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Watanabe, Yoshitaka
Research Institute for Information Technology, Kyushu University

Plum, Michael
Faculty of Mathematics, Karlsruhe University

Nakao, Mitsuhiro T.
Faculty of Mathematics, Kyushu University

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Y. Watanabe, M. Plum
M.T. Nakao

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

A computer-assisted instability proof for the Orr-Sommerfeld problem with Poiseuille flow

Yoshitaka Watanabe[†] Michael Plum[‡] Mitsuhiro T.Nakao[*]

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[†]Research Institute for Information Technology, Kyushu University
[‡]Faculty of Mathematics, Karlsruhe University
[*]Faculty of Mathematics, Kyushu University

Abstract

This paper presents a computer-assisted proof of solutions of the Orr-Sommerfeld equation describing hydrodynamic stability of Poiseuille flow. A numerical verification method for computing eigenpair enclosures for this non-selfadjoint eigenvalue problem is described. Some verification results confirm the effectiveness of the method.

1 The Orr-Sommerfeld model

Consider a two-dimensional flow of an incompressible viscous fluid between two infinite parallel plates at $y = y_1$ and $y = y_2$ (See Figure 1). The flow between the parallel plates

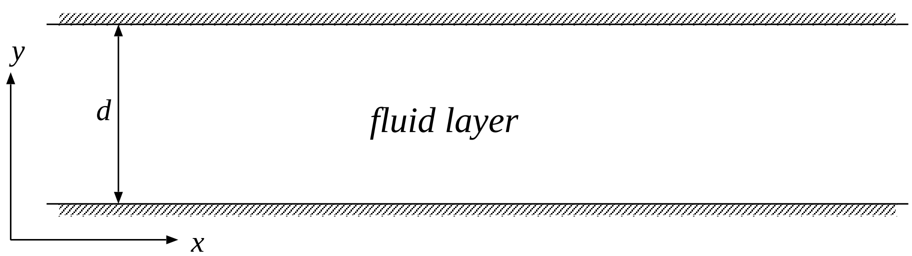

Figure 1: infinite parallel plates; $d := y_2 - y_1$

is described by the unsteady nonlinear incompressible non-dimensionalized Navier-Stokes equations:

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} & = & -\dfrac{\partial p}{\partial x} + \dfrac{1}{R}\Delta u, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} & = & -\dfrac{\partial p}{\partial y} + \dfrac{1}{R}\Delta v, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} & = & 0, \end{array} \right. \tag{1}$$

where, u, v and p are the velocity in the horizontal direction, the velocity in the vertical direction and the pressure field respectively, and R is the Reynolds number.

Let the basic (primary) flow be represented by

$$(u,v)=(U(y),0),\quad p=p_0+\frac{1}{R}\frac{d^2U(y)}{dy^2}x,\qquad 0\le x<\infty,\quad y_1\le y\le y_2. \tag{2}$$

Here, p_0 is a constant and U is a quadratic polynomial in y.

In order to study the linear stability of the system (1), we consider a small perturbation $(\hat{u},\hat{v},\hat{p})$ from the basic flow (2) such that

$$u=U+\hat{u},\quad v=\hat{v},\quad p=P+\hat{p},$$

where $U=U(y)$ and $P=p_0+\dfrac{1}{R}\dfrac{d^2U(y)}{dy^2}x$. Substituting for the equation (1) and disregarding the second-order terms involving products of the perturbations, the linearized equations:

$$\left\{\begin{array}{rcl}\dfrac{\partial\hat{u}}{\partial t}+U\dfrac{\partial\hat{u}}{\partial x}+\hat{v}\dfrac{dU}{dy} &=& -\dfrac{\partial\hat{p}}{\partial x}+\dfrac{1}{R}\Delta\hat{u},\\ \dfrac{\partial\hat{v}}{\partial t}+U\dfrac{\partial\hat{v}}{\partial x} &=& -\dfrac{\partial\hat{p}}{\partial y}+\dfrac{1}{R}\Delta\hat{v},\\ \dfrac{\partial\hat{u}}{\partial x}+\dfrac{\partial\hat{v}}{\partial y} &=& 0\end{array}\right. \tag{3}$$

are obtained.

Next, in order to satisfy the divergence free condition, the stream function $\psi(t,x,y)$ which satisfies

$$\hat{u}=\frac{\partial\psi}{\partial y},\qquad \hat{v}=-\frac{\partial\psi}{\partial x} \tag{4}$$

is introduced; note that the domain $(0,\infty)\times(y_1,y_2)$ is simply connected.

Cross-differentiating the equation (3) in order to eliminate the pressure term implies

$$\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial x^2}+\frac{\partial}{\partial t}\frac{\partial^2\psi}{\partial y^2}+U\frac{\partial^3\psi}{\partial x\partial y^2}+U\frac{\partial^3\psi}{\partial x^3}=\frac{d^2U}{dy^2}\frac{\partial\psi}{\partial x}+\frac{1}{R}\Delta^2\psi. \tag{5}$$

Here, when we impose a no-slip boundary condition at $y=y_1$ and $y=y_2$, the stream function ψ satisfies

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial y}=0,\qquad y=y_1,y_2. \tag{6}$$

In view of the independence of the basic flow $(U(y),0)$ on x, t, it makes sense to look for the following travelling wave form of the disturbance stream function $\psi(t,x,y)$:

$$\psi=\psi(t,x,y)=\phi(y)e^{ia(x-ct)}. \tag{7}$$

Here, $\phi(y)$ and $a>0$ mean the amplitude and wavenumber, respectively, and $c=c_r+ic_i$ is the complex wave speed; c_r represents the speed at which a wave propagates downstream, and ac_i characterizes the rate at which the disturbance grows or decays in time. If $c_i<0$, then ψ decays (i.e. the flow is stable), and if $c_i>0$, then ψ grows (the flow is unstable).

Substituting eq.(7) into eq.(5), and using $D := d/dy$, the equation

$$\frac{1}{R}(D^2 - a^2)^2\phi(y) = ia\left[(U - c)(D^2 - a^2)\phi(y) - \frac{d^2U}{dy^2}\phi(y)\right] \tag{8}$$

is obtained. From the no-slip boundary condition (6), $\phi(y)$ must satisfy

$$\phi = D\phi = 0, \qquad y = y_1, y_2.$$

This equation (8) is the well-known Orr-Sommerfeld problem derived by Orr [6] and Sommerfeld [9] for the disturbance eigenfunction $\phi(y)$, which in turn depends on the prescribed values of the wave number a and of the Reynolds number R.

