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A Computational Technique for Evaluating $L(1, \chi)$ and the Class Number of a Real Quadratic Field

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Abstract. A description is given of a method for estimating $L(1, \chi)$ to sufficient accuracy to determine the class number of a real quadratic field. This algorithm was implemented on an IBM/370-158 computer and the class number, regulator, and value of $L(1, \chi)$ were obtained for each real quadratic field $Q(\sqrt{D})$ ($D = 2, 3, \ldots, 149999$). Several tables, summarizing various results of these computations, are also presented.

1. Introduction. Recently Hendy [2] has calculated on a Burroughs B6700 computer the class numbers and number of genera for all the real quadratic fields $Q(\sqrt{D})$, with $10^3 \leqslant D \leqslant 10^5$ and D squarefree. The method he used to do this is a modification of Ince's [3] technique of counting periods. In this paper we describe an entirely different computational procedure for determining the class number of $Q(\sqrt{D})$ via its Dirichlet function $L(1, \chi)$. This algorithm was implemented on an IBM/370-158 computer and used to determine all the class numbers in the range $2 \leqslant D \leqslant 1.5 \times 10^5$.

Our method is based upon the formula

$$h = \sqrt{\Delta}\, L(1, \chi)/2R,$$

where h is the class number, R is the regulator and Δ is the discriminant of $Q(\sqrt{D})$. The Dirichlet series $L(s, \chi) = \Sigma_{n=1}^{\infty} \chi(n) n^{-s}$ has $\chi(n) = (\Delta | n)$ (the Kronecker Symbol). It should be remarked here that

$$\Delta = \begin{cases} D, & D \equiv 1 \pmod 4, \\ 4D, & D \equiv 2, 3 \pmod 4, \end{cases}$$

and $R = \log \epsilon$, where ϵ (> 1) is the fundamental unit of $Q(\sqrt{D})$.

In the next section we describe a means of evaluating R and in the third section a method of estimating $L(1, \chi)$ to sufficient accuracy to determine the integer h. Finally, we give some results of our computations in Section 4.

2. Evaluation of the Regulator. The regulator can be evaluated by using the continued fraction algorithm. Put $Q_0 = 1, P_0 = 0, q_0 = [\sqrt{D}]$*, $A_{-1} = B_{-2} = 1$, $A_{-2} = B_{-1} = 0$, and define

$$P_{n+1} = q_n Q_n - P_n, \quad Q_{n+1} = (D - P_{n+1}^2)/Q_n, \quad q_{n+1} = [(P_{n+1} + \sqrt{D})/Q_{n+1}],$$

$$A_{n+1} = q_{n+1} A_n + A_{n-1}, \quad B_{n+1} = q_{n+1} B_n + B_{n-1}.$$

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*We use the usual notation $[\alpha]$ to indicate the greatest integer $\leqslant \alpha$.

Let r be the least nonnegative integer such that $Q_{r+1} = 4$, and let s be the least nonnegative integer such that $Q_{s+1} = 1$ ($s + 1$ is the period length of the continued fraction for $\sqrt{D}$). It is well known that such a value of s always exists; and that if r exists, then $r < s/2$. With these definitions of r and s and $D \neq 5$ we have

$$\epsilon = \begin{cases} (A_r + \sqrt{D}\, B_r)/2 & \text{if } r \text{ exists,} \\ A_s + \sqrt{D}\, B_s & \text{otherwise.} \end{cases}$$

A faster and more convenient way to calculate the regulator, however, can be obtained by first using the following modification of the continued fraction algorithm to evaluate the P's and Q's. We put $Q_{-1} = D, R_0 = 0$ and use the formulas

$$P_{n+1} = [\sqrt{D}\,] - R_n, \qquad Q_{n+1} = Q_{n-1} + q_n(P_n - P_{n+1}),$$

$$q_{n+1} = [(P_{n+1} + [\sqrt{D}\,])/Q_{n+1}],$$

$$R_{n+1} = \text{remainder on dividing } P_{n+1} + [\sqrt{D}\,] \text{ by } Q_{n+1}.$$

