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Abstract

We study the convex hull of the intersection of a convex set $\mathcal{E}$ and a disjunctive set. This intersection is at the core of solution techniques for *Mixed Integer Convex Optimization*. We prove that if there exists a cone $\mathcal{K}$ (resp., a cylinder $\mathcal{C}$) that has the same intersection with the boundary of the disjunction as $\mathcal{E}$, then the convex hull is the intersection of $\mathcal{E}$ with $\mathcal{K}$ (resp., $\mathcal{C}$).

The existence of such a cone (resp., a cylinder) is difficult to prove for general conic optimization. We prove existence and unicity of a second order cone (resp., a cylinder), when $\mathcal{E}$ is the intersection of an affine space and a second order cone (resp., a cylinder). We also provide a method for finding that cone, and hence the convex hull, for the continuous relaxation of the feasible set of a Mixed Integer Second Order Cone Optimization (MISOCO) problem, assumed to be the intersection of an ellipsoid with a general linear disjunction. This cone provides a new conic cut for MISOCO that can be used in branch-and-cut algorithms for MISOCO problems.

1 Introduction

We consider the very general class of Mixed Integer Convex Optimization problems, which can be formulated as $\min\{c^\top x : x \in \mathcal{E}, x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}\}$, where $\mathcal{E}$ is a closed convex set. Solving such a problem often requires finding the convex hull of the intersection of $\mathcal{E}$ with a disjunction $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two half-spaces. In the first part of this paper, we prove that $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$ is the intersection of $\mathcal{E}$ with an appropriate cone $\mathcal{K}$.

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In the second part of this paper, we apply our result to a specific subclass of optimization problems where the set $\mathcal{E}$ is the intersection of an affine space and a second order cone. In order to establish a mindset that encompasses both mixed integer convex problems and mixed integer conic problems, we explicitly describe $\mathcal{E}$ as the intersection of a cone and an affine subspace. Note that any mixed integer convex problem can be described as a Mixed Integer Conic Optimization (MICO) problem and vice versa, since the former is a superset of the latter and any convex problem can be turned into a conic one by adding an auxiliary variable. Therefore, we consider problems of the form:

$$\begin{aligned} \text{minimize: } & c^\top x \\ \text{subject to: } & Ax = r \qquad\qquad \text{(MICO)} \\ & x \in \mathcal{K} \\ & x \in \mathbb{Z}^l \times \mathbb{R}^{n-l}, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\mathcal{K}$ is a convex cone, and the rows of A are linearly independent.

MICO problems comprise a wide range of discrete optimization problems. A very important class of MICO is the class of Mixed Integer Second Order Cone Optimization (MISOCO) problems, which find applications in engineering, finance and inventory problems [1, 10, 16, 20]. Theoretically, the integrality constraint can be tackled by means of a generic branch-and-bound algorithm. However, experience with Mixed Integer Linear Optimization (MILO) has shown that the development of methods for generating valid inequalities for the problem can improve the efficiency of the algorithm significantly [12]. The aim of this paper is the development of conic cuts for MICO problems.[1]

MICO problems are a class of non-convex optimization problems in which the non-convexity comes from the integrality of a subset of variables. Such non-convexity can be dealt with by means of disjunctive methods, which partition the set of feasible solutions into two or more feasible subsets. Disjunctive methods in mixed integer linear optimization have been studied extensively during the past decades [4, 5, 13, 17]. The contribution of this paper is twofold. First, we introduce conditions for the existence of a conic inequality arising as a disjunctive inequality for the general case of MICO that yields the convex hull of the intersection between a convex set and a disjunctive set (defined below). Second, we describe a procedure to find such a cut in the MISOCO case. The latter result allows us to generate second order cones for tightening the continuous relaxation of the MISOCO problem.

This paper is organized as follows. In §2 we present a brief review of the previous work done in MICO. Then, in §3 we derive conditions for the existence and unicity of the convex hull of the intersection between a disjunctive set and a closed convex set. In §4 we consider the special case of MISOCO: we introduce the disjunctive conic cut and a procedure to find it. We then compare our disjunctive cut with the conic cut introduced in [2] in §5. We provide some concluding remarks in §6.

Notation. Sets are denoted by script capital letters, matrices by capital letters, vectors by lower-case letters, and scalars by Greek letters. For a matrix M, M_{ij} is the (i, j) element, while M_j is the jth column. For vector v, its ith component is denoted as v_i.

[1] A cone is called a *conic cut* if it cuts off some non-integer solutions but none of the feasible integer solutions.

2 Literature review

There have been several attempts to extend some of the techniques developed for MILO to the case of MICO. For the MISOCO case, one approaches uses outer linear approximations of second order cones. Vielma et al. [25] used the polynomial-size polyhedral relaxation introduced by Ben-Tal and Nemirovski [9] in their "lifted linear programming" branch-and-bound algorithm for MISOCO problems. Krokhmal and Soberanis [19] generalized this approach for integer p-order conic optimization. Drewes [15] presented subgradient-based linear outer approximations for the second order cone constraints. This allows one to approximate the MISOCO problem by a mixed integer linear problem in a hybrid outer approximation branch-and-bound algorithm.

Stubbs and Mehrotra [24] generalized the lift-and-project algorithm of Balas et al. [6] for 0-1 MILO to 0-1 mixed integer convex problems. Later, Çezik and Iyengar [11] investigated the generation of valid convex cuts for 0-1 MICO problems and discussed how to extend the Chvátal-Gomory procedure for generating linear cuts for MICO problems and the extension of *lift-and-project* techniques for MICO problems. In particular, they showed how to generate linear and convex quadratic valid inequalities using the relaxation obtained by a project procedure. Later, Drewes [15] reviews the ideas proposed in [11] and [24] and applies them to MISOCO.

Atamtürk and Narayanan [2, 3] proposed two procedures for MISOCO problems that generate valid second order conic cuts. They first studied a generic lifting procedure for MICO [3], and then [2] extended the *mixed integer rounding* [22] procedure to the MISOCO case. The main idea in [2] is to reformulate a second order conic constraint using a set of two-dimensional second order cones. In this new reformulation the set of inequalities are called *polyhedral second-order conic constraint*. The authors used polyhedral analysis for studying these inequalities separately. This allowed the derivation of a mixed integer rounding procedure, which yields a *nonlinear conic mixed integer rounding*. A generalization of the use of polyhedral second-order conic constraints is presented by Masihabadi et al. [21].

Dadush et al. [14] studied the *split closure* of a strictly convex body and present a *conic quadratic inequality*. The conic quadratic inequality is introduced to present an example of a non-polyhedral split closure. In particular, the authors showed that it is necessary to consider conic quadratic inequalities in order to be able to describe the split closure of an ellipsoid. This independently obtained *conic quadratic inequality* coincides with the conic cut for MISOCO problems presented in §4.1.

3 The convex hull of a disjunctive convex set

We focus on the convex hull of the intersection of a full-dimensional closed convex set $\mathcal{E} \subseteq \mathbb{R}^n$, $n > 1$ with a *disjunctive set*. Consider a *disjunctive set* of the form

$$\mathcal{A} \cup \mathcal{B}, \tag{2}$$

where $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \geq \alpha\}$ and $\mathcal{B} = \{x \in \mathbb{R}^n \mid b^\top x \leq \beta\}$, are two half-spaces with $a, b \in \mathbb{R}^n$, and (a, α), (b, β) are not proportional, i.e., $\nexists \eta \in \mathbb{R}$ such that $a = \eta b$, $\alpha = \eta\beta$. This section presents a characterization of the convex hull of the set $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$.