Rewriting $y \to x$ and $u := \phi$, $\lambda := iaRc$ we have

$$\begin{cases} (-D^2 + a^2)^2 u + iaR[U(-D^2 + a^2) + U'']u = \lambda(-D^2 + a^2)u \quad \text{on} \quad \Omega = [x_1, x_2] \\ u(x_1) = u(x_2) = u'(x_1) = u'(x_2) = 0. \end{cases} \tag{9}$$

In this paper, we focus on the case of plane Poiseuille flow [4]

$$U = V := 1 - x^2, \quad x_1 = -1, \quad x_2 = 1. \tag{10}$$

The Orr-Sommerfeld equation (9) is a non-selfadjoint eigenvalue problem for the eigenpair (λ, u), and within the frame of linearized stability theory, the flow is stable if the spectrum is located in the right complex half-plane, otherwise unstable.

There are many numerical results for the Orr-Sommerfeld equation with Poiseuille flow. For example, Orszag [7] solved it numerically using expansions in Chebyshev polynomials and the QR matrix eigenvalue algorithm. He computed that the smallest value of R for which an unstable eigenmode exists (critical Reynolds number), according to "numerical evidence", is 5772.22 with $a \in [1.0255, 1.0257]$. Klein [1] proposed a method for eigenvalue inclusion using a generalization of Gerschgorin's theorem, however, he imposed some additional assumptions, and numerical results did not take into account effects of rounding error in floating point computation. Lahmann and Plum [2] gave a computer-assisted method for computing rigorous eigenvalue enclosures and applied it to the Orr-Sommerfeld problem with Blasius profile. However, concerning plane Poiseuille flow, a rigorous instability proof has never been given from the mathematical point of view.

In this paper, we propose a numerical verification procedure which encloses an eigenpair of the Orr-Sommerfeld equation with plane Poiseuille flow. The method uses numerical means, but all numerical errors are take into account, and hence the method implies a rigorous proof of all statements made. The method is based on a fixed-point theorem with some Newton-like operator. Especially, we are interested in whether the real part of the enclosed eigenvalue λ is negative or not, from the point of view in linearized stability theory.

The paper is organized as follows. In Section 2 we formulate a fixed-point equation in an infinite dimensional function space. Section 3 contains a study of a finite dimensional subspace and some constructive a priori error estimates for a projection onto it. Section 4 is concerned with a practical verification algorithm. In Section 5 we report on some verification results which prove the existence of eigenpairs in the computed regions, and in particular give rigorous instability proofs.

2 Fixed-point formulation

Setting

$$\tilde{\Delta} := -D^2 + a^2$$

and using real valued functions v, w and real values σ, μ such that

$$\begin{cases} u &= v + iw, \\ \lambda &= \sigma + i\mu, \end{cases} \tag{11}$$

equation (9) becomes

$$\begin{cases} \tilde{\Delta}^2 v - aR(V\tilde{\Delta} + V'')w = \sigma\tilde{\Delta}v - \mu\tilde{\Delta}w \quad \text{on } \Omega, \\ \tilde{\Delta}^2 w + aR(V\tilde{\Delta} + V'')v = \sigma\tilde{\Delta}w + \mu\tilde{\Delta}v \quad \text{on } \Omega, \\ v(-1) = v(1) = v'(-1) = v'(1) = 0, \\ w(-1) = w(1) = w'(-1) = w'(1) = 0. \end{cases} \tag{12}$$

Let $L^2(\Omega)$ be the real L^2 space on $\Omega = (-1,1)$ with the inner product $(\cdot,\cdot)_{L^2}$ and the norm $\|v\| := \sqrt{(v,v)_{L^2}}$, $\|v\|_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|$ the L^∞-norm on Ω, and for integers k, let $H^k(\Omega)$ denote the L^2-Sobolev space of order k on Ω with the norm $\|v\|_{H^k} := \sqrt{\sum_{j=0}^{k} \|d^j v/dx^j\|^2}$. Denoting

$$H_0^2(\Omega) := \left\{ v \in H^2(\Omega) \mid v(-1) = v'(-1) = v(1) = v'(1) = 0 \right\},$$

$\|v\|_{\tilde{\Delta}} := \|\tilde{\Delta}v\|$ is an equivalent norm for $\|v\|_{H^2}$ and $(\tilde{\Delta}v, \tilde{\Delta}w)_{L^2}$ can be chosen as the inner-product of $H_0^2(\Omega)$. We define a Banach space $X := H_0^2(\Omega) \times H_0^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ with the norm

$$\|[v,w,\sigma,\mu]^T\|_X := \sqrt{\|v\|_{\tilde{\Delta}}^2 + \|w\|_{\tilde{\Delta}}^2 + \sigma^2 + \mu^2}.$$

Since $\tilde{\Delta}$ has the properties

$$(\tilde{\Delta}v, w)_{L^2} = (v, \tilde{\Delta}w)_{L^2}, \qquad \forall v \in H_0^2(\Omega), \quad \forall w \in H^2(\Omega),$$

$$(\tilde{\Delta}v, \tilde{\Delta}w)_{L^2} = (\tilde{\Delta}^2 v, w)_{L^2}, \quad \forall v \in C_0^\infty(\Omega), \quad \forall w \in H_0^2(\Omega),$$

we can look for solutions for eq.(12), submitted to additional normalizing conditions for the eigenfunction, in the following weak formulation for $[v,w,\sigma,\mu]^T \in X$:

$$\begin{cases} (\tilde{\Delta}v, \tilde{\Delta}\xi)_{L^2} &= (aR(V\tilde{\Delta} + V'')w + \sigma\tilde{\Delta}v - \mu\tilde{\Delta}w, \xi)_{L^2}, \quad \forall \xi \in H_0^2(\Omega), \\ (\tilde{\Delta}w, \tilde{\Delta}\eta)_{L^2} &= (-aR(V\tilde{\Delta} + V'')v + \sigma\tilde{\Delta}w + \mu\tilde{\Delta}v, \eta)_{L^2}, \quad \forall \eta \in H_0^2(\Omega), \\ (v, v_0)_{L^2} &= \xi_R, \\ (w, w_0)_{L^2} &= \xi_I, \end{cases} \tag{13}$$

where $a, R, \xi_R, \xi_I \in \mathbb{R}$ and $v_0, w_0 \in H_0^2(\Omega)$ are given.

