A computer-assisted instability proof for the Orr-Sommerfeld equation with Blasius profile

Jan-Rainer Lahmann[1] and Michael Plum[2]

Dedicated to the memory of Friedrich Goerisch

Abstract

The Orr-Sommerfeld equation is one of the governing equations of hydrodynamic stability. Mathematically, it constitutes a non-selfadjoint eigenvalue problem. Depending on its spectrum being contained in the right complex half-plane or not, the underlying flow is stable or unstable under some given perturbation. Here, we focus on the Blasius profile modelling a flow along a wall. We present a computer-assisted method for computing eigenvalue enclosures for such non-selfadjoint problems. As a specific result, for a particular parameter constellation in the Orr-Sommerfeld equation (often used as test example in the engineering literature), we enclose an eigenvalue in a circle which is completely contained in the left half-plane. This constitutes the first rigorous proof of instability for the Orr-Sommerfeld equation with Blasius profile.

Keywords: Orr-Sommerfeld equation, Blasius profile, instability proof, computer-assisted proof

1 Introduction

The *Orr-Sommerfeld* equation

$$(-D^2 + a^2)^2 u + iaR[V(-D^2 + a^2)u + V''u] = \lambda(-D^2 + a^2)u \qquad (1.1)$$

is one of the central equations governing the linearized stability theory of incompressible flows. It is posed on some real interval I, with $D = d/dx$, $i^2 = -1$, and with R denoting the Reynolds number of an underlying fluid which moves in a stationary flow, perpendicular to I, with given real-valued flow profile $V \in C^2(I)$. This flow is exposed to a single-mode perturbation with wave number $a > 0$, and the question of stable or unstable reaction of the flow in response to this perturbation arises. Within the frame of linearized stability theory, the answer to this question is directly related to the *spectrum*

[1]IBM Deutschland, Am Keltenwald 1, 71139 Ehningen, Germany, jan.lahmann@de.ibm.com

[2]Mathematisches Institut I, Universität Karlsruhe, 76128 Karlsruhe, Germany, michael.plum@math.uni-karlsruhe.de

of the Orr-Sommerfeld equation (1.1) (with appropriate boundary conditions for the eigenfunction u, usually of Dirichlet type, posed at the endpoints of I): The flow is stable (with respect to the wave number a) if the spectrum is completely localized in the right complex half-plane, otherwise unstable. For more details on the physical background, see e.g. [9], [11], [14], [27].

In the present article, we will exclusively consider the case where $I = [0, \infty)$ (corresponding to a half-plane flow along a wall), the boundary conditions read

$$u(0) = u'(0) = \lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0, \tag{1.2}$$

and where the flow profile V is the *Blasius* profile given by $V = f'$, where f is the solution of the nonlinear boundary value problem

$$f''' + ff'' = 0,\ f(0) = f'(0) = 0,\ \lim_{x\to\infty} f'(x) = 1, \tag{1.3}$$

which can be shown to exist and to be unique (see also Subsection 3.1).

Numerical results, which can be found in particular in the engineering literature (see e.g. [7], [11], [13], [14], [15]), give rise to the conjecture that, for certain constellations of the parameters R and a, one eigenvalue λ of (1.1), (1.2) is contained in the left complex half-plane, which would imply *instability* of the underlying flow. However, this conclusion is critical since the numerical approximations for λ are very close to the imaginary axis, and of course undergo the usual numerical errors. An analytical instability (or stability) proof has never been given, so that, from the mathematical point of view, the stability question for this problem has been open up to now.

In this article, we will give a first strict mathematical instability proof for a specific parameter constellation which is some kind of "standard" example in the literature. For this purpose, we propose a computer-assisted method for computing rigorous *eigenvalue enclosures*, in form of circular discs in the complex plane which definitely contain eigenvalues. An application of this method to the Orr-Sommerfeld problem provides such a disc which is completely contained in the left complex half-plane, and thus, proves instability.

Clearly, before being able to formulate precise statements about any kind of spectral terms, we need an appropriate operator theoretical realization of the Orr-Sommerfeld problem (1.1), (1.2). The following choice seems to be the most natural and simple one:

Let $X := H^2(0,\infty) \cap H^1_0(0,\infty)$ and $Y := L^2(0,\infty)$; endowed with the norms

$$\|u\|_X := \|(-D^2 + a^2)u\|_{L^2(0,\infty)},\quad \|u\|_Y := \|u\|_{L^2(0,\infty)}, \tag{1.4}$$

both are Banach spaces, and $\|\cdot\|_X$ is equivalent to the usual H^2-norm (but better suited for later purposes). Furthermore, let $D(A) := H^4(0,\infty) \cap H^2_0(0,\infty)$, and define the operators $A : D(A) \subset X \to Y$ and $B : X \to Y$ by Au and Bu denoting the left-hand and the right-hand side (without λ) of

equation (1.1), respectively. By [10, Theorem IX.9.5], A is closed, and B is obviously bounded with norm 1.

Our formulation of (1.1), (1.2) now reads

$$Au = \lambda Bu, \tag{1.5}$$

constituting a non-selfadjoint eigenvalue problem. In [20], we determined the *essential spectrum* of (1.5) (defined as the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda B$ is not a Fredholm operator of index 0) to be the half line

$$\sigma_{\text{ess}} = \{\mu + a^2 + iaR : \mu \in [0, \infty)\}, \tag{1.6}$$

which is obviously contained in the right complex half-plane, so that the stability problem reduces to the question if the left half-plane contains *eigenvalues* of problem (1.5). In [20], rough enclosures for the set of *all* eigenvalues have also been derived, which however, except for very small Reynolds numbers, intersect with both half-planes and are therefore not suitable for deciding the stability problem.

Therefore, we need more detailed eigenvalue enclosures, which by nowadays techniques can only be obtained with the aid of computer-assistance. In the following sections, we will propose such an eigenvalue enclosure method for general non-selfadjoint problems of the abstract form (1.5), with A and B denoting a closed and a bounded linear operator between Banach spaces X and Y. The reason for treating the general equation (1.5) is primarily to cover other non-selfadjoint eigenvalue problems, e.g. Couette- or Poiseuille-flows (see e.g. [9], [27]), which however will not be further addressed in this article.

It is important to remark that variational methods (like the Rayleigh-Ritz method) which have proved to be powerful tools for computing eigenvalue bounds, cannot be used for problem (1.5) due to its non-selfadjointness. Nevertheless, we will apply such variational methods to a certain self-adjoint *auxiliary* problem arising in the course of our enclosure method.

We wish to remark, however, that our approach is *not* of perturbational type like the methods presented e.g. in [8], [24], [28]; our given problem is allowed to be "far away" from any selfadjoint or explicitly decomposable problem. Other approaches for computing eigenvalue enclosures for non-selfadjoint problems, which however will not be adressed in more detail in the present article, have been proposed e.g. in [1], [12], [29] (where complex function theoretical methods are used), and in [22], [23]. In [17], [18], eigenvalue enclosures and stability criteria for the Orr-Sommerfeld equation on a *bounded* interval (allowing compactness arguments) are derived.

The paper is organized as follows: In Section 2 we formulate the basic theorem on enclosure and local uniqueness of eigenpairs of problem (1.5), as well as some extensions; the proof is a bit technical and therefore postponed to

the appendix. Section 3 contains a description of the main numerical methods used to compute some of the auxiliar terms needed; we use combinations of standard floating point arithmetic, interval arithmetic [16], and MATHEMATICA [31] routines. Section 4 is concerned with the computation of another auxiliary term via a certain *selfadjoint* eigenvalue problem. In Section 5, we report on the application of the method to the Orr-Sommerfeld problem, providing the desired instability proof.

2 The basic enclosure theorem

Let X, Y denote complex Banach spaces, $D(A) \subset X$ a linear subspace, and let $A : D(A) \to Y$ be linear and closed, and $B : X \to Y$ linear and bounded. Consider the eigenvalue problem

$$AU = \lambda BU \tag{2.1}$$

for eigenpairs $(U, \lambda) \in (D(A)\backslash\{0\}) \times \mathbb{C}$. In this section, we will primarily formulate our basic theorem providing enclosures and local uniqueness of eigenpairs of problem (2.1).

Let $\widehat{X} := X \times \mathbb{C}$ and $\widehat{D(A)} := D(A) \times \mathbb{C}$. Endowed with the norm

$$\left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} := \sqrt{\|u\|_X^2 + |\sigma|^2}, \tag{2.2}$$

$\widehat{X}$ is a Banach space.

Now suppose that, for instance by numerical means, some *approximate* eigenpair $(\omega, \mu) \in \widehat{D(A)}$, $\omega \neq 0$, and a constant $\delta \geq 0$ bounding its *defect*

$$\|A\omega - \mu B\omega\|_Y \leq \delta, \tag{2.3}$$

have been computed. The computation of ω and δ will be discussed in Section 3. We make the additional assumption that

$$\mu \text{ is not an interior point of } \sigma_{\text{res}}\,, \tag{2.4}$$

where the residual spectrum σ_{res} is defined to consist of all $\lambda \in \mathbb{C}$ such that $A - \lambda B$ is one-to-one but $(A - \lambda B)(D(A))$ is not dense in Y. Since obviously $\sigma_{\text{res}} \subset \sigma_{\text{ess}}$ and, due to (1.6), σ_{ess} has no interior points in case of the Orr-Sommerfeld problem, condition (2.4) holds, in this case, for every $\mu \in \mathbb{C}$.

Besides δ satisfying (2.3), we need some more auxiliary terms: We choose some scaling parameter $\gamma \in \mathbb{C}\backslash\{0\}$ and some bounded linear functional $\varphi : X \to \mathbb{C}$ satisfying $\varphi(\omega) \neq 0$ (e.g., $\varphi(u) := \int \overline{\omega} u \, dx$ if X is contained in an L^2-space), and suppose that some constant $K > 0$ has been computed such that

$$\begin{aligned} \left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} \;\leq\; & K\|Au - \mu Bu - \gamma\sigma B\omega\|_Y \\ & \text{for all } \begin{pmatrix} u \\ \sigma \end{pmatrix} \in \widehat{D(A)} \text{ such that } \varphi(u) = 0. \end{aligned} \tag{2.5}$$

The practical computation of K is not an easy task and will be discussed in Section 4.

Finally, let $b > 0$ denote some constant such that

$$\|Bu\|_Y \le b\|u\|_X \text{ for all } u \in X. \tag{2.6}$$

By (1.4), $b := 1$ satisfies (2.6) in the Orr-Sommerfeld case.

Theorem 2.1 *Suppose that*

$$\beta := 2bK^2|\gamma|\delta < 1. \tag{2.7}$$

(a) Then there exists an eigenpair $(U,\lambda) \in \widehat{D(A)}$ *of* (2.1) *such that* $\varphi(U-\omega) = 0$ *and*

$$\left\| \begin{pmatrix} \gamma(U-\omega) \\ \lambda - \mu \end{pmatrix} \right\|_{\widehat{X}} \le \alpha := \frac{2K|\gamma|\delta}{1+\sqrt{1-\beta}}. \tag{2.8}$$

(b) If $(\widetilde{U},\widetilde{\lambda}) \in \widehat{D(A)}$ *is any eigenpair satisfying* $\varphi(\widetilde{U}-\omega) = 0$ *and*

$$\left\| \begin{pmatrix} \gamma(\widetilde{U}-\omega) \\ \widetilde{\lambda} - \mu \end{pmatrix} \right\|_{\widehat{X}} < \alpha + \frac{2}{Kb}\sqrt{1-\beta}, \tag{2.9}$$

then $\widetilde{U} = U$, $\widetilde{\lambda} = \lambda$ *(i.e.,* (U,λ) *is locally unique).*

(c) Finally, λ *is geometrically simple, and* $BU \notin \overline{(A-\lambda B)(D(A))}$ *(i.e.,* λ *is also "algebraically" simple).*

Sketch of proof: Here, we only refer to some of the essentials used in the proof; all details are postponed to the appendix.

