A Cone Complementarity Linearization Algorithm for Static Output-Feedback and Related Problems

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***Abstract*—This paper describes a linear matrix inequality (LMI)-based algorithm for the static and reduced-order output-feedback synthesis problems of nth-order linear time-invariant (LTI) systems with n_u (respectively, n_y) independent inputs (respectively, outputs). The algorithm is based on a "cone complementarity" formulation of the problem and is guaranteed to produce a stabilizing controller of order $m \leq n - \max(n_u, n_y)$, matching a generic stabilizability result of Davison and Chatterjee [7]. Extensive numerical experiments indicate that the algorithm finds a controller with order less than or equal to that predicted by Kimura's generic stabilizability result $(m \leq n - n_u - n_y + 1)$. A similar algorithm can be applied to a variety of control problems, including robust control synthesis.**

***Index Terms*— Complementarity problem, linear matrix inequality, reduced-order stabilization, static output feedback.**

I. Introduction

We consider the *reduced-order output-feedback* (ROF) stabilization problem. We are given an integer $m \geq 0$ and a linear time-invariant (LTI) system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

where $x \in \mathbf{R}^n, u \in \mathbf{R}^{n_u}$, and $y \in \mathbf{R}^{n_y}$, (A, B) is stabilizable, (C, A) detectable, B, C are full rank, and $m \leq n$. We seek to determine a dynamic output-feedback control law

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = K \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{2}$$

where $K \in \mathbf{R}^{(m+n_u)\times(m+n_y)}$ is a constant matrix such that the resulting closed-loop system is stable. When $m = 0$, the corresponding problem is referred to as the *static output-feedback* (SOF) stabilization problem. Note that the ROF problem is readily transformed into an SOF problem, using a well-known system augmentation technique (see Section II-D).

Despite its apparent simplicity, the ROF problem is still open. The complexity analysis of the problem is not quite complete yet. Anderson *et al.* have shown in [1] that the problem is decidable. A nice result of Blondel and Tsitsiklis [3] shows that the problem of finding an SOF controller with prespecified bounds on the controller matrix K is NP-hard. Of course, this does not prove that the SOF problem is NP-hard, since no *a priori* bounds on the controller matrix K are imposed.

A number of numerical procedures have been proposed for solving the problem. A survey was done by Syrmos *et al.* [22], and recent progress has been made for the related problem of pole placement; see [25], [19], and the references therein.

Among the recent approaches that have been proposed, those based on linear matrix inequalities (LMI's) are promising since the same framework can be used for related problems such as robust control (see Section III). (The book [21] describes a large number of such problems.) The LMI-based methods include the D-K iteration method mentioned in [20], the alternating projections method [12], the projection method of [18], the min–max algorithm [10], the potential reduction method [6], and the XY-centering algorithm [14]. This paper describes yet another LMI-based method for the problem. One of the strong points of the proposed algorithm is that it is guaranteed to find a controller with order less than or equal to $n - \max(n_u, n_y)$ and seems to consistently improve this number in practice.

The remainder of this paper is organized as follows. Section II describes a cone complementarity linearization algorithm. Section III shows how some robust control synthesis problems are amenable to the same method. Finally, Section IV provides extensive numerical results.

Notation: I_m denotes the $m \times m$ identity matrix; the size is omitted when it can be determined from context. $C_\perp$ (respectively, $B_\perp$) denotes the matrix with maximal rank such that $CC_\perp = 0$ (respectively, $B_\perp B = 0$). For a real matrix X, $\|X\|$ denotes the norm (largest singular value) of the matrix X, and $X > 0$ (respectively, $X \geq 0$) means X is symmetric and positive-definite (respectively, positive semidefinite).

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II. An LMI-Based Algorithm

A. LMI-Based Formulation

As seen in [8], there exists an mth-order state-feedback stabilizing controller if and only if there exist matrices X, S, and scalar $\beta > 0$ such that

$$\begin{aligned} B_\perp(AX + XA^T + 2\beta X)B_\perp^T &\leq 0 \\ C_\perp^T(A^T S + SA + 2\beta S)C_\perp &\leq 0 \\ \mathcal{K}(X, S) = \begin{bmatrix} X & I \\ I & S \end{bmatrix} &\geq 0 \end{aligned} \tag{3}$$

and $\mathbf{Rank}\mathcal{K}(X, S) \leq n + m$. We first seek to solve the above problem for $m = 0$ and for fixed $\beta > 0$. That is, we seek to obtain an SOF controller such that the closed-loop eigenvalues have a real part that is less than or equal to $-\beta$.

B. A Cone Complementarity Problem

For solving the problem above, we need to "saturate" the constraint $\mathcal{K}(X, S) \geq 0$ in (3). The idea is to associate to the SOF problem

$$\min \mathbf{Tr} XS \text{ subject to } (3). \tag{4}$$

There exists a β-stabilizing static-output feedback controller if and only if the global minimum of problem (4) is n.

