

A conformally flat realistic anisotropic model for a compact star

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Abstract A physically realistic stellar model with a simple expression for the energy density and conformally flat interior is found. The relations between the different conditions are used without graphic proofs. It may represent a real pulsar.

1 Introduction

The study of relativistic stellar structure is now more than 100 years old. It began with the discovery in 1916 by Karl Schwarzschild of the universal vacuum exterior solution [1] and the first interior stellar solution [2], which should be matched to the exterior one. For a long time the star interior was considered to be made of perfect fluid, which has equal radial (p_r) and tangential (p_t) pressures. This leads to the isotropic condition $p_r = p_t$, imposed on the Einstein equations.

The first attempts to consider pressure anisotropy in self-gravitating objects were made by Jeans within the context of Newtonian gravity [3]. Spherical symmetry demands only the equality of the two tangential pressures. In general relativity the first anisotropic model was proposed by Lemaitre in 1933 [4]. He discussed a model sustained solely by p_t and with constant energy density ρ . His work remained unnoticed for a long time.

In 1972 Ruderman [5] argued for the first time that nuclear matter at very high densities ρ of the order of 10^{15} g/cm³ may have anisotropic features and its interactions are relativistic. The work of Bowers and Liang [6] on building anisotropic models in 1974 gave start to a number of such solutions. Anisotropy may have a lot of sources [7]: a mixture of fluids of different types, presence of a superfluid, existence of a solid core, phase transitions, presence of magnetic field, viscosity, etc. Such models describe compact stellar objects like

neutron stars, strange stars, quark stars, boson stars, gravastars, dark stars and others.

The Einstein equations describe the effect of matter upon the metric of spacetime. For static, spherically symmetric fluid solutions the metric may be written in comoving canonical coordinates and has two components ν and λ . The energy-momentum tensor is represented by its diagonal components, mentioned above: ρ , p_r and p_t . There are only three equations for these five characteristics, so that two of them may be chosen freely. They should satisfy, however, a lot of regularity, stability and energy conditions for a realistic model. The situation with this undetermined system of differential equations is analogous to the one for charged isotropic star models [8]. This is not surprising since charge and other characteristics can be looked upon as an effective anisotropy of the model [9–11].

Different choices of the two given functions have been made. The simplest one is to propose ansatz for the two metric functions. One of the first was given in [12], where some of the Tolman isotropic solutions [13] were modified to become anisotropic. Other followed recently [14–20].

String theory has inspired embedding of branes like in the Randall–Sundrum model [21]. This rekindled the interest in stellar models embedded in five-dimensional flat spacetime (embedding class one). They must satisfy the Karmarkar condition [22]. It can be written as a relation between the metric functions and one of them can generate the whole solution. It is interesting that the isotropic condition, can be translated into a similar relation, giving different generating functions [23–27].

There are just two perfect fluid solutions of the Karmarkar condition – the interior Schwarzschild one and a cosmological one. When the fluid is anisotropic, a number of realistic solutions has been found in the last 2 years [28–47].

Conformally flat anisotropic spheres have a vanishing Weyl tensor. This leads to a differential equation for λ and ν , similar to the Karmarkar one or the isotropic condition. Early solutions were found in [48] where different ansatz

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for the mass function m were proposed. There is a simple relation between m and λ and from the condition for conformal flatness ν may be found. Then expressions for all other characteristics of the model are obtained. The first who integrated the vanishing Weyl condition was Ponce de Leon in 1987 [49], but no details were given. A relation between ν and λ is the outcome. Details were supplied in 2001 [50] and some models were discussed with $p_r = 0$ or prescribed λ . The authors worked in non-comoving coordinates and gave the general solution in another, but equivalent form, used later in [51]. Conformally flat spherically symmetric spacetimes were studied in different coordinate systems in [52]. A conformally flat model with polytropic equation of state was discussed in [53]. Other solutions were given too [54,55].

Closely related are solutions which admit conformal motion. They depend on the conformal factor. The first anisotropic solution of this kind was given in 1984 [56]. Time dependent solutions followed [57], as well as charged ones [58] with constant anisotropy, or constant charge density and a generalisation of the Lemaitre solution. Some recent solutions are [59–66]. The work [50] has been generalized to non-static solutions [54,55] and many new solutions were obtained.