Let bounded continuous maps f_1, f_2 from X to $L^2(\Omega)$ be denoted by

$$f_1[v, w, \sigma, \mu]^T := aR(V\tilde{\Delta} + V'')w + \sigma\tilde{\Delta}v - \mu\tilde{\Delta}w, \tag{14}$$

$$f_2[v, w, \sigma, \mu]^T := -aR(V\tilde{\Delta} + V'')v + \sigma\tilde{\Delta}w + \mu\tilde{\Delta}v. \tag{15}$$

Also by the Lax & Milgram Lemma, for any $g \in L^2(\Omega)$ there exists a unique solution $\omega \in H^4(\Omega) \cap H_0^2(\Omega)$ satisfying

$$\tilde{\Delta}^2 \omega = g. \tag{16}$$

For $g \in L^2(\Omega)$ let $(\tilde{\Delta}^2)^{-1}g$ be the solution of eq.(16), then the operator $(\tilde{\Delta}^2)^{-1} : L^2(\Omega) \longrightarrow H_0^2(\Omega)$ is compact due to the compactness of the imbedding $H^4(\Omega) \hookrightarrow H_0^2(\Omega)$.

Using f_1, f_2 and $(\tilde{\Delta}^2)^{-1}$, the operator $F : X \longrightarrow X$ defined by

$$F[v, w, \sigma, \mu]^T := \begin{bmatrix} (\tilde{\Delta}^2)^{-1} f_1[v, w, \sigma, \mu]^T \\ (\tilde{\Delta}^2)^{-1} f_2[v, w, \sigma, \mu]^T \\ \sigma - (\, v, v_0 \,)_{L^2} + \xi_R \\ \mu - (\, w, w_0 \,)_{L^2} + \xi_I \end{bmatrix}. \tag{17}$$

is also compact, and the weak problem (13) can be rewritten equivalently in the fixed-point form

$$F[v, w, \sigma, \mu]^T = [v, w, \sigma, \mu]^T.$$

In the following, for a general map A and a general set U, AU means

$$AU := \{Au \mid u \in U\}.$$

Then Schauder's fixed-point theorem asserts that if a nonempty, bounded, convex and closed set $U \subset X$ satisfies

$$FU \subset U$$

then there exists a fixed-point of F in U.

3 Finite dimensional subspace and projection error

In this section, we introduce a finite dimensional approximation subspace $S_h \subset H_0^2(\Omega)$, using basis functions constructed from piecewise cubic Hermite interpolating polynomials, and show a priori error estimates for a projection from $H_0^2(\Omega)$ onto S_h.

The interval Ω is divided into K equal parts:

$$-1 = x_0 < x_1 < \cdots < x_{K-1} < x_K = 1$$

with nodes $x_n = -1 + hn$ $(n = 0, \ldots, K)$, where $h := 2/K$. From standard functions $\Phi(x)$ and $\Psi(x)$ defined by

$$\Phi(x) = \begin{cases} (x+1)^2(1-2x) & -1 \le x \le 0 \\ (x-1)^2(1+2x) & 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \Psi(x) = \begin{cases} x(x+1)^2 & -1 \le x \le 0 \\ x(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

take

$$\phi_n(x) := \Phi\left(h^{-1}(x+1)-n\right), \quad \psi_n(x) := h\Psi\left(h^{-1}(x+1)-n\right) \quad n=1,\ldots,K-1.$$

Then these functions satisfy

$$\phi_n(x_m)=\delta_{nm},\ \ \phi_n'(x_m)=0,\ \ \psi_n(x_m)=0,\ \ \psi_n'(x_m)=\delta_{nm},\ \ 1\le n\le K-1,\ \ 0\le m\le K.$$

We define an approximation subspace $S_h \subset H_0^2(\Omega)$ as

$$S_h := \mathrm{span}\left\{\phi_n, \psi_n \ \mid\ n=1,\ldots,K-1\right\}.$$

By the well-definedness of the piecewise cubic Hermite interpolation, an interpolation operator

$$\mathcal{I}_H : H_0^2(\Omega) \longrightarrow S_h$$

can be defined by

$$\mathcal{I}_H f(x_j) = f(x_j), \qquad (\mathcal{I}_H f)'(x_j) = f'(x_j), \qquad 1\le j\le K-1,$$

and the following error estimates of interpolation:

$$\begin{aligned} \|(f-\mathcal{I}_H f)''\| &\le \pi^{-2}h^2\|f^{(iv)}\|, && (18)\\ \|(f-\mathcal{I}_H f)'\| &\le \pi^{-3}h^3\|f^{(iv)}\|, && (19)\\ \|f-\mathcal{I}_H f\| &\le \pi^{-4}h^4\|f^{(iv)}\| && (20)\end{aligned}$$

hold for all $f \in H^4(\Omega)\cap H_0^2(\Omega)$ [8].

Next, let $P_h : H_0^2(\Omega) \longrightarrow S_h$ be the orthogonal projection defined by

$$(\,\tilde{\Delta}(v-P_h v), \tilde{\Delta} v_h\,)_{L^2} = 0, \qquad \forall v_h \in S_h, \tag{21}$$

then P_h has the following property.

Lemma 1 For all $g \in L^2(\Omega)$, the difference between the solution ω of eq.(16) and its projection $P_h\omega$ satisfies constructive a priori estimates

$$\|\omega - P_h\omega\|_{\tilde{\Delta}} \le C\|g\|, \tag{22}$$

$$\|\omega - P_h\omega\| \le C^2\|g\|, \tag{23}$$

where

$$C := \frac{\sqrt{3}}{\pi^2}h^2\left(1+\frac{a^2}{\pi^2}h^2\right). \tag{24}$$

Proof. From estimates (18)–(20), we have

$$\begin{aligned} \|\omega - P_h\omega\|_{\tilde{\Delta}} &\le \|\omega - \mathcal{I}_H\omega\|_{\tilde{\Delta}}\\ &= (\|\omega''-\mathcal{I}_H\omega''\|^2 + 2a^2\|\omega'-\mathcal{I}_H\omega'\|^2 + a^4\|\omega-\mathcal{I}_H\omega\|^2)^{\frac{1}{2}}\\ &\le h^2\pi^{-2}(1+a^2h^2\pi^{-2})\|\omega^{(iv)}\|,\end{aligned}$$