Introducing the *error* terms $u = \gamma(U-\omega), \sigma = \lambda - \mu$, we easily recognize that problem (2.1) is equivalent to the equation

$$(A-\mu B)u - \gamma\sigma B\omega = \sigma Bu - \gamma(A\omega - \mu B\omega) \tag{2.10}$$

for (u,σ), with the side condition $\varphi(u) = 0$ (normalizing U). In the main case to be treated in the proof, the linear operator $\mathcal{L}$ given by $\mathcal{L}\left[\binom{u}{\sigma}\right] := (A-\mu B)u - \gamma\sigma B\omega$ (on an appropriate domain of definition) can be inverted from (2.10) to transform it into a *fixed-point equation* for the operator T given by

$$T\begin{pmatrix} u \\ \sigma \end{pmatrix} := \mathcal{L}^{-1}[\sigma Bu - \gamma(A\omega - \mu B\omega)]$$

on a suitable Banach space. Using (2.3) - (2.7) one can show that T maps

$$\mathcal{D} := \left\{ \begin{pmatrix} u \\ \sigma \end{pmatrix} \in \widehat{X} : \varphi(u) = 0,\ \left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} \le \alpha \right\}$$

into itself and is moreover contractive on $\mathcal{D}$, whence Banach's Fixed-Point-Theorem provides the desired fixed-point of T in $\mathcal{D}$.

Remarks: (i) The crucial condition (2.7) is obviously satisfied if K is "moderate" and if the defect bound δ satisfying (2.3) is sufficiently small, i.e., if the approximate eigenpair (ω, μ) has been computed with *sufficient accuracy* (measured by δ). Also the error bound α in (2.8) is small under these conditions. It does not however make sense to diminish δ by replacing ω by $\varepsilon\omega$ (with small $\varepsilon > 0$); see remark iii).

ii) The simplicity statement c) of Theorem 2.1 is "good news" since it provides additional information, but of course also "bad news" since it makes clear that our method is not able to enclose multiple eigenvalues.

iii) The parameter γ scales the eigenfunction and its approximation ω. The choice of an optimal γ (e.g. in order to minimize β or α) is not clear a priori and usually needs numerical testing. Observe that $\gamma \to 0$ is *not* useful since (2.5) yields $K \geq 1/(|\gamma| \ \|B\omega\|_Y)$ (by inserting $(u, \sigma) := (0, 1)$), so that $\beta \to \infty$ for $\gamma \to 0$. Also $|\gamma| \to \infty$ does not make sense since K does not tend to zero as $|\gamma| \to \infty$, whence again $\beta \to \infty$.

iv) The side condition $\varphi(u) = 0$ in (2.5) is very essential: Omitting it would allow one to insert $(u, \sigma) := (\omega, 0)$ in (2.5), implying (by (2.3)) that $K \geq \|\omega\|_X/\delta$. Remark iii) (and (2.6)) moreover yield $K \geq 1/(b|\gamma| \ \|\omega\|_X)$. Altogether, $\beta \geq 2 \max\{b|\gamma| \ \|\omega\|_X^2/\delta, \ \delta/(b|\gamma| \ \|\omega\|_X^2)\} \geq 2$, so that (2.7) could never be satisfied. However, choosing φ suitably such that ω (and the eigenfunction U we are looking for) are "away" from the subspace $\{u \in D(A) : \varphi(u) = 0\}$, there are good chances to find a "moderate" K (see Section 4).

The local uniqueness statement in part b) of Theorem 2.1 refers to *eigenpairs*, which does not directly provide local uniqueness of the *eigenvalue*. The following considerations and Theorem 2.2 show a way of closing this gap in the case where X and Y are *Hilbert* spaces with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively. Let (2.3) to (2.7) be satisfied, so that the assertions of Theorem 2.1 hold true. An essential aid to study local uniqueness of eigenvalues of problem (2.1) is the auxiliary *selfadjoint* eigenvalue problem

$$u \in D(A), \ \langle (A - \mu B)u, (A - \mu B)v \rangle_Y = \kappa \langle u, v \rangle_X \text{ for all } v \in D(A). \quad (2.11)$$

We assume that the bottom of the spectrum of (2.11) is a simple eigenvalue κ_1, and that the remaining spectrum is contained in $[\kappa_2, \infty)$, for some $\kappa_2 > \kappa_1$. Forming the Rayleigh quotient for (2.11) with test function $U \in D(A)$ (given by Theorem 2.1a)) provides $(0 \leq) \ \kappa_1 \leq |\lambda - \mu|^2 \|BU\|_Y^2 / \|U\|_X^2$, i.e., κ_1 is usually "small". Due to the simplicity statement in part c), there is however some hope that $\kappa_2 \gg \kappa_1$, unless there is another eigenvalue of (2.1) close to μ, i.e., if a local uniqueness statement (of the kind we are looking for) holds! The *variational* methods described in Section 4 (in the context of computing a constant K satisfying (2.5)), which provide eigenvalue enclosures for *selfadjoint* problems, constitute a tool for *proving* that κ_2 is "not too small", by computing a lower bound for it (at least if κ_2 is an

eigenvalue)! Here, we suppose now that such a "non-small" lower bound for κ_2 is known, and show that this indeed implies a local uniqueness statement for the eigenvalue λ.

Theorem 2.2 *Let $(\omega,\mu),\delta,\gamma,\varphi,K,b,\beta,\alpha$ as in (2.3) - (2.8), and let $\rho>0$ be such that*

$$\varepsilon(\rho) := \frac{1}{\sqrt{\kappa_2}}(\delta+\rho b\|\omega\|_X) < \frac{|\varphi(\omega)|}{\|\varphi\|_{X,\mathbb{C}}}, \tag{2.12}$$

$$\left\{\rho^2+|\gamma|^2\varepsilon(\rho)^2\left[1+\|\omega\|_X^2\left(\frac{|\varphi(\omega)|}{\|\varphi\|_{X,\mathbb{C}}}-\varepsilon(\rho)\right)^{-2}\right]\right\}^{\frac{1}{2}} \leq \alpha+\frac{2}{Kb}\sqrt{1-\beta}. \tag{2.13}$$

Then, the set $\{z\in\mathbb{C}:|z-\mu|<\rho\}$ does not contain any eigenvalue of (2.1) except (possibly) λ.

Proof: Let $(\widehat{U},\widetilde{\lambda})\in\widehat{D(A)}$ denote an eigenpair of (2.1) such that $|\widetilde{\lambda}-\mu|<\rho$. Let $\widehat{U}$ be normalized such that $\|\widehat{U}\|_X=1$, $\langle\omega,\widehat{U}\rangle_X\geq 0$, and let $V:=\omega-\langle\omega,\widehat{U}\rangle_X\widehat{U}$. First we prove that

$$\|V\|_X<\varepsilon(\rho). \tag{2.14}$$

This is trivial if ω and $\widehat{U}$ are linearly dependent. Otherwise, the min-max principle (see, e.g., [4]) applied to problem (2.11) yields

$$\begin{aligned}
\kappa_2 &\leq \max_{\substack{r,s\in\mathbb{C}\\ |r|^2+|s|^2>0}} \frac{\langle(A-\mu B)(r\omega+s\widehat{U}),(A-\mu B)(r\omega+s\widehat{U})\rangle_Y}{\langle r\omega+s\widehat{U},r\omega+s\widehat{U}\rangle_X}\\
&\leq \max_{\substack{r,s\in\mathbb{C}\\ |r|^2+|s|^2>0}} \frac{(|r|\delta+|s|\ |\widetilde{\lambda}-\mu|b)^2}{|r|^2\|\omega\|_X^2-2|r|\ |s|\langle\omega,\widehat{U}\rangle_X+|s|^2}\\
&= \sup_{0\leq t<\infty}\frac{(\delta+t|\widetilde{\lambda}-\mu|\ b)^2}{\|\omega\|_X^2-2t\langle\omega,\widehat{U}\rangle_X+t^2}.
\end{aligned}$$

Elementary differential calculus shows that the supremum is attained at

$$t=\frac{\delta\langle\omega,\widehat{U}\rangle_X+|\widetilde{\lambda}-\mu|b\|\omega\|_X^2}{\delta+|\widetilde{\lambda}-\mu|b\langle\omega,\widehat{U}\rangle_X},$$

with supremum value

$$\frac{\delta^2+(|\widetilde{\lambda}-\mu|b\|\omega\|_X)^2+2\delta|\widetilde{\lambda}-\mu|b\langle\omega,\widehat{U}\rangle_X}{\|\omega\|_X^2-\langle\omega,\widehat{U}\rangle_X^2}\leq\frac{(\delta+|\widetilde{\lambda}-\mu|b\|\omega\|_X)^2}{\|V\|_X^2}$$

which implies (2.14).

By (2.12), we obtain from (2.14) in particular that $\|V\|_X < \|\omega\|_X$, so that $\langle \omega, \widehat{U}\rangle_X \neq 0$, and thus, $\langle \omega, \widehat{U}\rangle_X > 0$ due to the normalization of $\widehat{U}$. Moreover, (2.14) implies

$$\left| \frac{\varphi(\widehat{U})}{\varphi(\omega)} - \frac{1}{\langle \omega, \widehat{U}\rangle_X} \right| = \frac{|\varphi(V)|}{|\varphi(\omega)|\langle \omega, \widehat{U}\rangle_X} < \frac{\|\varphi\|_{X,\mathbb{C}}\, \varepsilon(\rho)}{|\varphi(\omega)|\langle \omega, \widehat{U}\rangle_X}. \tag{2.15}$$

In particular this yields, by (2.12), that $\varphi(\widehat{U}) \neq 0$, so that we can define the eigenfunction $\widetilde{U} := (\varphi(\omega)/\varphi(\widehat{U}))\widehat{U}$, satisfying $\varphi(\widetilde{U} - \omega) = 0$.

