A Concise Sensitivity Analysis of the Quasi-Linear Viscoelastic Model Proposed by Fung

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Introduction

The quasi-linear viscoelastic theory introduced by Fung [1] states that the constitutive equation relating stress σ and strain ϵ for a soft biological tissue sample in simple elongation is given by the convolution integral[2]

$$\sigma(t) = \int_{\tau=0}^{t} G(t-\tau)\sigma_{,\epsilon}^{(e)} \epsilon_{,\tau} d\tau \qquad (1)$$

with $\sigma(t) = 0$ and $\epsilon(t) = 0$ for $t < 0$,

$$G(0) = 1.$$

The dependence of stress on both strain and time is separately described by the nonlinear elastic response $\sigma^{(e)}(\epsilon)$ and the reduced relaxation function $G(t)$, respectively.

In order to account for the remarkable strain rate insensitivity of both the stress-strain relation and the hysteresis loop of many soft biological tissues, Fung [1] proposed a specific relaxation spectrum of the form

$$S(\tau) = \begin{cases} C/\tau & \text{for } 0 < \tau_1 \leq \tau \leq \tau_2 \\ \\ 0 & \text{elsewhere} \end{cases} \qquad (2)$$

where C represents a dimensionless positive constant and τ_1 and τ_2 are time constants. This yields for the reduced relaxation function (RRF)

$$G(t) = [1 + C\{E_1(t/\tau_2) - E_1(t/\tau_1)\}]/[1 + C\ln(\tau_2/\tau_1)] \qquad (3)$$

with $E_1(x) = \int_{y=x}^{\infty} (\exp(-y)/y)dy$, the exponential integral function.

An important quantity for the description of the frequency-dependent behavior is the loss angle

$$\phi = \arctan(E_l/E_s) \qquad (4)$$
$$= \arctan(C\{\arctan(\omega\tau_2) - \arctan(\omega\tau_1)\}/\{1 + (C/2)\ln [\{1 + (\omega\tau_1)^2\}/\{1 + (\omega\tau_1)^2\}]\}).$$

E_l and E_s represent the loss and storage modulus, respectively; i.e., the imaginary and real parts of the complex modulus $E = E_s + iE_l$ which relates stress and strain during sinusoidal excitation in the steady state. The loss angle ϕ governs the description of the viscous losses: $\phi = 0$ and $\phi = \pi/2$ correspond with purely elastic and purely viscous behavior, respectively.

For the quantitative description of the viscoelastic properties of various types of soft biological tissues this quasi-linear viscoelastic model has yielded satisfactory results [2-9].

Using this model for the analysis of the relaxation behavior of aortic valve tissues in our laboratories the need was felt to obtain more insight into the importance of the parameters C, τ_1 and τ_2 for the description of certain viscoelastic phenomena.

Methods and Results

Before exploring the influence of the parameters C, τ_1 and τ_2 on the reduced relaxation function $G(t)$ and the loss angle $\phi(\omega)$, some remarks will be made on the general features of the model.

Although it is intended to describe viscoelastic phenomena, the question arises whether it is also capable of describing the extremes of the range covered by viscoelastic behavior, namely purely elastic and purely viscous behavior. For a purely elastic material will apply $G(t) = 1$ and $\phi(\omega) = 0$. From equations (3) and (4) is it seen that this requirement is met by setting $C = 0$. In the case of purely viscous behavior the complex modulus will be purely imaginary. This implies that the storage modulus equals zero

$$E_S = C_R[1 + (C/2)\ln\{(1 + (\omega\tau_2)^2)/(1 + (\omega\tau_1)^2)\}] = 0. \qquad (5)$$

C_R represents the stiffness exhibited by the material under quasi-static strain or load and will always be positive. Thus with $C_R \neq 0$ and $C > 0$ it follows that equation (5) can be satisfied only if $\tau_1 > \tau_2$. This is in contradiction with the requirement $\tau_2 > \tau_1$, stated in equation (2). Consequently it is concluded that the quasi-linear viscoelastic model based upon the relaxation spectrum (2) is not capable of describing purely viscous behavior.