then (22) follows from $\|\omega^{(iv)}\| \leq \sqrt{3}\,\|\tilde{\Delta}^2\omega\|$. This inequality can be obtained by partial integration for $f \in H^4(\Omega) \cap H_0^2(\Omega)$,

$$
\begin{array}{rcl}
\|\tilde{\Delta}^2 f\|^2 & = & (\, f^{(iv)}, f^{(iv)}\,)_{L^2} - 4a^2(\, f^{(iv)}, f''\,)_{L^2} + 2a^4(\, f^{(iv)}, f\,)_{L^2} + 4a^4(\, f'', f''\,)_{L^2} \\
 & & -4a^6(\, f'', f\,)_{L^2} + a^8(\, f, f\,)_{L^2} \\
 & = & (\, f^{(iv)}, f^{(iv)}\,)_{L^2} - 4a^2(\, f^{(iv)}, f''\,)_{L^2} + 6a^4(\, f'', f''\,)_{L^2} + 4a^6(\, f', f'\,)_{L^2} \\
 & & +a^8(\, f, f\,)_{L^2} \\
 & \geq & (\, f^{(iv)}, f^{(iv)}\,)_{L^2} - 4a^2(\, f^{(iv)}, f''\,)_{L^2} + 6a^4(\, f'', f''\,)_{L^2},
\end{array}
$$

and the inequality

$$
\begin{aligned}
(\, f^{(iv)}, f''\,)_{L^2} &\leq \|f^{(iv)}\|\;\|f''\| \\
&\leq \frac{1}{2}\left(\frac{1}{3a^2}\|f^{(iv)}\|^2 + 3a^2\|f''\|^2\right).
\end{aligned}
$$

The L^2-estimate (23) is derived by the usual Aubin-Nitsche technique. □

4 Verification condition

4.1 Computable algorithm

In this section, we propose a computable algorithm constructing a *candidate set* which is expected to satisfy a sufficient condition for Schauder's fixed-point theorem. Basically, this verification method is an extension of the one for solutions of second-order elliptic boundary value problems introduced by a part of the authors [5].

From now on, the identity maps on X, S_h and $H_0^2(\Omega)$ are denoted by the same symbol I. Define the finite dimensional subspace X_h of X by

$$X_h = S_h \times S_h \times \mathbb{R} \times \mathbb{R},$$

and the projection $\hat{P}_h$ from X to X_h by

$$\hat{P}_h[v, w, \sigma, \mu]^T = [P_h v, P_h w, \sigma, \mu]^T$$

using P_h from (21). Then any element $u = [v, w, \mu, \sigma]^T \in X$ can be uniquely decomposed into

$$[v, w, \mu, \sigma]^T = [\hat{v}, \hat{w}, \mu, \sigma]^T + [v_*, w_*, 0, 0]^T, \qquad [\hat{v}, \hat{w}, \mu, \sigma]^T \in X_h,\ [v_*, w_*, 0, 0]^T \in X_*,$$

where

$$X_* := \{[v_*, w_*, 0, 0] \in X \mid v_* = (I - P_h)v,\ w_* = (I - P_h)w,\ v \in H_0^2(\Omega),\ w \in H_0^2(\Omega)\} \subset X.$$

Therefore, the fixed-point equation $u = Fu$ on X is equivalently rewritten as

$$
\left\{
\begin{array}{rcl}
\hat{P}_h u & = & \hat{P}_h F u, \\
(I - \hat{P}_h) u & = & (I - \hat{P}_h) F u.
\end{array}
\right.
\tag{25}
$$

Now, we take an approximate solution $u_h = [v_h, w_h, \sigma_h, \mu_h]^T \in X_h$ obtained by some appropriate numerical method and, in order to accelerate contraction, apply a Newton-like method to the finite dimensional part in eq.(25). Let us define the Newton-like operator $\mathcal{N}_h : X \longrightarrow X_h$ by

$$\mathcal{N}_h u := \hat{P}_h u - [I - \hat{P}_h F'(u_h)]_h^{-1} \hat{P}_h (I - F) u.$$

Here $[I - \hat{P}_h F'(u_h)]_h^{-1} : X_h \longrightarrow X_h$ means the inverse of the restriction of the operator $\hat{P}_h(I - F'(u_h)) : X \longrightarrow X_h$ to X_h, where F' denotes the Fréchet derivative of F. Note that the existence of $[I - \hat{P}_h F'(u_h)]_h^{-1}$ is equivalent to the invertibility of a matrix, which is numerically checked in the actual verified computations. Since $\hat{P}_h u = \hat{P}_h \mathcal{N}_h u \Leftrightarrow \hat{P}_h u = \hat{P}_h F u$, using a compact map T on X defined by

$$Tu = \mathcal{N}_h u + (I - \hat{P}_h) F u,$$

we find that the two fixed-point problems: $u = Fu$ and $u = Tu$ are equivalent.

Next, for positive constants $\gamma, \delta, c_1, c_2, \alpha$ and β, set

$$U_h := \left\{ [\hat{v}_h, \hat{w}_h, \hat{\sigma}, \hat{\mu}]^T \in X_h \mid \|\hat{v}_h\|_{\tilde{\Delta}} \le \gamma, \ \|\hat{w}_h\|_{\tilde{\Delta}} \le \delta, \ |\hat{\sigma}| \le c_1, \ |\hat{\mu}| \le c_2 \right\} \subset X_h,$$

$$U_* := \left\{ [v_*, w_*, 0, 0]^T \in X_* \mid \|v_*\|_{\tilde{\Delta}} \le \alpha, \ \|w_*\|_{\tilde{\Delta}} \le \beta, \ \right\} \subset X_*,$$

and define a candidate set $U \subset X$ by

$$U := u_h + U_h + U_*.$$

Then a sufficient condition for the fixed-point theorem is as follows.

Theorem 1 When the two inclusions:

$$\begin{cases} \mathcal{N}_h U - u_h & \subset & U_h \\ (I - \hat{P}_h) F U & \subset & U_* \end{cases} \tag{26}$$

hold, there exists a fixed-point of T in U.