We can then avoid calculating the A's and B's by using a multiplicative formulation (see Smith [5]) wherein

$$(-1)^{n+1}(A_n - \sqrt{D}\, B_n) = \left(\prod_{i=1}^{n+1} (P_i + \sqrt{D})/Q_i \right)^{-1}.$$

Since $A_r^2 - DB_r^2 = 4(-1)^{r+1}$, we get

$$(A_r + \sqrt{D}\, B_r)/2 = 2 \prod_{i=1}^{r+1} (P_i + \sqrt{D})/Q_i.$$

Now during the development of the P's and Q's there must be an integer k such that $Q_k = Q_{k+1}$ or an integer j such that $P_j = P_{j+1}$. If k exists, then $s + 1 = 2k + 1$; if j exists, $s + 1 = 2j$; also, $r + 1 \leqslant k, j$. Using these facts to determine k or j, we can cut the calculation of $A_s + \sqrt{D}\, B_s$ in half by using the symmetry properties of the P's and Q's and the fact that $A_s^2 - DB_s^2 = (-1)^{s+1}$. We have

$$A_s + \sqrt{D}\, B_s = (P_{k+1} + \sqrt{D}) \left[\prod_{i=1}^{k} (P_i + \sqrt{D})/Q_i \right]^2 \quad \text{when } s + 1 = 2k + 1$$

or

$$A_s + \sqrt{D}\, B_s = Q_j \left[\prod_{i=1}^{j} (P_j + \sqrt{D})/Q_i \right]^2 \quad \text{when } s + 1 = 2j;$$

hence,

$$R = \log \epsilon = \begin{cases} \log 2 + \sum_{i=1}^{r+1} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ exists,} \\ \log(P_{k+1} + \sqrt{D}) + 2 \sum_{i=1}^{k} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ does not exist and } s + 1 = 2k + 1, \\ \log Q_j + 2 \sum_{i=1}^{j} \log(P_i + \sqrt{D})/Q_i & \text{when } r \text{ does not exist and } s + 1 = 2j. \end{cases}$$

3. Estimation of $L(1, \chi)$. In order to estimate $L(1, \chi)$ we make use of a device used by Barrucand [1]. We first note that $L(s, \chi)$ satisfies the functional equation

$$L(1-s, \chi) = A^{-s+1/2} \frac{\Gamma(s/2)}{\Gamma((1-s)/2)} L(s, \chi),$$

where $A = \pi/\Delta$. Putting $A = \alpha\beta$, we get

$$\alpha^{-s+1/2}\Gamma(s/2)L(s, \chi) = \beta^{s-1/2}\Gamma((1-s)/2)L(1-s, \chi).$$

If we let

$$B(x) = \sum_{n=1}^{\infty} \chi(n)e^{-n^2 x}$$

and use the method of [1], we obtain the functional equation

$$\sqrt{\alpha}\, B(\alpha^2 x) = \sqrt{\beta}\, x^{-1/2} B(\beta^2/x).$$

Hence,

$$\alpha^{-s}L(s, \chi)\Gamma(s/2) = \int_0^\infty B(\alpha^2 x)x^{s/2-1}\, dx$$

$$= \int_1^\infty B(\alpha^2 x)x^{s/2-1}\, dx + \sqrt{\beta/\alpha}\int_0^1 B(\beta^2/x)x^{(s-3)/2}\, dx$$

$$= \sum_{n=1}^{\infty} \chi(n) \int_1^\infty e^{-\alpha^2 n^2 x}x^{s/2-1}\, dx + \sqrt{\beta/\alpha} \sum_{n=1}^{\infty} \chi(n) \int_0^1 x^{(s-3)/2}e^{-\beta^2 n^2/x}\, dx.$$

If we put $\alpha = \beta = \sqrt{A}$, $s = 1$,

$$E(x) = \int_x^\infty e^{-t}/t\, dt, \qquad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt,$$

we get

$$L(1, \chi) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \chi(n)E(An^2) + \sum_{n=1}^{\infty} (\chi(n)/n)\operatorname{erfc}(n\sqrt{A}).$$

