

A Cone of Inhomogeneous Second-Order Polynomials*

Robert Erdahl

Department of Mathematics and Statistics, Queen's University,
 Kingston, Ontario K7L 3N6, Canada
 ErdahlR@QUCDN.Queens.U.CA

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Abstract. Let \mathcal{P}^n be the cone of quadratic functions

$$F1. f = f_0 + \sum f_i x_i + \sum f_{ij} x_i x_j, f_{ij} = f_{ji},$$

on \mathbb{R}^n that satisfy the additional condition

$$F2. f(z) \geq 0, z \in \mathbb{Z}^n,$$

where \mathbb{Z} denotes the integers. The coefficients and variables are assumed to be real and $1 \leq i, j \leq n$. The extent to which information on the convex structure of \mathcal{P}^n can be used to determine the integer solutions of the equation $f = 0$ for $f \in \mathcal{P}^n$ has been studied.

The **root figure** of $f \in \mathcal{P}^n$, denoted R_f , is the set of n -vectors $z \in \mathbb{Z}^n$ satisfying the equation $f(z) = 0$. The root figures relate to the convex structure of \mathcal{P}^n in an obvious way: if R is a root figure, then

$$F_R = \{q \in \mathcal{P}^n \mid R_q = R\}$$

is a relatively open face with closure $\{q \in \mathcal{P}^n \mid q(r) = 0, r \in R\}$. However, such formulas do not hold for all the relatively open and closed faces; this relates to some subtleties in the structure of \mathcal{P}^n .

Enumeration of the possible root figures is the central problem in the theory of \mathcal{P}^n . The group $G(\mathbb{Z}^n)$, of affine transformations on \mathbb{R}^n leaving \mathbb{Z}^n invariant, is the full symmetry group of \mathcal{P}^n . Classification of the root figures up to $G(\mathbb{Z}^n)$ -equivalence

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provides a complete solution to this problem, and this paper is concerned with some basic questions relating to such a classification.

The ideas in this study closely relate to the theory of L -polytopes in lattices as developed by Voronoi [V1], [V2], Delone [De1], [De2], and Ryshkov [RB]; L -polytopes, along with their circumscribing empty spheres (often referred to as holes in lattices), play a central role in the study of optimal lattice coverings of space. In addition, the theory of \mathcal{P}^n makes contact with: (1) the theory of finite metric spaces, in particular hypermetric spaces [DGL1], [DGL2], and (2) a significant problem in quantum mechanical many-body theory related to the theory of reduced density matrices [E2]–[E4].

1. Introduction

Suppose that $E \subset \mathbb{R}^n$ is an ellipsoid. Then E will be called **empty** if its interior contains no integer points. If, besides being empty, E contains sufficiently many elements from \mathbb{Z}^n so that these elements uniquely determine this quadratic surface, E will be called **perfect**.

An empty ellipsoid determines an entire ray of \mathcal{P}^n , for let f be some quadratic function such that the interior of E is given by $\{x \in \mathbb{R}^n \mid f(x) < 0\}$. Then f has nonnegative values on \mathbb{Z}^n and the ray generated by taking all positive scalar multiples of f belongs to \mathcal{P}^n ; this ray is uniquely determined by E . If E is perfect, then the root figure $E \cap \mathbb{Z}^n$ must contain at least $N - 1$ elements from \mathbb{Z}^n ; $N = \binom{n + 2}{2}$ is the dimension of the linear space of quadratic functions on \mathbb{R}^n . In the special case where E is perfect, f will be called a **perfect element** of \mathcal{P}^n .

More generally, an element $f \in \mathcal{P}^n$ will be called **perfect** if the subspace of quadratic functions given by

$$Q(R_f) = \{q \text{ quadratic} \mid q(r) = 0, r \in R_f\}$$

is one-dimensional. Elements which are not perfect will be referred to as **imperfect**.

The terms empty ellipsoid and perfect ellipsoid will often be used in the context of more general lattices $\Gamma \subset \mathbb{R}^n$, where $\dim \Gamma = 1, 2, \dots$. These notions are generalized by substituting $\text{aff}(\Gamma)$ (the affine hull of Γ) for \mathbb{R}^n and Γ for \mathbb{Z}^n in the original definitions. For example, an empty ellipsoid in Γ is an ellipsoid in $\text{aff}(\Gamma)$ containing no elements of Γ in its interior. Other notions, such as perfect element, can be generalized in a similar way.

Proposition 1.1. *$f \in \mathcal{P}^n$ is perfect if and only if R_f is a maximal, proper root figure.*

A **proper** root figure is neither empty nor equal to \mathbb{Z}^n . The root figures are partially ordered by containment and **maximal** refers to this ordering. It will be convenient to refer to maximal, proper root figures as perfect. Examples below show that perfect root figures can be either finite or infinite.

Proof. Assume that f is perfect. Then the linear space S_f spanned by

$$[1, r_1, \dots, r_n, r_1^2, 2r_1r_2, \dots, r_n^2], [r_1, \dots, r_n] \in R_f$$

has codimension one in the $\binom{n+2}{2}$ -dimensional space of coefficients. If $z \in \mathbb{Z}^n$ does not belong to R_f , then

$$s_z = [1, z_1, \dots, z_n, z_1^2, 2z_1z_2, \dots, z_n^2]$$

does not belong to this linear space, and the only vector orthogonal to S_f , s_z is the zero vector. That is, the only quadratic function q satisfying the equations $q(z) = 0$ and $q(r) = 0, r \in R_f$, is the zero function with improper root figure \mathbb{Z}^n . Therefore R_f is a maximal proper root figure.

That a maximal proper root figure is the root figure of a perfect element follows from Theorem 2.3. □

An element $f \in \mathcal{P}^n$ is said to be **extreme** if whenever $g, h \in \mathcal{P}^n$ are such that $f = g + h$, then both g, h are nonnegative scalar multiples of f .

Proposition 1.2. *The perfect elements of \mathcal{P}^n are extreme.*

Proof. Suppose that f is perfect and $f = g + h$ with $g, h \in \mathcal{P}^n$. By property F2, $R_f \subset R_g, R_h$. Since $\dim Q(R_f) = 1$, both g and h are scalar multiples of f . In fact, nonnegative scalar multiples since $g, h \in \mathcal{P}^n$. That is, f is extreme. □

There are perfect “ellipsoids” on the real line. An ellipsoid on \mathbb{R} amounts to a pair of points; an empty ellipsoid to a pair of points with no integer points in between. All perfect ellipsoids on \mathbb{R} consist of a pair of adjacent integer points $k, k + 1$; this follows from the fact that the linear space of quadratic functions on \mathbb{R} that are zero valued at $k, k + 1$ is one-dimensional.

Suppose that E is a perfect ellipsoid in \mathbb{Z}^n . Let $R = \mathbb{Z}^n \cap E$. In order that $Q(R) = \{q \text{ quadratic} \mid q(r) = 0, r \in R\}$ be one-dimensional, R must contain at least $\binom{n+2}{2} - 1$ elements. However, the number of elements of R cannot exceed 2^n by the following argument. The elements of \mathbb{Z}^n can be grouped into 2^n equivalence classes depending upon the oddness or evenness of their n components. Two distinct elements from the same equivalence class cannot belong to R since their average is integer and interior to E , thus violating the condition that E be empty. Since, for $n = 2, 3$,

$$2^n < \binom{n+2}{2} - 1$$

lattices having these dimensions cannot have perfect ellipsoids.

In fact, there are no perfect ellipsoids in \mathbb{R}^n when $2 \leq n \leq 5$ (Theorem 5.1). However, for $n = 6, 7$ there are examples of perfect ellipsoids (Theorem 5.3) which relate to the “deep holes” in the root lattices E_6 and E_7 . The conditions for perfection are severe and perfect ellipsoids occur infrequently in lattices. *It is this infrequent appearance of perfect ellipsoids that adds regularity, making \mathcal{P}^n amenable to investigation.*

Consider the function

$$f(x) = (a_1x_1 + \cdots + a_nx_n + k)(a_1x_1 + \cdots + a_nx_n + k + 1),$$

where $a_1, a_2, \dots, a_n, k \in \mathbb{Z}$ and $\gcd\{a_1, a_2, \dots, a_n\} = 1$. Since the coefficients are integer, the surface $f = 0$ is a pair of parallel hyperplanes with no integer points lying in between. That is, this surface is a degenerate empty ellipsoid and $f \in \mathcal{P}^n$. Because of the arithmetic condition on the coefficients a_1, a_2, \dots, a_n , the root figure consists of a pair of parallel sublattices of codimension one. It is an easy exercise to show that R_f is a maximal proper root figure. By Proposition 1.1, f is perfect. (In Section 2 the perfection of f is established by a different argument involving the main Theorem 2.1.)

There are other classes of extreme elements besides the perfect elements. For example, by Theorem 2.1 the function

$$f(x) = (\sqrt{2}x_1 + x_2 + \cdots + x_n + \sqrt{3})^2$$

is extreme. Since $1, \sqrt{2}, \sqrt{3}$ are linearly independent over the integers, the root figure R_f is empty.

For each $f \in \mathcal{P}^n$, let φ_f be the corresponding form. That is,

$$\varphi_f = \sum f_{ij}x_ix_j.$$

Since f satisfies condition F2, the form φ_f satisfies the condition $\varphi_f(z) \geq 0$ for all $z \in \mathbb{Z}^n$.

Proposition 1.3. φ_f is positive semidefinite.

Assume that φ_f is positive definite and that R_f is finite and nondegenerate. (A root figure R is **nondegenerate** if the affine hull of R is equal to \mathbb{R}^n .) Then f has a unique minimum at, say, $c \in \mathbb{R}^n$, and by completing the square can be written

$$f(x) = \varphi_f(x - c) - (\varphi_f(c) - f_0).$$

Since R_f was assumed nondegenerate $0 < \varphi_f(c) - f_0 = \rho^2$ and the surface $f = 0$ is an empty ellipsoid in \mathbb{R}^n . However, if φ_f is considered to be the metrical form

for a lattice another interpretation is possible: the surface $f = 0$ is an **empty sphere** of radius ρ in a lattice Γ_f determined by φ_f (for an expanded discussion of empty spheres see [ER]). By changing the metric, the ellipsoid becomes a sphere and the vectors $z \in \mathbb{Z}^n$ become coordinate vectors for elements of a lattice Γ_f with the basis defined as follows. Write the coefficient matrix for φ_f as

$$F = B'B,$$

where B is an $n \times n$ matrix; then take the columns of B as the basis for Γ_f (the basis is determined by F only up to orthogonal equivalence). The matrix F is the Gram matrix for the basis B , and c is the coordinate vector for the center of the empty sphere referred to this basis.