Proof. By definition, U is a non-empty, closed, convex and bounded set in X. For any $u \in U$, $\mathcal{N}_h u \in X_h$, $(I - \hat{P}_h) F u \in X_*$, and the decomposition $Tu = \mathcal{N}_h u + (I - \hat{P}_h) F u$ is unique. Hence by (26), we get $\mathcal{N}_h U + (I - \hat{P}_h) F U \subset u_h + U_h + U_*$ in X, namely, $TU \subset U$. Therefore, by the compactness of the operator T and Schauder's fixed-point theorem, the desired result is obtained. □

We now desribe a procedure to construct the candidate set U of X which is expected to satisfy the inclusion (26). Setting

$$\mathcal{N}_h U - u_h =: [V_h, W_h, \Sigma, M]^T \subset X_h,$$

the finite dimensional part of the inclusion, $\mathcal{N}_h U - u_h \subset U_h$, can be written as

$$\sup_{\bar{v}_h \in V_h} \|\bar{v}_h\|_{\tilde{\Delta}} \le \gamma, \quad \sup_{\bar{w}_h \in W_h} \|\bar{w}_h\|_{\tilde{\Delta}} \le \delta, \quad \sup_{\bar{\sigma} \in \Sigma} |\bar{\sigma}| \le c_1, \quad \sup_{\bar{\mu} \in M} |\bar{\mu}| \le c_2.$$

Details of the underlying computations will be explained in Subsection 4.1.

On the other hand, the infinite dimensional part of the inclusion, $(I-\hat{P}_h)FU \subset U_*$, means

$$\begin{bmatrix} (I-P_h)(\tilde{\Delta}^2)^{-1}f_1[v,w,\sigma,\mu]^T \\ (I-P_h)(\tilde{\Delta}^2)^{-1}f_2[v,w,\sigma,\mu]^T \\ 0 \\ 0 \end{bmatrix} \subset U_*$$

for any $u \in U$ such that $u = [v,w,\sigma,\mu]^T$. Setting

$$\hat{v}_* := (I-P_h)(\tilde{\Delta}^2)^{-1}f_1[v,w,\sigma,\mu]^T, \qquad \hat{w}_* := (I-P_h)(\tilde{\Delta}^2)^{-1}f_2[v,w,\sigma,\mu]^T,$$

Lemma 1 assures

$$\|\hat{v}_*\|_{\tilde{\Delta}} \le C\|f_1(u)\|, \quad \|\hat{w}_*\|_{\tilde{\Delta}} \le C\|f_2(u)\|, \quad \|\hat{v}_*\| \le C^2\|f_1(u)\|, \quad \|\hat{w}_*\| \le C^2\|f_2(u)\|.$$

Therefore, in order to satisfy $(I-\hat{P}_h)FU \subset U_*$, the conditions

$$C\sup_{\bar{u}\in U}\|f_1(\bar{u})\| \le \alpha, \qquad C\sup_{\bar{u}\in U}\|f_2(\bar{u})\| \le \beta$$

are sufficient. Note that C defined in (24) is small when h is chosen small.

From this we can derive the following theorem.

Theorem 2 With the notations defined before, if one can check the conditions:

$$\begin{aligned} \sup_{\bar{v}_h\in V_h}\|\bar{v}_h\|_{\tilde{\Delta}} &\le \gamma, \\ \sup_{\bar{w}_h\in W_h}\|\bar{w}_h\|_{\tilde{\Delta}} &\le \delta, \\ \sup_{\bar{\sigma}\in\Sigma}|\bar{\sigma}| &\le c_1, \\ \sup_{\bar{\mu}\in M}|\bar{\mu}| &\le c_2, \\ C\sup_{\bar{u}\in U}\|f_1(\bar{u})\| &\le \alpha, \\ C\sup_{\bar{u}\in U}\|f_2(\bar{u})\| &\le \beta, \end{aligned}$$

then there exists fixed-point of T in U.

Based on Theorem 2, we propose a verification algorithm in Figure 2.

The extension procedure involving ε occurring in this algorithm is called "ε-inflation" which is a kind of acceleration technique. The concrete value of $\varepsilon > 0$ should be adapted to the actual problem. Experimentally, the initial values of $\gamma^{(0)}, \delta^{(0)}, c_1^{(0)}, c_2^{(0)}, \alpha^{(0)}$ and $\beta^{(0)}$ are taken as machine epsilon.

Verification algorithm

- $k = 0$
  Set initial values $\gamma^{(0)}, \delta^{(0)}, c_1^{(0)}, c_2^{(0)}, \alpha^{(0)}, \beta^{(0)} > 0$.
- $k \geq 1$
  1. For a fixed small constant $\varepsilon > 0$ set
  $$\begin{aligned} \hat{\gamma}^{(k)} &:= (1+\varepsilon)\gamma^{(k-1)}, & \hat{\delta}^{(k)} &:= (1+\varepsilon)\delta^{(k-1)}, & \hat{c_1}^{(k)} &:= (1+\varepsilon)c_1^{(k-1)}, \\ \hat{c_2}^{(k)} &:= (1+\varepsilon)c_2^{(k-1)}, & \hat{\alpha}^{(k)} &:= (1+\varepsilon)\alpha^{(k-1)}, & \hat{\beta}^{(k)} &:= (1+\varepsilon)\beta^{(k-1)}. \end{aligned}$$
  2. The k-th candidate set $U^{(k)}$ is defined by
  $$\begin{aligned} U_h^{(k)} &:= \{[\hat{v}_h, \hat{w}_h, \hat{\sigma}, \hat{\mu}]^T \in X_h \mid \|\hat{v}_h\|_{\tilde{\Delta}} \leq \hat{\gamma}^{(k)},\ \|\hat{w}_h\|_{\tilde{\Delta}} \leq \hat{\delta}^{(k)},\ |\hat{\sigma}| \leq \hat{c_1}^{(k)},\ |\hat{\mu}| \leq \hat{c_2}^{(k)}\}, \\ U_*^{(k)} &:= \{[v_*, w_*, 0, 0]^T \in X_* \mid \|v_*\|_{\tilde{\Delta}} \leq \hat{\alpha}^{(k)},\ \|w_*\|_{\tilde{\Delta}} \leq \hat{\beta}^{(k)},\ \}, \\ U^{(k)} &:= u_h + U_h^{(k)} + U_*^{(k)}. \end{aligned}$$
  3. Evaluate $N_h U^{(k)} - u_h \subset X_h$ as
  $$[V_h^{(k)}, W_h^{(k)}, \Sigma^{(k)}, M^{(k)}]^T := \mathcal{N}_h U^{(k)} - u_h.$$
  4. Compute values of the k-th iteration by
  $$\begin{aligned} \gamma^{(k)} &:= \sup_{\bar{v}_h \in V_h^{(k)}} \|\bar{v}_h\|_{\tilde{\Delta}}, \\ \delta^{(k)} &:= \sup_{\bar{w}_h \in W_h^{(k)}} \|\bar{w}_h\|_{\tilde{\Delta}}, \\ c_1^{(k)} &:= \sup_{\bar{\sigma} \in \Sigma^{(k)}} |\bar{\sigma}|, \\ c_2^{(k)} &:= \sup_{\bar{\mu} \in M^{(k)}} |\bar{\mu}|, \\ \alpha^{(k)} &:= C \sup_{\bar{u} \in U^{(k)}} \|f_1(\bar{u})\|, \\ \beta^{(k)} &:= C \sup_{\bar{u} \in U^{(k)}} \|f_2(\bar{u})\|. \end{aligned}$$
  5. If $\gamma^{(k)} \leq \hat{\gamma}^{(k)}$, $\delta^{(k)} \leq \hat{\delta}^{(k)}$, $c_1^{(k)} \leq \hat{c_1}^{(k)}$, $c_2^{(k)} \leq \hat{c_2}^{(k)}$, $\alpha^{(k)} \leq \hat{\alpha}^{(k)}$, $\beta^{(k)} \leq \hat{\beta}^{(k)}$ hold then stop, and there exists a desired solution in $U^{(k)} \subset X$.
  6. Set $k := k+1$ and return to the step 1. If k reaches a maximum iteration number or some values exceed a criterion then stop, and the verification fails.