The main shortcoming of the existing model building is that the conditions for a realistic model are checked after the ansatz for the two free functions are made. The expressions for the different characteristics become very involved even for polynomial seeding functions and one has to turn to graphic proofs. Solutions usually have lots of constants in order to satisfy the set C1–C10, introduced in Sect. 3. One constant turns a 2-dimensional plot into a 3-dimensional one. With two and more constants only partial plots are possible.

Recently, [67] we have argued that the combination of free functions ρ , p_r is the right choice to reduce the number of graphic proofs. Another important fact is that the conditions C1–C10 are not independent. There are many relations between them and we have reduced the set to a couple of inequalities. Only they need in general a graphic proof in the concrete examples. To illustrate this formalism we have given a solution with simple energy density and linear equation of state (EoS) with bag constant.

In the present paper we apply the approach of [67] to conformally flat solutions with simple metric function λ , which leads to a simple ρ . We make a full analytic physical analysis of the solution and show that no graphic proofs are necessary. It implies that a certain constant of the model should fall in a particular range. A real pulsar is shown to satisfy this constraint.

In Sect. 2 the Einstein field equations are given, as well as the definitions of the main characteristics of a static anisotropic star. The Weyl condition and its solution are also introduced. In Sect. 3 we summarize the conditions for a physically realistic model. In Sect. 4 we present the model,

which depends on three constants. In Sect. 5 we perform a full physical analysis and find the range of the constants where C1–C10 are fulfilled. In Sect. 6 a real star is shown to satisfy these constraints and therefore is a candidate for a neutron star with conformally flat interior. Section 7 contains a discussion.

2 Field equations and definitions

The interior of static spherically symmetric stars is described by the canonical line element

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where λ and ν are dimensionless and depend only on the radial coordinate r . The Einstein equations read

$$kc^2\rho = \frac{1}{r^2}[r(1 - e^{-\lambda})]', \quad (2)$$

$$kp_r = -\frac{1}{r^2}(1 - e^{-\lambda}) + \frac{\nu'}{r}e^{-\lambda}, \quad (3)$$

$$kp_t = \frac{e^{-\lambda}}{4} \left(2\nu'' + \nu'^2 + \frac{2\nu'}{r} - \nu'\lambda' - \frac{2\lambda'}{r} \right), \quad (4)$$

where ρ is the matter density, p_r is the radial pressure, p_t is the tangential one, ' means a radial derivative and

$$k = \frac{8\pi G}{c^4}. \quad (5)$$

Here G is the gravitational constant and c is the speed of light.

The gravitational mass in a sphere of radius r is given by

$$m = \frac{kc^2}{2} \int_0^r \rho(\omega) \omega^2 d\omega. \quad (6)$$

Due to kc^2 , its dimension is length. Then Eq. (2) gives

$$e^{-\lambda} = 1 - \frac{2m}{r}. \quad (7)$$

The compactness of the star u is defined by

$$u = \frac{2m}{r} \quad (8)$$

and is dimensionless.

On the other side, the redshift Z depends on ν :

$$Z = e^{-\nu/2} - 1. \quad (9)$$

The field equations do not contain ν , but its first and second derivative. One can express ν' from Eqs. (2), (3), and (7) as

$$\nu' = \frac{krp_r + 2m/r^2}{1 - 2m/r}. \quad (10)$$

The second derivative ν'' may be excluded by differentiation of Eq. (3) and combination with the other field equations. The result is

$$p_r' = -\frac{1}{2}(\rho c^2 + p_r)v' + \frac{2\Delta}{r}, \tag{11}$$

where $\Delta = p_t - p_r$ is the anisotropic factor. Combining (10) and (11) one gets the well-known TOV (Tolman, Oppenheimer, Volkoff) equation [13, 68] of hydrostatic equilibrium in a relativistic star, found initially for isotropic solutions. Its anisotropic version was given by Bowers and Liang [6]:

$$p_r' = -(\rho c^2 + p_r) \frac{krp_r + 2m/r^2}{2(1 - 2m/r)} + \frac{2(p_t - p_r)}{r}. \tag{12}$$