Problem (4) can be called a "cone complementarity" problem (CCP), as it is an extension linear complementarity problems (LCP's) to the cone of positive semidefinite matrices; see, e.g., [26], [16], and [27].

To solve such a problem, a linearization method (originally proposed by Franke and Wolfe and described in [16] for LCP's) can be used. At a given point (X_0, S_0), a linear approximation of $\mathbf{Tr}XS$ takes the form

$$\phi_{\text{lin}}(X, S) = \text{constant } + \mathbf{Tr}(S_0 X + X_0 S).$$

The linearization algorithm is conceptually described as follows.

Algorithm 1:

1) Find a feasible point X_0, S_0. If there are none, exit. Set $k = 0$.
2) Set $V_k = S_k, W_k = X_k$, and find X_{k+1}, S_{k+1} that solve the LMI problem

$$\mathcal{P}_k : \text{minimize } \mathbf{Tr}(V_k X + W_k S) \text{ subject to (3).}$$

3) If a stopping criterion is satisfied, exit. Otherwise, set $k = k+1$ and go to Step 2).

The first step of the algorithm and every Step 2) are simple LMI problems. There are many algorithms that are available for this, especially interior-point methods, e.g., [24], [4], and [17]. In the sequel, we count each Step 2) as one outer iteration to make a distinction with the inner steps required to solve the LMI problems $\mathcal{P}_k$.

The following theorem shows that the algorithm converges.

Theorem 2.1: The sequence $t_k \triangleq \mathbf{Tr}(X_{k+1}S_k + S_{k+1}X_k), k \geq 0$ is bounded below by $2n$ and decreasing. Thus, the sequence (t_k) converges to some value $t_{\text{opt}} \geq 2n$. Equality holds if and only if $XS = I$ at the optimum.

Proof: Let $k > 0$. Since X_{k-1}, S_{k-1} are feasible, and (X_{k+1}, S_{k+1}) are optimal for problem $(\mathcal{P}_k)$, we have

$$t_k \leq \mathbf{Tr}(X_{k-1}S_k + S_{k-1}X_k) = t_{k-1}.$$

Now t_k is bounded below by $2n$, since

$$\begin{aligned} t_k = \mathbf{Tr}(X_{k+1}S_k + S_{k+1}X_k) &\geq \inf_{\mathcal{K}(V,W)\geq 0} \mathbf{Tr}(VX_k + WS_k) \\ &= 2\mathbf{Tr}\big(X_k^{1/2} S_k X_k^{1/2}\big)^{1/2} \geq 2n. \end{aligned}$$

The last inequality implies that if $t_{\text{opt}} = 2n$, then $XS = I$. □

The next theorem shows that from the first step of the algorithm, the method finds a controller of order that is less than or equal to the order predicted by the generic stabilizability results of Davison and Chatterjee [7].

Theorem 2.2: At every step k, we have

$$\mathbf{Rank}\begin{bmatrix} X_k & I \\ I & S_k \end{bmatrix} \leq 2n - \max(n_u, n_y).$$

Thus, the algorithm finds a controller of order that is less than or equal to $n - \max(n_u, n_y)$ at every step.

Proof: Let $k \geq 0$. The dual to problem $\mathcal{P}_k$ is (see [24])

$$\begin{aligned} \mathcal{D}_k : &\text{maximize } 2\mathbf{Tr}N \text{ subject to } Q \geq 0,\ P \geq 0, \\ &Z = A^T\big(B_\perp^T Q B_\perp\big) + \big(B_\perp^T Q B_\perp\big)A + 2\beta\big(B_\perp^T Q B_\perp\big) + V_k \\ &R = A\big(C_\perp P C_\perp^T\big) + \big(C_\perp P C_\perp^T\big)A^T + 2\beta\big(C_\perp P C_\perp^T\big) + W_k \\ &\tilde{Z} = \begin{bmatrix} Z & N \\ N^T & R \end{bmatrix} \geq 0. \end{aligned}$$

Pre- (respectively, post-) multiplying the first equality constraint in $\mathcal{D}_k$ by B^T (respectively, B), we obtain that every feasible Z satisfies

$$B^T Z B = B^T S_k B.$$

Since B is full rank and $S_k > 0$, we have $B^T S_k B > 0$, and thus $\mathbf{Rank}Z \geq n_u$. Similarly, $\mathbf{Rank}R \geq n_y$ for every feasible R.