Figure 2:

4.2 Detailed computation

Omitting iteration numbers' notation, in the verification step, given 6 parameters $\alpha, \beta, \gamma, \delta, c_1$ and $c_2 > 0$, we have to compute

$$\begin{aligned}
\hat{\gamma} &= \sup_{\bar{v}_h \in V_h} \|\bar{v}_h\|_{\tilde{\Delta}}, & \hat{\delta} &= \sup_{\bar{w}_h \in W_h} \|\bar{w}_h\|_{\tilde{\Delta}}, \\
\hat{c_1} &= \sup_{\bar{\sigma} \in \Sigma} |\bar{\sigma}|, & \hat{c_2} &= \sup_{\bar{\mu} \in M} |\bar{\mu}|, \\
\hat{\alpha} &= C \sup_{\bar{u} \in U} \|f_1(\bar{u})\|, & \hat{\beta} &= C \sup_{\bar{u} \in U} \|f_2(\bar{u})\|,
\end{aligned}$$

and confirm

$$\hat{\alpha} \le \alpha, \quad \hat{\beta} \le \beta, \quad \hat{\gamma} \le \gamma, \quad \hat{\delta} \le \delta, \quad \hat{c_1} \le c_1, \quad \hat{c_2} \le c_2.$$

In the actual computation, the candidate set U contains the infinite dimensional term U_*. Moreover, it is impossible to avoid the effect of rounding error of floating point arithmetic. However, by norm estimates, and interval arithmetic software taking into account effects of rounding error, we can obtain mathematically rigorous upper bounds for $\hat{\gamma}, \hat{\delta}, \hat{c_1}, \hat{c_2}, \hat{\alpha}$ and $\hat{\beta}$ with possible over-estimates. Let us describe these computations in more detail.

For any $u \in U$ such that

$$\begin{aligned}
u &= u_h + \hat{u}_h + u_*, \qquad \hat{u}_h \in U_h, \quad u_* \in U_* \\
&= \begin{bmatrix} v_h \\ w_h \\ \sigma_h \\ \mu_h \end{bmatrix} + \begin{bmatrix} \hat{v}_h \\ \hat{w}_h \\ \hat{\sigma} \\ \hat{\mu} \end{bmatrix} + \begin{bmatrix} v_* \\ w_* \\ 0 \\ 0 \end{bmatrix},
\end{aligned}$$

after some calculations we obtain

$$\begin{aligned}
\mathcal{N}_h u - u_h = \; & [I - \hat{P}_h F'(u_h)]_h^{-1} \begin{bmatrix} -v_h + P_h(\tilde{\Delta}^2)^{-1} f_1(u_h) \\ -w_h + P_h(\tilde{\Delta}^2)^{-1} f_2(u_h) \\ \xi_R - (\, v_h, v_0 \,)_{L^2} \\ \xi_I - (\, w_h, w_0 \,)_{L^2} \end{bmatrix} \\
+ \; & [I - \hat{P}_h F'(u_h)]_h^{-1} \begin{bmatrix} P_h(\tilde{\Delta}^2)^{-1}\{\hat{\sigma}\tilde{\Delta}\hat{v}_h - \hat{\mu}\tilde{\Delta}\hat{w}_h + f_1[v_*, w_*, \sigma_h + \hat{\sigma}, \mu_h + \hat{\mu}]^T\} \\ P_h(\tilde{\Delta}^2)^{-1}\{\hat{\sigma}\tilde{\Delta}\hat{w}_h + \hat{\mu}\tilde{\Delta}\hat{v}_h + f_2[v_*, w_*, \sigma_h + \hat{\sigma}, \mu_h + \hat{\mu}]^T\} \\ -(\, v_*, v_0 \,)_{L^2} \\ -(\, w_*, w_0 \,)_{L^2} \end{bmatrix}.
\end{aligned} \tag{27}$$

The first term

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} := [I - \hat{P}_h F'(u_h)]_h^{-1} \begin{bmatrix} -v_h + P_h(\tilde{\Delta}^2)^{-1} f_1(u_h) \\ -w_h + P_h(\tilde{\Delta}^2)^{-1} f_2(u_h) \\ \xi_R - (\, v_h, v_0 \,)_{L^2} \\ \xi_I - (\, w_h, w_0 \,)_{L^2} \end{bmatrix} \in X_h$$

on the right-hand side of eq.(27) is only constructed by the approximate solution and known functions; note that $P_h(\tilde{\Delta}^2)^{-1}f_i(u_h)$ is the solution of a finite dimensional linear problem. Therefore, each norm

$$\|r_1\|_{\tilde{\Delta}}, \quad \|r_2\|_{\tilde{\Delta}}, \quad |r_3|, \quad \text{and} \quad |r_4|$$

can be bounded by solving some linear algebraic systems with interval arithmetic.