Let $\mathcal{A}^= = \{x \in \mathbb{R}^n \mid a^\top x = \alpha\}$ and $\mathcal{B}^= = \{x \in \mathbb{R}^n \mid b^\top x = \beta\}$ denote the boundary hyperplanes of the half-spaces $\mathcal{A}$ and $\mathcal{B}$ respectively. Throughout this paper, we assume the following about the sets $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$:

Assumption 1. *$\mathcal{A} \cap \mathcal{B} \cap \mathcal{E}$ is empty.*

Assumption 2. *$\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are nonempty and bounded.*

3.1 Disjunctive conic cut

Let us recall the definition of a convex cone, as given by Barvinok [7, page 65].

Definition 1 (Convex Cone)**.** *A set $\mathcal{K} \subseteq \mathbb{R}^n$ is a convex cone if $0 \in \mathcal{K}$ and if for any two points $x, y \in \mathcal{K}$ and for any $\theta, \vartheta \geq 0$, we have $z = \theta x + \vartheta y \in \mathcal{K}$.*

Remark 1. *Observe that we can define a set $\hat{\mathcal{K}}$ as a translated cone if there exists a vector $x^* \in \hat{\mathcal{K}}$, called the* vertex *of $\hat{\mathcal{K}}$, such that for any $\theta, \vartheta \geq 0$ and $x, y \in \hat{\mathcal{K}}$, $x^* + (\theta(x - x^*) + \vartheta(y - x^*)) \in \hat{\mathcal{K}}$. One can use the translation $\mathcal{K} = \{y \in \mathbb{R}^n \mid y = x - x^*, x \in \hat{\mathcal{K}}\}$ to get a cone $\mathcal{K}$ in the sense of Definition 1. Although translated cones arise naturally in this setting, we assume w.l.o.g. that all cones have a vertex at the origin unless otherwise specified.*

Definition 2. *A closed convex cone $\mathcal{K} \subset \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is called a* disjunctive conic cut *(DCC) for the set $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$conv(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{E} \cap \mathcal{K}.$$

The following proposition gives a sufficient condition for a convex cone $\mathcal{K}$ to be a disjunctive conic cut for the set $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$.

Proposition 1. *A convex cone $\mathcal{K} \subset \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is a DCC for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$\mathcal{K} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad and \quad \mathcal{K} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=. \tag{3}$$

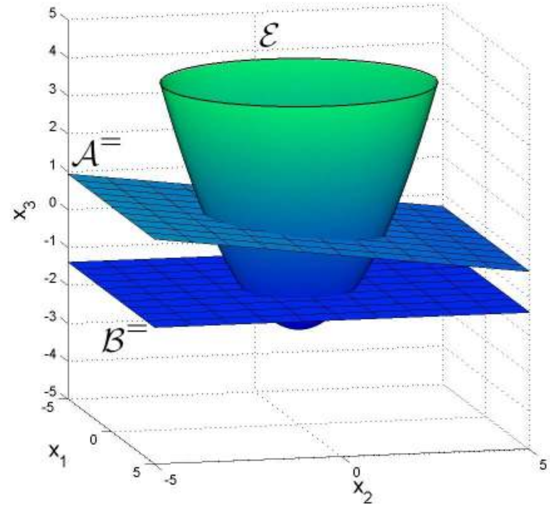
Figure 1 illustrates Proposition 1, where the set $\mathcal{E} \subset \mathbb{R}^3$ is the epigraph of a paraboloid. Before proving Proposition 1, we first provide a set of lemmas that will make the proof more compact. To begin, let us recall the definition of a *base of a cone* presented by Barvinok [7, page 66].

Definition 3 (Base of a cone)**.** *Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex cone. A set $\mathcal{L} \subset \mathcal{K}$ is called a* base *of $\mathcal{K}$ if $0 \notin \mathcal{L}$ and for every point $u \in \mathcal{K}$, $u \neq 0$, there is a unique $v \in \mathcal{L}$ and $\lambda > 0$ such that $u = \lambda v$.*

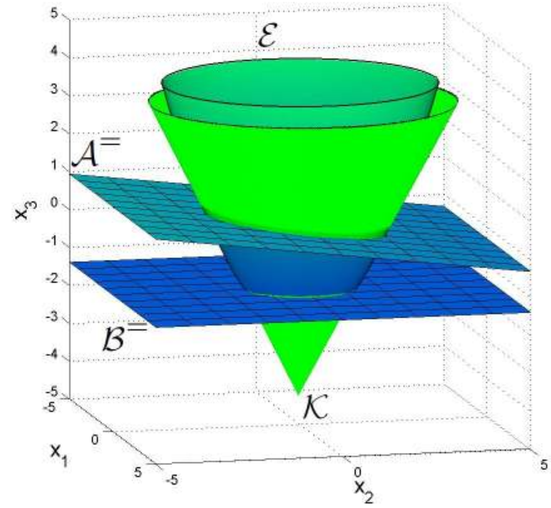
We can use Definition 3 to state Lemma 1, which shows a key relationship between the cone $\mathcal{K}$ and the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$.

Lemma 1. *Consider a half space $\mathcal{G} = \{x \in \mathbb{R}^n \mid g^\top x \leq \varrho\}$. Assume that $\mathcal{E} \cap \mathcal{G}^=$ is nonempty, bounded, and does not contain the origin 0. If there exists a convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, with $\dim(\mathcal{K}) > 1$ and $\mathcal{K} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$, then $\mathcal{E} \cap \mathcal{G}^=$ is a base of $\mathcal{K}$.*

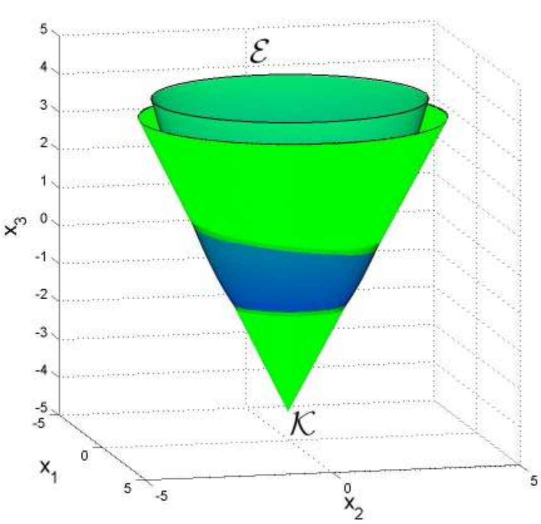
Proof. From the assumptions in the lemma, we have that $0 \notin \mathcal{K} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$. We may assume w.l.o.g. that $0 \in \mathcal{G}$. First, since $\mathcal{K} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$ is bounded we know that there exists no ray of $\mathcal{K}$ parallel to $\mathcal{G}^=$. Now, let us suppose that $\mathcal{E} \cap \mathcal{G}^=$ is not a base for $\mathcal{K}$. From Definition 3 we know that there must exist a point x such that $x \in \mathcal{K}$ but there exists no point $\hat{x} \in \mathcal{E} \cap \mathcal{G}^=$ for uniquely representing x as $\lambda \hat{x}$ for some $\lambda > 0$. Then, there is a ray in $\mathcal{K}$ parallel to the hyperplane $\mathcal{G}^=$. This implies that the set $\mathcal{K} \cap \mathcal{G}^=$ is unbounded, which contradicts the boundedness assumption of $\mathcal{E} \cap \mathcal{G}^=$. Therefore, $\mathcal{E} \cap \mathcal{G}^=$ is a base for $\mathcal{K}$. □ □