Both $\mathcal{P}_k$ and $\mathcal{D}_k$ are strictly feasible (for $\mathcal{D}_k$, set $Q = \epsilon I, P = \epsilon I$ for ϵ small enough, and $N = 0$). This guarantees the existence of optimal primal and dual points. At the optimum, we have (see [24])

$$\tilde{Z}\begin{bmatrix} X_{k+1} & I \\ I & S_{k+1} \end{bmatrix} = 0 \tag{5}$$

which implies

$$\tilde{Z} = \begin{bmatrix} I \\ -X_{k+1} \end{bmatrix} Z \begin{bmatrix} I \\ -X_{k+1} \end{bmatrix}^T = \begin{bmatrix} -S_{k+1} \\ I \end{bmatrix} R \begin{bmatrix} -S_{k+1} \\ I \end{bmatrix}^T.$$

We obtain $\mathbf{Rank}\tilde{Z} = \mathbf{Rank}R = \mathbf{Rank}Z \geq \max(n_u, n_y)$. From (5), we deduce

$$\mathbf{Rank}\begin{bmatrix} X_{k+1} & I \\ I & S_{k+1} \end{bmatrix} \leq 2n - \mathbf{Rank}\tilde{Z} \leq 2n - \max(n_u, n_y).$$

□

Remark 2.1: Algorithm 1 is based on the function $\phi(X, S) = \mathbf{Tr}XS$. It turns out that a whole class of algorithms can be devised with other choices for ϕ. For instance, we may work with the concave function $\phi_{1/2}(X, Y) = 2\mathbf{Tr}(Y^{1/2}XY^{1/2})^{1/2}$ by replacing V, W in Algorithm 1 by

$$\begin{aligned} V &= X^{-1/2}(X^{1/2}YX^{1/2})^{1/2}X^{-1/2} \\ W &= V^{-1} = Y^{-1/2}(Y^{1/2}XY^{1/2})^{1/2}Y^{-1/2}. \end{aligned}$$

In our experiments, we have found that this modified algorithm behaves similarly. For details, see [11].

C. System Augmentation

As stated in the introduction, the ROF problem can be addressed as an SOF problem for an augmented system. Define

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0_m \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 & B \\ I_m & 0 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} 0 & I_m \\ C & 0 \end{bmatrix}. \tag{6}$$

It can be shown that the ROF problem has a solution of order m if and only if the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ is SOF stabilizable. We may thus use the previous algorithm with the triple $(\tilde{A}, \tilde{B}, \tilde{C})$.

It is also possible to apply the algorithm directly to the triple (A, B, C). We conjecture that this direct approach yields the same answer (that is, the same controller order). If the conjecture is true, the direct approach is to be preferred, since it is less numerically demanding (it involves fewer variables and LMI's of smaller size).

D. Controller Reconstruction

Most LMI solvers require the boundness for the feasible set. For this, an *a priori* large bound on the variables can be imposed, e.g., $\mathbf{Tr}(X + S) \leq M$.

The following theorem can be used as a (heuristic) stopping criterion and provides details on how to construct the controller. (Its proof is in Appendix A.) In this theorem, the parameter α is interpreted as the desired stability degree of the closed-loop system, and $\beta > \alpha$ is a parameter needed to guarantee this stability degree strictly (if $\alpha > 0$, one may set $\beta = 2\alpha$).

Theorem 2.3: Suppose X, S satisfy conditions (3) and $\mathbf{Tr}(X + S) \leq M$. Let α, $0 \leq \alpha < \beta$. If there are $n - m$ eigenvalues of $X - S^{-1}$ that are less than or equal to

$$\epsilon_{\text{rof}} = \frac{(\beta - \alpha)}{M\|A + \alpha I\|} \tag{7}$$

then there exists an mth-order, dynamic output-feedback controller such that every eigenvalue of the closed-loop system has a real part that is less than or equal to $-\alpha$. The controller can be constructed as follows. Decompose $X - S^{-1}$ as

$$X - S^{-1} = RR^T + E \tag{8}$$

where $0 \leq E \leq \epsilon_{\text{rof}} I$, and $R \in \mathbf{R}^{n\times m}$ is either empty (when $m = 0$) or satisfies $R^T R > \epsilon_{\text{rof}} I$. Set $(\tilde{A}, \tilde{B}, \tilde{C})$ as in (6) and

$$\tilde{X} = \begin{bmatrix} X & R \\ R^T & I \end{bmatrix}.$$

Then, $\tilde{X} > 0$ and the LMI problem in K

$$(\tilde{A} + \tilde{B}K\tilde{C})\tilde{X} + \tilde{X}(\tilde{A} + \tilde{B}K\tilde{C})^T + 2\alpha\tilde{X} \leq 0 \tag{9}$$

has a solution. Alternatively, the analytic formulas of [13] can be used to reconstruct K.

III. Robust Output-Feedback Control

A similar algorithm can be applied for solving a number of other control problems. Here is a simple instance. Consider the parameter-dependent system

$$\begin{aligned}\dot{x} &= Ax + B_p p + B_u u \\ y &= C_y x \\ q &= C_q x + D_{qu} u \\ p &= \Delta(t) q, \quad \|\Delta\| \le 1, \quad \Delta \text{ diagonal}\end{aligned} \tag{10}$$

where A, B_u, B_p, C_q, C_y, and D_{qu} are constant matrices of appropriate size. The $N \times N$ time-varying matrix Δ, referred to as the perturbation, is bounded and structured (diagonal).