The hydrostatic force on the left F_h is balanced by the gravitational F_g and the anisotropic forces F_a on the right. This equation is not independent from the field equations, but is their consequence. It can replace one of them. It is also equivalent to the Bianchi identities $T_{\nu;\mu}^\mu = 0$, which in the static spherically symmetric case have only one non-trivial component [6, 69–71]. In CGS units $G = 6.674 \times 10^{-8} \text{ cm}^3/\text{g s}^2$, $c = 3 \times 10^{10} \text{ cm/s}$, $k = 2.071 \times 10^{-48} \text{ s}^2/\text{g cm}$, $kc^2 = 1.864 \times 10^{-27} \text{ cm/g}$. The mass in grams M is related to m by

$$m = \frac{GM}{c^2}. \tag{13}$$

From now on we set $G = c = 1$, passing to usual relativistic units. Then $k = 8\pi$. As a whole, we have three field equations for five unknown functions: λ , ν , ρ , p_r and p_t .

The space-time is conformally flat when its Weyl tensor vanishes. In our case this gives a relation between the two metric coefficients [49]

$$\frac{1 - e^\lambda}{r^2} - \frac{\nu'\lambda'}{4} - \frac{\nu' - \lambda'}{2r} + \frac{\nu''}{2} + \frac{\nu'^2}{4} = 0. \tag{14}$$

Similar relations arise in embeddings of class 1 [31], or in the case of isotropic pressure [27]. Equation (14) may be integrated. It appears that for the first time this was done in [49], but no details of the integration method were given. These were provided later in [50, 52, 55]. The result can be written as [50]

$$e^{\nu/2} = C_1 r \cosh\left(\int \frac{e^{\lambda/2}}{r} dr + C\right), \tag{15}$$

where C and C_1 are integration constants. This equation should be added to the three field equations, so one may choose freely one generating function to obtain solutions. The model will be physically realistic if a number of regularity, matching and stability conditions are satisfied too.

3 Conditions for a physically realistic model

A comparatively reasonable set of conditions includes

C1. The metric potentials are positive and should be finite and free from singularities in the star’s interior and at the centre should satisfy $e^{\lambda(0)} = 1$ and $e^{\nu(0)} = \text{const}$.

C2. Matching conditions. At the surface of the star $r = r_s$ the interior solution should match continuously to the exterior Schwarzschild solution [1],

$$ds^2 = \left(1 - \frac{2m_s}{r_s}\right) dt^2 - \left(1 - \frac{2m_s}{r_s}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{16}$$

where $m_s = m(r_s)$. This determines the metric at the surface

$$e^{\nu(r_s)} = e^{-\lambda(r_s)} = 1 - \frac{2m_s}{r_s}. \tag{17}$$

In addition, the radial pressure there vanishes, $p_{rs} = 0$. Neither the energy density nor the tangential pressure are obliged to do so.

C3. The interior redshift Z , given by Eq. (9), should decrease with the increase of r . The surface redshift and compactness are related, due to Eq. (17):

$$Z_s = (1 - u_s)^{-1/2} - 1. \tag{18}$$

They should be less than the universal bounds, found when different energy conditions hold (see C6). In the isotropic case they are 2 and 8/9 correspondingly [72]. In the anisotropic case, when DEC holds, they are 5.211 and 0.974. When TEC holds, one has the bounds 3.842 and 0.957 [73]. They are greater than those in the isotropic case, but not arbitrary as asserted in [6].

C4. The density and the pressures should be non-negative inside the star. At the centre they should be finite $\rho(0) = \rho_0$, $p_r(0) = p_{r0}$, $p_t(0) = p_{t0}$. Moreover, $p_{r0} = p_{t0}$ [73].

C5. They should reach a maximum at the centre, so $\rho'(0) = p_r'(0) = p_t'(0) = 0$ and should decrease monotonically outwards, $\rho' \leq 0$, $p_r' \leq 0$, $p_t' \leq 0$. The tangential pressure should remain bigger than the radial one, except at the centre, $p_t \geq p_r$.