Since for $0 < r < a$ and $z \in \mathbb{C}$,

$$|z - a| < r \Rightarrow \left| \frac{1}{z} - \frac{1}{a} \right| = \frac{|z-a|}{a|z|} < \frac{r}{a(a-r)},$$

we obtain from (2.15) and (2.12) that

$$\left| \frac{\varphi(\omega)}{\varphi(\widehat{U})} - \langle \omega, \widehat{U}\rangle_X \right| < \frac{\varepsilon(\rho)}{\frac{|\varphi(\omega)|}{\|\varphi\|_{X,\mathbb{C}}} - \varepsilon(\rho)} \langle \omega, \widehat{U}\rangle_X \leq \frac{\varepsilon(\rho)\|\omega\|_X}{\frac{|\varphi(\omega)|}{\|\varphi\|_{X,\mathbb{C}}} - \varepsilon(\rho)}. \tag{2.16}$$

Furthermore, since $V \perp \widehat{U}$,

$$\|\omega - \widetilde{U}\|_X^2 = \left\| V + \left(\langle \omega, \widehat{U}\rangle_X - \frac{\varphi(\omega)}{\varphi(\widehat{U})} \right) \widehat{U} \right\|_X^2 = \|V\|_X^2 + \left| \frac{\varphi(\omega)}{\varphi(\widehat{U})} - \langle \omega, \widehat{U}\rangle_X \right|^2.$$

Using here (2.14) and (2.16) on the right-hand side, we obtain that $\|(\gamma(\widetilde{U} - \omega), \widetilde{\lambda} - \mu)\|_{\widehat{X}}$ is less than the left-hand side of (2.13), so that (2.9) is satisfied. Theorem 2.1b) implies $\widetilde{\lambda} = \lambda$ (and $\widetilde{U} = U$). $\square$

We close this section by some remarks on eigenvalue *exclosures*, i.e., on the determination of subsets of $\mathbb{C}$ which do not contain eigenvalues of problem (2.1). First, we attack the question on a local basis. Let $\mu \in \mathbb{C}$ be such that we suspect that no eigenvalue of (2.1) is close to μ. To prove and to quantify this conjecture we consider the selfadjoint problem (2.11) again. Suppose that we have computed a positive lower bound for its bottom eigenvalue κ_1 (or, more generally, for the bottom value of its spectrum), by the variational methods mentioned earlier and to be described in Section 4. Then

$$\text{there is } \textit{no} \text{ eigenvalue } \widetilde{\lambda} \text{ of (2.1) such that } |\widetilde{\lambda} - \mu| < \frac{\sqrt{\kappa_1}}{b}, \tag{2.17}$$

since for any given eigenvalue $\widetilde{\lambda}$ we obtain by the min-max principle, with $\widetilde{u} \in D(A)$ denoting a corresponding eigenelement,

$$\kappa_1 \leq \frac{\langle (A - \mu B)\widetilde{u}, (A - \mu B)\widetilde{u}\rangle_Y}{\langle \widetilde{u}, \widetilde{u}\rangle_X} = \frac{|\widetilde{\lambda} - \mu|^2 \|B\widetilde{u}\|_Y^2}{\|\widetilde{u}\|_X^2} \leq |\widetilde{\lambda} - \mu|^2 b^2.$$

In principle, we can use local nonexistence statements of the form (2.17) to prove nonexistence of eigenvalues in arbitrary bounded subsets of $\mathbb{C}$, by

covering such a set by "local circles" obtained from the statements (2.17). Of course, it has to be admitted that for "big" bounded subsets, this way of proceeding requires an enormous computing time, since for each of the centers μ of the desired "local circles", a verified positive lower bound for the bottom eigenvlue κ_1 of problem (2.11) must be computed. On the other hand, it might appear natural, in comparison with other analytical problems, that nonexistence statements are comparatively hard to obtain.

For the *Orr-Sommerfeld* problem, rough enclosures for the set of *all* eigenvalues have been calculated (analytically) in [20]. These enclosing sets have a compact intersection with the left complex half-plane, so that the above considerations can in principle be used to prove, for certain parameter constellations, nonexistence of eigenvalues in the complete left half-plane, i.e., to prove *linearized stability* for these parameter constellations. However, due to the computing time problems mentioned above, such computations have, up to now, only been carried out approximately, without providing *rigorous* statements.

3 Numerical tools

In this section, we briefly report on the numerical tools we used to compute the terms needed for Theorem 2.1, i.e., an approximate eigenpair $(\omega, \mu) \in \widehat{D(A)}$ to problem (2.1), with A and B chosen according to (1.1), and its defect bound δ (see (2.3)). The computation of a constant K satisfying (2.5) is postponed to the following Section 4.

For computing (ω, μ) we need, in addition, an approximation $\widetilde{V}$ to the Blasius profile V generated by (1.3), and the computation of a (rigorous!) defect bound δ even requires an enclosure for V.

3.1 The Blasius profile

For *enclosing* V, we consider the following initial value problem associated with (1.3):

$$g''' + gg'' = 0,\ g(0) = g'(0) = 0,\ g''(0) = 1. \tag{3.1}$$

Using the integral equation

$$u(x) = \exp\left(-\frac{1}{2}\int_0^x (x-t)^2 u(t)dt\right) =: (Tu)(x) \tag{3.2}$$

for $u = g''$ (see [30]), one can show that the solution g of (3.1) exists on $[0, \infty)$, and that $\sigma := \lim\limits_{x\to\infty} g'(x) \in (0, \infty)$ exists.

Then,

$$f(x) := \sigma^{-\frac{1}{2}} g(\sigma^{-\frac{1}{2}} x) \tag{3.3}$$

solves (1.3) uniquely. By monotone iteration techniques (for the antitone operator T defined in (3.2)), an enclosure

$$\underline{u} \le u \le \overline{u} \tag{3.4}$$

for the solution u of (3.2) can be obtained. In practice, this iteration is carried out on a compact interval $[0, \widehat{x}_0]$ only, and the exponentials arising after each iteration step are replaced by bounding polynomials to facilitate the next iteration step, carried out by explicit integration using MATHEMATICA [31]. So (3.4) is obtained on $[0, \widehat{x}_0]$, with polynomials $\underline{u}$ and $\overline{u}$. By integration of $g'' = u$ (using MATHEMATICA again) and by use of the differential equation for g, we obtain enclosures for g, g', g'', g''' on $[0, \widehat{x}_0]$. Exploiting the equation (3.2) for $u = g''$ in more detail, these bounds can be used to enclose g, g', g'', g''' on $[0, \infty)$, with *linear* resp. *constant* bounds on $[\widehat{x}_0, \infty)$ (see [5], [19]), and to enclose the value $\sigma = \lim\limits_{x\to\infty} g'(x)$; the result is

$$\sigma \in 1.65519036023\mathrm{08}^{39}_{26},$$

with an obvious notation for upper and lower bounds. Using (3.3), we therefore obtain enclosures for f, f', f'', f''', and thus, for $V = f'$, V', V'', on $[0, \infty)$.

For computing an *approximation* $\widetilde{V} \in C^2[0,\infty) \cap H^{2,\infty}(0,\infty)$, let $x_0 \le \sqrt{\sigma}\, \widehat{x}_0$ (with approximate equality), and let $\widetilde{v}$ denote a polynomial satisfying $\widetilde{v}(x_0) = 0$ and approximating V'' (or, respectively, the arithmetic mean of the upper and the lower bound for V'' obtained before), on $[0, x_0]$. Then

$$\widetilde{V}(x) := \eta \int_0^x \int_{x_0}^t \widetilde{v}(s)ds\, dt \ (x \in [0, x_0]),\ \widetilde{V}(x) := 1 \ (x \in (x_0, \infty)) \tag{3.5}$$

(with η chosen such that $\widetilde{V}(x_0) = 1$) approximates V, and is polynomial on $[0, x_0]$. Note that $\eta \approx 1$ if $\widetilde{v}$ is a "good" approximation to V''.

3.2 Computation of an approximate eigenpair

For computing an approximate eigenpair $(\omega, \mu) \in \widehat{D(A)}$ to the Orr-Sommerfeld equation (i.e., (2.1) with A, B from (1.1)), we choose $M \in \mathbb{N}$, $N := 2M + 4$, and use the ansatz

$$\omega = \sum_{j=1}^{N} \beta_i \varphi_i, \tag{3.6}$$

with $\beta_1, \ldots, \beta_N \in \mathbb{C}$ to be determined, and with basis functions $\varphi_1, \ldots, \varphi_N \in D(A)$ defined by

$$\varphi_{j+4}(x) := \left\{ \begin{array}{cc} (x - x_0)^4 x^{j+1} & (x \in [0, x_0]) \\ 0 & (x \in (x_0, \infty)) \end{array} \right\},$$

$$\varphi_{j+M+4}(x) := \left\{ \begin{array}{cc} 0 & (x \in [0, x_0]) \\ (e^{-ax} - e^{-ax_0})^4 e^{-jax} & (x \in (x_0, \infty)) \end{array} \right\}$$

for $j = 1, \ldots, M$, and with $\varphi_1, \ldots, \varphi_4$ chosen linearly independent, such that, for $j = 1, \ldots, 4$,

$$\begin{aligned}&\varphi_j \mid_{[0,x_0]} \in \text{span}\,\{x^2, x^3, x^4, x^5\},\\ &\varphi_j \mid_{[x_0,\infty)} \in \text{span}\,\{e^{-ax}, e^{-2ax}, e^{-3ax}, e^{-4ax}\},\ \varphi_j \in C^3[0,\infty).\end{aligned}$$

Here, a is the wave number entering (1.1). The asymptotic behaviour of these basis functions is governed by e^{-ax}; this choice is inspired by Fischer [11], who proved that the exact eigenfunction decays asymptotically like e^{-ax}.

For computing μ and $\beta = (\beta_1, \ldots, \beta_N)$ in (3.6), we use a Ritz-Galerkin method

$$\beta \left(\langle \widetilde{A}\varphi_i, \varphi_j \rangle_Y \right)_{i,j=1,\ldots,N} = \mu\beta \left(\langle B\varphi_i, \varphi_j \rangle_Y \right)_{i,j=1,\ldots,N}, \tag{3.7}$$

where $\widetilde{A}$ is defined by replacing the coefficients V and V'' in A by their piecewise polynomial approximations $\widetilde{V}$ and $\widetilde{V}''$ from (3.5). The entries $\langle \widetilde{A}\varphi_i, \varphi_j \rangle_Y$ and $\langle B\varphi_i, \varphi_j \rangle_Y$ can therefore be computed by closed form integration, using MATHEMATICA; for our purpose, a special integration routine has been constructed for reasons of computing time.

The matrix eigenvalue problem (3.7) is solved approximately, first for some "moderate" N, by the built-in MATHEMATICA routine.

For improving the quality of the approximation (β, μ), we perform some inverse iteration steps for problem (3.7) with higher N, with shift parameter $\widetilde{\mu}$ taken from the rough calculation.

3.3 Computation of a defect bound

For the computation of a constant δ satisfying (2.3) (rigorously!), we use $\widetilde{A}$ as defined in the previous subsection to obtain

$$\begin{aligned}\|A\omega - \mu B\omega\|_Y &\le \|\widetilde{A}\omega - \mu B\omega\|_Y + \|A\omega - \widetilde{A}\omega\|_Y\\ &\le \|\widetilde{A}\omega - \mu B\omega\|_Y + aR\left[\|V - \widetilde{V}\|_\infty \|B\omega\|_Y + \|V'' - \widetilde{V}''\|_\infty \|\omega\|_Y\right].\end{aligned} \tag{3.8}$$

Here, the first term can be computed by closed form integration using MATHEMATICA, since all entries are polynomial on $[0, x_0]$ and polynomial in e^{-ax} on $[x_0, \infty)$.

For the same reason, $\|B\omega\|_Y$ and $\|\omega\|_Y$ are computable in closed form, so that the remaining second term in (3.8) can be bounded using the enclosures for V and V'' obtained according to Subsection 3.1.