In the following the partial derivatives with respect to C, τ_1 and τ_2 of the initial slope $(G_{,t})_{t=0}$ and the final (constant) value $G(\infty) = \lim_{t \to \infty} G(t)$ of $G(t)$ will be considered (Fig. 1).

The initial reduced relaxation rate r_0, i.e., the rate by which the RRF initially decreases, is given by

$$r_0 = -(G_{,t})_{t=0} = C(1/\tau_1 - 1/\tau_2)/[1 + C\ln(\tau_2/\tau_1)]. \qquad (6)$$

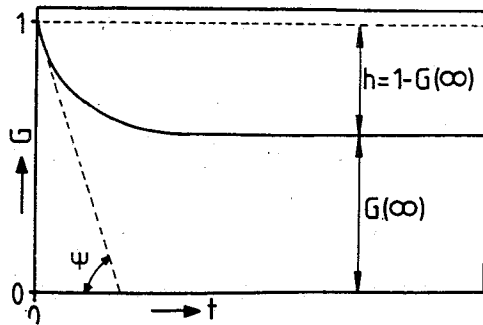From equation (3) the final relaxation fraction, i.e., the

Fig. 1 Outline of the reduced relaxation function G; $r_0 = -(G_{,t})_{t=0} = \tan(\psi)$: initial reduced relaxation rate; h: final relaxation fraction

fraction of the instantaneous stress response $\sigma(t=0)$ that has vanished after stress relaxation has ceased, is found to be

$$h = 1 - G(\infty) = 1 - [1 + C\ln(\tau_2/\tau_1)]^{-1}. \tag{7}$$

Using the abbreviations $\alpha = (1/\tau_1 - 1/\tau_2)$ and $\beta = \ln(\tau_2/\tau_1)$, the partial derivatives can be written as

$$r_{0,C} = \alpha/(1+\beta C)^2, \tag{8}$$

$$r_{0,\tau_1} = -C[1 + C\{\beta - 1 + \exp(-\beta)\}]/[\tau_1(1+\beta C)]^2, \tag{9}$$

$$r_{0,\tau_2} = C[1 + C\{\beta + 1 - \exp(\beta)\}]/[\tau_2(1+\beta C)]^2, \tag{10}$$

$$h_{,C} = \beta/(1+\beta C)^2, \tag{11}$$

$$h_{,\tau_1} = -C/[\tau_1(1+\beta C)^2], \tag{12}$$

and

$$h_{,\tau_2} = C/[\tau_2(1+\beta C)^2]. \tag{13}$$

Taking into consideration that $\alpha > 0$ and $\beta > 1$ because $0 < \tau_1 < \tau_2$, and that $C > 0$, it can be seen immediately from the foregoing equations that $r_{0,C} > 0$, $r_{0,\tau_1} < 0$, $h_{,C} > 0$, $h_{,\tau_1} < 0$ and $h_{,\tau_2} > 0$. Some more investigation is needed to decide on the sign of r_{0,τ_2}. The sign of r_{0,τ_2} is equal to that of the term between square brackets in the numerator of equation (10). From the values of this term for various soft tissues, determined from data in literature and listed in Table 1, it is readily concluded that $r_{0,\tau_2} < 0$. From equations (12) and (13) it follows with $\tau_1 < \tau_2$ that $|h_{,\tau_1}| > |h_{,\tau_2}|$, which means that the long-term relaxation behavior will be more affected by τ_1 than by τ_2.