C6. Energy conditions. The solution should satisfy the dominant energy condition (DEC) $\rho \geq p_r$, and $\rho \geq p_t$. The strong energy condition (SEC) [74] should be satisfied too, $\rho + p_r + 2p_t \geq 0$. Because of C4 it is trivial, as well as the null energy condition (NEC). It is desirable that even the trace energy condition (TEC) $\rho \geq p_r + 2p_t$ should be satisfied. Obviously, the latter is stronger than DEC.

C7. Causality condition. It says that the radial and tangential speeds of sound should not surpass the speed of light. The speeds of sound are defined as $v_r^2 = dp_r/d\rho$ and $v_t^2 = dp_t/d\rho$. Therefore this condition reads

$$0 < \frac{dp_r}{d\rho} \leq 1, \quad 0 < \frac{dp_t}{d\rho} \leq 1. \tag{19}$$

C8. The adiabatic index Γ as a criterion of stability. This index is the ratio of two specific heats and should be bigger than 4/3 for stability [7, 75, 76],

$$\Gamma = \frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho} \geq \frac{4}{3}. \tag{20}$$

C9. Stability against cracking. Cracking was introduced by Herrera [77] as a possibility of breaking of perturbed self-gravitating spheres. Abreu et al. [78] found a simple requirement for avoiding this to happen, namely the region of stability is

$$-1 \leq \frac{dp_t}{d\rho} - \frac{dp_r}{d\rho} \leq 0. \tag{21}$$

C10. The Harrison–Zeldovich–Novikov stability condition [79,80]. It implies that $dM(\rho_0)/d\rho_0 > 0$.

4 The model

We shall choose a simple ansatz for e^λ as a generating function, namely

$$e^{-\lambda} = (1-x)^2, \quad x = \frac{r^2}{b^2}, \tag{22}$$

where b is some constant of dimension length, so that x is dimensionless, as is the metric coefficient. Its range is from 0 to $x_s < 1$. Then Eqs. (7) and (8) give

$$m = \frac{1}{2}r(2x - x^2), \quad u = 2x - x^2. \tag{23}$$

The derivative of Eq. (6) yields

$$k\rho = \frac{2m'}{r^2} \tag{24}$$

and using this formula or Eq. (2) we obtain for the energy density

$$kb^2\rho = 6 - 5x, \tag{25}$$

which is very simple. In more general form it was used in the past, [67,81–92]. In the context of conformal flatness it was used in [48], Example 4, and [50], model III but only a partial physical analysis has been done. Eq. (25) clarifies the meaning of b :

$$b^2 = \frac{6}{k\rho_0}. \tag{26}$$

The zero index will be used for variables at the centre of the star. Thus b is related to the central density ρ_0 , whose value in CGS units is about $10^{15}g$ for compact neutron stars.

Let us introduce now the constants B and α instead of C

$$C = \frac{1}{2} \ln B^2, \quad \alpha = B^2 - 1. \tag{27}$$

Then Eq. (15) gives an expression for the other metric coefficient

$$e^\nu = \frac{C_1^2 b^2}{4(1+\alpha)} \frac{(1+\alpha x)^2}{1-x}, \tag{28}$$

which is obviously positive. The redshift Z throughout the star is obtained then from Eq. (9). The derivative of ν with respect to x , which enters the field equations, is

$$\nu_x = \frac{2\alpha}{1+\alpha x} + \frac{1}{1-x} \tag{29}$$

and is positive too. Eq. (3), which is an expression for the radial pressure, yields the formula

$$kb^2 p_r = -x + \frac{4\alpha(1-x)^2}{1+\alpha x}. \tag{30}$$

There is a general expression for the anisotropy factor Δ in conformally flat models, which follows from the field equations (3) and (4) and the requirement (14) [51,73]

$$\frac{k\Delta}{r} = -2 \left(\frac{m}{r^3} \right)'. \tag{31}$$

In the case of the simple ansatz (22) it becomes

$$kb^2\Delta = 2x, \quad kb^2\Delta_x = 2. \tag{32}$$

It makes the expression for the tangential pressure very similar to the one for the radial pressure

$$kb^2 p_t = x + \frac{4\alpha(1-x)^2}{1+\alpha x}. \tag{33}$$

Thus, the characteristics of the model are given by simple elementary functions. They depend on three constants b (or ρ_0), α and C_1 . They should be related to the mass m_s and the radius r_s of the star.