4 Computation of K

For computing a constant K satisfying (2.5) (and lower bounds for the eigenvalues κ_1, κ_2 of problem (2.11)) we restrict our considerations, as in the previous section, to the Orr-Sommerfeld problem specified by (1.3), (1.4) and

the definitions before and after that. For the linear functional φ used in (2.5) and Theorem 2.1 we make the special choice

$$\varphi(u) = \langle u, \phi \rangle_Y, \text{ where } \phi(x) := \left\{ \begin{array}{ll} 0 & \text{on } [0, x_0) \\ e^{-ax} & \text{on } [x_0, \infty) \end{array} \right\}, \tag{4.1}$$

with $x_0 \in (0, \infty)$ as in Section 3. (This φ is obviously bounded not only on X, but even on Y.)

Let $\widetilde{A} : D(A) \subset X \to Y$ be defined as in Section 3. Since $\widetilde{V} \in C^2[0, \infty) \cap H^{2,\infty}(0, \infty)$, $\widetilde{A}$ is closed (as A is) by [10, Theorem IX.9.5]. Thus, also the operators L and $\mathcal{L}$ defined by

$$\begin{array}{rcl} D(L) & := & \{u \in D(A) : \varphi(u) = 0\}, \; Lu := \widetilde{A}u - \mu B u, \\ D(\mathcal{L}) & := & D(L) \times \mathbb{C}, \; \mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix}\right] := Lu - \gamma \sigma B \omega \end{array} \tag{4.2}$$

are closed. (Note that the operator $\mathcal{L}$ introduced here involves $\widetilde{A}$ in place of A, in contrast to the operator $\mathcal{L}$ used - exclusively - in the proof of Theorem 2.1.)

For $u \in D(A)$, $\|Au - \widetilde{A}u\|_Y \le aR \left[\|\widetilde{V} - V\|_\infty + \; a^{-2} \|\widetilde{V}'' - V''\|_\infty \right] \cdot \|u\|_X =: \eta \|u\|_X$ (observe that $\|u\|_Y \le a^{-2} \|u\|_X$), so it is straightforward that (2.5) holds for $K := \widetilde{K}/(1 - \eta \widetilde{K})$ if $\widetilde{K}$ satisfies

$$\left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} \le \widetilde{K} \left\| \mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix} \right] \right\|_Y \text{ for all } \begin{pmatrix} u \\ \sigma \end{pmatrix} \in D(\mathcal{L}) \tag{4.3}$$

and $\eta \widetilde{K} < 1$. The latter condition is not problematic unless $\widetilde{K}$ is very large, since the tight enclosures for V and V'' provide a very small upper bound for η. By a similar treatment, A can be replaced by $\widetilde{A}$ also in problem (2.11). For computing $\widetilde{K}$, consider the eigenvalue problem

$$\langle \mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix} \right], \mathcal{L}\left[\begin{pmatrix} v \\ \tau \end{pmatrix} \right] \rangle_Y = \kappa \langle \begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix} \rangle_{\widehat{X}} \quad \text{for all } \begin{pmatrix} v \\ \tau \end{pmatrix} \in D(\mathcal{L}) \tag{4.4}$$

which is, due to the closedness of $\mathcal{L}$, equivalent to an eigenvalue problem for a selfadjoint operator in the Hilbert space $D(\mathcal{L})$, endowed with the graph norm. Consequently, the min-max principle holds for problem (4.4). In particular, (4.3) holds for

$$\widetilde{K} := \frac{1}{\sqrt{\kappa_1}}, \text{ with } \kappa_1 := \min\{\text{spectrum of (4.4)}\}, \tag{4.5}$$

provided that $\kappa_1 > 0$. So we are left with the computation of a positive lower bound for κ_1. If the spectrum is initially discrete, we therefore need *eigenvalue bounds* for the (selfadjoint!) eigenvalue problem (4.4). The self-adjointness opens the field for very powerful *variational* methods providing such bounds, the most important of which we will briefly describe in the following subsection.

4.1 Variational eigenvalue bounds

Variational methods for computing eigenvalue bounds will be described for the problem

$$M(u,v) = \kappa N(u,v) \quad \text{for all } v \in H, \tag{4.6}$$

with M and N denoting positive definite hermitian sesquilinear forms on the complex vector space H, such that $(H, M(\cdot,\cdot))$ is a separable Hilbert space and N is bounded. Under these hypotheses, problem (4.6) is equivalent to an eigenvalue problem for a selfadjoint operator in H. In this sense the usual spectral terms (like "essential spectrum" etc.) are well defined for problem (4.6). In particular, let $\inf \sigma_{\text{ess}} \in \mathbb{R} \cup \{+\infty\}$ denote the infimum of the essential spectrum of (4.6). The *Rayleigh extremal values* $\kappa_{R,j}$ of problem (4.6) are defined as

$$\kappa_{R,j} := \inf_{\substack{U \subset H \text{ subspace} \\ \dim U = j}} \max_{u \in U \setminus \{0\}} \frac{M(u,u)}{N(u,u)}. \tag{4.7}$$

(Here, the subscript R is chosen to distinguish Rayleigh extremal values from eigenvalues where necessary.) The sequence $(\kappa_{R,j})_{j \in \mathbb{N}}$ is monotonically nondecreasing, and the min-max principle (see [4]) states that *either* the whole sequence stays below $\inf \sigma_{\text{ess}}$ and it coincides with the sequence of all eigenvalues (counted by multiplicity) of (4.6) below $\inf \sigma_{\text{ess}}$, *or* there exists some $k \in \mathbb{N}$ such that $\kappa_{R,j} < \inf \sigma_{\text{ess}}$ for $j < k$, $\kappa_{R,j} = \inf \sigma_{\text{ess}}$ for $j \geq k$, and $\kappa_{R,1}, \ldots, \kappa_{R,k-1}$ coincide precisely with the eigenvalues of (4.6) below $\inf \sigma_{\text{ess}}$ (there is none if $k = 1$).

The min-max principle is the basis of obtaining *upper* eigenvalue bounds via the well-known

Rayleigh-Ritz method (*see, e.g.,* [26, Theorem 40.1 and Remarks 40.1, 40.2, 39.10]): *Let $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in H$ be linearly independent trial functions. Define the matrices*

$$A_0 := (M(v_i, v_j))_{i,j=1,\ldots,n}, \; A_1 := (N(v_i, v_j))_{i,j=1,\ldots,n} \tag{4.8}$$

and let $\widehat{\kappa}_1 \leq \widehat{\kappa}_2 \leq \cdots \leq \widehat{\kappa}_n$ denote the eigenvalues of

$$A_0 x = \widehat{\kappa} A_1 x. \tag{4.9}$$

Then, if $\widehat{\kappa}_n$ is below $\inf \sigma_{\text{ess}}$, there are at least n eigenvalues of (4.6) below $\inf \sigma_{\text{ess}}$, and the n smallest of these (counted by multiplicity), ordered by magnitude and denoted by $\kappa_1, \ldots, \kappa_n$, satisfy

$$\kappa_j \leq \widehat{\kappa}_j \; (j = 1, \ldots, n). \tag{4.10}$$

Since the matrix eigenvalues $\widehat{\kappa}_1, \ldots, \widehat{\kappa}_n$ can be enclosed by more direct methods combining numerical linear algebra ideas with interval analysis (see [2]), the Rayleigh-Ritz method provides a rather direct access to upper eigenvalue bounds.

For computing *lower* bounds we use the method given by the following theorem. In its original form, it is due to Lehmann [21], and later it has been considerably improved by Goerisch (see e.g. [3]) in its range of applicability. The following version (admitting essential spectrum) can be extracted from [32, Theorem 2.4].

Theorem 4.1 *Let $(X_G, b_G(\cdot,\cdot))$ denote a complex Hilbert space and $T : H \to X_G$ an isometric linear operator, i.e., $b_G(Tf, Tg) = M(f, g)$ for all $f, g \in H$. Let $v_1, \ldots, v_n \in H$ be linearly independent, as well as $w_1^0, \ldots, w_m^0 \in X_G$. Let $w_1^*, \ldots, w_n^* \in X_G$ and $w_1^0, \ldots, w_m^0$ satisfy*

$$b_G(Tf, w_j^*) = N(f, v_j),\ b_G(Tf, w_j^0) = 0 \text{ for all } f \in H. \tag{4.11}$$

In addition to A_0 and A_1 in (4.8), define the matrices

$$\begin{array}{rcl} A_2 & := & (b_G(w_j^*, w_k^*))_{j,k=1,\ldots,n},\ A_2^0 := (b_G(w_j^0, w_k^0))_{j,k=1,\ldots,m}, \\ F & := & (b_G(w_j^*, w_k^0))_{\substack{j=1,\ldots,n \\ k=1,\ldots,m}}, \end{array} \tag{4.12}$$

and let C be an approximation (the quality of which influences the quality of the eigenvalue bounds in (4.14)) to $-F(A_2^0)^{-1}$; note that A_2^0 is invertible since b_G is positive definite.

Suppose that the Rayleigh-Ritz method has been carried out (with trial functions $v_1, \ldots, v_n$) and that $\widehat{\kappa}_n < \inf \sigma_{\text{ess}}$, so that at least n eigenvalues $\kappa_1 \leq \cdots \leq \kappa_n$ (denoting w.l.o.g. the n smallest ones) below $\inf \sigma_{\text{ess}}$ exist. Suppose further that some $\rho > \widehat{\kappa}_n$ is known such that $(-\infty, \rho)$ does not contain any spectral point of problem (4.6) except $\kappa_1, \ldots, \kappa_n$.

Finally, let $\mu_1 \leq \cdots \leq \mu_n$ denote the eigenvalues of

$$(A_0 - \rho A_1)x = \mu(A_0 - 2\rho A_1 + \rho^2[A_2 + CF^* + FC^* + CA_2^0C^*])x \tag{4.13}$$

(here, the matrix on the left-hand side is negative definite, the one on the right-hand side is positive definite, whence $\mu_k < 0$ for all k). Then

$$\kappa_j \geq \rho - \frac{\rho}{1 - \mu_{n+1-j}} \text{ for } j = 1, \ldots, n. \tag{4.14}$$

When applying this theorem one has to solve the major general problem of finding a suitable parameter ρ with the properties required. If there is an $(n+1)$-st eigenvalue $\kappa_{n+1} < \inf \sigma_{\text{ess}}$ (such that $\kappa_1, \ldots, \kappa_{n+1}$ are the $n+1$ smallest ones), ρ obviously has to satisfy

$$\widehat{\kappa}_n < \rho \leq \kappa_{n+1}. \tag{4.15}$$

While the first of these two inequalities is easy to satisfy (since $\widehat{\kappa}_n$ is known), the second is problematic since it requires knowledge of a lower bound for κ_{n+1}, while our present considerations are just *aiming* at lower eigenvalue bounds. However, a rather *rough* lower bound for κ_{n+1} is sufficient in (4.15)

(to produce very *precise* bounds by Theorem 4.1). Such rough lower eigenvalue bounds can often be obtained via comparison problems or by a homotopy method using a chain of comparison problems, as explained in the next subsection.