To illustrate the foregoing the following set of parameter values was chosen: $C = 0.05$, $\tau_1 = 0.005$ s, $\tau_2 = 50$ s. These values were multiplied in turn with a constant c ($c = 0.1$ and 10), while the other two were kept at their original value (corresponding with $c = 1$). For each set r_0 and h were determined to get an indication on the sensitivity of r_0 and h as to changes in C, τ_1 and τ_2. The results are listed in Table 2 and illustrated in Fig. 2. It is readily observed that the initial reduced relaxation rate r_0 is mainly governed by τ_1 and C. An increase of τ_1 and C corresponds with a decrease and an increase of the relaxation rate, respectively. The effect of changes in τ_2 on r_0 is very small. The long-term relaxation behavior is most strongly affected by the value of C: increase of C results in an increase of stress relaxation. In comparison the influence of τ_1 and τ_2 is negligible.

The investigation of the loss angle $\phi(\omega)$ is mathematically more complicated. After rewriting equation (4) as

$$\phi = \arctan(\gamma) \tag{14}$$

with

Table 1 Values of the term between square brackets in the numerator of equation (10), determined for various tissues from data in literature

$1 + C(\beta + 1 - \exp(\beta))$	Tissue		Reference
-2.4408×10^7	Rabbit mesentery		Chen and Fung [3]
-48.80	Arch circumf. [a]	canine aorta	
-31.30	Arch axial	canine aorta	Tanaka and Fung [5]
-28.16	Prox.thor. [b] circumf.	canine aorta	
-19.13	Prox.thor. axial	canine aorta	
-2804	Articular cartilage		Woo, et al. [7]
-6557	Rabbit papillary muscle		Pinto and Patitucci [6]
-12759	Cat papillary muscle		
-3058	Leaflet circumf.		
-18590	Leaflet radial		
-441	Sinus circumf.	porcine aortic valve	
-6558	Sinus axial	porcine aortic valve	Sauren [9]
-17848	Aorta circumf.	porcine aortic valve	
-4776	Aorta axial		

[a] Circumferential

[b] Proximal thoracic

Table 2

(a) Effects upon the initial reduced relaxation rate r_0 of variations of C, τ_1 and τ_2 about the set of center values $C = 0.05$, $\tau_1 = 0.05$ (s) and $\tau_2 = 50$ (s). Between parentheses the quotient of the actual value and the center value of r_0 (corresponding with $c = 1$) is given.

c	0.1	1	10
C	0.96(0.14)	6.85(1)	17.84(2.60)
τ_1	63.46(9.26)		0.74(0.11)
τ_2	7.42(1.08)		6.35(0.93)