We seek a full-order controller of the form (2) such that the closed-loop system is stable for every admissible variation of Δ. We use the approach of quadratic stability combined with scaling (see, e.g., [5]).

Theorem 3.1: There exists a controller of the form (2), where $K \in \mathbf{R}^{(n+n_u)\times(n+n_y)}$ is a constant matrix such that the closed-loop system is stable for every admissible variation of Δ, if there exist matrices X, S, D, T such that

$$\begin{aligned}\mathcal{M}_\perp^T \begin{bmatrix} AX + XA^T + B_p D B_p^T & XC_q^T \\ C_q X & -D \end{bmatrix} \mathcal{M} &< 0 \\ \mathcal{N}^T \begin{bmatrix} A^T S + SA + C_q^T T C_q & SB_p \\ B_p^T S & -T \end{bmatrix} \mathcal{N} &< 0 \\ \begin{bmatrix} X & I \\ I & S \end{bmatrix} \ge 0, \quad \begin{bmatrix} D & I \\ I & T \end{bmatrix} &\ge 0 \\ D, T \text{ diagonal}, \quad DT &= I\end{aligned}$$

where $\mathcal{M} = [B_u^T \quad D_{qu}^T]$, $\mathcal{N} = \mathbf{diag}(C_{y\perp}, I)$.

The problem of finding matrices X, S, D, T feasible for the above constraints can be addressed using a similar approach by applying a linearization algorithm to minimize, e.g., $\mathbf{Tr} DT$, over diagonal matrices D, T. It is also possible to search for reduced-order robust controllers. Simply replace X and S by $\mathbf{diag}(X, D)$ and $\mathbf{diag}(S, T)$ in Algorithm 1.

IV. Numerical Experiments

For every run, we have chosen (unless otherwise stated) $M = 10^5$, $\alpha = 0.01, \beta = 2\alpha$. We have used the semidefinite programming code **SP** [23] and a MATLAB interface to **SP**, **LMITOOL** [9]. The **SP** parameters for absolute and relative convergence were both set to 10^{-10}.

The random tests presented next are based on generic stabilizability results given by Kimura [15]. In the sequel, we say that (A, B, C) satisfies the mth-order Kimura property if $m > n - n_u - n_y$. This property guarantees the existence of an output-feedback stabilizing controller of order m.

A. SOF

In Table I, we have generated 20 000 random triples (A, B, C) satisfying the 0th order Kimura property. The algorithm was *always successful* finding an SOF controller. In a large majority of cases (86%), at most two outer iterations were needed, and except for a few cases this number is less than the order of the plant. The mean CPU time for solving each one of these problems (on an HP-710 Workstation) was less than 3 s.

Next, we have generated 1000 random triples (A, B, C) and formed the augmented matrices $(\tilde{A}, \tilde{B}, \tilde{C})$ defined in (6), with $m = n$. We then know that the augmented triple is (generically) stabilizable by an SOF controller. Table II shows that the algorithm was again successful. The average number of outer iterations is (for about 90% of cases) less than the order of the (augmented) plant, as with the "direct" approach of the previous experiments.

TABLE I
Experiments with Random $A, B, C, n = 6, n_u = 4, n_y = 3$. Algorithm 1 Always Finds a Static Controller, as Predicted by Kimura's Result

outer iterations	number of experiments	Rate
1	12756	63.78 %
2	4392	21.96 %
3	1680	8.40 %
4	692	3.46 %
5	271	1.35 %
6	110	0.55 %
7-14	99	0.50 %
total	20000	100 %

TABLE II
Experiments with Augmented Matrices $\tilde{A}, \tilde{B}, \tilde{C}$, Where $\tilde{A}$ Is $2n \times 2n$, Formed with Random $A, B, C, n = 5, n_u = 2, n_y = 2$. Algorithm 1 Always Finds a Static Controller, as Predicted by the Generic Stabilizability (by a Full-Order Controller) of (A, B, C)

outer iterations	number of experiments	Rate
1	304	30.4 %
2	271	27.1 %
3	115	11.5 %
4	82	8.2 %
5	47	4.7 %
6	41	4.1 %
7	15	1.5 %
8	9	0.9 %
9	13	1.3 %
10	5	0.5 %
$10 < * \le 20$	52	5.2 %
$20 < * \le 30$	36	3.6 %
$30 < * \le 37$	10	1 %
total	1000	100 %