5 Physical analysis

Now we have to choose the free parameters of the model in such a way that the conditions C1–C10 are satisfied.

C1. Eq. (22) shows that e^λ is finite and positive and increases monotonically with r from 1 to $(1-x_s)^{-2}$. Equations (28) and (29) show that e^ν is also finite and positive and increases monotonically. Equation (17) shows that $e^{\nu(r_s)}$ is less than 1, hence $e^{\nu(0)}$ is also less than one.

C2. The matching condition for λ is fulfilled when $m_s = m(r_s)$. Thus Eq. (23) gives

$$m_s = \frac{b}{2} x_s^{3/2} (2 - x_s), \quad u_s = 2x_s - x_s^2. \tag{34}$$

Equations (17), (22) and (28) fix C_1

$$C_1^2 = \frac{4(1+\alpha)(1-x_s)^3}{b^2(1+\alpha x_s)^2} \tag{35}$$

in terms of α , b and r_s . The boundary condition $p_{r_s} = 0$, combined with Eq. (30) expresses α as a function of x_s

$$\alpha = \frac{x_s}{(2-x_s)(2-3x_s)}. \tag{36}$$

Thus α is positive, increases monotonically with x_s and is finite as long as $x_s < 2/3$. Then B^2 is positive as it should be.

C3. Equations (9) and (28) show that Z decreases monotonically throughout the star and at the surface

$$Z_s = (1 - x_s)^{-1} - 1, \tag{37}$$

due to Eqs. (18) and (34).

C4. Because of Eq. (25) ρ will be positive as long as $x < 6/5$. This inequality is true because $x_s < 2/3$. The energy-density is finite at the centre and taken to be about 10^{15} g/cm^3 . This defines b according to Eq. (26). Both terms in the expression for p_r in Eq. (30) decrease monotonically when r increases. We have arranged that $p_{rs} = 0$ (Eq. 36). Hence, in the interior p_r decreases monotonically to zero and is positive. This is confirmed by its derivative

$$kb^2 p_{rx} = -1 - \frac{4\alpha(1-x)(2+\alpha+\alpha x)}{(1+\alpha x)^2}. \tag{38}$$

It is obviously negative and since $x' = 2r/b^2$, p'_r is also negative. Finally, due to Eq. (32), $\Delta = p_t - p_r \geq 0$ and $p_t \geq p_r$. Therefore p_t is also positive and at $r = 0$ coincides with p_r . Their value at the centre is given by Eq. (30)

$$p_{t0} = p_{r0} = \frac{4\alpha}{kb^2}. \tag{39}$$

C5. Equation (25) gives

$$kb^2 \rho_x = -5. \tag{40}$$

Equations (32), (38) and (40) combine to deliver $\rho'(0) = p'_r(0) = p'_t(0) = 0$ and the monotonic decrease of ρ and p_r . It remains to prove that $p'_t \leq 0$. Before doing that let us turn to C7.

C7a. Causality condition for $dp_r/d\rho$. This ratio can be written as p_{rx}/ρ_x . We have just proved that the numerator and the denominator are negative, so their ratio is positive. Hence, the left inequality of the first part of Eq. (19) is true. The right inequality demands

$$-kb^2 p_{rx} \leq 5, \tag{41}$$

because of Eq. (40). Let us go now to C9.