For the application of the Rayleigh-Ritz- and the Lehmann-Goerisch method to our problem (4.4) we choose

$$\begin{aligned} &H := D(\mathcal{L}), \qquad N\left(\begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix}\right) := \langle \begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix} \rangle_{\widehat{X}}, \\ &M\left(\begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix}\right) := \langle \mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix}\right], \mathcal{L}\left[\begin{pmatrix} v \\ \tau \end{pmatrix}\right] \rangle_Y + c\, \langle \begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} v \\ \tau \end{pmatrix} \rangle_{\widehat{X}}, \end{aligned} \tag{4.16}$$

with $c > 0$ denoting some spectral shift to be chosen later, e.g. on an experimental basis,

$$\begin{aligned} &X_G := Y \times Y \times \mathbb{C}, \qquad T\begin{pmatrix} u \\ \sigma \end{pmatrix} := \left(\mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix}\right], Bu, \sigma\right), \\ &b_G((w_1, w_2, w_3), (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)) := \langle w_1, \widetilde{w}_1 \rangle_Y + c\langle w_2, \widetilde{w}_2 \rangle_Y + c\, w_3\, \overline{\widetilde{w}}_3, \end{aligned}$$

which clearly satisfy the general assumptions (recall that $\langle u, v\rangle_X = \langle Bu, Bv\rangle_Y$). Let $v_1 = (u_1, \sigma_1), \ldots, v_n = (u_n, \sigma_n) \in D(\mathcal{L})$ be chosen for forming A_0 and A_1 (see (4.8)), with trial functions $u_1, \ldots, u_n$ similar to those in Subsection 3.2, but with the additional property that $\varphi(u_j) = 0$ $(j = 1, \ldots, n)$, obtained e.g. in the form $u_j = \widetilde{u}_j - (\varphi(\widetilde{u}_j)/\varphi(\widetilde{u}_0))\widetilde{u}_0$, with linearly independent $\widetilde{u}_0, \ldots, \widetilde{u}_n, \varphi(\widetilde{u}_0) \neq 0$. Then, condition (4.11) is satisfied for $w_j^* = (w_{j,1}^*, w_{j,2}^*, w_{j,3}^*)$ $(j = 1, \ldots, n)$, as can be derived in a straightforward way, if $w_{j,1}^* \in D(\mathcal{L}^*), w_{j,2}^* \in H^2(0, \infty)$, and

$$\mathcal{L}^*[w_{j,1}^*] + c\begin{pmatrix} (-D^2 + a^2)w_{j,2}^* \\ w_{j,3}^* \end{pmatrix} = \begin{pmatrix} (-D^2 + a^2)^2 u_j \\ \sigma_j \end{pmatrix} + \tau\begin{pmatrix} \phi \\ 0 \end{pmatrix} \tag{4.17}$$

(with ϕ given by (4.1)), where $\tau \in \mathbb{C}$ is arbitrary and $\mathcal{L}^* : D(\mathcal{L}^*) \subset Y \to Y_0 \times \mathbb{C}$ is the adjoint of $\mathcal{L} : D(\mathcal{L}) \subset Y_0 \times \mathbb{C} \to Y$ (with $Y_0 := \{u \in Y : \varphi(u) = 0\}$). Direct calculations provide $D(\mathcal{L}^*) = D(A)$ and

$$\mathcal{L}^*[u] = \begin{pmatrix} (\widetilde{A}^* - \overline{\mu}B)u - \frac{1}{\langle \phi, \phi\rangle_Y}\, \langle (\widetilde{A}^* - \overline{\mu}B)u, \phi\rangle_Y\, \phi \\ -\overline{\gamma}\, \langle u, B\omega\rangle_Y \end{pmatrix},$$

with $\widetilde{A}^*$ denoting the adjoint of $\widetilde{A} : D(A) \subset Y \to Y$ (see also [10, Theorem IX.9.5]).

(4.17) shows that we can choose

$$w_j^* := \left(0, \frac{1}{c}(-D^2 + a^2)u_j, \frac{1}{c}\sigma_j\right) \ (j = 1, \ldots, n). \tag{4.18}$$

According to (4.11), also $w_j^0 = (w_{j,1}^0, w_{j,2}^0, w_{j,3}^0)$ $(j = 1, \ldots, m)$ have to satisfy (4.17), now with (u_j, σ_j) replaced by $(0, 0)$. Therefore, we choose

$$w_1^0 := (0, w_{1,2}^0, 0) \text{ with } (-D^2 + a^2)w_{1,2}^0 = \phi, \ w_2^0 := \left(0, e^{-ax}, 0\right), \tag{4.19}$$

and, with $w^0_{3,1}, \ldots, w^0_{m,1} \in D(A)$ chosen linearly independent,

$$w^0_j := (w^0_{j,1}, w^0_{j,2}, w^0_{j,3}) \text{ with } ((-D^2 + a^2)w^0_{j,2}, w^0_{j,3}) = -\frac{1}{c}\mathcal{L}^*[w^0_{j,1}], \quad \text{for } j = 3, \ldots, m. \tag{4.20}$$

Choosing $u_1, \ldots, u_n$ and $w^0_{3,1}, \ldots, w^0_{m,1}$ in the numerical trial function space described in Section 3, we obtain from (4.18), (4.19), (4.20) that the first two components of w^*_j and w^0_j are polynomial on $[0, x_0]$ and polynomial in e^{-ax} on $[x_0, \infty)$; observe that also the differential equation solving required in (4.19), (4.20) (with arbitrary fixing of the free parameters) can be carried out in closed form and does not lead out of this function class. Consequently, all matrices in (4.8) and (4.12) can be calculated in closed form by MATHEMATICA. The matrix eigenvalue problems (4.9) and (4.13) (with ρ for the moment assumed to be at hand) are then treated by the method described in [2], providing verified bounds for the eigenvalues, so that (4.10) and (4.14) yield enclosures for the eigenvalues $\kappa_1, \ldots, \kappa_n$ of (4.6), and thus, after subtracting the spectral shift c, for the first n eigenvalues of (4.4).

4.2 A homotopy method

Here, we are aiming at a method for obtaining a suitable parameter ρ satisfying the crucial condition (4.15) needed for Theorem 4.1. Besides problem (4.6), consider a *comparison problem* with H, M, N replaced by corresponding terms H_0, M_0, N_0 satisfying the same general assumptions, and moreover,

$$H_0 \supset H, \ \frac{M_0(u,u)}{N_0(u,u)} \leq \frac{M(u,u)}{N(u,u)} \text{ for all } u \in H. \tag{4.21}$$

Suppose in addition that, like problem (4.6), the comparison problem also has at least $n+1$ eigenvalues (the $n+1$ smallest of which we denote by $\kappa^0_1, \ldots, \kappa^0_{n+1}$, ordered by magnitude) below its essential spectrum. By (4.21), the min-max-principle immediately provides

$$\kappa^0_j \leq \kappa_j \text{ for } j = 1, \ldots, n+1, \tag{4.22}$$

so that for (4.15) we can choose $\rho := \kappa^0_{n+1}$, provided that κ^0_{n+1} (or at least some "close" lower bound for it) is *known* and moreover, $\kappa^0_{n+1} > \widehat{\kappa}_n$. While the latter condition (implying $\kappa^0_{n+1} > \kappa_n$) requires the comparison problem to be "not too far away" from problem (4.6), the former can often only be satisfied by choosing a comparison problem deviating substantially from (4.6).

To escape this dilemma, a *homotopy method* can be applied: Again one chooses a problem involving H_0, M_0, N_0 (now called *base problem*) with all the properties required before, except that the condition $\kappa^0_{n+1} > \widehat{\kappa}_n$ is dropped now, so that the base problem is allowed to be "far away" from problem (4.6). Besides the base problem, a family $(H_s, M_s, N_s)_{s \in [0,\widehat{s}]}$ of problems (all satisfying our general assumptions) has to be chosen which connects the

base problem monotonically to the given problem (4.6), i.e., $(H_{\widehat{s}}, M_{\widehat{s}}, N_{\widehat{s}}) = (H, M, N)$ and, for $0 \le s \le t \le \widehat{s}$,

$$H_s \supset H_t, \ \frac{M_s(u,u)}{N_s(u,u)} \le \frac{M_t(u,u)}{N_t(u,u)} \text{ for all } u \in H_t. \tag{4.23}$$

By the min-max principle, the eigenvalues κ_j^s are then monotonically nondecreasing in s, for each fixed $j \in \{1, \dots, n+1\}$.

The general homotopy idea is now to compute eigenvalue enclosures for several values $0 < s_1 < s_2 < \cdots < s_L = \widehat{s}$ of the homotopy parameter s. Here, for each $j = 1, \dots, L$, the lower bound ρ for $\kappa_{n_j+1}^{s_j}$, needed for computing lower bounds to $\kappa_1^{s_j}, \dots, \kappa_{n_j}^{s_j}$ by Theorem 4.1, can be obtained from the lower bound computed for $\kappa_{n_j+1}^{s_{j-1}}$ in the *previous* homotopy step, since $\kappa_{n_j+1}^{s_j} \ge \kappa_{n_j+1}^{s_{j-1}}$; this means that the problem at s_{j-1} serves as a comparison problem for the problem at s_j. In general, n_j decreases, in a well-controlled way, in the course of the homotopy, so that one has to start (at the base problem) with some more eigenvalues than finally needed. For more details on the homotopy method, see [25]. An additional remark to be made here is that, by (4.23), the bottom $\inf \sigma_{\text{ess}}^s$ of the essential spectrum is also monotonically nondecreasing in s, so that the condition $\widehat{\kappa}_n < \inf \sigma_{\text{ess}}$, required for the Rayleigh-Ritz method and for Theorem 4.1, is easy to control during the homotopy.

In the rest of this section, we will briefly describe the special homotopy chosen for our concrete problem (4.4) or for its shifted version introduced in (4.16), respectively. The homotopy consists of four parts, described here in reversed order, for simpler presentation: The given problem is connected to a simpler base problem, which however is still not solvable in closed form; this problem is then further connected to the next base problem, still not solvable in closed form, and so on. Finally, we arrive at the "real" base problem admitting eigenvalue enclosures by elementary interval analytical methods.

i) By (4.2), we obtain for M defined in (4.16), for any $\varepsilon \in (0,1)$,

$$\begin{aligned} M\left(\begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} u \\ \sigma \end{pmatrix}\right) &= \langle Lu - \gamma\sigma B\omega, Lu - \gamma\sigma B\omega\rangle_Y + c\langle \begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} u \\ \sigma \end{pmatrix}\rangle_{\widehat{X}} \\ &\ge (1-\varepsilon)\langle Lu, Lu\rangle_Y - \left(\frac{1}{\varepsilon} - 1\right)|\gamma|^2 \|B\omega\|_Y^2 \sigma\overline{\sigma} + c\langle \begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} u \\ \sigma \end{pmatrix}\rangle_{\widehat{X}} \\ &=: M_0\left(\begin{pmatrix} u \\ \sigma \end{pmatrix}, \begin{pmatrix} u \\ \sigma \end{pmatrix}\right), \end{aligned}$$

so that the base problem (with M_0 in place of M, and with H and N unchanged) has at least $n+1$ (smallest) eigenvalues

$$\kappa_1^0 = -\left(\frac{1}{\varepsilon} - 1\right)|\gamma|^2\|B\omega\|_Y^2 + c, \ \kappa_{j+1}^0 = (1-\varepsilon)\nu_j + c \ (j = 1, \dots, n)$$

(with eigenfunctions (u_j, σ_j) such that $u_1 \equiv 0$ and $\sigma_j = 0$ for $j \geq 2$) below its essential spectrum, provided that there are at least n (smallest) eigenvalues $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$ of the problem

$$u \in D(L),\ \langle Lu, Lv\rangle_Y = \nu\langle u, v\rangle_X (\equiv \nu\langle Bu, Bv\rangle_Y) \text{ for all } v \in D(L) \quad (4.24)$$

(receiving further treatment in ii)) below its essential spectrum.