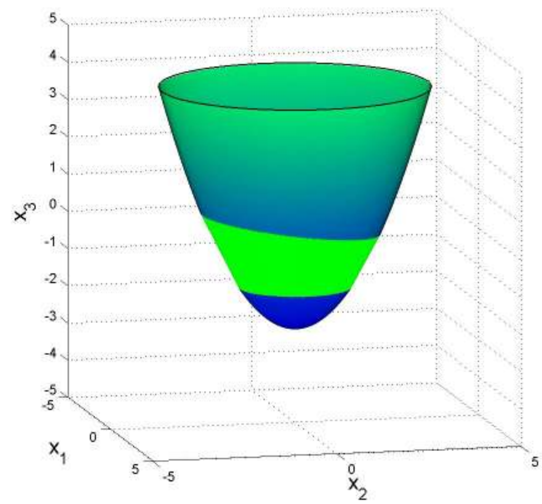
(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and $\mathcal{E}$



(b) The cone yielding $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$



(c) $\mathcal{E} \cap \mathcal{K}$



(d) $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

Figure 1: Illustration of a disjunctive conic cut as specified in Proposition 1

The result of Lemma 1 allows us to show that $\mathcal{K}$ is a pointed cone, which is an important result for our further development.

Lemma 2. *Any convex cone $\mathcal{K}$ satisfying Lemma 1 must be pointed.*

Proof. Assume that $\mathcal{K}$ is not pointed. This means that $\mathcal{K}$ contains a line. Hence, there exist two vectors $\hat{r}, \bar{r} \in \mathcal{K} \setminus \{0\}$ such that $\hat{r} = -\bar{r}$. Additionally, we have that $\mu\hat{r} + \nu\bar{r} \in \mathcal{K}$, for any $\mu, \nu > 0$. Now, since $\mathcal{E} \cap \mathcal{G}^=$ is a base of $\mathcal{K}$, there exists a point $\hat{x} \in \mathcal{E} \cap \mathcal{G}^=$ in the ray defined by $\hat{r}$ such that $\hat{x} = \mu\hat{r}$, for some $\mu > 0$. Similarly, there exists a point $\bar{x} \in \mathcal{E} \cap \mathcal{G}^=$ in the ray defined by $\bar{r}$ and

$\nu > 0 \in \mathbb{R}$ such that $\bar{x} = \nu\bar{r}$. Given that $\mathcal{G}^=$ is an affine set, we have

$$\gamma\hat{x} + (1-\gamma)\bar{x} \in \mathcal{G}^=, \quad \forall\gamma \in \mathbb{R}.$$

Expressing $\hat{x}$ and $\bar{x}$ in term of $\hat{r}$ and $\bar{r}$ gives

$$\begin{aligned}\gamma\hat{x} + (1-\gamma)\bar{x} &= \gamma(\mu\hat{r}) + (1-\gamma)(\nu\bar{r}) \\ &= -\gamma(\mu\bar{r}) + (1-\gamma)(\nu\bar{r}) \\ &= \nu\bar{r} - \gamma(\mu+\nu)\bar{r}.\end{aligned}$$

Hence, if $\gamma = 0$ then $\nu\bar{r} \in \mathcal{K}$. On the other hand, if $\gamma < 0$ we get that $\nu\bar{r} - \gamma(\mu+\nu)\bar{r} \in \mathcal{K}$, since it is a point on the ray defined by $\bar{r}$. Finally, if $\gamma > 0$ then $\nu\bar{r} - \gamma(\mu+\nu)\bar{r} = \nu\bar{r} + \gamma(\mu+\nu)\hat{r} \in \mathcal{K}$, since it is a positive combination of two points in the cone $\mathcal{K}$. Hence, $\mathcal{K} \cap \mathcal{G}^=$ contains a whole line, which contradicts the assumption that $\mathcal{K} \cap \mathcal{G}^=$ is bounded. $\square$ $\square$

We can now prove that the vertex of the cone $\mathcal{K}$ belongs exclusively to either $\mathcal{A}$ or $\mathcal{B}$. Observe that this does not mean that the set $\mathcal{A} \cap \mathcal{B}$ is empty, but that the vertex of $\mathcal{K}$ is not contained in it even when $\mathcal{A} \cap \mathcal{B}$ is nonempty.

Lemma 3. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex cone, with $\dim(\mathcal{K}) > 1$, such that $\mathcal{E} \cap \mathcal{A}^= = \mathcal{K} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^= = \mathcal{K} \cap \mathcal{B}^=$. Then the origin $x^* = 0$ is either in $\mathcal{A}$ or in $\mathcal{B}$, but not in $\mathcal{A} \cap \mathcal{B}$.*

Proof. First, consider the case when $x^* \in \mathcal{A}^=$. Then, we have that $x^* \in \mathcal{E} \cap \mathcal{A}^=$, since $\mathcal{E} \cap \mathcal{A}^= = \mathcal{K} \cap \mathcal{A}^=$. Hence, from Assumption 1, we have that $x^* \notin \mathcal{B}$. Similarly, we have that if $x^* \in \mathcal{B}^=$, then $x^* \notin \mathcal{A}$.

Second, assume that neither $\mathcal{A}^=$ nor $\mathcal{B}^=$ contain x^*. By Lemma 1 we have that $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are bases of the cone $\mathcal{K}$. Additionally, by Lemma 2 we know that the cone $\mathcal{K}$ is pointed. Let x be a unit length vector defining a ray of $\mathcal{K}$. Then, there are two points $\hat{x} \in \mathcal{E} \cap \mathcal{A}^=$ and $\bar{x} \in \mathcal{E} \cap \mathcal{B}^=$ such that $\hat{x} = \mu x$ and $\bar{x} = \nu x$ for some $\mu, \nu > 0$.

We prove first that $x^* \in \mathcal{A} \cup \mathcal{B}$. Let us assume to the contrary that $x^* \in \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$, where the bar denotes the complement set. Let $y = \gamma x$ for $\gamma \geq 0$ be a point in the ray defined by x. Then, for any $\gamma < \min\{\nu, \mu\}$ we have that $y \in \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$, and w.l.o.g. we may assume that $\nu < \mu$. Note that we cannot have $\nu = \mu$ as, by Assumption 1, $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E} = \emptyset$. Additionally, for any $\gamma \geq \nu$ we have that $y \in \mathcal{B}$, so the point $\hat{x}$ is contained in the half-space $\mathcal{B}$, and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E} \neq \emptyset$, which contradicts Assumption 1.

Now, we prove that $x^* \notin \mathcal{A} \cap \mathcal{B}$. Let us assume to the contrary that $x^* \in \mathcal{A} \cap \mathcal{B}$, and let $y = \gamma\bar{x} + (1-\gamma)\hat{x}$ for some $0 \leq \gamma \leq 1$. Then, we have that $y \in \mathcal{A}$ or $y \in \mathcal{B}$. When $\nu < \mu$ and $\gamma = 1$ implies that $y = \bar{x}$ and we have $y \in \mathcal{A} \cap \mathcal{B}^= \cap \mathcal{E}$. Similarly, when $\mu < \nu$ and $\gamma = 0$ implies that $y = \hat{x}$ and we have $y \in \mathcal{A}^= \cap \mathcal{B} \cap \mathcal{E}$. Hence, $x^* \in \mathcal{A} \cap \mathcal{B}$ implies $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E} \neq \emptyset$, which contradicts Assumption 1. $\square$ $\square$

We are able now to show that $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}) \subset \mathcal{K}$. This will facilitate the proof of the relation $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) \subseteq \mathcal{E} \cap \mathcal{K}$.