B. ROF

In Table III, we have sought to stabilize a mass-spring system consisting of N unit masses connected by linear springs of unit spring constant. The input acts on the left mass, and the position of the right mass is measured. Using Algorithm 1, we have sought a low-order stabilizing controller for this system such that the closed-loop eigenvalues have a negative real part greater than $\alpha > 0$. For instance, the three-mass system has the plant matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

In order to guarantee the closed-loop eigenvalues to be sufficiently stable, we have set $\alpha = 0.1$. The resulting third-order controller matrix is

$$K = \begin{bmatrix} 0.5283 & -0.4405 & -1.8749 & 0.7618 \\ 0.4405 & 0.0170 & 0.3971 & -0.1767 \\ 1.8750 & 0.3971 & -1.4453 & 1.3969 \\ -0.7618 & -0.1767 & 1.3969 & -1.1984 \end{bmatrix}.$$

TABLE III
REDUCED-ORDER α-STABILIZING CONTROLLERS FOR MASS/SPRING SYSTEMS. AS THE REQUIRED CLOSED-LOOP DECAY RATE GROWS, THE ATTAINABLE CONTROLLER ORDER INCREASES

number of masses	controller order	
	$\alpha = .1$	$\alpha = .001$
1	1	1
2	2	2
3	3	3
4	5	5
5	6	6
6	7	7
7	9	8
8	13	9
9	15	11
10	17	11

The closed-loop eigenvalues of the augmented three-mass system are

$$s = \begin{bmatrix} -0.1000 \pm 1.7870\,i \\ -0.1000 \pm 1.4753\,i \\ -0.1000 \pm 0.9072\,i \\ -0.1000 \pm 0.4690\,i \\ -0.1000 \end{bmatrix}.$$

Table IV presents 20 000 experiments, with random (A, B, C) satisfying the first-order Kimura condition. Our numerical results match the fact that every system is generically stabilizable with a first-order controller.

C. Robust Output Feedback

Consider a more "realistic" model taken from [2]. The plant matrices are given by

$$A(p_1, p_2) = \begin{bmatrix} -.0366 & .0271 & .0188 & -.4555 \\ .0482 & -1.01 & .0024 & -4.0208 \\ .1002 & p_1 & -.707 & p_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$B_u(p_3) = \begin{bmatrix} .4422 & .1761 \\ p_3 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}$$
$$C = [0 \quad 1 \quad 0 \quad 0].$$

The three uncertain parameters p_1, p_2, p_3 are within the bounds $|p_1 - 0.3681| \le .05$, $|p_2 - 1.4200| \le .01$, and $|p_3 - 3.5446| \le .04$. We set A, B_u to be the nominal values and write the above model as (10) with the following matrices:

$$B_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.04 \\ 0.05 & 0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C_q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$C_y^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad D_{qu} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We seek a robust, low-order controller for the system above, with A replaced by $A + \alpha I$. We apply our algorithm to the CCP with $\mathbf{diag}(X, D)$ and $\mathbf{diag}(S, T)$. In other words, we have sought to minimize

$$\mathbf{Tr} XD + \mathbf{Tr} ST$$

TABLE IV
EXPERIMENTS WITH RANDOM A, B, C $(n = 6, n_u = 3, n_y = 3)$. ALGORITHM 1 FINDS A CONTROLLER OF ORDER 0 IN MORE THAN 99% OF THE CASES AND OF ORDER 1 AT MOST

controller order	Number of experiments	Rate
0	19887	99.43 %
1	113	0.57 %
total	20000	100%

using the linearization Algorithm 1. For a required decay rate of $\alpha = 0.1$, we have found a robust, stabilizing order controller of zeroth-order after two outer iterations. The result is

$$S = \begin{bmatrix} 0.8902 & -0.2499 & -0.4479 & -0.4212 \\ -0.2499 & 1.5753 & 0.7388 & 0.4740 \\ -0.4479 & 0.7388 & 1.3595 & 0.7208 \\ -0.4212 & 0.4740 & 0.7208 & 1.5059 \end{bmatrix}$$
$$D = \mathbf{diag} \begin{bmatrix} 0.9985 \\ 1.5145 \\ 0.9997 \end{bmatrix}$$
$$X = \begin{bmatrix} 1.4033 & -0.0084 & 0.3449 & 0.2300 \\ -0.0084 & 0.8564 & -0.4342 & -0.0641 \\ 0.3449 & -0.4342 & 1.2886 & -0.3837 \\ 0.2300 & -0.0641 & -0.3837 & 0.932 \end{bmatrix}$$
$$T = \mathbf{diag} \begin{bmatrix} 1.0015 \\ 0.6603 \\ 1.0003 \end{bmatrix}.$$

We can check that $DT = I$ and that $XS = I$. The static robust controller, which stabilizes this above system, can be computed by solving a solution K to LMI's that express quadratic stability of the closed-loop system. We obtained a feasible K

$$K = \begin{bmatrix} -0.4357 \\ 9.5652 \end{bmatrix}.$$

D. Comparison with Other Algorithms

In Table V, we compared the behavior of our algorithm with other existing algorithms: the D-K iteration method [20] and the min–max algorithm [10] (labeled in Table V by GSS). One-thousand (A, B, C)'s satisfying the 0th-order Kimura condition were chosen randomly. Our algorithm always finds a static controller in less than eight outer iterations. In more than 70% of the cases, the algorithm is successful after only one outer iteration. The D-K iteration algorithm fails in the vast majority of cases. The GSS algorithm behaves much better than the D-K iteration algorithm, but it fails sometimes or takes many more iterations than our algorithm. Note that all three algorithms require approximately the same amount of work at each (outer) iteration.