Assuming that (lower) bounds for $\nu_1, \ldots, \nu_n$ are at hand, we therefore have the desired bounds for $\kappa_1^0, \ldots, \kappa_{n+1}^0$. Items ii) to iv) below, and the numerical computations finally yield $\widehat{\kappa}_n < \kappa_{n+1}^0 < \inf \sigma_{\text{ess}}$ for $\varepsilon := 10^{-4}$ and $n := 3$, so that the base problem serves directly as a comparison problem, without need for a homotopy.

ii) For proving the existence of the eigenvalues $\nu_1, \ldots, \nu_n$ below the essential spectrum of (4.24), and for computing lower bounds to them, we make use of the idea of *Neumann decoupling* proposed by Davies [6]. Clearly, for fixed $x_0 \in (0, \infty)$ (in (4.1)),

$$\begin{aligned} D(L) \ \subset\ & \{u \in L^2(0,\infty) : u|_{(0,x_0)} \in H^4(0, x_0),\ u|_{(x_0,\infty)} \in H^4(x_0, \infty), \\ & u(0) = u'(0) = 0,\ \varphi(u) = 0\} =: S_1 \oplus S_2, \end{aligned} \quad (4.25)$$

where

$$S_1 := \{u \in H^4(0, x_0) : u(0) = u'(0) = 0\},\ S_2 := \{u \in H^4(x_0, \infty) : \varphi(u) = 0\}$$

(observe that, due to (4.1), φ "acts" only on (x_0, ∞), which is essential for the above decoupling!). According to the min-max principle, (4.25) implies that lower bounds for the Rayleigh extremal values $\nu_{R,j}$ of (4.24) are given by the Rayleigh extremal values $\nu_{R,k}^{(1)}, \nu_{R,k}^{(2)}$ (to be ordered, in total, by magnitude) of the corresponding "decoupled" eigenvalue problems on $(0, x_0)$ and on (x_0, ∞), respectively, with "Neumann" boundary conditions at x_0, i.e., of the problems

$$u \in S_1, \qquad \langle L^{(1)}u, L^{(1)}v\rangle_{L^2(0,x_0)} = \nu^{(1)}\langle B^{(1)}u, B^{(1)}v\rangle_{L^2(0,x_0)} \quad \text{for all } v \in S_1, \quad (4.26)$$

$$u \in S_2, \qquad \langle L^{(2)}u, L^{(2)}v\rangle_{L^2(x_0,\infty)} = \nu^{(2)}\langle B^{(2)}u, B^{(2)}v\rangle_{L^2(x_0,\infty)} \quad \text{for all } v \in S_2, \quad (4.27)$$

with $L^{(i)}, B^{(i)}$ defined on S_i, representing the same differential expressions as L and B, respectively.

Problem (4.27) has constant coefficients since, due to (3.5), $\widetilde{V} = 1$ (and $\widetilde{V}'' = 0$) on $[x_0, \infty)$. Using [10, Theorem IX.9.6], it can be shown that problem (4.27) has essential spectrum $[\sigma^*, \infty)$, where $\sigma^* := (\max\{0, a^2 - \text{Re}\mu\})^2 + (aR - \text{Im}\mu)^2 = \text{dist}(\mu, \sigma_{\text{ess}}^{\text{o.s.}})^2$ (with $\sigma_{\text{ess}}^{\text{o.s.}}$ denoting the essential spectrum of the original Orr-Sommerfeld problem, given by (1.6)). Moreover, in the case $\sigma^* > 0$ (i.e., $\mu \notin \sigma_{\text{ess}}^{\text{o.s.}}$), problem (4.27) has a simple eigenvalue 0 (with eigenfunction $(a+p)e^{-p(x-x_0)} - 2ae^{-a(x-x_0)}$ where $p^2 = a^2 + iaR - \mu$, $\text{Re}\, p > 0$),

and direct (MATHEMATICA-supported) calculations show that there is no further eigenvalue below σ^*. Consequently, the Rayleigh extremal values satisfy $\nu^{(2)}_{R,1} = 0,\ \nu^{(2)}_{R,k} = \sigma^*$ for $k \geq 2$. Because the Rayleigh extremal values $\nu^{(2)}_{R,k}$ of (4.27) and $\nu^{(1)}_{R,k}$ of (4.26) (which are *eigenvalues* $\nu^{(1)}_k$ since (4.26) is posed on a compact interval), in union, are lower bounds for the Rayleigh extremal values $\nu_{R,j}$ of (4.24), we therefore obtain

$$\nu_{R,1} \geq 0,\ \nu_{R,j+1} \geq \nu^{(1)}_j \text{ for all } j \in \mathbb{N} \text{ such that } \nu^{(1)}_j \leq \sigma^*. \tag{4.28}$$

In our numerical application, a Rayleigh-Ritz computation for problem (4.26) shows that the side condition $\nu^{(1)}_j \leq \sigma^*$ in (4.28) is satisfied for $j \leq 7$.

Since the lower bounds in (4.28), the computation of which is considered in iii) below, are too coarse for using the decoupled problem directly as a comparison problem for problem (4.24), we need a homotopy here: For $0 \leq s \leq \infty$, we define $N_s(u,v) := \langle B^{(1)}u, B^{(1)}v\rangle_{L^2(0,x_0)} + \langle B^{(2)}u,\ B^{(2)}v\rangle_{L^2(x_0,\infty)}$ (independent of s), and

$$\begin{aligned} M_s(u,v) := {} & \langle L^{(1)}u, L^{(1)}v\rangle_{L^2(0,x_0)} + \langle L^{(2)}u, L^{(2)}v\rangle_{L^2(x_0,\infty)} + c_0 N_s(u,v) \\ & + s\sum_{j=0}^{3} \left(u^{(j)}(x_0-0) - u^{(j)}(x_0+0)\right)\ \overline{(v^{(j)}(x_0-0) - v^{(j)}(x_0+0))}, \end{aligned}$$

with $c_0 > 0$ denoting a spectral shift facilitating the application of Goerisch's method in the course of the homotopy. Moreover, let $H_s := S_1 \oplus S_2$ for $0 \leq s < \infty$, and $H_\infty := D(L)$. The homotopy defined in this way clearly satisfies (4.23), and it connects (up to the spectral shift) the decoupled problem ($s = 0$) to problem (4.24) ($s = \infty$).

Accompanying Rayleigh-Ritz computations provide, in each homotopy step, a sufficient number of eigenvalues below σ^*, and thus, due to the min-max principle, below the respective essential spectrum. In this way, we end up with bounds for the first three eigenvalues of problem (4.24) (below its essential spectrum).

iii) For computing lower bounds to the eigenvalues $\nu^{(1)}_j$ of problem (4.26) needed for (4.28), we use an additional homotopy. Direct estimates provide, for $u \in S_1$,

$$\begin{aligned} & \langle L^{(1)}u, L^{(1)}u\rangle_{L^2(0,x_0)} \\ & \quad = \ \|(-D^2+a^2)^2 u + (iaR\widetilde{V} - \mu)(-D^2+a^2)u + iaR\widetilde{V}''u\|^2_{L^2(0,x_0)} \\ & \quad \geq \ C_1\|(-D^2+a^2)^2 u\|^2_{L^2(0,x_0)} + C_2\|(-D^2+a^2)u\|_{L^2(0,x_0)}, \end{aligned} \tag{4.29}$$

with easily computable constants $C_1 > 0$, $C_2 \in \mathbb{R}$. Here, C_2 involves (a positive lower bound for) the smallest eigenvalue τ_1 of the problem

$$\begin{aligned} u \in S_1,\ \langle(-D^2+a^2)u, (-D^2+a^2)v\rangle_{L^2(0,x_0)} = \tau\langle u, v\rangle_{L^2(0,x_0)} \\ \text{for all } v \in S_1, \end{aligned} \tag{4.30}$$

which will be addressed in iv) below. Now we define, for $s \in [0,1]$, $H_s = S_1$, $N_s(u,v) = \langle B^{(1)}u, B^{(1)}v\rangle_{L^2(0,x_0)}$,

$$\begin{aligned} M_s(u,v) &:= s\langle L^{(1)}u, L^{(1)}v\rangle_{L^2(0,x_0)} \\ &\quad +(1-s)\left[C_1\langle(-D^2+a^2)^2u, (-D^2+a^2)^2v\rangle_{L^2(0,x_0)}\right. \\ &\qquad \left. +C_2\langle(-D^2+a^2)u, (-D^2+a^2)v\rangle_{L^2(0,x_0)}\right] \end{aligned}$$

for $u, v \in S_1$. Clearly, (4.29) shows that this homotopic family satisfies (4.23). Moreover, the homotopy ends (for $s = 1$) at problem (4.26), and it starts at a *constant coefficient* problem which is, up to normalization, equivalent to

$$\begin{aligned} u \in S_1, \quad &\langle(-D^2+a^2)^2u, (-D^2+a^2)^2v\rangle_{L^2(0,x_0)} \\ &= \zeta\langle(-D^2+a^2)u,\ (-D^2+a^2)v\rangle_{L^2(0,x_0)} \quad \text{for all } v \in S_1. \end{aligned} \tag{4.31}$$

iv) The remaining task is the computation of lower eigenvalue bounds for the constant coefficient problems (4.30) and (4.31). This is carried out in the following way by a MATHEMATICA notebook, supported by interval arithmetic (we give a description for problem (4.31) only): First the *strong* formulation of (4.31) is derived; in particular, this calculation provides six additional boundary conditions (besides $u(0) = u'(0) = 0$), two of which are ζ-dependent. Next, a (ζ-dependent) *fundamental system* $(\psi_1^{(\zeta)}, \ldots, \psi_8^{(\zeta)})$ of the strong eigenvalue equation is calculated; since this differential equation involves even order derivatives only, a closed form calculation is indeed possible. Then, with $R_1^{(\zeta)}, \ldots, R_8^{(\zeta)}$ denoting the boundary operators, the function $d(\zeta) := \det(R_j^{(\zeta)}[\psi_k^{(\zeta)}])$ is put up, the zeroes of which are the eigenvalues we are looking for. Finally, the (first n) zeroes of $d(\zeta)$ are enclosed by means of interval analysis; in particular, the interval Newton method and interval bisection are used.

5 Enclosure results

Here, we report on the concrete enclosures for a "critical" eigenpair of the Orr-Sommerfeld problem (2.1) (with A, B from (1.1)), which we obtained from Theorem 2.1 after performing the numerical approximation and enclosure procedures described in the previous sections.

The Blasius profile V (and V', V'') have been enclosed and approximated, as described in Subsection 3.1, using two different choices for the point x_0, and two different degrees N_V of the polynomials both in the monotone iteration procedure and for the approximation process. The results are contained in the following Table 1.