(b) Idem for the final relaxation fraction h

c	0.1	1	10
C	0.04(0.12)	0.32(1)	0.82(2.56)
τ_1	0.36(1.12)		0.26(0.81)
τ_2	0.19(0.59)		0.31(0.97)
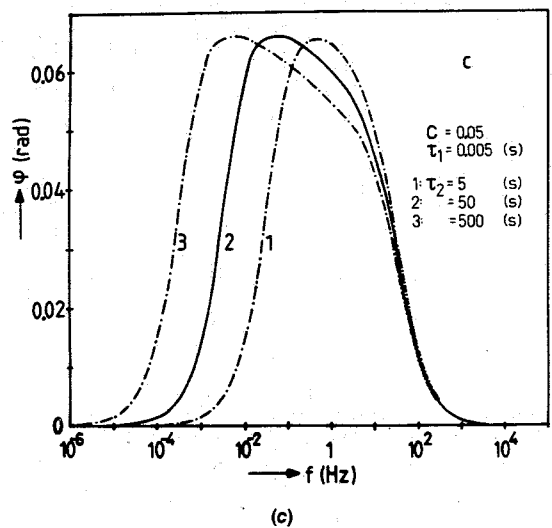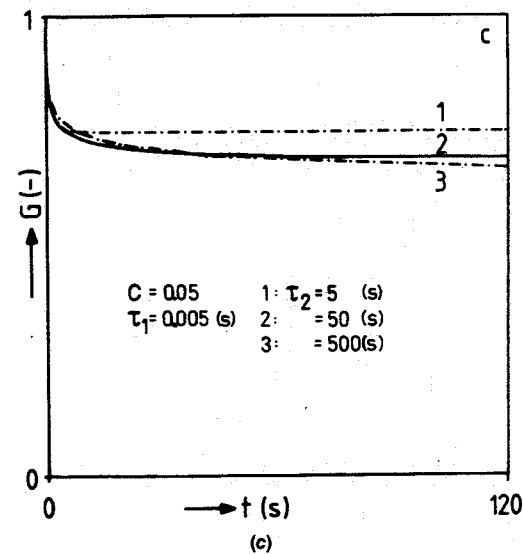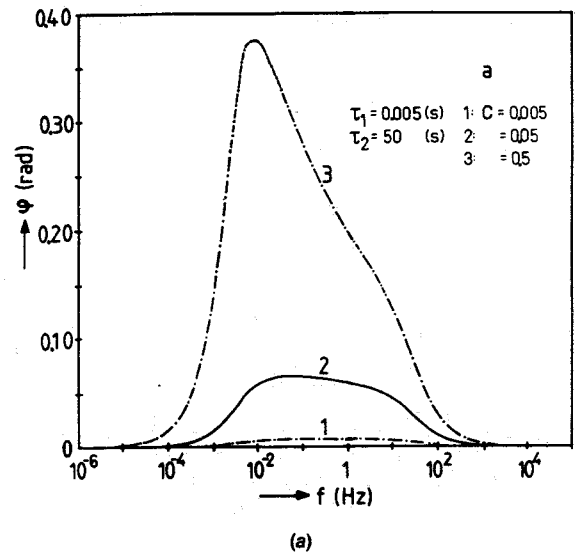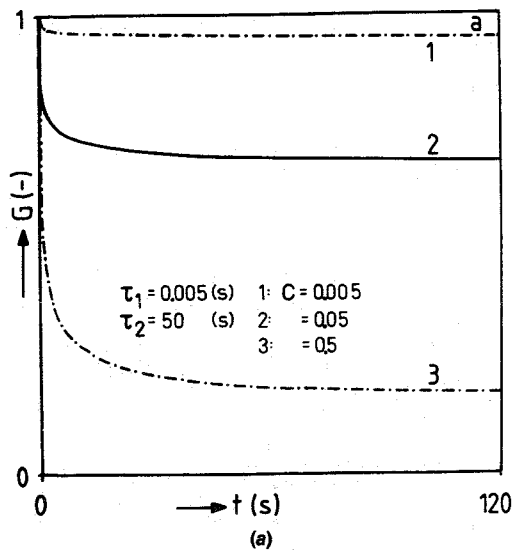
Fig. 2 Graphical illustration of the influence of parameter variations about a set of center values (see Table 2) upon the reduced relaxation function. The figures (a) through (c) show the effects of changes in C, τ_1 and τ_2, respectively. The solid curve in each figure represents the characteristic corresponding with the set of center values.
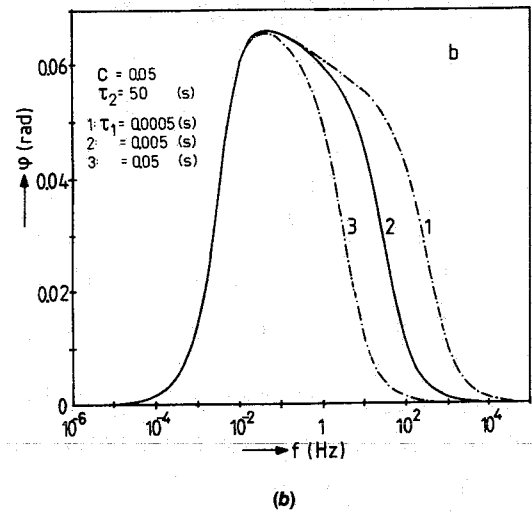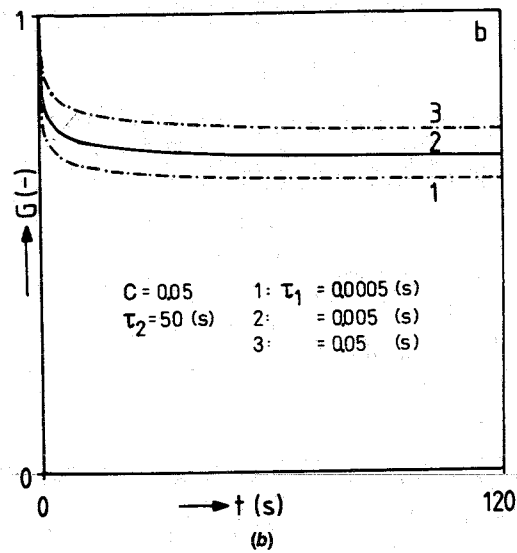
Fig. 3 Graphical illustration of the influence of parameter variations upon the loss angle. For further explanation see text of Fig. 2. Note the different scaling factors on the vertical axes.

