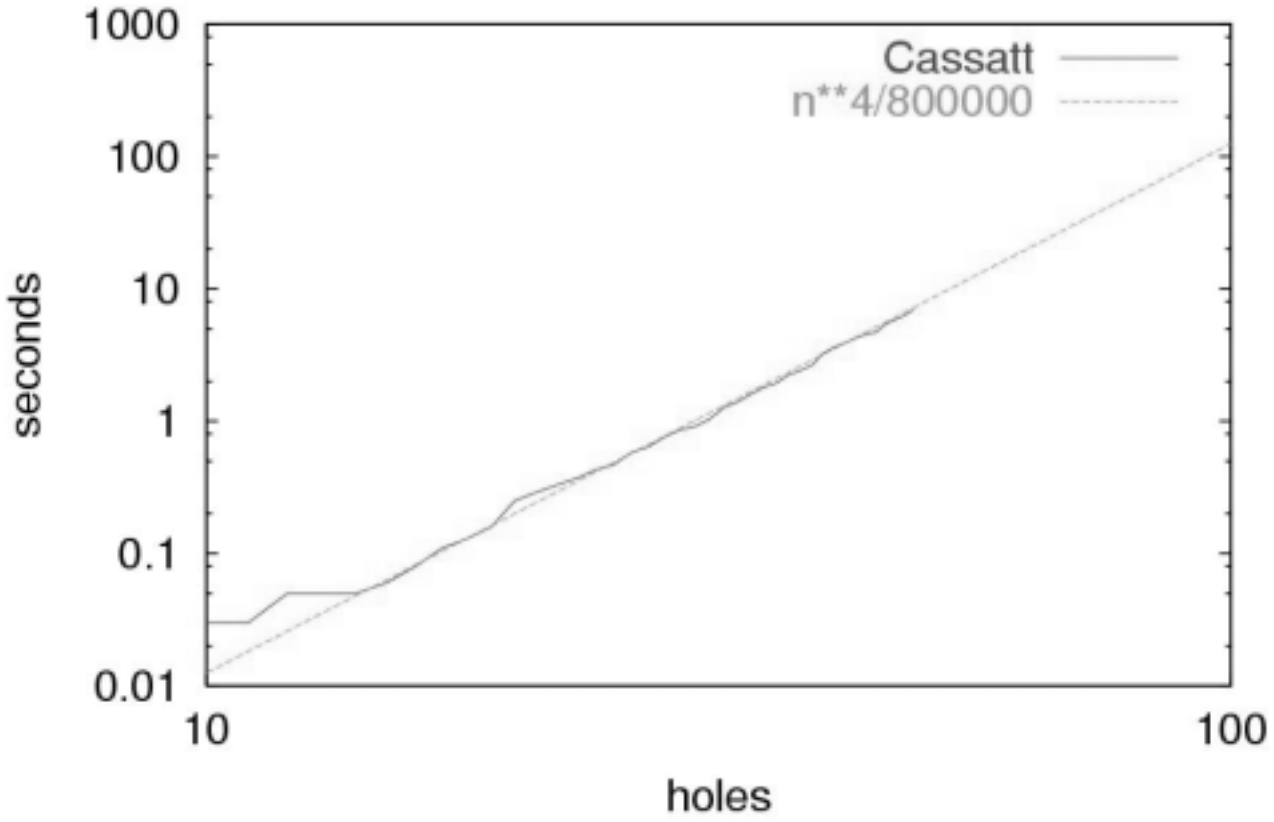
Find the set A of activated clauses

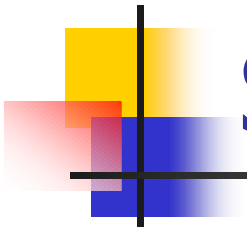
`Z ← CartesianProduct(Z, A)`

Results


Cassatt
Chaff
ZRes
GRASP
seconds
holes
0
5
10
15
20
25
30
35
40
5
10
15
20
25
30

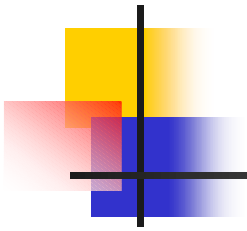
Results





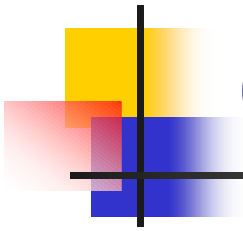
Summary of Results

- Proposed a novel algorithm for SAT
  - BFS with compression
  - Efficiency is due to exponential compression via ZDDs
- Implementation and empirical results
  - Solves pigeon-hole instances in poly-time
    - Outperforms Zres of Simon and Chatalic
  - Beats best DLL solvers on Urquhart instances
    - not better than Zres
  - Reasonable but not stellar performance on DIMACS benchmarks



Future Work

- Improved efficiency via Boolean Constraint Propagation
  - BCP is a part of all leading-edge SAT solvers
- Exploring the effects of clause and variable ordering on memory/runtime
- Implications of Cassatt in terms of proof systems



Questions?