A Concise Proof of the Littlewood-Richardson Rule

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Abstract

We give a short proof of the Littlewood-Richardson rule using a sign-reversing involution.

Introduction.

The Littlewood-Richardson rule is one of the most important results in the theory of symmetric functions. It provides an explicit combinatorial rule for expressing either a skew Schur function, or a product of two Schur functions, as a linear combination of (non skew) Schur functions. Since Schur functions in n variables are the irreducible polynomial characters of $GL_n(\mathbf{C})$, the Littlewood-Richardson rule amounts to a tensor product rule for $GL_n(\mathbf{C})$.

The rule was first formulated in a 1934 paper by Littlewood and Richardson [**LR**], but the first complete proofs were not published until the 1970's. (For a historical account of the evolution of the rule and its proofs, see the recent survey paper of van Leeuwen [**vL**].) There are now many proofs available, such as those based on the Robinson-Schensted-Knuth correspondence, *jeu de taquin*, or the plactic monoid. In this note, we present a very simple, self-contained proof of the rule; the argument also proves at the same time the "bi-alternant" formula for Schur functions—the formula originally used by Cauchy to define Schur functions.

We obtained this proof by specializing a crystal graph argument that works in much greater generality (see Theorem 2.4 of [**S**]). The fact that crystal graphs (or the closely related Path Model of Littelmann) may be used to prove the Littlewood-Richardson rule, as well as tensor product rules for other semisimple Lie groups, is well-known (see [**KN**] or [**L**]), but we believe that it is not widely understood that there exist versions of these proofs that are self-contained, with no need to appeal to a general theory.

The proof we present here is not the first short proof. Alternatives include proofs by Berenstein and Zelevinsky [**BZ**], Remmel and Shimozono [**RS**], and Gasharov [**G**]. Furthermore, aside from the differences in language between semistandard tableaux and Gelfand patterns, the sign-reversing involution we use here is essentially a translation of the one used by Berenstein and Zelevinsky.

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The Details.

Let $\mathcal{P}$ denote the set of nonnegative integer sequences of the form $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ with finitely many nonzero terms; i.e., the set of partitions. We let $\mathcal{P}_n$ denote the set of partitions with at most n nonzero terms, viewed (by truncation) as a subset of $\mathbf{Z}^n$.

Now regard n as fixed, and set $\rho = (n-1, \ldots, 1, 0)$ and $\varnothing = (0, \ldots, 0) \in \mathcal{P}_n$.

For each $\lambda \in \mathbf{Z}^n$, define $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and $a_\lambda = \det[x_i^{\lambda_j}] = \sum_{w \in S_n} \operatorname{sgn}(w) x^{w\lambda}$.

Given $\mu, \nu \in \mathcal{P}$, let $D(\mu, \nu) = \{(i,j) \in \mathbf{Z}^2 : 1 \leq i \leq n,\ \nu_i < j \leq \mu_i\}$. Assuming $\nu \leq \mu$ (meaning $\nu_i \leq \mu_i$ for all i), define $\mathcal{S}(\mu/\nu)$ to be the set of semistandard tableaux of shape μ/ν; i.e., the set of mappings $T : D(\mu,\nu) \to [n]$ with increasing columns ($T(i,j) < T(i+1,j)$) and weakly increasing rows ($T(i,j) \leq T(i,j+1)$). The weight of T is $\omega(T) = (\omega_1(T), \ldots, \omega_n(T)) \in \mathbf{Z}^n$, where $\omega_k(T) = |T^{-1}(k)|$ denotes the number of k's in T. The generating series $s_{\mu/\nu} = \sum_{T \in \mathcal{S}(\mu/\nu)} x^{\omega(T)}$ is a skew Schur function.

There is a well-known set of involutions $\sigma_1, \ldots, \sigma_{n-1}$ on $\mathcal{S}(\mu/\nu)$, due to Bender and Knuth [**BK**], with the property that σ_k acts by changing certain entries of $T \in \mathcal{S}(\mu/\nu)$ from k to $k+1$ and vice-versa in such a way that $\omega(\sigma_k(T)) = s_k \omega(T)$, where s_k denotes the transposition $(k, k+1) \in S_n$. The existence of these involutions proves that $s_{\mu/\nu}$ is a symmetric function of $x_1, \ldots, x_n$.

To explicitly describe the action of σ_k on $T \in \mathcal{S}(\mu/\nu)$, declare an entry k or $k+1$ to be *free* in T if there is no corresponding $k+1$ or k (respectively) in the same column. It is easy to check that the free entries in a given row must occur in consecutive columns; moreover, the entries in the free positions may be arbitrarily changed from k to $k+1$ and vice-versa without violating semistandardness as long as the free positions remain weakly increasing by row. The tableau $\sigma_k(T)$ is obtained by reversing the numbers of free k's and $k+1$'s within each row; i.e., if there are a_i free k's and b_i free $k+1$'s in row i of T, then there should be b_i free k's and a_i free $k+1$'s in row i of $\sigma_k(T)$.