$$\gamma = C\{\arctan(\omega\tau_2) - \arctan(\omega\tau_1)\}$$
$$/\{1 + (C/2)\ln[\{1 + (\omega\tau_2)^2\}/\{1 + (\omega\tau_1)^2\}]\}$$

the partial derivative $\phi_{,x}$ appears to be of the form

$$\phi_{,x} = \gamma_{,x}/(1 + \gamma^2) \quad \text{with} \quad 0 < 1/(1 + \gamma^2) \le 1. \tag{15}$$

The sign of $\phi_{,x}$ will be equal to that of $\gamma_{,x}$. Therefore an examination of the partial derivatives of γ will be sufficient. Using the abbreviation

$$g = 1 + (C/2)\ln[\{1 + (\omega\tau_2)^2\}/\{1 + (\omega\tau_1)^2\}] \tag{16}$$

we get

$$\gamma_{,C} = \{\arctan(\omega\tau_2) - \arctan(\omega\tau_1)\}/g^2, \tag{17}$$

$$\gamma_{,\tau_1} = C\omega(\omega\tau_1\gamma - 1)/[g\{1 + (\omega\tau_1)^2\}] \tag{18}$$

and

$$\gamma_{,\tau_2} = C\omega(1 - \omega\tau_2\gamma)/[g\{1 + (\omega\tau_2)^2\}]. \tag{19}$$

With $\tau_1 < \tau_2$ it follows immediately from equation (17) that always will apply $\gamma_{,C} \ge 0$. However, from equations (18) and (19) it is seen that the sign of the derivatives $\gamma_{,\tau_1}$ and $\gamma_{,\tau_2}$ will depend in general on the actual values of C, τ_1, τ_2 and ω. A detailed mathematical analysis of the equations (18) and (19) would go far beyond the scope of this paper. Instead we will restrict ourselves to a graphical illustration of the effects on ϕ of variations in τ_1 and τ_2 (see Fig. 3). The loss angle is highly sensitive to changes in C. A decrease of C results not only in a decrease of ϕ at a particular frequency (as indicated by equation (17)), but also causes a flattening of the ϕ curve: the material behavior becomes more and more elastic with decreasing C. Variations of C and τ_1 have a negligible influence upon the value of the frequency at which ϕ is maximum, whereas an increase of τ_2 results in a considerable shift of this frequency towards lower values. The maximum value of ϕ, however, is insensitive to variations of the time constants τ_1 and τ_2 but is strongly affected by the parameter C. The curves in Figs. 3(b) and (c) illustrate clearly the pronounced influence of τ_1 and τ_2 upon, respectively, the parts on the ϕ curves beyond and below their maximum: "slow" viscous phenomena are mainly affected by τ_2, whereas τ_1 governs the "fast" phenomena.

The foregoing may be summarized as follows: The parameter C is the most important as it determines the degree to which viscous effects are present. The time constants τ_1 and τ_2 govern, respectively, the "fast" and "slow" viscous phenomena.

Acknowledgments

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