On the other hand, when we set

$$[t_1, t_2, t_3, t_4]^T = [I - \hat{P}_h F'(u_h)]_h^{-1}[s_1, s_2, s_3, s_4]^T \in X_h,$$

for $[s_1, s_2, s_3, s_4]^T \in X_h$, it can be shown that

$$\begin{cases} \|t_1\|_{\tilde{\Delta}} & \le \rho_1\|s_1\|_{\tilde{\Delta}} + \rho_2\|s_2\|_{\tilde{\Delta}} + \|L^T(G_{13}^{-1}s_3 + G_{14}^{-1}s_4)\|_E, \\ \|t_2\|_{\tilde{\Delta}} & \le \rho_3\|s_1\|_{\tilde{\Delta}} + \rho_4\|s_2\|_{\tilde{\Delta}} + \|L^T(G_{23}^{-1}s_3 + G_{24}^{-1}s_4)\|_E, \\ |t_3| & \le \rho_5\|s_1\|_{\tilde{\Delta}} + \rho_6\|s_2\|_{\tilde{\Delta}} + |G_{33}^{-1}s_3 + G_{34}^{-1}s_4|, \\ |t_4| & \le \rho_7\|s_1\|_{\tilde{\Delta}} + \rho_8\|s_2\|_{\tilde{\Delta}} + |G_{43}^{-1}s_3 + G_{44}^{-1}s_4|, \end{cases}$$

where

$$G := \begin{bmatrix} A_1 - \sigma_h A_3 & -A_2 + \mu_h A_3 & -A_3\boldsymbol{v}_h & A_3\boldsymbol{w}_h \\ A_2 - \mu_h A_3 & A_1 - \sigma_h A_3 & -A_3\boldsymbol{w}_h & -A_3\boldsymbol{v}_h \\ \boldsymbol{v}_0^T A_4 & 0 & 0 & 0 \\ 0 & \boldsymbol{w}_0^T A_4 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2K\times 2K},$$

$$\begin{aligned} [A_1]_{nm} :=& (\,\tilde{\Delta}\hat{\phi}_m, \tilde{\Delta}\hat{\phi}_n\,)_{L^2}, \\ [A_2]_{nm} :=& aR(\,(V\tilde{\Delta} + V'')\hat{\phi}_m, \hat{\phi}_n\,)_{L^2}, \\ [A_3]_{nm} :=& (\,\tilde{\Delta}\hat{\phi}_m, \hat{\phi}_n\,)_{L^2} \\ [A_4]_{nm} :=& (\,\hat{\phi}_m, \hat{\phi}_n\,)_{L^2}, \end{aligned}$$

L is the Cholesky factor of A_1: $A_1 = LL^T$,

$$C_1 := \sqrt{\frac{\lambda_{\max}(A_4)}{\lambda_{\min}(A_1)}}, \quad G^{-1} =: \begin{bmatrix} G_{11}^{-1} & G_{12}^{-1} & G_{13}^{-1} & G_{14}^{-1} \\ G_{21}^{-1} & G_{22}^{-1} & G_{23}^{-1} & G_{24}^{-1} \\ G_{31}^{-1} & G_{32}^{-1} & G_{33}^{-1} & G_{34}^{-1} \\ G_{41}^{-1} & G_{42}^{-1} & G_{43}^{-1} & G_{44}^{-1} \end{bmatrix},$$

$$\rho_1 := \|L^T G_{11}^{-1} L\|_M, \quad \rho_2 := \|L^T G_{12}^{-1} L\|_M, \quad \rho_3 := \|L^T G_{21}^{-1} L\|_M, \quad \rho_4 := \|L^T G_{22}^{-1} L\|_M,$$
$$\rho_5 := \|(G_{31}^{-1} L)^T\|_E, \quad \rho_6 := \|(G_{32}^{-1} L)^T\|_E, \quad \rho_7 := \|(G_{41}^{-1} L)^T\|_E, \quad \rho_8 := \|(G_{42}^{-1} L)^T\|_E,$$

and $\|\cdot\|_M$ and $\|\cdot\|_E$ mean the usual matrix and vector 2-norms. Evaluations of ρ_1, ρ_2, ρ_3 and ρ_4 can be reduced to the computation of the maximum singular value of a matrix.

Therefore, norm bounds for the second term on the right-hand side of eq.(27) are obtained from norm bounds for

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} := \begin{bmatrix} P_h(\tilde{\Delta}^2)^{-1}\{\sigma\tilde{\Delta}\hat{v}_h - \mu\tilde{\Delta}\hat{w}_h + f_1[v_*, w_*, \sigma_h + \hat{\sigma}, \mu_h + \hat{\mu}]^T\} \\ P_h(\tilde{\Delta}^2)^{-1}\{\sigma\tilde{\Delta}\hat{w}_h + \mu\tilde{\Delta}\hat{v}_h + f_2[v_*, w_*, \sigma_h + \hat{\sigma}, \mu_h + \hat{\mu}]^T\} \\ -(\, v_*, v_0 \,)_{L^2} \\ -(\, w_*, w_0 \,)_{L^2} \end{bmatrix},$$

which in turn can be computed as

$$\begin{aligned} \|s_1\|_{\tilde{\Delta}} &\leq C_1(\hat{c_1}\ \hat{\gamma} + \hat{c_2}\ \hat{\delta}) + C((\rho_9 + \tau_2)\hat{\beta} + \tau_1\ \hat{\alpha}), \\ \|s_2\|_{\tilde{\Delta}} &\leq C_1(\hat{c_1}\ \hat{\delta} + \hat{c_2}\ \hat{\gamma}) + C((\rho_9 + \tau_2)\hat{\alpha} + \tau_1\ \hat{\beta}), \\ |s_3| &\leq C\hat{\alpha}\rho_{10}, \\ |s_4| &\leq C\hat{\beta}\rho_{11}, \end{aligned}$$

where

$$\tau_1 := \sup_{|\hat{\sigma}| \leq \hat{c_1}} |\sigma_h + \hat{\sigma}|, \qquad \tau_2 := \sup_{|\hat{\mu}| \leq \hat{c_2}} |\mu_h + \hat{\mu}|,$$

$$\rho_9 := aR\|V\|_\infty + \sqrt{2}R\|V'\|_\infty + \frac{2R}{a}\|V''\|_\infty, \qquad \rho_{10} := \|v_0\|, \qquad \rho_{11} := \|w_0\|.$$