At the 1924 International Congress of Mathematicians in Toronto, Delone gave an elegant definition for **L-polytope** [De1]: *a polytope with vertices belonging to some lattice Γ (and having the same dimension as Γ) is called an L-polytope in Γ if it can be circumscribed by an empty sphere*. Accordingly, R_f gives the coordinates of the vertices of an L-polytope in the lattice Γ_f with basis B . Voronoi introduced the notion of L-polytope in the second of his two famous mémoires “Nouvelles applications des paramètres continus à la théorie des formes quadratiques” [V1], [V2]; these two papers formed the basis for later work on the theory of optimal lattice coverings. In their book *Sphere Packings, Lattices and Groups* Conway and Sloane refer to the empty spheres circumscribing L-polytopes as **holes** in lattices [CS].

All of the L-polytopes in lattices of dimension $n = 1, 2, \dots, 5$ can be studied uniformly and efficiently with **dual systems of integer vectors**. For $n \geq 6$ the family of L-polytopes that can be studied is large, but does not exhaust the entire set of L-polytopes. Dual systems were recently introduced in a series of three papers [RE3]–[RE5], the basic ideas and definitions being motivated by earlier work on the structure of \mathcal{P}^n [E1]. Details on how \mathcal{P}^n relates to the method of dual systems are given in Section 6. This discussion is in the context of the more general question of how root figures and therefore \mathcal{P}^n relate to the theory of L-polytopes.

All of the L-polytopes in lattices of dimension $n \leq 4$ have been described earlier using other methods [ER], but these polytopes can be studied more efficiently using dual systems [RE5].

There are two other applications of \mathcal{P}^n which are worth mentioning. The theory of finite metric spaces and \mathcal{P}^n have recently been linked. In two interesting papers on hypermetrics and L-polytopes, Deza *et al.* establish a one-to-one correspondence between hypermetric spaces and L-polytopes [DGL1], [DGL2].

The study of \mathcal{P}^n was originally taken up to shed light on a fundamental problem in quantum mechanical many-body theory. In his paper entitled “The structure of fermion density matrices,” Coleman initiated the study of the mathematical properties of reduced density matrices [Col] (see also [Y]); the cone \mathcal{P}^n was introduced to study some of the discrete geometric aspects of this theory [E1]. Additional details on how this physical theory relates to the cone \mathcal{P}^n can be found in the literature [E2]–[E4] (see also [Da] and [DM]).

2. The Extreme Elements

The main result on \mathcal{P}^n is (a preliminary version can be found on p. 62 of [E1]):

Structure Theorem 2.1. *An element $f \in \mathcal{P}^n$ is extreme if and only if it satisfies one of the three mutually exclusive conditions:*

- E1. f is a positive constant.
- E2. $f = (\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + \eta)^2$ where $\alpha = [\alpha_1, \dots, \alpha_n]$ is not proportional to an integer vector.
- E3. f is perfect.

If f is perfect its root figure is necessarily of the form

$$R_f = R_0 + \Gamma,$$

where

- R1. R_0 is the collection of lattice points lying on a perfect ellipsoid in some lattice $\Gamma_0 \subset \mathbb{Z}^n$; $1 \leq \dim \Gamma_0 \leq n$.
- R2. Γ is a second sublattice of \mathbb{Z}^n which together with Γ_0 satisfy the conditions $\mathbb{Z}^n = \Gamma_0 + \Gamma$, $(\Gamma_0 - \Gamma_0) \cap \Gamma = 0$.

Moreover, any subset of \mathbb{Z}^n of the form $R_0 + \Gamma$, where R_0, Γ satisfy conditions R1 and R2, is the root figure of a perfect element of \mathcal{P}^n .

The proof is left until Section 4. Condition R2 ensures that elements $z \in \mathbb{Z}^n$ can be written uniquely as $z = \gamma_0 + \gamma$ with $\gamma_0 \in \Gamma_0, \gamma \in \Gamma$.

If p is perfect, the surface $p = 0$ is the cartesian product of a perfect ellipsoid in Γ_0 with the subspace $V = \text{span } \Gamma$ having dimension 0, 1, 2, ...; an ellipsoid, cylinder or generalized cylinder, depending on the dimension of V . The "axis" V of this cylinder is also given by $\{x \in \mathbb{R}^n \mid \varphi_p(x) = 0\}$, where $\varphi_p = \sum p_{ij} x_i x_j$ is the associated quadratic form. That V has a basis consisting of integer vectors plays an important role in the following sections.

Consider again the function

$$p(x) = (a_1 x_1 + \dots + a_n x_n + k)(a_1 x_1 + \dots + a_n x_n + k + 1),$$

where $a_1, a_2, \dots, a_n, k \in \mathbb{Z}$ and $\text{gcd}\{a_1, a_2, \dots, a_n\} = 1$. Choose some transverse line L intersecting the parallel hyperplanes of the surface $p = 0$ at, say, $z_1, z_2 \in \mathbb{Z}^n$. Then

$$R_p = R_0 + \Gamma,$$

where $R_0 = \{z_1, z_2\}$ and $\Gamma = [a_1, a_2, \dots, a_n]^\perp \cap \mathbb{Z}^n$. However, R_0 is a pair of adjacent lattice points in the one-dimensional lattice $\Gamma_0 = L \cap \mathbb{Z}^n$ and therefore a perfect "ellipsoid" in Γ_0 (see the discussion in Section 1), but Γ_0, Γ clearly satisfy condition R2, so by Theorem 2.1 p is perfect.

The group $G(\mathbb{Z}^n)$ induces an action on \mathcal{P}^n leaving it invariant, and the perfect

elements of the form $(a_1x_1 + \dots + a_nx_n + k)(a_1x_1 + \dots + a_nx_n + k + 1)$ all lie in a single orbit under this action. This is a special case of:

Corollary 2.2. *For a given n , there is a one-to-one correspondence between the $G(\mathbb{Z}^n)$ -orbits of perfect elements and the $G(\mathbb{Z}^k)$ -equivalence classes of perfect ellipsoids in lattices of dimension $k = 1, 2, \dots, n$.*

There are several immediate consequence of Theorem 2.1.

Theorem 2.3. *Any root figure $R \subset \mathbb{Z}^n$ can be represented as*

$$R = P_1 \cap P_2 \cap \dots \cap P_k,$$

where P_i is a perfect root figure.

Proof. It is easy to show that \mathcal{P}^n has a base consisting of a compact convex set. Carathéodory's theorem [G, Theorem 1.7.1] can be applied and any element $f \in \mathcal{P}^n$ can be written as

$$f = e_1 + e_2 + \dots + e_m,$$

where e_i is extreme and $m \leq n$. Assuming that it is not empty the root figure then has the form

$$R_f = E_1 \cap E_2 \cap \dots \cap E_m,$$

where E_i is the root figure of e_i .

Suppose that e_1, e_2, \dots, e_p ($p \leq m$) are perfect. Let g be the sum of the remaining extreme elements, which by Theorem 2.1 have the form $(\alpha_1x_1 + \alpha_2x_2 + \dots + \alpha_nx_n + \eta)^2$. When g is zero the theorem obviously holds with $k = m = p$. When g is nonzero the surface $g = 0$ is a proper affine subspace of \mathbb{R}^n , and R_g is either a single point or a proper sublattice of \mathbb{Z}^n . The root figure R_g can then be represented as the set of integer solutions of a system of linear equations, each of which has the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n + k = 0,$$

where $a_1, a_2, \dots, a_n, k \in \mathbb{Z}$, $\gcd\{a_1, a_2, \dots, a_n\} = 1$. Since

$$(a_1x_1 + \dots + a_nx_n + k)^2 = \frac{1}{2}(a_1x_1 + \dots + a_nx_n + k)(a_1x_1 + \dots + a_nx_n + k + 1) + \frac{1}{2}(a_1x_1 + \dots + a_nx_n + k)(a_1x_1 + \dots + a_nx_n + k - 1),$$

R_g can also be represented as the intersection of a finite number, say q , of perfect root figures. This finishes the proof since, then, R_f can be written as the intersection of $k = p + q$ perfect root figures. □

The study of root figures quickly reduces to the study of nondegenerate root figures where $\dim R = n$. For suppose that $R \subset \mathbb{Z}^n$ is a root figure and $\dim R = k < n$. Write \mathbb{Z}^n as $\mathbb{Z}^k \otimes \mathbb{Z}^{n-k}$ and let $\gamma \in G(\mathbb{Z}^n)$ be such that $\gamma(R) \subset \mathbb{Z}^k \otimes 0^{n-k}$. Then $\gamma(R)$ can be interpreted as a nondegenerate root figure in \mathbb{Z}^k . The statement of Theorem 2.3 is easily strengthened when R is nondegenerate.

Representation Theorem 2.4. *Suppose that $f \in \mathcal{P}^n$ has nondegenerate root figure R_f . Then f can be represented as*

$$f = p_1 + p_2 + \dots + p_k,$$

where $p_i \in \mathcal{P}^n$ is perfect and $k \leq n$; R_f has the corresponding representation

$$R_f = P_1 + P_2 + \dots + P_k,$$

where P_i is the root figure of p_i .