C9. The anti-cracking condition may be written as

$$-1 + \frac{dp_r}{d\rho} \leq \frac{p'_t}{\rho'} \leq \frac{p'_r}{\rho'}. \tag{42}$$

We suppose that inequality (41) holds. Then the left hand side of Eq. (42) is negative. Let us multiply Eq. (42) by ρ' , which was shown to be negative. We have

$$\rho' \left(-1 + \frac{dp_r}{d\rho} \right) \geq p'_t \geq p'_r. \tag{43}$$

Now the left hand side is positive. Combining this chain of inequalities with the inequality $p'_t \leq 0$, that we have to prove, we obtain

$$0 \geq p'_t \geq p'_r. \tag{44}$$

Then we shall finish the proof of C5 and C9. These are the same inequalities, derived in [67], Eq. (26). In addition, since Eq. (44) may be written as

$$0 \leq \frac{dp_t}{d\rho} \leq \frac{dp_r}{d\rho} \leq 1, \tag{45}$$

we also prove C7b, the causality condition for $dp_t/d\rho$. Thus, C5 about p'_t , C7 and C9 are reduced to Eqs. (41) and (44). Moving to x -derivatives and subtracting p_{rx} from Eq. (44) we obtain

$$-p_{rx} \geq \Delta_x \geq 0. \tag{46}$$

Utilizing Eq. (32), we transform the above two inequalities into one:

$$-kb^2 p_{rx} \geq 2. \tag{47}$$

Finally, Eqs. (41) and (47) yield

$$2 \leq -kb^2 p_{rx} \leq 5. \tag{48}$$

To solve these two inequalities we use Eq. (38). Then the left inequality becomes

$$5\alpha^2 x^2 + 10\alpha x + 1 - 4\alpha(2+\alpha) \leq 0, \tag{49}$$

while the right inequality transforms into

$$2\alpha^2 x^2 + 4\alpha x + 1 - \alpha(2+\alpha) \geq 0. \tag{50}$$

The terms containing x in Eq. (50) increase with x , hence, it is enough to prove it for $x = 0$. Then it becomes an inequality for α

$$\alpha^2 + 2\alpha - 1 \leq 0. \tag{51}$$

The l.h.s. increases with α starting from -1 , therefore α should be less or equal than the positive root of the corresponding equation

$$\alpha \leq \alpha_1 = -1 + \sqrt{2} = 0.414. \tag{52}$$

Equation (36) is quadratic for x_s with α as parameter. The root less than one should be used to express x_s , namely

$$x_{s1} = \frac{8\alpha + 1 - \sqrt{16\alpha^2 + 16\alpha + 1}}{6\alpha}. \tag{53}$$

As we mentioned after Eq. (36), x_s decreases with α , hence $x_s \leq x_{s1}(\alpha_1)$ or

$$x_s \leq \frac{8\sqrt{2} - 7 - \sqrt{33 - 16\sqrt{2}}}{6(\sqrt{2} - 1)} = 0.439. \tag{54}$$

The terms containing x in Eq. (49) increase with x , hence, it is enough to prove it for x_s . Then, due to Eq. (36), it becomes a fourth degree equation for x_s . Going back to Eq. (38), one can write Eq. (49) as

$$(1 + \alpha x_s)^2 \leq 4\alpha(1 - x_s)(2 + \alpha + \alpha x_s). \tag{55}$$

Replacing α with its expression from Eq. (36) we get

$$4(1 - x_s)^4 \leq x_s(1 - x_s)^2(8 - 7x_s). \tag{56}$$

The fourth degree inequality surprisingly becomes a quadratic one, when we divide both sides by the common multiplier, and thus much easier to be solved. We have

$$11x_s^2 - 16x_s + 4 \leq 0. \tag{57}$$

The derivative of the l.h.s. is negative, so it decreases from 4 and becomes negative at the point, where the inequality becomes an equality. We solve this quadratic equation and take the root that is less than 1. The solution reads

$$x_s \geq \frac{8 - 2\sqrt{5}}{11} = 0.321. \tag{58}$$

Combining Eqs. (54) and (58) we obtain the range of x_s

$$0.321 = \frac{8 - 2\sqrt{5}}{11} \leq x_s \leq \frac{8\sqrt{2} - 7 - \sqrt{33 - 16\sqrt{2}}}{6(\sqrt{2} - 1)} = 0.439. \tag{59}$$

In this range C5, C7 and C9 hold.

Let us discuss next the energy conditions C6. The left part of the proven Eq. (19) may be written as

$$(p_r - \rho)' \geq 0, \tag{60}$$

since $\rho' \leq 0$. A definite integral of the l.h.s. is also positive,

$$\int_r^{r_s} (p_r - \rho)' dr = -p_r - \rho_s + \rho \geq 0, \tag{61}$$

which proves DEC for p_r , because $\rho_s \geq 0$.