V. CONCLUDING REMARKS

This paper describes an algorithm for the ROF problem. The algorithm is guaranteed to generate a controller of order that is less than or equal to $n - \max(n_u, n_y)$. Although the algorithm may not be able to find the smallest-order controller in all cases, numerical experiments indicate a very satisfactory behavior, matching in particular Kimura's generic stabilizability result. The approach can be applied to a large number of other rank-minimization problems over LMI's that arise in control theory (see [21] for many examples).

The behavior of the algorithm can be intuitively understood in parallel to primal-dual interior-point methods for LMI problems, as follows. Our algorithm is a natural extension of a classical algorithm,

TABLE V
COMPARISON WITH OTHER ALGORITHMS WITH RANDOM A, B, C ($n = 5$, $n_u = 3$, $n_y = 3$). THE TABLE SHOWS THE NUMBER OF SUCCESSES VERSUS THE NUMBER OF ITERATIONS. FOR THE GSS ALGORITHM, FAIL MEANS THAT IT GENERATES THE UNBOUNDED SEQUENCE ($\|X_k\| > 10^6$) AND PROVIDES NO SOLUTION. FOR THE D-K ALGORITHM, FAIL MEANS THAT THE STATIONARY POINT OBTAINED FAILED TO STABILIZE THE SYSTEM

Outer Iterations	Algorithm 1	GSS	D-K
1	717	129	168
2	161	126	4
3	74	31	2
4	28	47	0
5	11	44	0
6	6	49	0
7	2	81	0
8	1	89	0
9	0	97	0
$10 \le * \le 50$	0	260	0
Fail	0	47	826
total	1000	1000	1000

originally devised for linear complementarity problems to CCP's. The CCP's arise in particular in primal-dual formulations of LMI problems, as seen in, e.g., [24]. In a CCP constructed from an LMI problem, we have a pair of symmetric matrix variables X, S, each subject to equality and LMI constraints. By construction, the constraints on X and those on S are "dual" so that the primal-dual gap $\mathbf{Tr}XS$ is actually *linear* in X, S on the feasible set. The most efficient algorithms for LMI problems to date use the above fact and work with the CCP formulation, using both primal and dual variables simultaneously. Our algorithm is an adaptation of this idea to a problem where the LMI constraints on X, S are not necessarily "dual" to each other. As in LMI problems, working with both primal and dual variables simultaneously seems to be comparatively efficient; this was illustrated in Table V.

Note that the LMI constraints (3) of the ROF problem can also be viewed as "dual" to each other, in the sense given to duality in control theory. The first constraint is related to stabilizability, and the second is related to detectability. The connection between convex duality and control duality may play an important role in the answer to the ROF problem.

APPPENDIX A
PROOF OF THEOREM 2.3

We first prove the theorem for $m = 0$ in which case the triple $(\tilde{A}, \tilde{B}, \tilde{C})$ defined in (6) coincides with (A, B, C).

For $\beta > 0$, introduce a linear operator Γ defined on the set of $n \times n$ symmetric matrices by

$$\Gamma_\beta(H) = B_\perp(AH + HA^T + 2\beta H)B_\perp^T. \tag{11}$$

Since $B_\perp$ is orthogonal, a bound on the norm of Γ_β is

$$\max_{\|H\|=1} \|\Gamma_\beta(H)\| \le 2\|A + \beta I\|. \tag{12}$$

Assume that X, S are feasible for the LMI constraints (3). Let $H = S^{-1} - X$ and $0 \le \alpha < \beta$. With Γ defined by (11), we have

$$\begin{aligned}\Gamma_\alpha(S^{-1}) &= \Gamma_\alpha(X) + \Gamma_\alpha(H)\\ &\le -2(\beta - \alpha)B_\perp X B_\perp^T + \Gamma_\alpha(H).\end{aligned}$$

From (12), we have $\|\Gamma_\alpha(H)\| \le 2\|A + \alpha I\|\|H\|$. Thus, the criterion

$$\|H\| \le \frac{2(\beta - \alpha)\lambda_{\min}\left(B_\perp X B_\perp^T\right)}{\|A + \alpha I\|}$$

guarantees that

$$\begin{aligned}B_\perp(AS^{-1} + S^{-1}A^T + 2\alpha S^{-1})B_\perp^T &\le 0\\ C_\perp^T(A^T S + SA + 2\alpha S)C_\perp &\le 0.\end{aligned}$$

From the elimination lemma (see, e.g., [5]), we obtain that problem (9) is feasible. The proof of the theorem for $m = 0$ then follows from the fact that $\mathcal{K}(X, S) \ge 0$ together with $\mathbf{Tr}(X + S) \le M$ implies $\lambda_{\min}(B_\perp X B_\perp^T) \ge M^{-1}$.