Proof. The proof follows the same line as that for Theorem 2.3. The key point is that when f is written as a sum of extreme elements, each of these must be perfect. Among the extreme elements it is only the perfect elements that have nondegenerate root figures. The bound on k is a consequence of Caratheodory's theorem. □

Corollary 2.5 [ER, Theorem 2.1]. *An arbitrary nondegenerate root figure R has the form*

$$R_0 + \Gamma,$$

where

1. R_0 is the collection of lattice points lying on some empty ellipsoid in $\Gamma_0 \subset \mathbb{Z}^n$; $1 \leq \dim R_0 = \dim \Gamma_0 \leq n$.
2. $\Gamma \subset \mathbb{Z}^n$ is a second sublattice of \mathbb{Z}^n which together with Γ_0 form a direct sum decomposition of \mathbb{Z}^n ; $\mathbb{Z}^n = \Gamma_0 + \Gamma$, $(\Gamma_0 - \Gamma_0) \cap \Gamma = 0$.

R is finite if and only if $\dim \Gamma = 0$.

Proof. By Theorem 2.4 an arbitrary nondegenerate root figure R is the root figure of an element

$$p = p_1 + p_2 + \dots + p_k,$$

where p_i is perfect. Let φ_p be the form associated with p and let φ_i be the form associated with p_i ($i = 1, 2, \dots, k$). Only the case where φ_p is indefinite need be considered since the result trivially holds when φ_p is positive definite.

By Theorem 2.1 the surface $p_i = 0$ is a cylinder or generalized cylinder with "axis" $V_i = \{x \in \mathbb{R}^n \mid \varphi_i(x) = 0\}$, and V_i has a basis consisting of integer vectors. The

(nonzero) subspace

$$V = \{x \in \mathbb{R}^n \mid \varphi_p(x) = 0\}$$

is equal to $V_1 \cap V_2 \cap \dots \cap V_k$, so also has a basis consisting of integer vectors. Since p must satisfy condition F2 it follows immediately that $p(x + v) = p(x)$ for $v \in V$.

Let $\Gamma = \mathbb{Z}^n \cap V$ and introduce a second sublattice Γ_0 so that Γ_0, Γ satisfy statement 2 of the theorem. The root figure R can then be written as

$$R = R_0 + \Gamma,$$

where $R_0 = \Gamma_0 \cap R$.

Since R was assumed to be nondegenerate, $\dim R_0 = \dim \Gamma_0$. Since $V = \text{span } \Gamma$, the surface $p = 0$ restricted to $\text{aff } \Gamma_0$ is an empty ellipsoid E . Since $R_0 = E \cap \Gamma_0$, R_0 is as in statement 1 of the theorem and is finite. It follows that R is finite if and only if $\dim \Gamma = 0$. □

In the last step of the proof the following result was established.

Corollary 2.6. *Suppose that R_f is finite and nondegenerate. Then φ_f is positive definite.*

By Corollary 2.5 (and the discussion above) the study of root figures reduces to the study of finite nondegenerate root figures. By Corollary 2.6 and the comments in the introduction, this is equivalent to the problem of classifying L -polytopes in lattices.

Density Theorem 2.7. *The perfect elements of \mathcal{P}^n are dense in the extreme elements.*

Proof. By Theorem 2.1 all of the extreme elements which are not perfect have the form

$$(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + \eta)^2.$$

These can be approximated with arbitrary accuracy by perfect elements of the form

$$\alpha(a_1 x_1 + \dots + a_n x_n + k)(a_1 x_1 + \dots + a_n x_n + k + 1),$$

where $\alpha \in \mathbb{R}$, $\{a_1, a_2, \dots, a_n, k\} \subset \mathbb{Z}$, and $\text{gcd}\{a_1, a_2, \dots, a_n\} = 1$. □

3. The Decomposition Lemma

This section is about two linear subspaces of \mathbb{R}^n . Let

$$V = \{x \in \mathbb{R}^n \mid \varphi_f(x) = 0\},$$

where φ_f is the form associated with a typical element $f \in \mathcal{P}^n$. What turns out to be crucial is whether V has a basis of integer vectors; if it does, V and $\mathbb{Z}^n \cap V$ have a common basis. The second subspace is V_1 , the minimal linear subspace containing V that has a basis of integer vectors. The subspace V_1 is uniquely determined by V , and using orthogonal complements can be written as

$$V_1 = (\mathbb{Z}^n \cap V^\perp)^\perp.$$

Of course, if V has a basis consisting of integer vectors (or is 0) $V_1 = V$.

Decomposition Lemma 3.1. *Let $f \in \mathcal{P}^n$. Then f can be written as*

$$f = g + h,$$

where $g, h \in \mathcal{P}^n$ and have the following properties: $\{x \in \mathbb{R}^n \mid \varphi_g(x) = 0\} = V_1$; $h(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Some preliminary results are needed for the proof.

Proposition 3.2. *Let $f \in \mathcal{P}^n$. If $V \neq 0$, then f is constant on the translates of V by elements $x \in \mathbb{R}^n$: $f(x + v) = f(x)$, $v \in V$.*

Proof. Let B be the closed ball of radius $\rho = \sqrt{n}/2$ centered at the origin and let $M = \max_{x \in B} f(x)$.

For fixed $x \in \mathbb{R}^n$, the function $f(x + v) - f(x)$ is linear on V . Therefore, if the conclusion of the proposition does not hold there are elements $v \in V$, $x \in \mathbb{R}^n$ satisfying the condition

$$M + f(x + v) - f(x) < 0.$$

Assume that this is the case.

However, ρ is the covering radius for \mathbb{Z}^n , so an element $z \in \mathbb{Z}^n$ can be chosen so that $z - v \in B$ and $f(z - v) \leq M$. The contradiction is achieved using the fact that f must satisfy condition F2:

$$\begin{aligned} 0 \leq f(z) &= f(z - v + v) = f(z - v) + f(x + v) - f(x) \\ &\leq M + f(x + v) - f(x) < 0. \end{aligned}$$

(For the second equality the equation $\varphi_f(v) = 0$ has been used.) □

Proposition 3.3. *Let $f \in \mathcal{P}^n$. Then*

$$M = \{m \in \mathbb{R}^n \mid f(m) \leq f(m + v), v \in V_1\}$$

is an affine subspace of \mathbb{R}^n satisfying the conditions $\mathbb{R}^n = M + V_1$ and $(M - M) \cap V_1 = V$.

Proof. Let $x \in \mathbb{R}^n$, $v \in V_1$. By Taylor's formula $f(m + tv) = f(m) + t(\nabla f(m))'v + t^2\varphi_f(v)$ and $m \in M$ if and only if $\nabla f(m) \perp V_1$. Being the solution set of a system of linear equations, M is an affine subspace of \mathbb{R}^n .

For $x \in \mathbb{R}^n$ let $V_1(x)$ be the translate, $x + V_1$. By Proposition 3.2, the restriction of f to $V_1(x)$ has a minimizing subset $M(x)$ so $V_1(x) = M(x) + V_1$. By the definition of M , $M(x) \subset M$ so $V_1(x) \subset M + V_1$. Since the union of all the translates $V_1(x)$ covers \mathbb{R}^n , $\mathbb{R}^n = M + V_1$.

By Proposition 3.2, $M(x)$ is a translate of V so $V = M(x) - M(x) \subset M - M$. Since $V \subset V_1$, $V \subset (M - M) \cap V_1$. Now choose $v \in (M - M) \cap V_1$ arbitrarily. If $m \in M$, $t \in \mathbb{R}$, then $m + tv \in M$ and the following two inequalities hold:

$$f(m) \leq f(m + tv),$$

$$f(m + tv) \leq f(m + tv - tv) = f(m).$$

Then $f(m + tv) = f(m)$ for all real t and $v \in V$. It follows that $(M - M) \cap V_1 = V$, which completes the proof of the proposition. □

Proposition 3.4. *Let $f \in \mathcal{P}^n$. Then f is nonnegative on the translates of V_1 by elements $z \in \mathbb{Z}^n$: $f(z + v) \geq 0$, $v \in V_1$.*

To prove this statement the following result is needed.

Lemma. *Suppose that V is properly contained in V_1 . Let*

$$K(\varepsilon) = \{v \in V_1 | d(v, V) \leq \varepsilon\},$$

an infinite, closed convex set in V_1 . Then translates of $K(\varepsilon)$ by arbitrary elements $v \in V_1$ contain integer elements; $\{v + K(\varepsilon)\} \cap \mathbb{Z}^n \neq \emptyset$. Here $d(\cdot, \cdot)$ denotes the Euclidean distance.

Proof of the Lemma. Consider the lattice $\Gamma_1 = \mathbb{Z}^n \cap V_1$. For $\beta > 0$ the convex set $\beta K(\varepsilon) \subset V_1$ is symmetric about 0 and has infinite volume in V_1 . Therefore, by Minkowski's celebrated Convex Body Theorem [GL, p. 40], the intersection $\Gamma_1 \cap \beta K(\varepsilon)$ contains at least one nonzero element. The essential step of the proof is to show that $\dim \Gamma_1 \cap \beta K(\varepsilon) = \dim V_1$. For if this is true, an arbitrary element $v \in V_1$ can be expanded in a basis, $\{u_1, u_2, \dots, u_k\}$ for V_1 , chosen from $\Gamma_1 \cap \beta K(\varepsilon)$:

$$v = \xi_1 u_1 + \xi_2 u_2 + \dots + \xi_k u_k.$$

If integers z_1, z_2, \dots, z_k are chosen so that $|z_i - \xi_i| \leq \frac{1}{2}$, then

$$\gamma = z_1 u_1 + z_2 u_2 + \dots + z_k u_k \in \Gamma_1 \subset \mathbb{Z}^n$$

and

$$\gamma - v = (z_1 - \xi_1)u_1 + (z_2 - \xi_2)u_2 + \dots + (z_k - \xi_k)u_k \in \frac{k}{2} \beta K(\varepsilon).$$

With β now set equal to $2/k$, $\gamma \in v + K(\varepsilon)$ and $(v + K(\varepsilon)) \cap \mathbb{Z}^n \neq \emptyset$.