Moreover, estimates for $\|f_1(u)\|$ and $\|f_2(u)\|$ are obtained by

$$\begin{aligned} \|f_1(u)\| &\leq \rho_{12} + \tau_3\hat{\delta} + \rho_{13}C_1\hat{\delta} + \tau_1\hat{\gamma} + \tau_4 + \tau_3\hat{\beta} + \rho_{13}C\hat{\beta} + \tau_1\hat{\alpha}, \\ \|f_2(u)\| &\leq \rho_{14} + \tau_3\hat{\gamma} + \rho_{13}C_1\hat{\gamma} + \tau_1\hat{\delta} + \tau_5 + \tau_3\hat{\alpha} + \rho_{13}C\hat{\alpha} + \tau_1\hat{\beta}, \end{aligned}$$

where

$$\begin{aligned} \tau_3 &:= \sup_{|\hat{\mu}| \leq \hat{c_2}} \|aRV - \hat{\mu}_h - \hat{\mu}\|_\infty \\ \tau_4 &:= \sup_{|\hat{\sigma}| \leq \hat{c_1}, |\hat{\mu}| \leq \hat{c_2}} \|\hat{\sigma}v_h - \hat{\mu}w_h\|_{\tilde{\Delta}}, \\ \tau_5 &:= \sup_{|\hat{\sigma}| \leq \hat{c_1}, |\hat{\mu}| \leq \hat{c_2}} \|\hat{\sigma}w_h + \hat{\mu}v_h\|_{\tilde{\Delta}} \end{aligned}$$

$$\rho_{12} := \|f_1(u_h)\|, \quad \rho_{13} := \|aRV''\|_\infty, \quad \rho_{14} := \|f_2(u_h)\|.$$

5 Verification results

We now show some verification results. It is well known that the discretization of the Orr-Sommerfeld equation yields a stiff system. The quadruple precision interval arithmetic in each verification step was implemented using Sun ONE Studio 7, Compiler Collection Fortran 95 on FUJITSU PRIMEPOWER850 (CPU: SPARC64-GP 1.3GHz, OS: Solaris8). The approximate solutions were obtained by a Newton-Raphson method using usual floating point arithmetic with quadruple precision.

5.1 Result 1

For $R = 5774$ and $a = 1.02$, the verification algorithm executed successfully with $K = 1000$ in the following candidate set:

$$U = u_h + U_h + U_*, \quad U_h = [V_h, W_h, \Sigma, M]^T, \quad U_* = [V_*, W_*, 0, 0]^T,$$

where

$$\|V_h\|_{\tilde{\Delta}} \le 5.518 \times 10^{-4}, \quad \|W_h\|_{\tilde{\Delta}} \le 5.383 \times 10^{-4},$$
$$\|V_*\|_{\tilde{\Delta}} \le 3.868 \times 10^{-3}, \quad \|W_*\|_{\tilde{\Delta}} \le 6.578 \times 10^{-3}.$$

Especially, an eigenvalue can be enclosed within the complex interval

$$\lambda \in [-0.03745, 0.00347] + i[1554.34370, 1554.38555].$$

5.2 Result 2

For $R = 5775$ and $a = 1.02$, the verification algorithm also executed successfully with $K = 1000$ in the following candidate set:

$$U = u_h + U_h + U_*, \quad U_h = [V_h, W_h, \Sigma, M]^T, \quad U_* = [V_*, W_*, 0, 0]^T,$$

where

$$\|V_h\|_{\tilde{\Delta}} \le 5.388 \times 10^{-4}, \quad \|W_h\|_{\tilde{\Delta}} \le 5.523 \times 10^{-4},$$
$$\|V_*\|_{\tilde{\Delta}} \le 6.587 \times 10^{-3}, \quad \|W_*\|_{\tilde{\Delta}} \le 3.867 \times 10^{-3}.$$

Especially, an eigenvalue can be enclosed within the complex interval

$$\lambda \in [-0.04719, -0.00625] + i[1554.56608, 1554.60797].$$

As mentioned in Section 1, within the frame of linearized stability theory, we can therefore conclude that the flow is unstable because at least one spectral point is located in the left complex half-plane.

Figure 3 shows the minimum Reynolds number R for which the verification algorithm assures that the real part of an eigenvalue is strictly negative for the corresponding wave number a. Therefore, it is expected that the critical curve $Re(\lambda) = 0$ should be located below these dots.

6 Conclusion

For some fixed Reynolds number and wave number $[a, R]$ we can enclose an eigenpair for the Orr-Sommerfeld equation with Poiseuille flow from hydrodynamic stability. We cannot say for certain whether the enclosed eigenvalue has the smallest real part or not, and we also cannot enclose the critical curve. These questions must be solved in our future work. We wish to remark that in principle, a computer-assisted *stability* proof could be given with the aid of [3], where a box has been computed which contains all eigenvalues of the Orr-Sommerfeld problem, and which has a compact intersection with the left complex half-plane.

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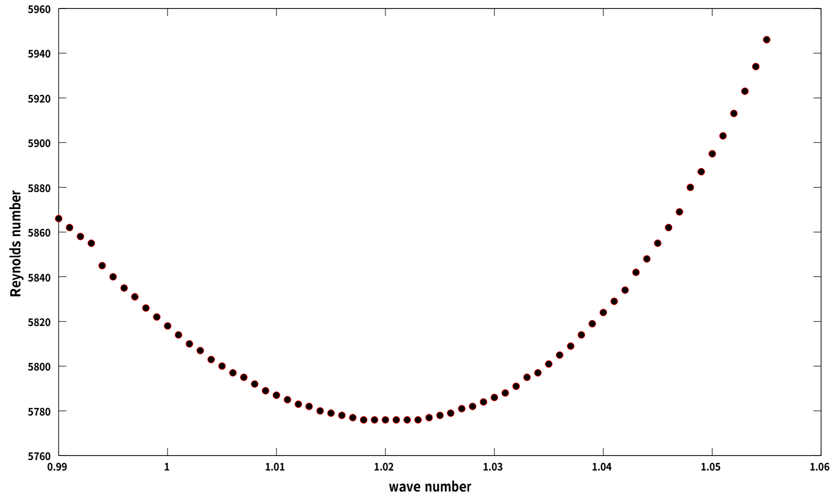

Figure 3: $[a, R]$ with $Re(\lambda) < 0$

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