Let

$$C(m) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{m} \chi(n)E(An^2) + \sum_{n=1}^{m} (\chi(n)/n)\operatorname{erfc}(n\sqrt{A}),$$

$$T(m) = \frac{1}{\sqrt{\Delta}} \sum_{n=m+1}^{\infty} \chi(n)E(An^2) + \sum_{n=m+1}^{\infty} (\chi(n)/n)\operatorname{erfc}(n\sqrt{A}).$$

Since we wish to approximate $L(1, \chi)$ by the partial sum $C(m)$, we want to be able to determine m such that the remainder $T(m)$ is small enough that the integer h can be determined unequivocally. Now if $x > 0$,

$$0 < \operatorname{erfc}(x) < \frac{1}{x\sqrt{\pi}} e^{-x^2} \quad \text{and} \quad 0 < E(x) < e^{-x}/x;$$

thus, since $|\chi(n)| \leqslant 1$, we get

$$|T(m)| < \frac{2\sqrt{\Delta}}{\pi} \sum_{n=m+1}^{\infty} e^{-An^2}/n^2 < \frac{2\sqrt{\Delta}}{\pi}\int_m^\infty e^{-At^2}/t^2\, dt < \frac{\Delta^{3/2}}{\pi^2} e^{-Am^2}/m^3.$$

If we let $h^* = \sqrt{\Delta}\, C(m)/2R$, then

$$|h - h^*| < \frac{\sqrt{\Delta}}{2R} \, |T(m)| < \frac{A^2}{2Rm^3} \, e^{-Am^2}.$$

Since h is an integer and $2^{t-\lambda} | h$, where t is the number of distinct prime factors of Δ and

$$\lambda = \begin{cases} 1 & \text{if all prime divisors of } D \text{ are congruent to } 1, 2 \pmod 4, \\ 2 & \text{otherwise,} \end{cases}$$

we must find m such that $A^2 e^{-Am^2}/m^3 < 2^{t-\lambda}R$. When this is done, h is the unique integer, which is divisible by $2^{t-\lambda}$, in the interval $[h^* - 2^{t-\lambda-1}, h^* + 2^{t-\lambda-1}]$.

If we put $c = \sqrt{A}\, m$ and $l = \log(\sqrt{A}/2^{t-\lambda}R)$, we must find c such that $c^2 + 3\log c > l$. If $l > 1$ and $\Delta < 10^7$, then $1 < c < 2$; thus, $c = 1 + \epsilon$, $\epsilon < 1$ and $\log c > \epsilon - \epsilon^2/2$. If $l < 1$, then $\frac{1}{2} < c < 1$ and $c = 1 - \epsilon > 1 - 2\epsilon^2 > e^{-2\epsilon}$; hence, $\log c > 2(c - 1)$. If we use these results, we see that we may put $m = [c\sqrt{\Delta}/\sqrt{\pi}] + 1$, where

$$c = \begin{cases} 6 - \sqrt{27 - 2l}, & l > 1, \\ \sqrt{15 + l} - 3, & l < 1. \end{cases}$$

For $\Delta < 6 \times 10^5$ it is rarely necessary to go beyond 500 terms in the series $C(m)$ in order to evaluate h.

4. Results of the Computations. The program which evaluated $C(m)$ and then h, was written, using double precision, in FORTRAN. A special subroutine to evaluate $E(x)$ in double precision was written in assembler language, and the FORTRAN function DERFC was used to evaluate $\text{erfc}(x)$. In about seven hours of CPU time the computer calculated the class numbers of $Q(\sqrt{D})$ for all squarefree D such that $1 < D \leqslant 1.5 \times 10^5$. These class numbers for $10^3 \leqslant D \leqslant 10^5$ probably agree entirely with those in Hendy's table since the number of D, between these limits, having a given h agree in the two tables. Once h had been calculated for $Q(\sqrt{D})$, the value of $L(1, \chi)$ was calculated more precisely by using $L(1, \chi) = 2Rh/\sqrt{\Delta}$.