Assume that $\dim \Gamma_1 \cap \beta K(\epsilon) < \dim V_1$. Then V_* , the subspace generated by the elements of $\Gamma_1 \cap \beta K(\epsilon) \subset \mathbb{Z}^n$, is properly contained in V_1 . Since V_1 is the uniquely determined minimal subspace containing V which is spanned by its integer elements, V cannot be a subset of V_* . Choose $a \in V, a \notin V_*$ and consider the infinite cylinder $C(a, \delta) = \{x \in V_1 \mid d(x, l(a)) \leq \delta\}$, where $l(a)$ is the line through a and 0. Assume that $\delta > 0$ is chosen small enough so that $C(a, \delta)$ is contained in $\beta K(\epsilon)$ but contains no elements of $V_* \cap \mathbb{Z}^n$. A second application of Minkowski's theorem shows that $C(a, \delta)$ must contain at least one nonzero element of Γ_1 . Then V_* does not contain $\Gamma_1 \cap K(\epsilon)$, and we have arrived at the contradiction. It follows that $\dim \Gamma_1 \cap \beta K(\epsilon) = \dim V_1$. \square

Proof of Proposition 3.4. If $V = V_1$, the result is easily established by applying Proposition 3.2. If V is properly contained in V_1 , the lemma is applied showing that $\{v + K(\epsilon)\} \cap \mathbb{Z}^n \neq \emptyset$ and hence that $\{z + v + K(\epsilon)\} \cap \mathbb{Z}^n \neq \emptyset$, when $z \in \mathbb{Z}^n, v \in V_1$. Choose a sequence $\epsilon_i > 0, i = 1, 2, \dots$, converging to zero, and select $z_i \in z + v + K(\epsilon_i)$. By Proposition 3.2 and the continuity of $f, \lim_{i \rightarrow \infty} f(z_i) = f(z + v)$. Since $f(z_i) \geq 0$ it follows that $f(z + v) \geq 0$ and the proof is complete. \square

Proof of the Decomposition Lemma. If $V = V_1$ the Decomposition Lemma clearly holds: choose $g = f$ and $h = 0$. The other possibility, that V is properly contained in V_1 , requires more work.

Because of the properties of M quoted in Proposition 3.3, there are many ways to choose an affine subspace $N \subset M$ so that N, V_1 satisfy the conditions $\mathbb{R}^n = N + V_1$ and $(N - N) \cap V_1 = 0$. Assuming such a choice has been made, elements $x \in \mathbb{R}^n$ can be written uniquely in the form

$$x = n + v, \quad n \in N, \quad v \in V_1,$$

and functions P_N, P_V on \mathbb{R}^n can be defined so that $P_N(x) = n, P_V(x) = v$. It is easy to show that P_N, P_V are affine.

Consider the quadratic functions defined by

$$\begin{aligned} g(x) &= f(P_N(x)), \\ h(x) &= \varphi_f(P_V(x)). \end{aligned}$$

In the proof of Proposition 3.3 it was established that $m \in M$ if and only if $\nabla f(m) \perp V_1$. Since $N \subset M$,

$$\begin{aligned} f(n + v) &= f(n) + \nabla f(n)'v + \varphi_f(v) \\ &= f(n) + \varphi_f(v) \\ &= g(x) + h(x). \end{aligned}$$

Being positive on $\mathbb{R}^n, h \in \mathcal{P}^n$. That g belongs to this cone can be seen as follows. Let $z \in \mathbb{Z}^n$ be chosen arbitrarily. Then $g(z) = f(P_N(z)) = f(z - P_V(z)) \geq 0$, the last inequality following from Proposition 3.4.

It is easy to see from the definition of g that

$$V_1 = \{x \in \mathbb{R}^n \mid \varphi_g(x) = 0\}.$$

This completes the proof of the Decomposition Lemma. □

4. Proof of the Structure Theorem

Proceedings are initiated by noting that the Decomposition Lemma implies that an extreme element $f \in \mathcal{P}^n$ must satisfy at least one of the following two conditions:

- C1. $f(x) \geq 0$ for $x \in \mathcal{P}^n$,
- C2. $V = V_1$.

We first consider the extreme elements of \mathcal{P}^n that satisfy condition C1. For this it is convenient to introduce the subcone $\mathcal{P}_0^n \subset \mathcal{P}^n$ of nonnegative quadratic functions on \mathbb{R}^n . The extreme elements of \mathcal{P}_0^n are easily shown to be the positive constants and functions of the form

$$(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \eta)^2,$$

where $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_n] \neq 0$. We must determine which from this list are extreme in \mathcal{P}^n .

E1. Suppose that f is a positive constant; that is, f satisfies condition E1 in the statement of Theorem 2.1. If $f = p + q$, with $p, q \in \mathcal{P}^n$, then $\varphi_p = \varphi_q = 0$ (see Proposition 1.3). By Proposition 3.2, p, q are nonnegative constant functions and therefore nonnegative scalar multiples of f . Therefore f is extreme.

It is worth noting that the positive constants also satisfy C2. It will turn out that these are the only class of extreme elements satisfying both C1 and C2.

E2. Consider $f(x) = (\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \eta)^2 = (\langle \alpha, x \rangle + \eta)^2$, where $\alpha = [\alpha_1, \dots, \alpha_n]$ is a nonzero integer vector with $\gcd\{\alpha_1, \dots, \alpha_n\} = 1$. Choose $k \in \mathbb{Z}$ so that $k - \frac{1}{2} \leq \eta \leq k + \frac{1}{2}$. Then a short computation shows that the real constants a, b, c , defined by the equality

$$f(x) = a(\langle \alpha, x \rangle + k)(\langle \alpha, x \rangle + k - 1) + b(\langle \alpha, x \rangle + k)(\langle \alpha, x \rangle + k + 1) + c,$$

are nonnegative. Therefore f is not extreme. Since scaling by a positive number does not change this situation, $(\alpha_1 x_1 + \cdots + \alpha_n x_n + \eta)^2$ is not extreme if α is proportional to a nonzero integer vector.

Now consider the case where α is *not* proportional to an integer vector; that is, f satisfies condition E2 in the statement of Theorem 2.1. Then $V_1 = (\mathbb{Z}^n \cap V^\perp)^\perp = (\mathbb{Z}^n \cap \alpha)^\perp = \mathbb{R}^n$. Assume that $f = p + q$ with $p, q \in \mathcal{P}^n$. Let V_p, V_q be the subspaces where φ_p, φ_q take the value zero. Since $\varphi_f = \varphi_p + \varphi_q$, $V \subset V_p, V_q$ and $(\mathbb{Z}^n \cap V_p^\perp)^\perp = (\mathbb{Z}^n \cap V^\perp)^\perp = \mathbb{R}^n$. An application of Proposition 3.4 shows that p, q are nonnegative and therefore elements of \mathcal{P}_0^n . Since f is extreme in \mathcal{P}_0^n , p and q

are nonnegative scalar multiples of f . It follows that $f = (\alpha_1 x_1 + \dots + \alpha_n x_n + \eta)^2$ is extreme in \mathcal{P}^n .

E3. To complete the exhaustive description of extreme elements we consider the third mutually exclusive class. These are the extreme elements satisfying condition C2 but not C1, and turn out to be the perfect elements. The argument proceeds by establishing the truth of three separate statements which yield two alternative characterizations of the perfect elements.

A. *If f is extreme and satisfies C2 but not C1, then*

$$R_f = R_0 + \Gamma,$$

where R_0, Γ satisfy conditions R1 and R2 of the theorem.

Let $\Gamma = \mathbb{Z}^n \cap V$ and introduce a second sublattice Γ_0 so that $\mathbb{Z}^n = \Gamma_0 + \Gamma$, $(\Gamma_0 - \Gamma_0) \cap \Gamma = 0$.

Let E be the restriction of the surface $f = 0$ to $V_0 = \text{aff}(\Gamma_0)$. Since f satisfies condition C2, $V = \text{span } \Gamma$, from which is drawn the following two conclusions. By Proposition 3.2 and the fact that f does not satisfy C1, there is a region of V_0 where f assumes negative values. Secondly, the surface E bounds this region and is necessarily an ellipsoid. By condition F2, E is an empty ellipsoid in Γ_0 . By again using Proposition 3.2 it follows that

$$R_f = R_0 + \Gamma,$$

where $R_0 = E \cap \Gamma_0$ is the collection of lattice points lying on the empty ellipsoid E in Γ_0 .

To complete the proof of statement A it must be shown that E is perfect in Γ_0 . However, if E were imperfect there would be a second quadratic surface $S \subset \text{aff } \Gamma_0$, distinct from E , such that $R_0 \subset \Gamma_0 \cap S$. Let g be a quadratic function on \mathbb{R}^n with the property that the surface $g = 0$ coincides with $S + V$. Then, with $\varepsilon > 0$ sufficiently small, the two surfaces $f + \varepsilon g = 0, f - \varepsilon g = 0$ are generalized cylinders bounding regions of \mathbb{R}^n void of integer elements. (These surfaces are slightly disturbed copies of the original surface $f = 0$ which can be written as $E + V$.) Then $f + \varepsilon g, f - \varepsilon g \in \mathcal{P}^n$. Since

$$f = \frac{1}{2}(f + \varepsilon g) + \frac{1}{2}(f - \varepsilon g)$$

it follows that f is not extreme. Since this contradicts one of our hypotheses, E must be perfect. □

B. *If $R = R_0 + \Gamma$ is a subset of \mathbb{Z}^n such that R_0, Γ satisfy conditions R1 and R2 of the theorem, then R is the root figure of a perfect element of \mathcal{P}^n .*

Let E be the perfect ellipsoid in Γ_0 containing R_0 and let $V = \text{span } \Gamma$. Choose a nonzero quadratic function f which has negative values only on the interior of

the generalized cylinder $E + V$. Since this region is void of integer elements, $f \in \mathcal{P}^n$. For statement B to hold, f must be perfect.