Now consider the general case $m \ge 0$. Let $\tilde{X}$ be defined as in (8) and define

$$\tilde{S} = \begin{bmatrix} X - E & R \\ R^T & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -R^T & I \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -R \\ 0 & I \end{bmatrix}.$$

It is easily seen that both $\tilde{X}, \tilde{S}$ are positive definite. Moreover, for every $\alpha, 0 \le \alpha \le \beta$

$$\begin{aligned}&\tilde{B}_\perp(\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T + 2\alpha\tilde{X})\tilde{B}_\perp^T\\ &\quad = B_\perp(AX + XA^T + 2\alpha X)B_\perp^T \le 0\\ &\tilde{C}_\perp^T(\tilde{A}^T\tilde{S} + \tilde{S}\tilde{A} + 2\alpha\tilde{S})\tilde{C}_\perp\\ &\quad = C_\perp^T(A^T S + SA + 2\alpha S)C_\perp \le 0.\end{aligned}$$

Finally

$$\tilde{X} - \tilde{S}^{-1} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \ge 0$$

which implies $\|\tilde{X} - \tilde{S}^{-1}\| = \|E\| \le \epsilon_{\text{rof}}$. We obtain that $\tilde{X}, \tilde{S}$ satisfy the conditions of the theorem with $m = 0$, for the augmented system $(\tilde{A}, \tilde{B}, \tilde{C})$. This achieves the proof.

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REFERENCES

[1] B. D. Anderson, N. K. Bose, and E. I. Jury, "Output feedback stabilization and related problems—Solution via decision methods," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 53–66, 1975.

[2] S. P. Bhattacharyya, *Robust Stabilization Against Structured Perturbations*, vol. 99 of Lecture Notes in Control and Information Sciences. New York: Springer-Verlag, 1987.

[3] V. Blondel and J. N. Tsitsiklis, "NP-hardness of some linear control design problems," *European J. Contr.*, vol. 1, 1995.

[4] S. Boyd and L. El Ghaoui, "Method of centers for minimizing generalized eigenvalues," *Linear Algebra Appl.*, Special Issue on Numerical Linear Algebra Methods Contr., Signals and Syst., vol. 188, July 1993.

[5] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of Studies in Applied Mathematics. Philadelphia, PA: SIAM, 1994.

[6] J. David, "Algorithms for analysis and design of robust controllers," Ph.D. dissertation, EE Dept., K. U. Leuven, ESAT, EE Dept., Leuven, Belgium, Mar. 1994.

[7] H. J. Davison and R. Chatterjee, "A note on pole assignment in linear systems with incomplete state-feedback," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 98–99, Feb. 1971.

[8] J. Doyle, A. Packard, and K. Zhou, "Review of LFT's, LMI's and μ," in *Proc. IEEE Conf. Decision Contr.*, Brighton, Dec. 1991, vol. 2, pp. 1227–1232.

[9] L. El Ghaoui, R. Nikoukhah, and F. Delebecque, *LMITOOL: A Front-End for LMI Optimization*, User's Guide, Feb. 1995; ftp to ftp.ensta.fr, under /pub/elghaoui/lmitool.

[10] J. C. Geromel, C. C. de Souza, and R. E. Skelton, "LMI numerical solution for output feedback stabilization," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1995, pp. 40–44.

[11] L. El Ghaoui, F. Oustry, and M. Ait Rami, "An LMI-based linearization algorithm for static output-feedback and related problems," *Lab. Appl. Math., Ecole Nat. Sup. Techniques Avancées*, Paris, France, Tech. Rep., Nov. 1995.

[12] K. M. Grigoriadis and R. E. Skelton, "Low order control design for lmi problems using alternating projection methods," *Automatica*, June 1995.

[13] T. Iwasaki and R. E. Skelton, "All controllers for the general $\mathbf{H}_\infty$ control problem: LMI existence conditions and state space formulas," *Automatica*, vol. 30, no. 8, pp. 1307–1317, Aug. 1994.

[14] ——, "The XY-centering algorithm for the dual LMI problem: A new approach to fixed order control desgin," *Int. J. Contr.*, vol. 62, no. 6, pp. 1257–1272, 1995.