A large table, listing for each of the 91189 $Q(\sqrt{D})$, the regulator, the class number, and the value of $L(1, \chi)$ has been deposited in the UMT file. In this section we present some excerpts from that table. In Table 1 we give each value of h which occurs in the large table, the frequency $f(h)$ with which this h occurs, and the least value of D such that h is the class number for $Q(\sqrt{D})$.

Denote by $R(d)$ the regulator of $Q(\sqrt{d})$ and by $L(1, \chi_\delta)$ the value of $L(1, \chi)$ when $\chi(n) = (\delta | n)$. In Table 2 we give those values of D and $R(D)$ where $R(D)$ attains a new maximum:

$$R(D) > R(d) \qquad \text{for all } 2 \leqslant d < D.$$

In his examination of Littlewood's bounds on $L(1, \chi)$, Shanks [4] considered the function

$$L_{-\delta}(1) = \sum_{m=1}^{\infty} (4\delta | m) m^{-1}.$$

TABLE 1

h	f(h)	least D	h	f(h)	least D
1	20574	2	2	26427	10
3	2677	79	4	18573	82
5	943	401	6	3453	235
7	462	577	8	6898	226
9	311	1129	10	1237	1111
11	176	1297	12	2434	730
13	124	4759	14	563	1534
15	115	9871	16	1970	2305
17	62	7054	18	385	4954
19	48	15409	20	788	3601
21	43	7057	22	163	4762
23	20	23593	24	838	9634
25	30	24859	26	110	13321
27	20	8761	28	324	5626
29	16	49281	30	113	11665
31	4	97753	32	397	15130
33	11	55339	34	47	19882
35	8	25601	36	165	18226
37	7	24337	38	33	19834
39	6	41614	40	179	16899
41	1	55966	42	30	47959
43	3	14401	44	82	11026
45	7	32401	46	14	49321
47	1	78401	48	92	21610
49	1	70969	50	8	54769
51	1	69697	52	28	23410
53	1	69694	54	8	49834
55	1	106537	56	38	39999
57	2	41617	58	7	27226
60	18	78745	61	1	126499
62	3	68179	63	1	57601
64	23	71290	66	3	87271
68	12	53362	70	5	56011
72	11	45511	74	1	38026
76	7	93619	78	1	136159
80	3	94546	84	3	77779
86	2	110926	87	2	90001
88	3	56170	94	2	99226
96	4	50626	100	2	131770
108	1	140626	110	1	125434
116	1	116554			

Here we have

$$L_{-\delta}(1) = \begin{cases} \tfrac{1}{2} L(1, \chi_\delta), & \delta \equiv 1 \pmod 8, \\ 3L(1, \chi_\delta)/2, & \delta \equiv 5 \pmod 8, \\ L(1, \chi_\delta), & \text{otherwise.} \end{cases}$$

In Table 3 we give the values of D, $L_{-D}(1)$ (also the "Upper Littlewood Index" ULI [4]) such that

$$L_{-D}(1) > L_{-\delta}(1) \quad \text{for all } 2 \leqslant \delta < D.$$

This gives us an extension of Shanks' table of Hichamps [4, Table 6].