The r.h.s. of Eq. (19) may be written as

$$(p_t - \rho)' \geq 0 \tag{62}$$

and the same integral of this inequality gives

$$\rho - p_t \geq \rho_s - p_{ts}. \tag{63}$$

Hence DEC for p_t holds in the interior, if it holds at the surface of the star. In [67] a sufficient universal condition was given, $u_s \leq 0.8$. For our model Eqs. (34) and (59) give

$$0.539 \leq u_s \leq 0.685, \tag{64}$$

so that the sufficient condition is satisfied in the whole range of x_s . We can also use the expressions for ρ (Eq. 25) and p_t (Eq. 33) to find

$$kb^2(\rho_s - p_{ts}) = 2(1 - x_s)(3 - 2\alpha + 5\alpha x_s). \tag{65}$$

Equations (36) and (59) show that $\alpha \in [0.184, 0.414]$. In this interval $3 - 2\alpha$ is positive and consequently the r.h.s. of the above equation is positive too. This proves that DEC holds for p_t as well.

Let us prove, finally, that TEC is true. Eq. (61) gives $\rho \geq p_r + \rho_s$. If

$$\rho_s \geq 2p_{r0}, \tag{66}$$

TEC follows from the chain of inequalities

$$\rho \geq p_r + 2p_{r0} = p_r + 2p_{r0} \geq p_r + 2p_t. \tag{67}$$

This chain is true due to Eq. (39) (the two pressures are equal at the centre of the star) and Eq. (44) (the tangential pressure decreases towards the stellar surface). Using Eqs. (25) and (30), Eq. (66) becomes

$$6 \geq 5x_s + 8\alpha. \tag{68}$$

As we know α increases with x_s till 0.414. Inserting this and the maximum of $x_s = 0.439$ in the above inequality, yields $6 \geq 5.567$, which obviously is true. Hence, TEC holds for the whole range of x_s . Thus C6 holds in its entirety.

Condition C8. In [67] a sufficient condition was given for Eq. (20) to hold, namely TEC and a lower limit for the radial speed of sound

$$\frac{dp_r}{d\rho} \geq \frac{1}{3}. \tag{69}$$

For our model TEC holds and

$$\frac{dp_r}{d\rho} = \frac{kb^2 p_{rx}}{kb^2 \rho_x} = -\frac{1}{5} kb^2 p_{rx}, \tag{70}$$

so that Eq. (69) becomes

$$-kb^2 p_{rx} \geq \frac{5}{3}. \tag{71}$$

This is true because of Eq. (48). Hence, the range of x_s , given by Eq. (59) allows to prove that C1–C9 are true.

The final condition C10. Combining Eqs. (26) and (34) yields

$$6m_s = k\rho_0 r_s^3 - \frac{1}{12} k^2 \rho_0^2 r_s^5. \tag{72}$$

The derivative of the total stellar mass with respect to the central density, when the star radius is kept constant, reads

$$\frac{dm_s(\rho_0)}{d\rho_0} = \frac{1}{6} k r_s^3 (1 - x_s) \tag{73}$$

and is obviously positive. Thus C10 is true and the whole set C1–C10 is true as long as x_s belongs to the range given by Eq. (59), $x_s \in [0.321, 0.439]$.

6 Model of a real star

The astronomers collect data about the radius r_s and the mass M of real stars. Usually the ratio β to the solar mass M_{sol} is used

$$\frac{M}{M_{sol}} = \frac{m_s}{m_{sol}} \equiv \beta. \tag{74}$$

It is known that in relativistic units $m_{sol} = 1.474$ km. Then we can find the compactness from Eq. (8)

$$u_s = 2m_{sol} \frac{\beta}{r_s} = 2.948 \frac{\beta}{r_s}, \tag{75}$$

where r_s is in km. The conformally flat solution is physically realistic when $x_s \in [0.321, 0.439]$. Equations (7) and (22) or Eq. (23) give a relation between u_s and x_s

$$x_s = 1 - \sqrt{1 - u_s}. \tag{76}$$

Equation (64) may be used too, $u_s \in [0.539, 0.685]$. Then

$$0.183 \leq \frac{\beta}{r_s} \leq 0.232. \tag{77}$$

Thus, it is very easy to find whether a real star may be described by our model. Only the surface compactness of the star matters or equivalently, its surface redshift. Equation (37) gives the limits $Z_s \in [0.473, 0.782]$. These are somewhat higher than the redshifts of many other models, discussed in the literature.