[15] H. Kimura, "Pole placement by gain output feedback," *IEEE Trans. Automat. Contr.*, vol. 20, pp. 509–516, 1975.

[16] O. L. Mangasarian and J. S. Pang, "The extended linear complementarity problem," *SIAM J. Matrix Anal. Appl.*, vol. 2, pp. 359–368, Jan. 1995.

[17] Yu. Nesterov and A. Nemirovsky, *Interior Point Polynomial Methods in Convex Programming: Theory and Applications*. Philadelphia, PA: SIAM, 1994.

[18] P. L. D. Peres, J. C. Geromel, and S. R. Souza, "$\mathbf{H}_\infty$ robust control by static output feedback," in *Proc. Amer. Contr. Conf.*, San Francisco, CA, June 1993, vol. 1, pp. 620–621.

[19] J. Rosenthal and X. A. Wang, "Output feedback pole placement with dynamic compensators," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 830–843, June 1996.

[20] M. A. Rotea and T. Iwasaki, "An alternative to the D-K iteration?," in *Proc. Amer. Contr. Conf.*, June 1994, pp. 53–57.

[21] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, "A unified approach to linear control design," to be published.

[22] V. L. Syrmos, C. Abdallah, and P. Dorato, "Static output feedback: A survey," in *Proc. IEEE Conf. Decision Contr.*, Dec. 1994, pp. 837–842.

[23] L. Vandenberghe and S. Boyd, **SP**, *Software for Semidefinite Programming, User's Guide*, Dec. 1994; ftp to isl.stanford.edu under /pub/boyd/semidef_prog.

[24] ——, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, Mar. 1996.

[25] X. A. Wang, "Grassmanian, central projection, and output feedback pole assignment of linear systems," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 786–794, June 1996.

[26] Y. Ye, "A fully polynomial-time algorithm for computing a stationary point for the general linear complementarity problem," *Math. Oper. Res.*, vol. 18, pp. 334–345, 1993.

[27] Y. Zhang, "On the convergence of a class of infeasible interior-point methods for the horizontal linear complementarity problem," *SIAM J. Optimization*, vol. 4, no. 1, pp. 208–227, Feb. 1994.

A Study of the Gap Between the Structured Singular Value and Its Convex Upper Bound for Low-Rank Matrices

Roy Smith

***Abstract*—The size of the smallest structured destabilizing perturbation for a linear time-invariant system can be calculated via the structured singular value (μ). The function μ can be bounded above by the solution of a convex optimization problem, and in general there is a gap between μ and the convex bound. This paper gives an alternative characterization of μ which is used to study this gap for low-rank matrices. The low-rank characterization provides an easily computed bound which can potentially be significantly better than the standard convex bound. This is used to find new examples with larger gaps than previously known.**

***Index Terms*—Robust control, structured singular value.**

I. Introduction

The structured singular value—commonly denoted by μ—was introduced by Doyle [1] as a means of analyzing the stability of systems under structured perturbations. It gives the size of the smallest destabilizing perturbation and is therefore a measure of the robustness of a system.

In application, upper and lower bounds for μ are calculated. The upper bound is formulated as a convex optimization over scalings for a singular value problem [1], [2]. This has an equivalent linear matrix inequality (LMI) formulation (see for example, the work of Beck and Doyle [3] and Boyd *et al*. [4]). Lower bounds are typically calculated via a power algorithm [5].

This paper provides a reformulation of μ and uses it to study a class of problems where there is a known gap between the standard convex upper bound and μ. The extent of this gap, and concomitant conservativeness of the easily calculated convex bound, has remained an open problem. In the case where some of the perturbations are constrained to be real-valued, there are examples of the relative gap being arbitrarily large [6]. Furthermore, the decision problem "is $\mu(M) < 1$?" is known to be NP-hard, even for the purely complex μ problem [7]. In light of this result we cannot expect to provide a definitive quantification of the gap in all cases. The results given here do provide a bound which, in certain cases, gives a tighter answer to the above decision problem than the convex bound. The quantification of the possible extent of this gap has also been studied by Megretski [8].

II. Notation and Technical Background

Given a complex valued matrix, M, M^T is the transpose of M and $\sigma_{\max}(M)$ denotes its maximum singular value. The i, j element of M is given by m_{ij}.

Consider block diagonal complex-valued matrices with the size of the blocks specified by a set of integers, $k_1, \cdots, k_n$. The set of all such block diagonal matrices will be denoted by $\boldsymbol{\Delta}$. More formally

$$\boldsymbol{\Delta} = \{\mathrm{diag}\,(\Delta_1, \cdots, \Delta_n) | \Delta_i \in \mathcal{C}^{k_i \times k_i}\}.$$

More general block structures are possible, and the reader is referred to Packard and Doyle [5] for further discussion.

For an illustration of the distinction between the structured singular value and its upper bound, it is sufficient to consider scalar valued

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