Lemma 4. *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex cone, with $\dim(\mathcal{K}) > 1$, for which (3) holds. Then:*

$$(\mathcal{E} \cap \mathcal{A}) \subset \mathcal{K} \quad \textit{and} \quad (\mathcal{E} \cap \mathcal{B}) \subset \mathcal{K}.$$

Proof. We start showing that if $\mathcal{E}\cap\mathcal{A}^=$ is a single point, then $\mathcal{E}\cap\mathcal{A}\subseteq\mathcal{K}$. First, if $\mathcal{E}\cap\mathcal{A}^=$ is a single point, then $0\in\mathcal{E}\cap\mathcal{A}^=$, otherwise $\dim(\mathcal{K})=1$. The last statement follows from the assumption $\mathcal{E}\cap\mathcal{A}^= = \mathcal{K}\cap\mathcal{A}^=$. Henceforth, we obtain in this case that $\mathcal{E}\cap\mathcal{A}^= = \{0\}$. Now, it is clear that $\mathcal{E}\cap\mathcal{A}^=\subseteq\mathcal{E}\cap\mathcal{A}$, hence we need to show that $\mathcal{E}\cap\mathcal{A}\subseteq\mathcal{E}\cap\mathcal{A}^=$. Assume to the contrary that there exists a point $x\in\mathcal{E}\cap\mathcal{A}$ such that $x\notin\mathcal{E}\cap\mathcal{A}^=$. Additionally, consider a point $y\in\mathcal{E}\cap\mathcal{B}^=$, which implies that $y\notin\mathcal{E}\cap\mathcal{A}$. Then, from convexity of $\mathcal{K}$ we have that for any $\theta,\gamma\geq 0$ the point $\theta x+\gamma y\in\mathcal{K}$, which contradicts that $\mathcal{E}\cap\mathcal{A}^=$ is a single point. Henceforth, we obtain that $\mathcal{E}\cap\mathcal{A}=\mathcal{E}\cap\mathcal{A}^=\subseteq\mathcal{K}$. Similarly, if $\mathcal{E}\cap\mathcal{B}^=$ is a single point, then one can show that $\mathcal{E}\cap\mathcal{B}=\mathcal{E}\cap\mathcal{B}^=\subseteq\mathcal{K}$. Note that $\mathcal{E}\cap\mathcal{A}^=$ and $\mathcal{E}\cap\mathcal{B}^=$ cannot be single points simultaneously, which follows from Assumption 1 and $\dim(\mathcal{K})>1$.

Now, we prove that if $\mathcal{E}\cap\mathcal{A}^=$ is not a single point then $(\mathcal{E}\cap\mathcal{A})\subseteq\mathcal{K}$. Let us assume to the contrary that there exists a vector x such that $x\in(\mathcal{E}\cap\mathcal{A})$ but $x\notin\mathcal{K}$. First, by the separation theorem[2], there exists a hyperplane $\mathcal{H}$ separating x and $\mathcal{K}$ that contains a ray of $\mathcal{K}$, denoted by $\mathcal{K}_r$, and does not contain x. Here the assumption of $\dim(\mathcal{K})>1$ is needed, since the hyperplane $\mathcal{H}$ does not exist when $\dim(\mathcal{K})=1$.

From Lemma 3 we know that $0\in\mathcal{A}$ or $0\in\mathcal{B}$. On one hand, if $0\notin\mathcal{E}\cap\mathcal{B}^=$, then it follows from (3), Assumption 2, Lemma 1, that the sets $\mathcal{E}\cap\mathcal{A}^=$ and $\mathcal{E}\cap\mathcal{B}^=$ are bases for the cone $\mathcal{K}$. Hence, there exists a vector $w\in\mathcal{E}\cap\mathcal{B}^=$ that defines the ray $\mathcal{K}_r$. On the other hand, if $0\in\mathcal{E}\cap\mathcal{B}^=$ then we know that $\mathcal{E}\cap\mathcal{B}^= = \{0\}$. In this case, one can take $w=0$, since $0\in\mathcal{K}_r$.

Given that the set $\mathcal{E}$ is convex, $\lambda x+(1-\lambda)w\in\mathcal{E}$ for all $0\leq\lambda\leq 1$. On the other hand, since w is a point on a face of $\mathcal{K}$, we have that $\lambda x+(1-\lambda)w\notin\mathcal{K}$ for $0<\lambda\leq 1$. Furthermore, since $x\in(\mathcal{E}\cap\mathcal{A})$ and $\mathcal{A}\cap\mathcal{B}\cap\mathcal{E}=\emptyset$, we have $a^\top x\geq\alpha$ and $a^\top w<\alpha$. Hence, from the equation $a^\top(\lambda x+(1-\lambda)w)=\lambda a^\top x+(1-\lambda)a^\top w$, there exists a $\lambda\in(0,1]$ such that $a^\top(\lambda x+(1-\lambda)w)=\alpha$. Therefore, there is a vector $\hat{x}=\lambda x+(1-\lambda)w$ for some $\lambda\in(0,1]$, such that $\hat{x}\in\mathcal{E}\cap\mathcal{A}^=$, but $\hat{x}\notin\mathcal{K}$, which contradicts condition (3). Hence, $(\mathcal{E}\cap\mathcal{A})\subseteq\mathcal{K}$. Analogously, one can prove that $(\mathcal{E}\cap\mathcal{B})\subseteq\mathcal{K}$ when $(\mathcal{E}\cap\mathcal{B})$ is not a single point.

Recall that the sets $\mathcal{E}\cap\mathcal{A}^=$ and $\mathcal{E}\cap\mathcal{B}^=$ are disjoint and nonempty. Then, by condition (3) we have that $\mathcal{E}\cap\mathcal{A}\neq\mathcal{K}$ and $\mathcal{E}\cap\mathcal{B}\neq\mathcal{K}$, and the result of the lemma follows. □ □

Now we present the proof of Proposition 1.

Proof of Proposition 1. First, we prove that if $\mathcal{E}\cap\mathcal{A}^= = \mathcal{K}\cap\mathcal{A}^=$ and $\mathcal{E}\cap\mathcal{B}^= = \mathcal{K}\cap\mathcal{B}^=$ then $\mathcal{K}$ is a DCC. Consider a point $x\in(\mathcal{E}\cap\mathcal{A})\cup(\mathcal{E}\cap\mathcal{B})$. Then, from Lemma 4 we have that $x\in\mathcal{E}\cap\mathcal{K}$. Now, consider any two points $x,y\in(\mathcal{E}\cap\mathcal{A})\cup(\mathcal{E}\cap\mathcal{B})$. Then, since both $\mathcal{K}$ and $\mathcal{E}$ are convex, for any $0\leq\lambda\leq 1$ we have $\lambda x+(1-\lambda)y\in\mathcal{E}\cap\mathcal{K}$. Hence, $\operatorname{conv}(\mathcal{E}\cap(\mathcal{A}\cup\mathcal{B}))\subseteq\mathcal{E}\cap\mathcal{K}$.