An important characteristic is the central density ρ_0 . We suppose that it may be written as

$$\rho_0 = a \times 10^{15} \text{ g/cm}^3, \tag{78}$$

where a is some constant close to 1. Equations (26) and (22) in CGS units give

$$\rho_0 = \frac{6x_s}{kc^2 r_s^2}. \tag{79}$$

Thus a is given by

$$a = \frac{6 \times 10^{-15} x_s}{kc^2 r_s^2} = 321.9 \frac{x_s}{r_s^2}, \tag{80}$$

where we have used $kc^2 = 1.864 \times 10^{-27}$ cm/g and r_s is given in km.

Let us apply these formulas to some stars, described in a recent paper [93]. There a model with given λ and Δ was used. The pulsar 4U1820-30 has $\beta = 1.58$ and $r_s = 9.1$ km. Then $\beta/r_s = 0.174$ and is out of range. The pulsar Cen X-3 has $\beta = 1.49$ and $r_s = 9.178$ km. Again $\beta/r_s = 0.162$ is too small. However, the pulsar PSR J1614-2230 has $\beta = 1.97$ and $r_s = 9.69$ km and $\beta/r_s = 0.203$ which satisfies Eq. (77) and may be a candidate for a compact neutron star with a conformally flat interior. We get from Eq. (75) $u_s = 0.598$. Equation (76) yields $x_s = 0.366$. Then we obtain from Eq. (80) $a = 1.258$, which is realistic. The surface redshift is comparatively high, $Z_s = 0.577$ (see Eqs. (18) or (37)), but is less than the known limits [72, 73]. The other characteristics of this star may be found from the formulas in the previous sections.

7 Discussion

We have followed in this paper the approach of [67]. Although, instead of ρ and p_r , we chose an ansatz for λ and the condition of conformal flatness, we have been able to satisfy all physical conditions without using graphic proofs. After all, C1–C10 involve inequalities, which, in principle, may be proved by algebra and calculus. Their use is limited to solving quadratic equations and integrating derivatives. The relations between the different conditions, that we found in the above reference allowed us to come to the same basic couple of inequalities, Eq. (44). It is interesting that the sufficient conditions for the other realistic features of the model are contained in them and do not restrict further the range of the main parameter x_s . A remarkable fact is that only the compactness (or the surface redshift) of the star is necessary to determine, whether it may have a conformally flat interior. The mass and the radius of the star are enough to determine all of its characteristics.

In the perfect fluid case the Buchdahl bounds [72] on the compactness and the redshift are $8/9 = 0.889$ and 2 (see C3). They are saturated by the Schwarzschild interior solution [2], which is an incompressible sphere with constant density. It is unphysical, because the speed of sound is infinite. Something more, the saturation occurs when the pressure is infinite at the centre [73]. This model is the unique conformally flat one for perfect (isotropic) fluids. This is one of the reasons to study conformally flat solutions in the anisotropic case. It may provide an explanation for the intermediate ranges of compactness and redshifts of realistic anisotropic solutions.

In the literature, in many papers the real strong energy condition (SEC) is used, which, however, is trivial. In many others, the trace energy condition (TEC) is called SEC and made use of. It is really strong, because it requires that the energy density (which in CGS units is multiplied by c^2) should be bigger than the sum of the radial and the two equal tangential pressures. Thus, it is even stronger than the dominant energy condition (DEC). We have tried to clarify this misuse of notation.

Finally, the proofs of C1–C10 were considerably simplified by the simple expression for the anisotropy factor Δ . In the case of embeddings of class one, the Karmarkar condition leads to a more sophisticated form for Δ . Therefore, the present paper may be considered also as a preparation to attack this case.

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