TABLE 2

D	R(D)	D	R(D)
2	0.88137359	9619	239.95274415
3	1.31695790	10399	255.84851576
6	2.29243167	10651	270.87206891
7	2.76865938	12919	283.24482085
11	2.99322285	13126	298.64260332
14	3.40008441	15031	303.73613093
19	5.82893697	16699	306.31406366
22	5.97634447	16879	318.45171155
31	8.01961269	17494	335.65693960
43	8.84850928	17614	336.73823980
46	10.79281810	18379	367.19773204
67	11.48949306	21319	392.01026227
94	15.27100210	23566	397.85610155
109	16.69360526	23599	400.38847076
139	18.85975147	25939	415.06367196
151	21.96346336	26959	423.40328648
199	24.20550214	27934	433.05457646
211	27.04530804	28414	447.08037149
214	27.96084155	31606	456.89547593
331	36.25638320	32839	458.83193172
379	37.79233938	32971	502.24984001
526	46.57116319	34654	508.58627196
571	47.33886269	38119	525.24115870
631	52.93846995	42046	532.11987985
739	53.63256141	42571	538.69114448
751	57.94214806	43726	550.86782494
886	58.00204637	46006	585.68371690
919	64.36292549	48799	619.57038850
991	68.80184250	53299	645.02269054
1291	69.42731847	55819	646.58791328
1366	77.50745983	56611	647.63115251
1699	77.68763324	58774	649.84304001
1726	91.48344937	60811	653.01948610
1999	91.88716165	61051	700.82741506
2011	100.53300453	67846	725.32530214
2311	110.30316856	72934	737.97224426
2326	111.22886783	76651	754.29325713
2566	114.05602902	78094	795.25078126
2671	119.59493590	78439	813.56346791
3019	127.49681351	82471	817.86184239
3259	132.12648968	84991	822.16136682
3691	137.39824137	85999	826.11841497
3931	147.25726673	87151	841.01248095
4174	153.01734303	90931	867.19521570
4846	162.46487523	98011	867.76246000
4951	166.65898164	100291	879.44151133
5119	172.50838882	102859	894.92275682
6211	174.49073086	104311	907.83373877
6379	175.31521179	106591	922.69477242
6406	188.37917309	111094	971.04549162
6451	196.13099779	122719	982.46753897
7606	215.68131176	132694	986.17025244
8254	221.19253648	133519	1029.90983807
8779	231.75791826	139591	1063.78482684

TABLE 3

D	$L_{-D}(1)$	ULI	D	$L_{-D}(1)$	ULI
2	0.62322524	0.4780	2146	2.31103546	0.5888
3	0.76034600	0.4690	2479	2.31375725	0.5853
6	0.93588131	0.4544	2599	2.34972306	0.5931
7	1.04645488	0.4881	3826	2.45090149	0.6074
10	1.15008652	0.4947	5014	2.45196600	0.6003
19	1.33724985	0.5122	5251	2.47339296	0.6044
31	1.44036496	0.5142	7459	2.53759015	0.6108
34	1.45715183	0.5140	8551	2.54931771	0.6102
46	1.59131421	0.5410	9454	2.57982512	0.6150
79	1.71299181	0.5495	10651	2.62463570	0.6227
106	1.74607012	0.5446	13666	2.67064271	0.6275
151	1.78736130	0.5404	18379	2.70856368	0.6293
211	1.86187579	0.5479	22234	2.76477214	0.6380
214	1.91136378	0.5619	32971	2.76601001	0.6295
274	1.91926263	0.5538	39274	2.77587028	0.6279
331	1.99283105	0.5673	45046	2.79115252	0.6285
394	2.06430094	0.5806	48799	2.80469210	0.6299
631	2.10744721	0.5748	61051	2.83638181	0.6324
751	2.11433901	0.5706	62386	2.84175734	0.6332
919	2.12313701	0.5662	74299	2.85281091	0.6321
991	2.18556256	0.5803	78439	2.90486139	0.6425
1054	2.24501069	0.5940	84319	2.91010275	0.6423
1486	2.26360155	0.5878	111094	2.91336081	0.6376
1654	2.27963311	0.5886	127906	2.93902278	0.6406

No further information was found to add to Shanks' table of Lochamps [4, Table 4]. There is no value of D in the interval $2 \times 10^3 < D < 1.5 \times 10^5$ for which

$$L_{-D}(1) < L_{-398}(1) = 0.33494376.$$

That is, $D = 398$ is such a strong Lochamp that it cannot be beaten for $D < 1.5 \times 10^5$. In that respect, it is similar to the imaginary quadratic field with the notorious $D = -163$; see [4, Table 3].

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