N_V	$\|V - \widetilde{V}\|_\infty \le$	$\|V' - \widetilde{V}'\|_\infty \le$	$\|V'' - \widetilde{V}''\|_\infty \le$	x_0
10	$2.04 \cdot 10^{-3}$	$6.69 \cdot 10^{-4}$	$1.59 \cdot 10^{-3}$	6.43
60	$8.57 \cdot 10^{-15}$	$3.64 \cdot 10^{-15}$	$7.99 \cdot 10^{-15}$	10.29

Table 1: Enclosures of the Blasius profile

For the wave number a and the Reynolds number R, we choose the following values which are standard in the engineering as well as in the more theoretical literature on stability (the factor $\sqrt{2}$ is due to our scaling of the Blasius equation (1.3)):

$$a = \sqrt{2} \cdot 0.179, \ R = \sqrt{2} \cdot 580. \tag{5.1}$$

Using a polynomial degree $N = 74$ for computing an approximate eigenpair (ω, μ) according to Subsection 3.2, and bounding its defect according to Subsection 3.3, we obtained the preliminary results (for $x_0 = 6.43$):

$$\mu = -1.652772126 + 75.60630560\, i, \ \delta \leq 0.021. \tag{5.2}$$

With this approximate eigenpair (ω, μ), a constant K satisfying (2.5) is computed, according to the methods described in Section 4, to be

$$K = 0.2299. \tag{5.3}$$

After some numerical testing, the scaling parameter $\gamma := R/20$ has been chosen. From (5.2), (5.3), Theorem 2.1 provides the error bound

$$\left\| \begin{pmatrix} \gamma(U - \omega) \\ \lambda - \mu \end{pmatrix} \right\|_{\widehat{X}} \leq 0.202 \tag{5.4}$$

for an eigenpair (U, λ), so that in particular,

$$|\lambda - \mu| \leq 0.202, \tag{5.5}$$

which proves, according to (5.2), that λ is in the left half-plane, and therefore, the *instability of the Orr-Sommerfeld problem* for the parameter constellation (5.1).

For improving the quality of the bounds (5.4), (5.5), we used the high-degree results for the Blasius profile in Table 1 ($N_V = 60$), and computed new approximate eigenpairs (ω, μ), this time with higher polynomial degrees N, and with $x_0 = 10.29$; see Table 2. For saving computing time, the "new" constants K are *not* computed using the "new" approximate eigenpairs in the method of Section 4 (which would imply that all eigenvalue calculations in the various homotopy steps would have to be performed again for each new (ω, μ), with increasing polynomial degrees), but with a perturbation argument implying, with K^{old} from (5.3),

$$K \leq \frac{K^{\text{old}}}{1 - \varepsilon K^{\text{old}}},$$

with ε involving differences between "old" and "new" Orr-Sommerfeld eigenpairs (ω, μ) and between "old" ($N_V = 10$; see Table 1) and "new" ($N_V = 60$) Blasius profiles.

In this way, we obtained the results contained in Table 2. The error bounds in the last column hold as well with $|\lambda - \mu|$ replaced by the full error $\left\| \begin{pmatrix} \gamma(U - \omega) \\ \lambda - \mu \end{pmatrix} \right\|_{\widehat{X}}$ (compare (5.4), (5.5)).

N	μ	$\delta \leq$	$\lvert\lambda - \mu\rvert \leq$
114	-1.652756336792 $+ 75.606472219516\, i$	$2.58 \cdot 10^{-3}$	$2.6 \cdot 10^{-2}$
134	-1.652756336777 $+ 75.606472219493\, i$	$5.62 \cdot 10^{-5}$	$5.6 \cdot 10^{-4}$
154	-1.652756336777 $+ 75.606472219493\, i$	$7.83 \cdot 10^{-7}$	$7.8 \cdot 10^{-6}$
174	-1.652756336777 $+ 75.606472219493\, i$	$7.65 \cdot 10^{-9}$	$7.7 \cdot 10^{-8}$
194	-1.652756336777 $+ 75.606472219493\, i$	$6.02 \cdot 10^{-11}$	$6.0 \cdot 10^{-10}$
214	-1.652756336777 $+ 75.606472219493\, i$	$4.24 \cdot 10^{-12}$	$4.3 \cdot 10^{-11}$

Table 2: Eigenvalue enclosures for a and R from (5.1)

The final Table 3 contains results for values of a and R different from (5.1). Here, the respective constants K (not listed in the table) have not been computed with final rigor: Instead of performing the homotopies described in Section 4, we used Theorem 4.1 with an *estimate* for the number ρ (obtained from the Rayleigh-Ritz computations), so that (4.15) has not been proved rigorously. Up to this slight lack of rigor, the first line of this table yields the upper bound 426.59 for the so-called critical Reynolds number R_c. In [7], the approximation $R_c \approx 426.585$ has been computed.

a	R	μ	$\delta \leq$	$\lvert\lambda - \mu\rvert \leq$
0.25	426.59	-0.000009327 $+ 42.3111953177\, i$	$3.83 \cdot 10^{-9}$	$3.1 \cdot 10^{-8}$
0.2	500	-0.0622471354 $+37.1835973223\, i$	$2.23 \cdot 10^{-9}$	$2.3 \cdot 10^{-8}$
0.2	1500	-5.1130791417 $+ 96.1250646277\, i$	$8.15 \cdot 10^{-6}$	$1.1 \cdot 10^{-4}$
0.2	3000	-7.2354492532 $+ 175.407270321\, i$	$8.31 \cdot 10^{-4}$	$1.8 \cdot 10^{-2}$
0.2	4000	-4.8863887472 $+ 225.813011574\, i$	$4.13 \cdot 10^{-4}$	$1.1 \cdot 10^{-2}$
0.1	5000	-10.202113420 $+ 114.984407775\, i$	$8.11 \cdot 10^{-6}$	$2.8 \cdot 10^{-4}$
0.1	10000	-23.666502784 $+ 207.439464400\, i$	$3.95 \cdot 10^{-4}$	$2.3 \cdot 10^{-2}$
0.1	15000	-34.202444724 $+ 293.760528163\, i$	$1.87 \cdot 10^{-4}$	$9.9 \cdot 10^{-3}$

Table 3: Eigenvalue enclosures for other values of a and R

6 Appendix: Proof of Theorem 2.1

For proving Theorem 2.1, we assume that (2.3) to (2.7) hold, and define

$$\widehat{X}_0 := \left\{ \begin{pmatrix} u \\ \sigma \end{pmatrix} \in \widehat{X} : \varphi(u) = 0 \right\}, \ D(\mathcal{L}) := \widehat{D(A)} \cap \widehat{X}_0, \tag{6.1}$$
$$\mathcal{L}\left[\begin{pmatrix} u \\ \sigma \end{pmatrix} \right] := Au - \mu Bu - \gamma\sigma B\omega.$$

(Note that $\mathcal{L}$ defined here involves, in contrast to (4.2), A in place of $\widetilde{A}$, which however will not cause confusion since $\widetilde{A}$ does not occur in this Appendix.) We start with two lemmata:

Lemma 6.1 *If $(A-\mu B)(D(A))$ is dense in Y, then $\mathcal{L}(D(\mathcal{L})) = Y$, and $\mathcal{L}^{-1} : Y \to \widehat{X}_0$ exists and is bounded (where $\widehat{X}_0$ is endowed with $\|\cdot\|_{\widehat{X}}$).*

Proof: Since A is closed, and B and φ are bounded, $\mathcal{L}$ is closed. Due to (2.5), $\mathcal{L}$ is moreover one-to-one, and $\mathcal{L}^{-1} : \mathcal{L}(D(\mathcal{L})) \subset Y \to \widehat{X}_0$ is bounded. The closedness of $\mathcal{L}$ implies closedness of $\mathcal{L}^{-1}$. Therefore, $\mathcal{L}(D(\mathcal{L}))$ is closed in Y. So we are left to show that $\mathcal{L}(D(\mathcal{L}))$ is dense in Y.

Thus, let $r \in Y$ be given. The denseness assumption for $(A-\mu B)(D(A))$ provides sequences (u_n) and (v_n) in $D(A)$ such that, as $n \to \infty$,

$$(A-\mu B)u_n \to r, \ (A-\mu B)v_n \to \gamma B\omega \text{ in } Y. \tag{6.2}$$

Here, $\varphi(v_n) \nrightarrow 0$, because otherwise $w_n := v_n - (\varphi(v_n)/\varphi(\omega))\omega$ would yield $(w_n, 1) \in D(\mathcal{L})$ and $\mathcal{L}\left[\binom{w_n}{1}\right] = (A-\mu B)v_n - \gamma B\omega - \frac{\varphi(v_n)}{\varphi(\omega)}(A-\mu B)\omega \to 0$, contradicting (2.5). Passing to a subsequence of (v_n) (but without changing (u_n)!) we can therefore achieve that $(\varphi(v_n))$ is bounded away from 0, and that $\|(A-\mu B)v_n - \gamma B\omega\|_Y \le [n(1+|\varphi(u_n)|)]^{-1}$ for all n (note (6.2)), so that $\varphi(u_n)[(A-\mu B)v_n - \gamma B\omega] \to 0$. Thus, for $\sigma_n := -\varphi(u_n)/\varphi(v_n)$ and $z_n := u_n + \sigma_n v_n$, we obtain $(z_n, \sigma_n) \in D(\mathcal{L})$ and $\mathcal{L}\left[\binom{z_n}{\sigma_n}\right] = (A-\mu B)u_n + \sigma_n[(A-\mu B)v_n - \gamma B\omega] \to r$ according to (6.2), so that $r \in \overline{\mathcal{L}(D(\mathcal{L}))}$. $\square$

Lemma 6.2 *Let $\varepsilon \ge 0$, $Kb\varepsilon < 1$, and let $(\omega_\varepsilon, \mu_\varepsilon) \in \widehat{D(A)}$ such that $\varphi(\omega_\varepsilon - \omega) = 0$ and $\left\| \binom{\gamma(\omega_\varepsilon - \omega)}{\mu_\varepsilon - \mu} \right\|_{\widehat{X}} \le \varepsilon$. Then,*

$$\left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} \le \frac{K}{1-Kb\varepsilon} \|Au - \mu_\varepsilon Bu - \gamma\sigma B\omega_\varepsilon\|_Y \text{ for all } \begin{pmatrix} u \\ \sigma \end{pmatrix} \in D(\mathcal{L}).$$

Proof: Using (2.5) and (2.6) we obtain, for all $\binom{u}{\sigma} \in D(\mathcal{L})$,

$$\begin{aligned} \left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} &\le K\|Au - \mu Bu - \gamma\sigma B\omega\|_Y \\ &\le K\|Au - \mu_\varepsilon Bu - \gamma\sigma B\omega_\varepsilon\|_Y \\ &\quad + Kb[|\mu_\varepsilon - \mu| \, \|u\|_X + |\gamma| \, |\sigma| \, \|\omega_\varepsilon - \omega\|_X] \\ &\le K\|Au - \mu_\varepsilon Bu - \gamma\sigma B\omega_\varepsilon\|_Y + Kb\varepsilon \left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}}, \end{aligned}$$

whence the assertion since $Kb\varepsilon < 1$. $\square$

Proof of Theorem 2.1: First we strengthen assumption (2.4) by requiring that $\mu \notin \sigma_{\rm res}$.

ad a) To prove the first part (existence and enclosure) of Theorem 2.1 we have to separate the (easy) case where μ is an exact eigenvalue of (2.1): In this case, let $\widehat{U} \in D(A)$ denote a corresponding eigenelement. Then, $\varphi(\widehat{U}) \neq 0$, since otherwise $(\widehat{U}, 0) \in D(\mathcal{L})$ and $\mathcal{L}\left[\binom{\widehat{U}}{0}\right] = 0$, contradicting (2.5). Thus, $U := (\varphi(\omega)/\varphi(\widehat{U}))\widehat{U}$ is an eigenelement (corresponding to μ) satisfying $(\gamma(U-\omega), 0) \in D(\mathcal{L})$, and (2.5), (2.3) yield $\left\|\binom{\gamma(U-\omega)}{0}\right\|_{\widehat{X}} \leq K|\gamma|\,\|A\omega - \mu B\omega\|_Y \leq K|\gamma|\delta \leq \alpha$, i.e., the assertion (2.8).