Let $q \in Q(R)$, the linear space of quadratic functions which are zero valued on R . For arbitrary $r \in R_0$, $\gamma \in \Gamma$, the function q is zero valued on the line through r , $r + \gamma$. Since E is perfect in Γ_0 , R_0 has full dimension in Γ_0 . Therefore, for arbitrary $x \in \text{aff}(\Gamma_0)$, $v \in V$, q is constant valued on the line through x , $x + v$. Therefore the dimension of $Q(R)$ is equal to that of the linear space of quadratic functions on $\text{aff}(\Gamma_0)$ which are zero valued on R_0 . Since by hypothesis this linear space is one-dimensional, f is perfect. \square

C. *The perfect elements of \mathcal{P}^n are extreme and satisfy condition C2, but not C1.*

By Proposition 1.2 the perfect elements of \mathcal{P}^n are extreme. By the Decomposition Lemma we infer that they must satisfy one of the conditions C1 or C2. We have seen that the extreme elements satisfying C1 also satisfy either condition E1 or E2. However, these classes have degenerate root figures and are therefore imperfect. Thus perfect elements satisfy condition C2 but not C1. \square

Statements A, B, and C can be used to verify the assertions of the theorem regarding the perfect elements. This completes the proof of Theorem 2.1. \square

Suppose e is some element of a cone $C \subset \mathbb{R}^n$ with vertex at the origin. Then e is said to be **exposed** if there is a supporting hyperplane H such that $H \cap C$ contains only the ray generated by e . Exposed elements are necessarily extreme, but extreme elements need not be exposed. By working in the space of coefficients, additional information can be obtained on the extreme elements of \mathcal{P}^n : *The extreme elements belonging to class 1 and 3 of Theorem 2.1 are exposed. The extreme elements belonging to class 2 are exposed if and only if the corresponding root figure is not empty.* (Details are given in [E1].)

5. Perfect Ellipsoids

Theorem 5.1. *There are no perfect ellipsoids in lattices of dimension 2, 3, 4, or 5.*

Since the proof is long and uses ideas considerably different than those developed here it will be published separately. (The result for the case $n = 4$ is implicit in [ER]; see also Theorem 5 of [RE2].) As an immediate consequence, it follows that (see Theorem 2.1 and the discussion following):

Corollary 5.2. *When $n \leq 5$ all perfect elements are of the form*

$$f(x) = \alpha(a_1x_1 + \cdots + a_nx_n + k)(a_1x_1 + \cdots + a_nx_n + k + 1),$$

where $\alpha \in \mathbb{R}$ is positive, $a_1, \dots, a_n, k \in \mathbb{Z}$, and $\text{gcd}\{a_1, \dots, a_n\} = 1$.

The remainder of this section is a report on initial efforts to study perfect ellipsoids for the cases $n \geq 6$. The collection of integer points lying on a perfect ellipsoid is necessarily finite and nondegenerate, but by Corollary 2.6, and the comments immediately following it, such root figures can be studied by examining L -polytopes in lattices. We have investigated the indecomposable root lattices, searching for L -polytopes contained in **perfect holes**, that is, the L -polytopes circumscribed by empty spheres which are perfect.

The **deep holes** in a lattice are those with maximum radius, all others are called **shallow**. It is the deep holes that are of greatest interest, the radii being equal to the covering radius of the lattice. (The center of a deep hole is a point at maximum distance from the lattice.)

Theorem 5.3. *With the exception of the deep holes in A_1, E_6, E_7 all of the holes in the indecomposable root lattices A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7, E_8 are imperfect. The perfect holes in E_6 and E_7 contain L -polytopes with 27 and 56 vertices, respectively; these are the Gosset polytopes 2_{21} and 3_{21} .*

The notation $2_{21}, 3_{21}$ for these Gosset polytopes was introduced by Coxeter; they are described on pp. 202–204 of *Regular Polytopes* [Cox2] (see also pp. 21, 26 and 33 of [Cox4]).

In their most recent paper [DGL2], *et al.* report perfect ellipsoids in dimensions 15 and 16 related to the Barnes–Wall lattice and dimensions 22 and 23 related to the Leech lattice.

We will need a description of the holes in the indecomposable root lattices A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7, E_8 . The development below is very close to that reported in [TD]; an alternate approach to obtain many of these results can be found in Chapter 21 of [CS]. For additional details on root lattices see Chapter 4 of [CS].

For any lattice Γ , the holes can be classified using the L -polytopes they contain, holes being considered different if they contain isometrically distinct species of L -polytopes. The collection of all L -polytopes having a vertex at the origin is called the **star of L -polytopes** at the origin. Each species of L -polytope is represented in this star, the number of representatives being a positive integral multiple of the number of vertices. This follows since each vertex of a given L -polytope can be translated to the origin yielding distinct representatives; the positive multiple exceeds one only when there are translationally inequivalent representatives of the same species.

The L -polytopes in the star at the origin fit together so that adjacent L -polytopes have a common facet. More generally, the collection of all L -polytopes form a tiling of space in which adjacent L -polytopes are glued together along common facets. These tilings are called **L -decompositions**.

The centers of the empty spheres circumscribing the L -polytopes in the star have the following interesting characterization: *they are the vertices of the Voronoi polytope V of 0*. It is easy to show that this description of the **Voronoi polytope** is consistent with the usual one, that V is the portion of space as close to the origin as to any other lattice point. Translating V by $\gamma \in \Gamma$ produces the Voronoi polytope

of the lattice point γ ; the tiling formed by all the Voronoi polytopes is dual to the L -decomposition.

If an integral lattice Γ , containing the origin, is generated by its vectors of squared length two it is called a **root lattice**. These minimal vectors are called **roots**, and the set of all roots Φ is called a **root system**. By the Cauchy-Schwarz inequality, roots α, β , when $\alpha \neq \pm\beta$, must satisfy the condition $\langle \alpha, \beta \rangle \in \{0, \pm 1\}$.

The root system Φ (and corresponding root lattice Γ) is **decomposable** if Φ can be written as an orthogonal direct sum, $\Phi = \Phi_1 \oplus \Phi_2$, of root systems Φ_1, Φ_2 . If Φ is not decomposable it is **indecomposable**.

We will need the following result on root lattices: *if Γ is a root lattice, then*

$$V = \{x \in \mathbb{R}^n \mid 0 \leq |\alpha - x|^2 - |x|^2, \alpha \in \Phi\}.$$

By replacing $\alpha \in \Phi$ by $\alpha \in \Gamma$ this converts to the usual formula for the Voronoi polytope at the origin for a general lattice, but involves an infinite system of inequalities. The simplification for root lattices follows almost directly from Coxeter's celebrated characterization of the fundamental region of an indecomposable reflection group as a simplex. (See the argument appearing on pp. 456-459 of [CS].)

Notice that since $|\alpha - x|^2 - |x|^2 = |\alpha|^2 - 2\langle x, \alpha \rangle = 2(1 - \langle x, \alpha \rangle)$, $x \in V$ if and only if $\langle x, \alpha \rangle \leq 1$ for all $\alpha \in \Phi$. Therefore V can be identified as the dual of the set of roots Φ , and borrowing notation from convexity theory we write $V = \Phi^0$.

Let Γ be a root lattice, and let R be the vertex set of an L -polytope in the star at the origin. Denote by Γ_R the lattice generated by the elements of R . Then $\Gamma_R \subseteq \Gamma$; if equality holds the L -polytope, or corresponding hole, will be called **generating**.

Proposition 5.4. *The edge vectors of $\text{conv } R$ are roots, and Γ_R is a root lattice. If Γ is indecomposable, then so is Γ_R .*

Proof. Let x be the vertex of V corresponding to the center of the empty sphere circumscribing R . Since $V = \Phi^0$, x is also dual to the facet

$$F = \{\alpha \in \Phi \mid \langle \alpha, x \rangle = 1\}$$

of Φ . It is easy to see that $\text{conv } R$ intersects $\text{conv } \Phi$ in a pyramid with the apex at the origin and the base equal to $\text{conv } F$. This shows that the edges of $\text{conv } R$ which emanate from the origin, are precisely the roots F .

Additional copies of $\text{conv } R$ in the star are generated by translating successively the nonzero vertices of R to the origin. Application of the above argument then shows that the edges emanating from any vertex, hence all edges, are roots. Since Γ_R can be generated by the edges of $\text{conv } R$ it is necessarily a root lattice.

We now consider the case where Γ is indecomposable. To establish that Γ_R is also indecomposable we argue by contradiction, assuming that Γ_R is decomposable and can be written $\Gamma_R = \Gamma_1 \oplus \Gamma_2$. There is then a corresponding decomposition for the facet $F = R \cap \Phi$: $F = F_1 \oplus F_2$. Since Γ was assumed to be indecomposable

there must be roots $\alpha \in F_1, \beta \in F_2, \gamma \in \Phi$ satisfying the conditions $\langle \alpha, \gamma \rangle, \langle \beta, \gamma \rangle = \pm 1$. By possibly replacing γ by one of the roots $-\gamma, \alpha - \gamma, \beta - \gamma$, the signs of the scalar products can be adjusted so that $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle = 1$. If δ is the root $\alpha + \beta - \gamma$ (the squared length of this vector is two), then

$$\frac{1}{2}(\alpha + \beta) = \frac{1}{2}(\gamma + \delta),$$

a formula showing that α, β do not generate an edge of $\text{conv } \Phi$. This is the contradiction that was sought: α, β do generate an edge of F . It follows that Γ_R must be indecomposable. □

The dual lattice Γ^* of a lattice $\Gamma \subset \mathbb{R}^n$ is given by

$$\Gamma^* = \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \gamma \rangle \in \mathbb{Z}, \gamma \in \Gamma\}.$$

Notice that if Γ is an integral lattice, for example a root lattice, then $\Gamma \subset \Gamma^*$.

Theorem 5.5. *Assume that the n -dimensional root lattice $\Gamma \subset \mathbb{R}^n$ is indecomposable. Then $x \neq 0$ is a vertex of V if and only if $x \in \Lambda^* \cap V$, where $0 \in \Lambda \subseteq \Gamma$ is some n -dimensional, indecomposable root lattice. If R is the vertex set of an L -polytope with such a center $x \in \Lambda^* \cap V$, then R generates Λ .*

Using this theorem useful information can be obtained on the vertices of the Voronoi polytope of an indecomposable root lattice Γ . For example, suppose that $\Lambda \subseteq \Gamma$ ($0 \in \Lambda$) is an indecomposable root lattice of full dimension n . Since $\Gamma \subseteq \Lambda^*$, the elements of Λ^* may be grouped into Γ -translation classes

$$[0], [1], [2], \dots, [d - 1],$$

the number of classes d being equal to the index of Γ in Λ^* . The translation class $[0]$ is equal to Γ and is uninteresting. If $d \geq 2$ the sets $[1] \cap V, \dots, [d - 1] \cap V$ are nonempty and by Theorem 5.5 are Γ -translation classes of vertices of V ; each corresponds to a distinct Γ -translation class of L -polytope. By considering all such lattices $\Lambda \subseteq \Gamma$, including Γ itself, all of the Γ -translation classes of vertices and L -polytopes are generated.