Consider a point $x\in\mathcal{E}\cap\mathcal{K}$. First, if $x\in\mathcal{E}\cap\mathcal{A}$ or $x\in\mathcal{E}\cap\mathcal{B}$, we have that $x\in\operatorname{conv}(\mathcal{E}\cap(\mathcal{A}\cup\mathcal{B}))$. Assume then that $x\notin(\mathcal{E}\cap\mathcal{A})\cup(\mathcal{E}\cap\mathcal{B})$, which implies $x\in(\bar{\mathcal{A}}\cap\bar{\mathcal{B}}\cap\mathcal{K})$. Furthermore, by Lemma 1 there are two vectors $\hat{x}\in\mathcal{E}\cap\mathcal{A}^=$ and $\bar{x}\in\mathcal{E}\cap\mathcal{B}^=$ such that, for some $\mu,\nu\geq 0$, $\hat{x}=\mu x$ and $\bar{x}=\nu x$. From Lemma 3, the vertex of the cone is either in $\mathcal{A}$ or $\mathcal{B}$ but not in both. Assume w.l.o.g. that the vertex of the cone is in $\mathcal{B}$. Then, $\nu<1<\mu$ and there exists a $\gamma\in(0,1)$ such that $\gamma\nu+(1-\gamma)\mu=1$. Hence, we can write

$$\begin{aligned}\gamma\bar{x}+(1-\gamma)\hat{x}&=\gamma\nu x+(1-\gamma)\mu x\\&=(\gamma\nu+(1-\gamma)\mu)x\\&=x.\end{aligned}$$

[2] Lemma 8.2 in Barvinok [7, page 65] and Theorems 11.3 and 11.7 in Rockafeller [23, pages 97 and 100].

Therefore, x can be expressed as a linear convex combination of two points in $(\mathcal{E} \cap \mathcal{A}^=) \cup (\mathcal{E} \cap \mathcal{B}^=)$. Hence, any point $x \in (\mathcal{E} \cap \mathcal{K})$ can be written as a linear convex combination of two points in $(\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$. Thus, $(\mathcal{E} \cap \mathcal{K}) \subseteq \operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. Finally, since the subset relation is valid in both directions, this proves that $(\mathcal{E} \cap \mathcal{K}) = \operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. Finally, since $(\mathcal{E} \cap \mathcal{A}^=)$ and $(\mathcal{E} \cap \mathcal{B}^=)$ are compact sets, then it follows from Lemma 1 and Lemma 8.6 in Barvinok [7, page 67] that $\mathcal{K}$ is closed. □ □

We close the analysis by showing that if a disjunctive conic cut exist, then it is unique.

Lemma 5. *If a closed convex cone $\mathcal{K}$ exists, with $\dim(\mathcal{K}) > 1$, satisfying property (3), then $\mathcal{K}$ is unique.*

Proof. Assume to the contrary that there are two different cones $\mathcal{K}_1$ and $\mathcal{K}_2$ that satisfy property (3). Let $v^1 \in \mathcal{K}_1$ be the vertex of $\mathcal{K}_1$ and $v^2 \in \mathcal{K}_2$ be the vertex of $\mathcal{K}_2$. Now, we may assume w.l.o.g. that $v^1 = 0$.

First, we prove that if either $\mathcal{E} \cap \mathcal{A}^=$ or $\mathcal{E} \cap \mathcal{B}^=$ is a single point, then $\mathcal{K}_1 = \mathcal{K}_2$. Since $\dim(\mathcal{K}_1) > 1$ and $\dim(\mathcal{K}_2) > 1$, we have that $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ cannot be both single point sets. Let $u \in \mathcal{E}$, and assume that $\mathcal{E} \cap \mathcal{A}^= = \{u\}$, then $\mathcal{K}_1 \cap \mathcal{A}^= = \{u\}$ and $\mathcal{K}_2 \cap \mathcal{A}^= = \{u\}$. Now, if $u \neq v^1$, then we have that $\mathcal{K}_1 = \{\theta v^1 \mid \theta \geq 0\}$, which implies that the set $\mathcal{E} \cap \mathcal{B}^=$ is a single point. Thus, we have that $u = v^1$. On the other hand, if $u \neq v^2$, then we have that $\mathcal{K}_2 = \{y \in \mathbb{R}^n \mid y = v^2 + \theta(u - v^2), \theta \geq 0\}$, which also implies that the set $\mathcal{E} \cap \mathcal{B}^=$ is a single point. Hence, we have that $u = v^2$. Therefore, we have that $v^1 = v^2$, and since $\mathcal{E} \cap \mathcal{B}^= = \mathcal{K}_1 \cap \mathcal{B}^= = \mathcal{K}_2 \cap \mathcal{B}^=$, we obtain that $\mathcal{K}_1 = \mathcal{K}_2$. A symmetric argument would show that $\mathcal{K}_1 = \mathcal{K}_2$ if $\mathcal{E} \cap \mathcal{B}^= = \{z\}$.

Second, we show that if $\{v^1, v^2\} \cap (\mathcal{A}^= \cup \mathcal{B}^=) = \emptyset$, then $v^1 \in \mathcal{K}_2$ and $v^2 \in \mathcal{K}_1$. Assume to the contrary that $v^1 \notin \mathcal{K}_2$. Here use a similar argument to the one in the proof of Lemma 4. By the separation theorem, there exists a hyperplane $\mathcal{H}$ separating v^1 and $\mathcal{K}_2$ properly such that $v^1 \notin \mathcal{H}$. From Lemma 1, we know that the sets $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are bases for $\mathcal{K}_2$. Hence, there exists a vector $w \in \mathcal{E} \cap \mathcal{B}^=$ such that the extreme ray $\mathcal{R}_w = \{v^2 + \gamma(w - v^2) \mid \gamma \geq 0\}$ of $\mathcal{K}_2$ is in $\mathcal{H}$. Additionally, by Lemma 3 we have that v^1 is either in $\mathcal{A}$ or $\mathcal{B}$ but not in $\mathcal{A} \cap \mathcal{B}$. We may assume w.l.o.g. that $v^1 \in \mathcal{A}$. Given that $\mathcal{K}_1$ is convex, $\lambda v^1 + (1-\lambda)w \in \mathcal{K}_1$ for all $0 \leq \lambda \leq 1$. On the other hand, since w is a vector on an exposed face of $\mathcal{K}_2$, for $0 < \lambda \leq 1$ we have $\lambda v^1 + (1-\lambda)w \notin \mathcal{K}_2$. Furthermore, since $v^1 \in \mathcal{A}$, by Assumption 1 we have $a^\top v^1 \geq \alpha$ and $a^\top w < \alpha$. Hence, from the equation $a^\top(\lambda v^1 + (1-\lambda)w) = \lambda a^\top v^1 + (1-\lambda)a^\top w$, we may obtain $0 < \lambda \leq 1$ such that $a^\top(\lambda v^1 + (1-\lambda)w) = \alpha$. Therefore, there exists a vector $u = \lambda v^1 + (1-\lambda)w$ for some $0 < \lambda \leq 1$, such that $u \in \mathcal{K}_1 \cap \mathcal{A}^=$, but $u \notin \mathcal{K}_2$, which contradicts $\mathcal{K}_1 \cap \mathcal{A}^= = \mathcal{K}_2 \cap \mathcal{A}^=$. Hence, we obtain that $v^1 \in \mathcal{K}_2$. Using a similar argument one can prove that $v^2 \in \mathcal{K}_1$.