In the following, $T_{\geq j}$ denotes the subtableau of T formed by the entries in columns $j, j+1, \ldots$, and we use similar notations such as $T_{<j}$ and $T_{>j}$ in the obvious way.

Theorem. *For all $\lambda \in \mathcal{P}_n$ and all $\mu, \nu \in \mathcal{P}$ such that $\nu \leq \mu$, we have*

$$a_{\lambda+\rho} s_{\mu/\nu} = \sum a_{\lambda+\omega(T)+\rho},$$

where the sum ranges over all $T \in \mathcal{S}(\mu/\nu)$ such that $\lambda + \omega(T_{\geq j}) \in \mathcal{P}_n$ for all $j \geq 1$.

Proof. As noted above, we know that $s_{\mu/\nu}$ is symmetric, so for each $w \in S_n$, the quantities $w(\lambda+\rho) + \omega(T)$ and $w(\lambda+\rho+\omega(T))$ are identically distributed as T varies over $\mathcal{S}(\mu/\nu)$. Hence,

$$a_{\lambda+\rho} s_{\mu/\nu} = \sum_{w \in S_n} \sum_{T \in \mathcal{S}(\mu/\nu)} \operatorname{sgn}(w) x^{w(\lambda+\rho+\omega(T))} = \sum_{T \in \mathcal{S}(\mu/\nu)} a_{\lambda+\omega(T)+\rho}. \tag{1}$$

We declare T to be a Bad Guy if $\lambda + \omega(T_{\geq j})$ fails to be a partition for some j; i.e.,

$$\lambda_k + \omega_k(T_{\geq j}) < \lambda_{k+1} + \omega_{k+1}(T_{\geq j})$$

for some pair k, j. Among all such pairs k, j, choose one that maximizes j, and among those, choose the smallest k. It must be the case that $\lambda + \omega(T_{>j})$ is a partition, and since $\omega_k(T_{\geq j}) - \omega_{k+1}(T_{\geq j})$ can change by at most one if we increment or decrement j, there must be a $k+1$ in column j of T (and no k), and

$$\lambda_k + \omega_k(T_{\geq j}) + 1 = \lambda_{k+1} + \omega_{k+1}(T_{\geq j}). \tag{2}$$

Let T^* denote the tableau obtained from T by applying the Bender-Knuth involution σ_k to the subtableau $T_{<j}$, leaving the remainder of T unchanged. Since this involves changing some subset of the entries of $T_{<j}$ from k to $k+1$ and vice-versa, and column j has a $k+1$ but no k, it is easy to see that T^* is semistandard. Furthermore, $(T^*)_{\geq j}$ and $T_{\geq j}$ are identical, so $T \mapsto T^*$ is an involution on the set of Bad Guys. In comparing the contributions of T and T^* to (1), note that $s_k\omega(T_{<j}) = \omega(T^*_{<j})$, whereas (2) implies that s_k fixes $\lambda + \omega(T_{\geq j}) + \rho$, whence $s_k(\lambda + \omega(T) + \rho) = \lambda + \omega(T^*) + \rho$ and

$$a_{\lambda+\omega(T)+\rho} = -a_{\lambda+\omega(T^*)+\rho}.$$

The contributions of Bad Guys may therefore be canceled from (1). □

For the shape $\mu = \mu/\varnothing$, we have $\omega(T_{\geq j}) \in \mathcal{P}_n$ for all j only if every entry in row i of T is i; thus, there is a unique such T, it has weight μ, and hence $a_\rho s_\mu = a_{\mu+\rho}$, or

Corollary (The Bi-Alternant Formula)**.** *For all $\mu \in \mathcal{P}_n$, we have $s_\mu = a_{\mu+\rho}/a_\rho$.*

Corollary. *For all $\lambda \in \mathcal{P}_n$ and all $\mu, \nu \in \mathcal{P}$ such that $\nu \leq \mu$, we have*

$$s_\lambda s_{\mu/\nu} = \sum s_{\lambda+\omega(T)},$$

where the sum ranges over all $T \in \mathcal{S}(\mu/\nu)$ such that $\lambda + \omega(T_{\geq j}) \in \mathcal{P}_n$ for all $j \geq 1$.

This corollary is Zelevinsky's extension of the Littlewood-Richardson rule [**Z**].

Taking the specialization $\lambda = \varnothing$, we obtain the decomposition of $s_{\mu/\nu}$ into Schur functions; it is simpler than the traditional formulation of the Littlewood-Richardson rule as found (e.g.) in [**M**], since it does not involve converting tableaux to words and imposing the "lattice permutation" condition. However, it still involves counting semistandard tableaux of shape μ/ν satisfying certain properties, and it is a not-too-difficult exercise to show that these two formulations count the *same* tableaux.

Via the specialization $\nu = \varnothing$, we obtain yet another formulation of the Littlewood-Richardson rule— in this case involving the decomposition of $s_\lambda s_\mu$ into Schur functions.

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