If no such n -dimensional lattices Λ are contained properly in Γ , then all of the holes of Γ are generating (see Proposition 5.4). On the other hand, if Γ is unimodular (and therefore equal to E_8) none of the holes in Γ are generating.

Using some of the ideas in the following proof this last statement can be strengthened: if Γ is integral, even and unimodular, then none of its holes are generating. A small but general deformation of such a lattice yields a general lattice in which all of the L -polytopes are simplicial. Since the resulting simplicial decomposition is a refinement of the L -decomposition for Γ , none of these L -simplexes are generating. In other words, this establishes the existence of a remarkable type of general lattice in which none of the L -simplexes are standard. The edge vectors of the individual L -simplexes generate sublattices of index 2,

3, ... and are conveniently referred to as L -simplexes with double, triple, ... volume. (A simplex of volume k has volume $k/n!$ times that of a fundamental parallelepiped.)

Proof. Assume that x is a vertex of V . If R is the vertex set of the corresponding L -polytope, then $\gamma \in R$ must satisfy the equation

$$|\gamma - x|^2 - |x|^2 = |\gamma|^2 - 2\langle \gamma, x \rangle = 0.$$

Since Γ is even it follows that $\langle \gamma, x \rangle \in \mathbb{Z}$, which in turn implies that $x \in \Gamma_R^*$. By Proposition 5.4 Γ_R is an indecomposable root lattice containing the origin. So if Λ is taken equal to Γ_R the *only if* side of the proof is settled.

Now assume that $x \neq 0$ belongs to $V \cap \Lambda^*$, where $\Lambda \subseteq \Gamma$ is an n -dimensional, indecomposable root lattice containing the origin. Notice that if $\alpha \in \Phi_\Lambda$ (Φ_Λ is the corresponding root system), then $\langle x, \alpha \rangle \in \{0, \pm 1\}$. It will prove convenient to write

$$\Phi_\Lambda = \Phi + \Phi_0 + \Phi_+,$$

where $\Phi_+ = \{\alpha \in \Phi_\Lambda \mid \langle x, \alpha \rangle = 1\}$, etc. Clearly, $\Phi = -\Phi_+$; it is argued below that $\Phi_0 = \Phi_+ - \Phi_+$. These formulas are significant because they establish that Λ is generated by Φ_+ . Therefore Φ_+ has full dimension n and x is a vertex of V . In addition, it follows that the L -polytope centered at x generates the lattice Λ .

Suppose that $\delta \in \Phi_0$ is not orthogonal to Φ_+ . Then there is a root $\alpha \in \Phi_+$ such that $|\langle \alpha, \delta \rangle| = 1$. In this case $\beta = \alpha - \langle \alpha, \delta \rangle \delta$ is a second root belonging to Φ_+ and the formula $\langle \alpha, \delta \rangle \delta = \alpha - \beta$ holds. Therefore $\langle \alpha, \delta \rangle \delta$, hence δ belong to the difference set $\Phi_+ - \Phi_+$. Since Φ_Λ was assumed to be indecomposable, it then follows that $\Phi_0 \cap \Phi_+^\perp$ is empty and $\Phi_0 = \Phi_+ - \Phi_+$. □

Suppose that R is the vertex set of an L -polytope in the star of L -polytopes at the origin for some general lattice Γ . The following serves as a useful test to determine whether the hole R is imperfect.

Proposition 5.6. *Assume that the dimension of Γ exceeds one. If there is a vector $\delta \in \Gamma^*$ satisfying the condition $\langle \delta, \rho \rangle = 0, 1$ for all vertices $\rho \in R$, then R is imperfect.*

Proof. Since $\langle \delta, x \rangle$ is integer valued for $x \in \Gamma$, the quadratic function

$$f(x) = (\delta_1 x_1 + \dots + \delta_n x_n)(\delta_1 x_1 + \dots + \delta_n x_n - 1)$$

is nonnegative on Γ , and by hypothesis zero valued on R . Suppose that g is quadratic and such that the empty sphere circumscribing R is given by the equation $g = 0$. Then f, g are linearly independent (the hypothesis $n \geq 2$ is used here) and zero valued on R . The dimension of $Q(R) = \{q \text{ quadratic} \mid q(x) = 0\}$ is at least two and R is imperfect. □

It is now time to consider the details for the indecomposable root lattices.

Lattice A_n . For $n \geq 1$

$$A_n = \{x \in \mathbb{Z}^{n+1} \mid x_1 + x_2 + \dots + x_{n+1} = 0\};$$

$n + 1$ coordinates are used to describe these n -dimensional lattices [CS, pp. 108–117]. There are no n -dimensional, indecomposable root lattices Λ properly contained in A_n . Therefore all the holes of A_n are generating and the vertices of V are elements of A_n^* . The lattice A_n has index $n + 1$ in A_n^* and representatives of the various A_n -translation classes of A_n^* , chosen in V , are given by

$$[0] = [0^{n+1}], \quad [1] = \frac{1}{n+1} [n, -1^n], \quad [2] = \frac{1}{n+1} [(n-1)^2, -2^{n-1}], \quad \dots,$$

$$[n] = \frac{1}{n+1} [1^n, -n].$$

When $i \neq 0$ the general formula is given by

$$[i] = \frac{1}{n+1} [j^i, -i^j], \quad i + j = n + 1;$$

the vertices of the corresponding L -polytope are generated by independently permuting the first i and last j coordinates of

$$[1^k, 0^{i-k}, -1^k, 0^{j-k}],$$

where $1 \leq k \leq i, j$. Any other L -polytope is translationally equivalent to one of these.

As mentioned in the introduction, all of the holes in A_1 , the one-dimensional lattice, are perfect. For $n \geq 2$ the vector

$$\delta = \frac{1}{n+1} [n, -1^n] = [1, 0, \dots, 0] - \frac{1}{n+1} [1, 1, \dots, 1]$$

belongs to the dual lattice A_n^* (in fact, is a vertex of V). If ρ is a vertex of any of the representative L -polytopes listed above, then $\langle \delta, \rho \rangle = 0, 1$, and by Proposition 5.6 all of the holes in $A_n, n \geq 2$, are imperfect.

Lattice D_n . For $n \geq 4$

$$D_n = \{x \in \mathbb{Z}^n \mid x_1 + x_2 + \dots + x_n \text{ even}\}$$

[CS, pp. 117–120], [RS]). There are no n -dimensional, indecomposable root lattices properly contained in D_n , so again all holes are generating and the vertices of V are given by the formula $D_n^* \cap V$. The lattice D_n has index 4 in D_n^* , and

representatives in V of the various translation classes of the elements of D_n^* are given by

$$\begin{aligned}
 [0] &= [0^n], & [1] &= \frac{1}{2}[1, 1, \dots, 1], \\
 [2] &= \frac{1}{2}[1, 1, \dots, -1], & [3] &= [0^{n-1}, 1].
 \end{aligned}$$

The L -polytopes centered at $[1], [2]$ are isometrically equivalent and correspond to deep holes. The vertices of the L -polytope centered at $[1]$ are all $(0, 1)$ -vectors whose entries have even sum; these 2^{n-1} vertices form the alternate vertices of an n -cube with edge length one. The other species of hole at $[4]$ is shallow, the L -polytope being a cross polytope with $2n$ vertices given by

$$[4] \pm e_i \quad (i = 1, \dots, n),$$

where $e_i = [0, \dots, 0, 1, 0, \dots, 0]$ is the unit vector along the i th coordinate axis. The vector $\delta_1 = [0^{n-1}, 1]$ belongs to D_n^* and satisfies the condition $\langle \delta_1, \rho \rangle = 0, 1$ when ρ is a vertex of the L -polytope centered at $[1]$. Therefore this hole and all the deep holes of D_n are imperfect. The vector $\delta_3 = \frac{1}{2}[1, 1, \dots, 1]$ belongs to D_n^* and satisfies the condition $\langle \delta_3, \rho \rangle = 0, 1$ when ρ is a vertex of the L -polytope centered at $[3]$. Therefore the shallow holes, hence all the holes of D_n , are imperfect.

When $n = 4$ the radii of the deep and shallow holes coincide, both equaling one. The alternate vertices of a 4-cube are also the vertices of a cross polytope, and all of the L -polytopes in D_4 are isometrically equivalent. The L -decomposition of \mathbb{R}^4 into copies of a single cross polytope is the regular honeycomb $\{3, 3, 4, 3\}$ [Cox2, Sections 7.8 and 8.1]. The Voronoi polytope is the regular polytope $\{3, 4, 3\}$ with 24 vertices and 24 octahedral facets; the 24-cell.

Lattice E_n ($n = 6, 7, 8$). In his paper entitled "Extreme forms," Coxeter replaces Euclidean n -space with Minkowskian $(n + 1)$ -space when describing the lattices $E_n, n = 6, 7, 8$ [Cox 1, p. 419]. The Minkowski scalar product is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n - x_{n+1} y_{n+1},$$

and E_n can be realized by taking all the integer Minkowski vectors, x , satisfying the equation $\langle t, x \rangle = q$, where q is an integer and t is the time-like vector $[1, 1, \dots, 1; 3]$. It will prove convenient to use the copy of E_n determined by the equation

$$x_1 + x_2 + \dots + x_n - 3x_{n+1} = -1.$$

The n vectors obtained by permuting the first n components of $[-1 \ 0^{n-1}; 0]$ belong to E_n and form the vertices of a regular $(n - 1)$ -dimensional simplex S_{n-1} . This simplex forms the glue joint for a pair of neighboring L -polytopes in E_n . Table 1 describes the L -polytopes lying to the "right" and "left" of S_{n-1} ; S_{n-1} is a facet of both.