Third, we show that if $v^1 \neq v^2$, then they cannot be both in $\mathcal{A}$ or in $\mathcal{B}$. Assume to the contrary that $v^1 \in \mathcal{A}$ and $v^2 \in \mathcal{A}$. Note that if $\alpha > 0$, then we have $v^1 \notin \mathcal{A}$, thus we assume that $\alpha \leq 0$. On one hand, since $v^1 \in \mathcal{K}_2$ we have that $\mathcal{R}_{v^1} = \{(1-\theta)v^2 \mid \theta \geq 0\} \subseteq \mathcal{K}_2$. Hence, if $a^\top v^2 \leq 0$, then $\mathcal{R}_{v^1} \subseteq \mathcal{A}$ which implies that $\mathcal{R}_{v^1}$ is parallel to $\mathcal{A}^=$, and we obtain that $\mathcal{A}^= \cap \mathcal{K}_2$ is unbounded. On the other hand, since $v^2 \in \mathcal{K}_1$ we have that $\mathcal{R}_{v^2} = \{\theta v^2 \mid \theta \geq 0\} \subseteq \mathcal{K}_1$. Hence, if $a^\top v^2 \geq 0$, then $\mathcal{R}_{v^2} \subseteq \mathcal{A}$, which implies that $\mathcal{R}_{v^2}$ is parallel to $\mathcal{A}^=$, and we obtain that $\mathcal{A}^= \cap \mathcal{K}_1$ is unbounded. Hence, if $v^1 \in \mathcal{A}$ and $v^2 \in \mathcal{A}$, then we obtain a contradiction to Assumption 2. Similarly, we can prove that v^1 and v^2 cannot be simultaneously in $\mathcal{B}$.

Finally, we show that if v^1 and v^2 are in different halfspaces and $\{v^1, v^2\} \cap (\mathcal{A}^= \cup \mathcal{B}^=) = \emptyset$, then this contradicts the assumption that $\mathcal{K}_1 \cap \mathcal{A}^= = \mathcal{K}_2 \cap \mathcal{A}^=$ and $\mathcal{K}_1 \cap \mathcal{B}^= = \mathcal{K}_2 \cap \mathcal{B}^=$. Assume that $v^1 \in \mathcal{A}$ and $v^2 \in \mathcal{B}$. Recall that in this case $v^1 \in \mathcal{K}_2$ and $v^2 \in \mathcal{K}_1$, thus the set $\mathcal{R}_{v^1} =$

$\{(1-\theta)v^2 \mid \theta \geq 0\} \subseteq \mathcal{K}_2$ and $\mathcal{R}_{v^2} = \{\theta v^2 \mid \theta \geq 0\} \subseteq \mathcal{K}_1$. Now, since $\dim(\mathcal{K}_1) > 1$ and $\mathcal{B}^= \cap \mathcal{K}_1$ is a base of $\mathcal{K}_1$, there is at least one extreme ray $\mathcal{R}_w = \{\gamma w \mid \gamma \geq 0\}$ of $\mathcal{K}_1$ such that $v^2 \notin \mathcal{R}_w$ and $w \in \mathcal{K}_1 \cap \mathcal{B}^= = \mathcal{K}_2 \cap \mathcal{B}^=$ is a vector in the boundary of $\mathcal{K}_1$. Now, if $w \in \mathrm{ri}(\mathcal{K}_2)$, then, since $\mathcal{K}_2 \cap \mathcal{B}^=$ is bounded and is a base of $\mathcal{K}_2$, we have that $w \in \mathrm{ri}(\mathcal{K}_2 \cap \mathcal{B}^=)$. Thus, in this case there exists a vector $u \in \mathcal{K}_2 \cap \mathcal{B}^=$ such that $u \notin \mathcal{K}_1 \cap \mathcal{B}^=$, which contradicts $\mathcal{K}_1 \cap \mathcal{B}^= = \mathcal{K}_2 \cap \mathcal{B}^=$.

Assume now that w is a vector on the boundary of $\mathcal{K}_2$. Since $w \in \mathcal{K}_2$, we have that $\{v^2 + \gamma(w - v^2) \mid \gamma \geq 0\} \subseteq \mathcal{K}_2$. Moreover, since $v^2 \in \mathcal{B}$ and $w \in \mathcal{B}^=$, there exists a $\hat{\gamma} > 1$ such that $a^\top(v^2 + \hat{\gamma}(w - v^2)) = \alpha$. However, since w is on the extreme ray $\mathcal{R}_w$ of $\mathcal{K}_1$ and $v^2 \notin \mathcal{R}_w$, then the vector $(v^2 + \hat{\gamma}(w - v^2)) \notin \mathcal{K}_1$. This contradicts the assumption $\mathcal{K}_1 \cap \mathcal{A}^= = \mathcal{K}_2 \cap \mathcal{A}^=$. A symmetric argument is valid if we assume that $v^1 \in \mathcal{B}$ and $v^2 \in \mathcal{A}$. Hence, since v^1 and v^2 cannot be in different halfspaces, then $v^1 = v^2$. In conclusion, we have that $\mathcal{K}_1 = \mathcal{K}_2$, since $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are bases for $\mathcal{K}_1$ and $\mathcal{K}_2$, which proves that the disjunctive conic cut is unique. □ □

Figure 2 illustrates how Lemma 5 fails when the intersections $\mathcal{E} \cap \mathcal{A}^=$ or $\mathcal{E} \cap \mathcal{B}^=$ are unbounded. In this case, one can see that the $\mathcal{K} \cap \mathcal{E}$ is the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. The other two cones $\mathcal{K}_1$ and $\mathcal{K}_2$ have the same intersections with $\mathcal{A}^=$ and $\mathcal{B}^=$ as the convex set $\mathcal{E}$. However, the intersections $\mathcal{K}_1 \cap \mathcal{E}$ and $\mathcal{K}_2 \cap \mathcal{E}$ fail to give $\mathrm{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$.

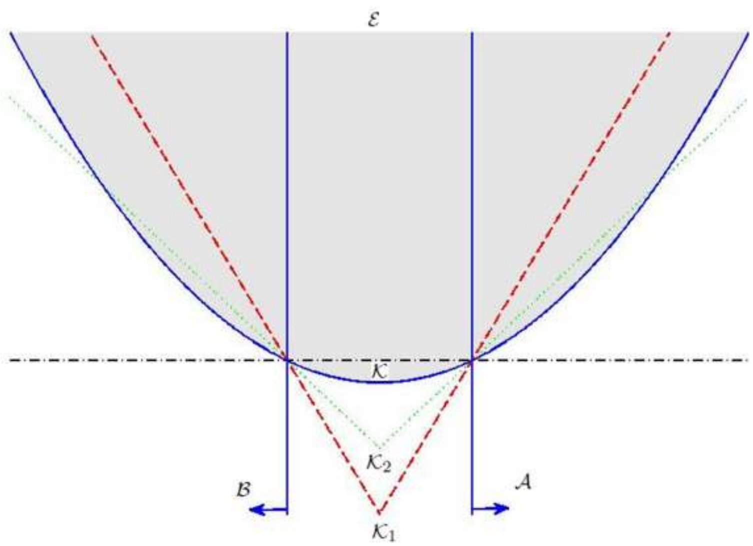
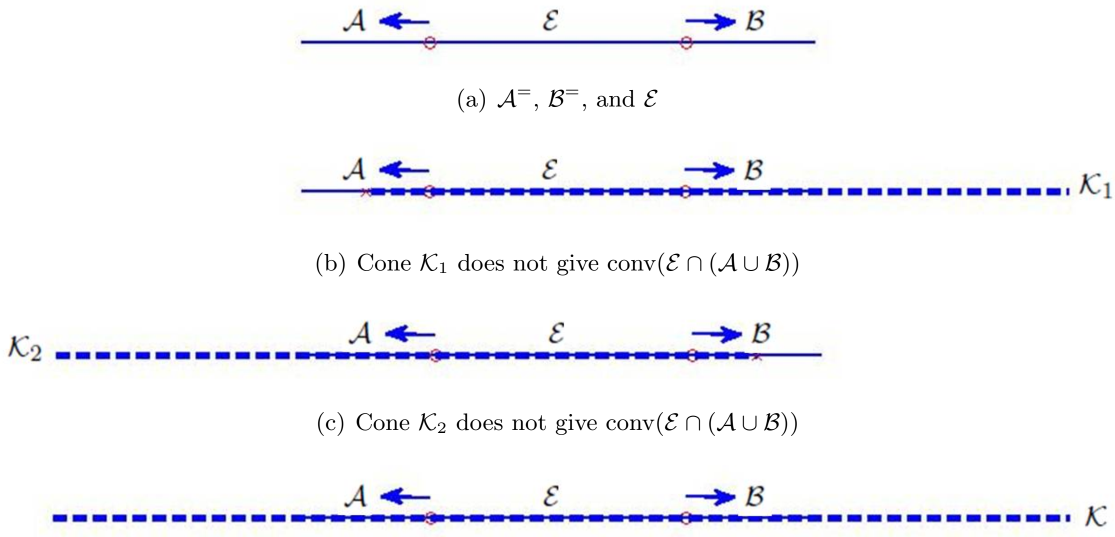