Now let μ not be an eigenvalue. Since $\mu \notin \sigma_{\rm res}$, $(A-\mu B)(D(A))$ is then dense in Y. According to Lemma 6.1, $\mathcal{L}^{-1} : Y \to \widehat{X}_0$ is well-defined and bounded, so that the (nonlinear!) operator

$$T : \widehat{X}_0 \to \widehat{X}_0,\ T\binom{u}{\sigma} := \mathcal{L}^{-1}[\sigma B u - \gamma(A\omega - \mu B\omega)] \tag{6.3}$$

is well defined. T maps the closed subset

$$\mathcal{D} := \left\{ \binom{u}{\sigma} \in \widehat{X}_0 : \left\|\binom{u}{\sigma}\right\|_{\widehat{X}} \leq \alpha \right\} \tag{6.4}$$

into itself, since (6.3), (2.5), (2.6), (2.3), (2.7) provide, for $(u,\sigma) \in \mathcal{D}$,

$$\begin{aligned} \left\|T\binom{u}{\sigma}\right\|_{\widehat{X}} &\leq K\|\sigma B u - \gamma(A\omega - \mu B\omega)\|_Y \\ &\leq K[b|\sigma|\,\|u\|_X + |\gamma|\delta] \leq K\left[\frac{1}{2}b\alpha^2 + |\gamma|\delta\right] = \alpha, \end{aligned} \tag{6.5}$$

where the last equality follows by elementary calculations. Moreover, T is contractive on $\mathcal{D}$, since (6.3), (2.5), (2.6), (6.4) imply, for $(u_1,\sigma_1), (u_2,\sigma_2) \in \mathcal{D}$,

$$\begin{aligned} \left\|T\binom{u_1}{\sigma_1} - T\binom{u_2}{\sigma_2}\right\|_{\widehat{X}} &\leq K\|\sigma_1 B u_1 - \sigma_2 B u_2\|_Y \\ &= \frac{1}{2}K\|(\sigma_1+\sigma_2)B(u_1-u_2) + (\sigma_1-\sigma_2)B(u_1+u_2)\|_Y \\ &\leq \frac{1}{2}bK\big[|\sigma_1+\sigma_2|\,\|u_1-u_2\|_X + |\sigma_1-\sigma_2|\,\|u_1+u_2\|_X\big] \\ &\leq \frac{1}{2}bK\left\|\binom{u_1}{\sigma_1} + \binom{u_2}{\sigma_2}\right\|_{\widehat{X}} \left\|\binom{u_1}{\sigma_1} - \binom{u_2}{\sigma_2}\right\|_{\widehat{X}} \\ &\leq bK\alpha\left\|\binom{u_1}{\sigma_1} - \binom{u_2}{\sigma_2}\right\|_{\widehat{X}}, \end{aligned}$$

and $bK\alpha = \beta/(1+\sqrt{1-\beta}) < 1$ due to (2.7).

Consequently, Banach's Fixed-Point-Theorem provides a fixed-point $(u,\sigma) \in \mathcal{D}$ of T which is unique in $\mathcal{D}$. The fixed-point equation (and (6.3)) imply all assertions of part (a) for $(U,\lambda) := (\omega + \gamma^{-1}u, \mu + \sigma)$; in particular, $U \neq 0$ since $\varphi(U) = \varphi(\omega) \neq 0$, and (2.8) holds since $(u,\sigma) \in \mathcal{D}$.

ad b) Let $(\widetilde{U},\widetilde{\lambda}) \in \widehat{D(A)}$ denote an eigenpair satisfying $\varphi(\widetilde{U}-\omega) = 0$ and (2.9). Let $(\widetilde{u},\widetilde{\sigma}) := (\gamma(\widetilde{U}-\omega), \widetilde{\lambda}-\mu)$. A straightforward calculation shows that $(\widetilde{u},\widetilde{\sigma})$ is a fixed-point of T. Similarly as in (6.5) we obtain

$$\left\| \begin{pmatrix} \widetilde{u} \\ \widetilde{\sigma} \end{pmatrix} \right\|_{\widehat{X}} = \left\| T \begin{pmatrix} \widetilde{u} \\ \widetilde{\sigma} \end{pmatrix} \right\|_{\widehat{X}} \leq K \left[\frac{1}{2} b \left\| \begin{pmatrix} \widetilde{u} \\ \widetilde{\sigma} \end{pmatrix} \right\|_{\widehat{X}}^2 + |\gamma|\delta \right],$$

so that $\|(\widetilde{u},\widetilde{\sigma})\|_{\widehat{X}} \notin (t_1,t_2)$, with $t_1 < t_2$ denoting the zeroes of the polynomial $\frac{1}{2}Kbt^2 - t + K|\gamma|\delta$, i.e., $t_1 = \alpha$ and $t_2 = \alpha + \frac{2}{Kb}\sqrt{1-\beta}$. Due to (2.9), we conclude that $\|(\widetilde{u},\widetilde{\sigma})\|_{\widehat{X}} \leq \alpha$, i.e., $(\widetilde{u},\widetilde{\sigma}) \in \mathcal{D}$. Since the fixed-point of T is unique within $\mathcal{D}$, we obtain $(\widetilde{u},\widetilde{\sigma}) = (u,\sigma)$, and thus, $(\widetilde{U},\widetilde{\lambda}) = (U,\lambda)$.

ad c) By (2.8), we can apply Lemma 6.2 with $\varepsilon := \alpha$ (note that $Kb\alpha = \beta/(1+\sqrt{1-\beta}) = 1-\sqrt{1-\beta} < 1$) and $(\omega_\varepsilon,\mu_\varepsilon) := (U,\lambda)$ to obtain

$$\left\| \begin{pmatrix} u \\ \sigma \end{pmatrix} \right\|_{\widehat{X}} \leq \frac{K}{\sqrt{1-\beta}} \|Au - \lambda Bu - \gamma\sigma BU\|_Y \text{ for all } \begin{pmatrix} u \\ \sigma \end{pmatrix} \in D(\mathcal{L}). \quad (6.6)$$

This implies $\varphi(\widehat{U}) \neq 0$ for every eigenelement $\widehat{U} \in D(A)$ corresponding to λ, since otherwise $(u,\sigma) := (\widehat{U},0) \in D(\mathcal{L})$ and the right-hand side of (6.6) vanishes for this (u,σ), contradicting (6.6). Therefore, no linearly independent eigenelements $U_1, U_2 \in D(A)$ corresponding to λ can exist, because then $\widehat{U} := \varphi(U_2)U_1 - \varphi(U_1)U_2$ would be an eigenelement satisfying $\varphi(\widehat{U}) = 0$. Thus, λ is geometrically simple.

Assuming for contradiction that $BU \in \overline{(A-\lambda B)(D(A))}$ we obtain a sequence (v_n) in $D(A)$ such that $(A-\lambda B)v_n \to BU$ in Y. Inserting $(u,\sigma) := (v_n - [\varphi(v_n)/\varphi(U)]U, \gamma^{-1}) \in D(\mathcal{L})$ into (6.6) we obtain that the right-hand side tends to zero (as $n \to \infty$) but the left-hand side does not, a contradiction.

The theorem is now proved under the additional assumption $\mu \notin \sigma_{\text{res}}$. Dropping this assumption and replacing it by (2.4), i.e. assuming merely that μ is not an interior point of σ_{res}, we obtain a sequence (μ_n) in $\mathbb{C}$ such that $\mu_n \notin \sigma_{\text{res}}$ and $\varepsilon_n := |\mu_n - \mu| \to 0$. We may assume that $Kb\varepsilon_n < 1$ for $n \in \mathbb{N}$, whence Lemma 6.2 provides, with $\varepsilon := \varepsilon_n$ and $(\omega_\varepsilon,\mu_\varepsilon) := (\omega,\mu_n)$, that (2.5) holds with μ_n and $K_n := K/(1-Kb\varepsilon_n)$ in place of μ and K, respectively. Moreover, (2.3) holds with μ_n and $\delta_n := \delta + \varepsilon_n\|B\omega\|_Y$ in place of μ and δ. Clearly, $K_n \to K$ and $\delta_n \to \delta$ as $n \to \infty$. Thus, re-starting the sequence at some sufficiently high index (and re-indexing) we can achieve, using (2.7), that $\beta_n := 2bK_n^2|\gamma|\delta_n$ (tending to β) satisfies $\beta_n < 1$, and that, for $\alpha_n := 2K_n|\gamma|\delta_n / \left(1+\sqrt{1-\beta_n}\right)$ (converging to α),

$$\alpha_1 - \alpha_n + |\mu_1 - \mu_n| < \frac{2}{K_n b}\sqrt{1-\beta_n} \quad \text{for } n \in \mathbb{N}. \quad (6.7)$$

According to the version of the theorem proved so far, we obtain a sequence $(U_n, \lambda_n) \in \widehat{D(A)}$ of eigenpairs of (2.1) such that the assertions a), b), c) hold, for each $n \in \mathbb{N}$, with the corresponding n-dependent terms. In particular, by (6.7)

$$\begin{aligned} \left\| \begin{pmatrix} \gamma(U_1 - \omega) \\ \lambda_1 - \mu_n \end{pmatrix} \right\|_{\widehat{X}} &\leq \left\| \begin{pmatrix} \gamma(U_1 - \omega) \\ \lambda_1 - \mu_1 \end{pmatrix} \right\|_{\widehat{X}} + |\mu_1 - \mu_n| \leq \alpha_1 + |\mu_1 - \mu_n| \\ &< \alpha_n + \frac{2}{K_n b}\sqrt{1 - \beta_n}, \end{aligned}$$

i.e., $(\widetilde{U}, \widetilde{\lambda}) := (U_1, \lambda_1)$ satisfies the n-dependent condition (2.9) (and $\varphi(\widetilde{U} - \omega) = 0$), whence $U_1 = U_n,\ \lambda_1 = \lambda_n$. The eigenpair sequence (U_n, λ_n) is therefore *constant*, equal to $(U_1, \lambda_1) =: (U, \lambda)$.

Part a) of the theorem now follows immediately by letting $n \to \infty$ in the corresponding n-dependent statement. To prove part b) let $(\widetilde{U}, \widetilde{\lambda}) \in \widehat{D(A)}$ satisfy $\varphi(\widetilde{U} - \omega) = 0$ and (2.9). Then, also the n-dependent statement (2.9) holds for n sufficiently large, implying $\widetilde{U} = U_n = U, \widetilde{\lambda} = \lambda_n = \lambda$. Finally, part c) follows immediately from the corresponding n-dependent statement and the constancy of the sequence (U_n, λ_n). □

Acknowledgement: The authors are grateful to three anonymous referees for very helpful remarks and suggestions.

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