Table 1. Typical holes in E_n .

Lattice	E_6	E_7	E_8
(the L -polytope to the left of S_{n-1})			
Center	$\frac{1}{3}[-2^6; -3]$	$\frac{1}{4}[-1^7; -1]$	$\frac{1}{2}[-1^8; -2]$
Radius	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{7}}{8}$	1
Vertices	$[-2 \ -1^5; -2] \times 6$ $[-1^4 \ 0^2; -1] \times 15$ $[-1 \ 0^5; 0] \times 6$	$[-1^7; -2] \times 1$ — $[-1 \ 0^6; 0] \times 7$	$[-1^7 \ 0; -2] \times 8$ — $[-1 \ 0^7; 0] \times 8$
Description	Gosset polytope 2_{21}	Simplex with double volume	Cross polytope with double volume
(the L -polytope to the right of S_{n-1})			
Center	$\frac{1}{3}[1^6; 3]$	$\frac{1}{2}[1^7; 3]$	$\frac{1}{3}[0^8; 1]$
Radius	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{8}}{9}$
Vertices	$[-1 \ 0^5; 0] \times 6$ $[0^4 \ 1^2; 1] \times 15$ $[0 \ 1^5; 2] \times 6$	$[-1 \ 0^7; 0] \times 7$ $[0^5 \ 1^2; 1] \times 21$ $[0^2 \ 1^5; 2] \times 21$ $[1^6 \ 2; 3] \times 7$	$[-1 \ 0^7; 0] \times 8$ — $[1^8; 3] \times 1$
Description	Gosset polytope 2_{21}	Gosset polytope 3_{21}	Simplex with tripple volume

The lattice E_6 is spectacular: *all of the holes are equivalent*, each containing a copy of the Gosset polytope 2_{21} with 27 vertices. In the L -decomposition these polytopes glue together along facets which are either regular simplexes (as illustrated in the table) or regular cross polytopes (which can easily be checked); this Gosset polytope has 72 simplicial facets and 27 facets which are cross polytopes [Cox2, pp. 202–204].

The lattice E_6 has index 3 in E_6^* . Since there is only one type of L -polytope, it follows from Theorem 5.5 that 2_{21} has its center in E_6^* and is generating; more importantly, the vertices of the Voronoi polytope belong to E_6^* and fall into two distinct E_6 -translation classes. (It is also possible to deduce from this theorem that there are no proper six-dimensional sublattices in E_6 which are indecomposable root lattices.) Since 2_{21} has 27 vertices, each translation class contains 27 elements; the L -polytopes of these two classes are related by inversion. The Voronoi polytope is the reciprocal of the polytope 1_{22} , described on p. 24 of [Cox4]. There are $54 (= 27 \times 2)$ Gosset polytopes in the star of L -polytopes at the origin.

There are two types of holes in E_7 . The deep holes contain copies of the Gosset polytope 3_{21} with 56 vertices [Cox2, pp. 202–204]. Each Gosset polytope is glued to 126 equivalent Gosset polytopes along facets which are regular cross polytopes (that these six-dimensional cross polytopes serve as glue joints for a pair of Gosset polytopes is easily checked). In addition, each Gosset polytope is glued to 576 regular simplexes along simplicial facets (as shown in the table). These simplicial L -polytopes are contained in the shallow holes of E_7 .

As is apparent from the table, the L -simplex generates a sublattice of index two in E_7 . Since the simplicial holes in A_n are generating this sublattice can be identified

with A_7 (see Theorem 5.5); these simplicial holes have minimal radius in A_7 and correspond to the translation classes $[1], [n]$ in the discussion of A_n . The total number of such L -simplexes in the star at the origin of E_7 is given by $|E_7|/|A_6| = 576$; A_6 is the stability group of any one of these L -simplexes at the origin. Since each copy of A_7 accounts for $16 = 2 \times 8$ simplexes, there are 36 distinct ways in which A_7 appears as a sublattice containing the origin in E_7 .

The lattice E_7 has index two in E_7^* . By Theorem 5.5 this accounts for a single translation class of generating L -polytope in the star of L -polytopes at the origin. This is the translation class of the Gosset polytope 3_{21} , and it follows that there are 56 such polytopes in the star at the origin. Since there is only one translation class it can be inferred that 3_{21} has a center of symmetry. (This is also apparent from the representative of 3_{21} recorded in the table.)

The Voronoi polytope for E_7 has $632 = 56 + 576$ vertices and is the reciprocal of the polytope 2_{31} [Cox4, p. 27].

The L -polytope lying in a typical deep hole of E_8 is a regular cross polytope. Each of these cross polytopes is glued to 256 regular simplexes along each of its simplicial facets (as shown in the table). These L -simplexes are contained in the shallow holes, the only other type of hole in E_8 .

Each cross polytope generates a copy of D_8 with index two in E_8 . The star of L -polytopes at the origin contains $|E_8|/|D_7| = 2160$ such cross polytopes; there are 135 ($= 2160/16$) distinct copies of D_8 containing the origin in E_8 . Each simplex generates a copy of A_8 with index three in E_8 . There are $|E_8|/|A_7| = 17,280$ L -simplexes in the star of L -polytopes at the origin; there are 96 ($= 1728/16$) distinct copies of A_8 containing the origin in E_8 . The Voronoi polytope is the reciprocal of the polytope 4_{21} [Cox2, p. 204], [Cox4, p. 36].

Since all of the holes in E_8 and the shallow holes in E_7 contain polytopes with an insufficient number of vertices $\left(< \binom{n+2}{2} - 1 \right)$, these holes are imperfect. (That these holes are imperfect also follows from the discussion of A_n and D_n .) This leaves the deep holes in E_6 and E_7 as the only possible candidates for perfect holes.

Both types of hole in E_8 and the shallow hole in E_7 have been considerable upstaged in this discussion by the deep holes in E_6 and E_7 , but these underdog holes are also distinguished; the edge vectors generate proper sublattices. As has been pointed out, the edge vectors of the simplex in E_7 generate a sublattice of index two (a simplex of double volume). The first instance of such an L -simplex is in five dimensions (see the discussion on p. 796 of Part I of [ER] and Sections 7 and 10 of [RB]), lattices in lower dimensions having only standard simplexes. There are examples showing that in dimension $2k + 1$ ($k \geq 2$) there are lattices with L -simplexes which generate sublattices of index k ; when $k \geq 3$ it is not known whether this dimension is lowest possible for this phenomena. There are no examples of cross polytopes generating sublattices of index two in dimensions less than eight.

Completion of the proof of Theorem 5.3. We have just seen that, with the exception of the deep holes in $A_1, E_6,$ and E_7 , all of the holes in the irreducible

root lattices are imperfect. Since there is only one type of hole in A_1 and it is perfect, all that remains is to verify the perfection of the deep holes in E_6 and E_7 .

Notice that the restriction map establishes a linear isomorphism between the homogeneous quadratic forms on \mathbb{R}^{n+1} and the quadratic functions on the plane

$$x_1 + x_2 + \dots + x_n - 3x_{n+1} = -1$$

(which contains the copy of E_n under investigation). Then, $R \subset E_n$ is the vertex set of a perfect L -polytope if and only if the linear space of forms on \mathbb{R}^{n+1} which are zero valued on R is one-dimensional. An equivalent characterization of perfection is that the matrices $r'r$, $r \in R$ (with r a row vector, $r'r$ is an $(n+1) \times (n+1)$ matrix), generate a linear space of codimension one in the space of real symmetric matrices. This last criterion is satisfied if the matrices $r'r$, along with the metric matrix M for the Minkowski metric, span this linear space.

It is convenient to denote the vertex sets for the Gosset polytopes 2_{21} , 3_{21} by

$$\begin{aligned} a_1, a_2, \dots, a_n & \quad \text{with} \quad a_1 = [-1 \ 0^{n-1}; \ 0]; \\ b_{12}, b_{13}, \dots, b_{n-1n} & \quad \text{with} \quad b_{12} = [1^2 \ 0^{n-2}; \ 1]; \\ c_1, c_2, \dots, c_6 & \quad \text{with} \quad c_1 = [0 \ 1^5; \ 2] \quad (2_{21} \text{ only}); \\ c_{12}, c_{13}, \dots, c_{67} & \quad \text{with} \quad c_{12} = [0^2 \ 1^5; \ 2] \quad (3_{21} \text{ only}); \\ d_1, d_2, \dots, d_7 & \quad \text{with} \quad d_1 = [2 \ 1^6; \ 3] \quad (3_{21} \text{ only}). \end{aligned}$$

Let E_{km} , $1 \leq k \leq m \leq n+1$, be the standard basis for the real symmetric matrices:

$$\begin{aligned} E_{kk} &= \{\delta_{ik}\delta_{jk}\}, \\ E_{km} &= \{\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}\}. \end{aligned}$$

Let

$$\begin{aligned} B &= \sum_{k < m} b'_{km} b_{km}, \\ B_1 &= \sum_{1 < k < m} b'_{km} b_{km}, \\ C &= \begin{cases} \sum_k c'_k c_k & (2_{21} \text{ only}), \\ \sum_{k < m} c'_{km} c_{km} & (3_{21} \text{ only}), \end{cases} \\ C_1 &= \begin{cases} c'_1 c_1 & (2_{21} \text{ only}), \\ \sum_{1 < k} c'_{1k} c_{1k} & (3_{21} \text{ only}). \end{cases} \end{aligned}$$

Then the following formulas hold:

$$\begin{aligned}
 E_{11} &= a'_1 a_1, \\
 E_{n+1, n+1} &= -M + \sum_k a'_k a_k, \\
 E_{1, n+1} &= \begin{cases} \frac{1}{10}[4B - C - 5B_1 + 5C_1 - 15E_{11} - 6E_{77}] & (2_{21} \text{ only}), \\ \frac{1}{30}[10B - C - 12B_1 - 3C_1 - 45E_{11} - 18E_{88}] & (3_{21} \text{ only}), \end{cases} \\
 E_{12} &= b'_{12} b_{12} - E_{11} - E_{22} - E_{n+1, n+1} - E_{1, n+1} - E_{2, n+1}.
 \end{aligned}$$

By permuting the indices $\{1, 2, \dots, n\}$ all of the basis elements are obtained. Thus the deep holes of E_6 and E_7 are perfect. □

That the deep holes in E_6 are perfect has been known for a long time [E1]; the author is thankful to S. S. Ryshkov who was the first to notice that the deep holes in E_7 are also perfect.