Figure 2: Example of unbounded intersections

Another important case to consider here is when the set $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$ is of dimension $n = 1$. Figure 3(a) illustrates this case. In particular, we can see that the uniqueness in Lemma 5 fails in this case too. Observe the cone $\mathcal{K}_1$ in Figure 3(b) and the cone $\mathcal{K}_2$ in Figure 3(c), which are given by two half-lines. These two cones have the same intersections with $\mathcal{A}^=$ and $\mathcal{B}^=$ as the set $\mathcal{E}$. However, the intersections $\mathcal{E} \cap \mathcal{K}_1$ and $\mathcal{E} \cap \mathcal{K}_2$ differ from $\mathrm{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. In this case, the cone $\mathcal{K}$ in Figure 3(c), given by a line, is such that $\mathcal{E} \cap \mathcal{K} = \mathrm{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$.

3.2 Disjunctive cylindrical cut

Let us now present the definition of a convex cylinder.



(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and $\mathcal{E}$

(b) Cone $\mathcal{K}_1$ does not give $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

(c) Cone $\mathcal{K}_2$ does not give $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

(d) $\mathcal{K} \cap \mathcal{E} = \text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

Figure 3: Example when the set $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$ has dimension $n = 1$

Definition 4 (Convex Cylinder). *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex set and $d_0 \in \mathbb{R}^n$ a vector. Then, the set $\mathcal{C} = \{x \in \mathbb{R}^n | x = d + \sigma d_0, d \in \mathcal{D}, \sigma \in \mathbb{R}\}$ is a convex cylinder in $\mathbb{R}^n$.*

Definition 5. *A closed convex cylinder $\mathcal{C}$ is a* disjunctive cylindrical cut *for the set $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$\mathit{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{C} \cap \mathcal{E}.$$

The following proposition gives a sufficient condition for a convex cylinder $\mathcal{C}$ to be a disjunctive cylindrical cut. The result and proofs for the cylinder case are similar to the cone case, still we provide them for completeness.

Proposition 2. *A convex cylinder $\mathcal{C}$ is a disjunctive cylindrical cut for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if*

$$\mathcal{C} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad \textit{and} \quad \mathcal{C} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=. \tag{4}$$

Figure 4 illustrates Proposition 2, where the set $\mathcal{E}$ is the epigraph of a paraboloid. Before proving Proposition 2 we first provide a set of lemmas that will ease to understand the proof. First, let us define the *base* of a cylinder in a similar way as the base of a cone is defined in [7].

Definition 6 (Base of a cylinder). *Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex cylinder. A set $\mathcal{D} \subset \mathcal{C}$ is called a* base *of $\mathcal{C}$ if, for every point $x \in \mathcal{C}$, there is a unique $d \in \mathcal{D}$ and $\sigma \in \mathbb{R}$ such that $x = d + \sigma d_0$.*

Lemma 6. *Consider a half space $\mathcal{G} = \{x \in \mathbb{R}^n \mid g^\top x \le \varrho\}$. Assume that $\mathcal{E} \cap \mathcal{G}^=$ is nonempty and bounded. If $\mathcal{C} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$, then $\mathcal{E} \cap \mathcal{G}^=$ is a base for $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a cylinder such that $\mathcal{C} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$. Observe that if $g^\top d_0 = 0$ then for any $\hat{x} \in \mathcal{C} \cap \mathcal{G}^=$ we have that $\{y \in \mathbb{R}^n \mid y = \hat{x} + \sigma d_0, \sigma \in \mathbb{R}\} \subseteq \mathcal{C} \cap \mathcal{G}^=$, which is an unbounded set. Hence, $g^\top d_0 \neq 0$ because $\mathcal{C} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$ is bounded. Now, let us assume that $\mathcal{E} \cap \mathcal{G}^=$ is not a base for $\mathcal{C}$. Then, from Definition 6 we know that there exists a point $x \in \mathcal{C}$ such that there exists no point $\bar{x} \in \mathcal{E} \cap \mathcal{G}^=$ that represents x as $\bar{x} + \sigma d_0$ for some $\sigma \in \mathbb{R}$. Thus, $\{y \in \mathbb{R}^n \mid y = x + \sigma d_0, \sigma \in \mathbb{R}\} \cap \mathcal{E} \cap \mathcal{G}^= = \emptyset$. However, with $\hat{\sigma} = (\varrho - g^\top x)/g^\top d_0$ we obtain that $x + \hat{\sigma} d_0 \in \mathcal{C} \cap \mathcal{G}^= = \mathcal{E} \cap \mathcal{G}^=$ whenever $g^\top d_0 \neq 0$. Therefore, the relation $\{y \in \mathbb{R}^n \mid y = x + \sigma d_0, \sigma \in \mathbb{R}\} \cap \mathcal{E} \cap \mathcal{G}^= = \emptyset$ is true only if $g^\top d_0 = 0$. Hence, if $\mathcal{E} \cap \mathcal{G}^=$ is not a base for $\mathcal{C}$, then we have that $\mathcal{E} \cap \mathcal{G}^=$ is unbounded, which contradicts the boundedness assumption of $\mathcal{E} \cap \mathcal{G}^=$. Therefore, $\mathcal{E} \cap \mathcal{G}^=$ is a base for $\mathcal{C}$. □ □

The next lemma states the relationship between cylinder $\mathcal{C}$ and the intersections of $\mathcal{E}$ with the half spaces $\mathcal{A}$ and $\mathcal{B}$.