6. Holes in Lattices

The problem of classifying root figures reduces to that of classifying the finite, nondegenerate root figures (see Corollary 2.5). Suppose that R is such a root figure. Consider the relatively open face of \mathcal{P}^n given by

$$F_R = \{f \in \mathcal{P}^n \mid R_f = R\}.$$

When $f \in F_R$, φ_f is the metrical form for a lattice for which R gives the coordinates of the vertices of an L -polytope (see Corollary 2.6 and the discussion in the introduction on L -polytopes). Define the **domain of R** , Φ_R , to be the cone of all such metrical forms. The linear mapping $\Pi: f \rightarrow \varphi_f$ maps F_R into Φ_R . In fact, Π establishes a one-to-one correspondence between the elements of F_R and Φ_R . Suppose that $\varphi \in \Phi_R$. If c is the coordinate vector of the center of the L -polytope with vertices given by R , and ρ is the radius, then

$$f(x) = \varphi(x - c) - \rho^2$$

belongs to F_r and Π is invertible.

Affine equivalence played a fundamental role in Voronoi's theory of L -types of lattices, the crowning achievement of his two mémoires [V2, Part IV]. He showed that the cone of positive definite forms can be partitioned into disjoint, relatively open convex subcones of varying dimension, called L -type domains, with the following properties:

- A. On each of these subcones the affine structure of the L -decomposition of the corresponding lattices is constant. Subcones are arithmetically equivalent if and only if the affine structure of corresponding L -decompositions is identical.

- B. When the L -type domain has full dimension $N = \binom{n+1}{2}$ it is called general and the corresponding L -decomposition is simplicial. These domains are polyhedral. For each value of n there are only a finite number of arithmetically inequivalent general L -type domains.
- C. L -type domains of lesser dimension are called special and nonsimplicial L -polytopes appear in the corresponding L -decompositions. These are the faces of varying dimension of the general L -type domains, each being a face of two or more general domains.

The domain Φ_R is the union of all those L -type domains for which R gives the coordinates of the vertex set of an L -polytope in the corresponding L -decomposition. In general, this union contains both general and special L -type domains.

Lemma 6.1. Φ_R is the union of a finite number of L -type domains.

Proof. By properties B and C the L -type domains of Φ_R are partitioned into a finite number of arithmetic classes. In turn, these classes can be partitioned into a finite number of **R -equivalence classes** which we now proceed to define.

Suppose that $\varphi_1, \varphi_2 \in \Phi_R$. If Γ_1, Γ_2 are lattices determined by these forms, then R gives the coordinates of the vertices of two L -polytopes, $P_1 \in \Gamma_1, P_2 \in \Gamma_2$. Define φ_1 to be **R -equivalent** to φ_2 , if there is an affine transformation T , mapping Γ_1 onto Γ_2 preserving the structure of the L -decomposition, and such that $T(P_1) = P_2$. By property A, the notion of R -equivalence extends to the L -type domains of Φ_R , and is a refinement of arithmetic equivalence for these domains. For any given lattice, there are only a finite number of L -polytopes inequivalent under the group of lattice translations. It follows that the arithmetic classes of L -type domains of Φ_R partition into a finite number of R -equivalence classes.

More can be said about the R -equivalence classes of Φ_R . Suppose that φ_1, φ_2 are R -equivalent and let T be as above, mapping P_1 onto P_2 . Then this transformation induces an affine transformation of coordinate vectors for the two lattices, g_T , leaving R invariant. That is, $g_T \in G(\mathbb{Z}^n)$ and $g_T(R) = R$. Moreover, g_T is equal to the identity transformation if and only if φ_1, φ_2 belong to the same L -type domain. The transformation g_T belongs to the invariance group $G_R \subset G(\mathbb{Z}^n)$ of R which must be finite, since by assumption R is finite and nondegenerate. It follows that the R -equivalence classes of L -type domains in Φ_R are finite. Since there are only a finite number of R -inequivalent L -type domains in Φ_R , there can only be a finite number of L -type domains in total. □

Theorem 6.2. Assume that R is a finite, nondegenerate root figure. Then the relatively open cones F_R and Φ_R are polyhedral and linearly equivalent.

Proof. That F_R and Φ_R are linearly isomorphic was argued above. That Φ_R , hence F_R , is polyhedral follows from Lemma 6.1 and statements B and C above. □

By Theorem 2.4 the extreme rays of F_R are perfect. Since F_R is open and polyhedral, an arbitrary element $f \in F_R$ can be written

$$f = \sum_{P \ni R} \lambda_P P,$$

where $\lambda_P > 0$. The summation is over the perfect root figures containing R . That is, over the perfect elements belonging to F_R (P is the root figure of the perfect element p). The corresponding formula for elements of Φ_R is given in:

Corollary 6.3. *Assume that R is a finite, nondegenerate root figure. Then an arbitrary form $\varphi \in \Phi_R$ can be written as*

$$\varphi = \sum_{P \supseteq R} \lambda_P \varphi_P,$$

where $\lambda_P > 0$.

Suppose that $R \subset \mathbb{Z}^n$ is some collection of integer vectors (for the moment not a root figure). Define the **dual** of R , R^* by

$$R^* = \{r^* \in \mathbb{Z}^n \mid 0 \leq \langle r, r^* \rangle \leq 1, r \in R\},$$

where $\langle r, r^* \rangle$ is the usual Euclidian scalar product. If R satisfies the condition that

$$R = (R^*)^*,$$

then R, R^* is called a **dual system of integer vectors**. It is convenient to say that the dual system is nondegenerate if both R and R^* are nondegenerate [RE3]–[RE5].

If R, R^* is a dual system, then both R and R^* contain the zero vector. If the system is nondegenerate, then both R, R^* must be finite.

For each vector $r \in R$, let φ_r be the form given by

$$\varphi_r(x) = (r_1x_1 + r_2x_2 + \dots + r_nx_n)^2;$$

φ_r has unit rank.

A finite nondegenerate root figure containing the origin gives the coordinates of the vertices of an L -polytope in the star of L -polytopes at the origin.

Theorem 6.4. *Assume that $n \leq 5$. Assume that R is a finite, nondegenerate root figure containing 0. Then R is one of the partners of a nondegenerate dual system and the domain Φ_R is the collection of all forms*

$$\sum_{r \in R^*} \lambda_r \varphi_r,$$

where $\lambda_r > 0$.

Proof. By Theorem 2.4 and Corollary 5.2, when $n \leq 5$, the extreme rays of F_R are generated by the elements

$$p(x) = (a_1x_1 + \dots + a_nx_n + k)(a_1x_1 + \dots + a_nx_n + k + 1)$$

satisfying the condition $p(r) = 0, r \in R$ ($a_1, a_2, \dots, a_n, k \in \mathbb{Z}$ and $\gcd\{a_1, \dots, a_n\} = 1$).

Since $0 \in R$, k must equal 0 or -1 . However, the first of these cases converts to the second by the replacement $a = [a_1, \dots, a_n] \rightarrow -a$. Assuming that $k = -1$, $p(r) = 0$ if and only if $0 \leq \langle a, r \rangle \leq 1$. Since this last condition must hold for all $r \in R$, it follows that $a \in R^*$. From this correspondence between the extreme rays of F_R and the elements of R^* , it immediately follows that Φ_R is the collection of forms

$$\sum_{r \in R^*} \lambda_r \varphi_r,$$

where $\lambda_r > 0$.

The condition $r \in R$ if and only if $f(r) = 0$, for $f \in F_R$, can now be restated as: $r \in R$ if and only if $0 \leq \langle a, r \rangle \leq 1$ for all $a \in R^*$. That is, $R = (R^*)^*$ and R, R^* form a dual system.

Since, for $f \in F_R$, φ_f is positive definite (Corollary 2.5), R^* is nondegenerate. Since R was assumed to be nondegenerate, the dual system R, R^* is nondegenerate. \square

This theorem serves as the basis for a new method to enumerate L -polytopes in lattices. When the dimension n satisfies the condition $n \leq 5$ all of the possible L -polytopes can be determined by first establishing a list of the nondegenerate dual systems up to arithmetic equivalence. (If R, R^* is a dual system and $U \in GL(n, \mathbb{Z})$, $U^* = (U^{-1})'$, then UR, U^*R^* is an arithmetically equivalent dual system.) Such a program has been carried out for $n \leq 4$ and the results are now available [RE4]. There are 16 arithmetically inequivalent, nondegenerate dual systems that can occur when $n = 4$. Each partner gives the vertex set of an L -polytope, but there is one self-dual case where R is arithmetically equivalent to R^* . The 31 L -polytopes that result, are then grouped into 19 $G(\mathbb{Z}^n)$ -equivalence classes, representing the various species of L -polytope that can occur in four-dimensional lattices.

Dual systems account for a large number of L -polytopes in higher-dimensional lattices. If R, R^* is a nondegenerate dual system, then both R and R^* give the coordinates of the vertices of an L -polytope no matter what the dimension. However, the formula of Theorem 6.4 for Φ_R need not hold when $n \geq 6$. For all values of n , the L -polytopes in the root lattices A_n, D_n can be represented as partners of dual systems; the simplexes of double and triple volume in E_7 and E_8 can also be represented as partners. The Gosset polytopes belong to the deep holes in E_6 and E_7 are obvious examples of L -polytopes that cannot be represented as partners (see Theorem 5.3). The cross polytope of double volume in E_8 is an example of an imperfect L -polytope which cannot be represented as a partner of a dual system (and will be reported on elsewhere).

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