Lemma 7. *Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex cylinder $\mathcal{C}$, for which* (4) *holds. Then*

$$(\mathcal{E} \cap \mathcal{A}) \subset \mathcal{C} \quad \text{and} \quad (\mathcal{E} \cap \mathcal{B}) \subset \mathcal{C}.$$

Proof. We prove first that $(\mathcal{E} \cap \mathcal{A}) \subseteq \mathcal{C}$. Let us assume to the contrary that there exists a $x \in (\mathcal{E} \cap \mathcal{A})$ such that $x \notin \mathcal{C}$. First, by the separation theorem, there exists a hyperplane $\mathcal{H} = \{y \in \mathbb{R}^n \mid h^\top y = \eta\}$ separating x from $\mathcal{C}$. From the definition of $\mathcal{C}$ we have that $h^\top d_0 = 0$. Now, let $\mathcal{H}$ be a supporting hyperplane of $\mathcal{C}$, which implies that $\mathcal{H} \cap \mathcal{C}$ is an exposed face of $\mathcal{C}$. Note that for any $\hat{y} \in \mathcal{H} \cap \mathcal{C}$ the inclusion $\{y \in \mathbb{R}^n \mid y = \hat{y} + \sigma d_0, \sigma \in \mathbb{R}\} \subseteq \mathcal{H} \cap \mathcal{C}$ must hold. Additionally, according to Definition 6, by Assumption 2, and Lemma 6, the sets $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are bases for $\mathcal{C}$. Hence, there exists a point $w \in \mathcal{E} \cap \mathcal{B}^=$ such that $w \in \mathcal{H}$, and w is in an exposed face of $\mathcal{C}$.

Convexity of $\mathcal{E}$ implies $\lambda x + (1-\lambda) w \in \mathcal{E}$ for any $\lambda \in [0,1]$. On the other hand, the point w is in an exposed face of $\mathcal{C}$, so $\lambda x + (1-\lambda) w \notin \mathcal{C}$ for $0 < \lambda \leq 1$. Since $x \in (\mathcal{E} \cap \mathcal{A})$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E} = \emptyset$, we have that $a^\top x \geq \alpha$ and $a^\top w < \alpha$. Hence, from the equation $a^\top(\lambda x + (1-\lambda) w) = \lambda a^\top x + (1-\lambda) a^\top w$, there must exists a value $0 < \lambda \leq 1$ such that $a^\top(\lambda x + (1-\lambda) w) = \alpha$. Therefore, for some $0 < \lambda \leq 1$ there is a point $\hat{x} = \lambda x + (1-\lambda) w$, such that $\hat{x} \in \mathcal{E} \cap \mathcal{A}^=$, but $\hat{x} \notin \mathcal{C}$, which contradicts condition 4. Hence, $(\mathcal{E} \cap \mathcal{A}) \subseteq \mathcal{C}$. One can prove $(\mathcal{E} \cap \mathcal{B}) \subseteq \mathcal{C}$ analogously.

Recall that the sets $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are disjoint and nonempty. Then, condition 4 implies that $\mathcal{E} \cap \mathcal{A} \neq \mathcal{C}$ and $\mathcal{E} \cap \mathcal{B} \neq \mathcal{C}$, and the result of the lemma follows. □ □

Now we can present the proof of Proposition 2.

Proof of Proposition 2. First, consider a vector $x \in (\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$. Then, Lemma 7 implies that $x \in \mathcal{E} \cap \mathcal{C}$. Consider any two points $x, y \in (\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$. Then, since both $\mathcal{C}$ and $\mathcal{E}$ are convex, for all $0 \leq \lambda \leq 1$ the convex combination $\lambda x + (1-\lambda) y \in \mathcal{E} \cap \mathcal{C}$. Hence, $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) \subseteq (\mathcal{E} \cap \mathcal{C})$.

Consider now a point $x \in (\mathcal{E} \cap \mathcal{C})$. First, if $x \in (\mathcal{E} \cap \mathcal{A})$ or $x \in (\mathcal{E} \cap \mathcal{B})$, we have that $x \in \operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. Suppose then that $x \notin (\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$. Then, $x \in (\bar{\mathcal{A}} \cap \bar{\mathcal{B}} \cap \mathcal{C})$. Furthermore, by Lemma 6 there are two vectors $\hat{x} \in \mathcal{E} \cap \mathcal{A}^=$ and $\bar{x} \in \mathcal{E} \cap \mathcal{B}^=$ such that $x = \hat{x} + \mu d_0$ and $x = \bar{x} + \nu d_0$, for some $\mu, \nu \in \mathbb{R}$. Thus, given that $x \notin (\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$ we can assume w.l.o.g. that $\nu > 0$ and $\mu < 0$. Then, we have that $x = \lambda \hat{x} + (1-\lambda)\bar{x}$, where $\lambda = \nu/(\nu - \mu)$ and $0 < \lambda < 1$. In other words, x is a convex combination of $\hat{x}$ and $\bar{x}$. Since x is an arbitrary point we have that any point $x \in (\mathcal{E} \cap \mathcal{C})$ can be written as a convex combination of two points in $(\mathcal{E} \cap \mathcal{A}) \cup (\mathcal{E} \cap \mathcal{B})$. As a conclusion, we have that $(\mathcal{E} \cap \mathcal{C}) \subseteq \operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. Finally, since $(\mathcal{E} \cap \mathcal{A}^=)$ and $(\mathcal{E} \cap \mathcal{B}^=)$ are compact sets, then it follows from Lemma 6 and Lemma 11 that $\mathcal{C}$ is closed. □ □

Lemma 8. *If a convex cylinder $\mathcal{C}$ with property* (4) *exists, then $\mathcal{C}$ is unique.*

Proof. Assume that there exist two different cylinders $\mathcal{C}_1 = \{x \in \mathbb{R}^n \mid x = d^1 + \gamma d_0^1, d^1 \in \mathcal{D}_1, \gamma \in \mathbb{R}\}$ and $\mathcal{C}_2 = \{x \in \mathbb{R}^n \mid x = d^2 + \sigma d_0^2, d^2 \in \mathcal{D}_2, \sigma \in \mathbb{R}\}$ that satisfy Definition 5. Then, we have that $\mathcal{C}_1 \cap \mathcal{A}^= = \mathcal{C}_2 \cap \mathcal{A}^=$ and $\mathcal{C}_1 \cap \mathcal{B}^= = \mathcal{C}_2 \cap \mathcal{B}^=$.

Given that $\mathcal{C}_1 \neq \mathcal{C}_2$ there must exist a point $\hat{x}$ that belongs only to one cylinder, and w.l.o.g. we assume that $\hat{x} \in \mathcal{C}_1$ and $\hat{x} \notin \mathcal{C}_2$. Observe that if $\hat{x} \in \mathcal{A} \cap \mathcal{B}$, then there exists a point $\bar{x} \in \mathcal{C}_1$ such that either $\bar{x} \in \mathcal{A}^= \cap \mathcal{B}$ or $\bar{x} \in \mathcal{A} \cap \mathcal{B}^=$, which implies that $\bar{x} \in \mathcal{E} \cap \mathcal{A} \cap \mathcal{B}$, contradicting Assumption 1.

Let us begin assuming that $\hat{x} \in \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$. Then, given that $\mathcal{E} \cap \mathcal{A}^=$ is a base for both cylinders there exists a $\hat{\gamma} \in \mathbb{R}$ such that $\hat{x} = \hat{d}^1 + \hat{\gamma} d_0^1$ for some $\hat{d}^1 \in \mathcal{E} \cap \mathcal{A}^= = \mathcal{C}_1 \cap \mathcal{A}^= = \mathcal{C}_2 \cap \mathcal{A}^=$. On the other hand, since $\mathcal{E} \cap \mathcal{B}^=$ is a base for $\mathcal{C}_1$, there exists $\bar{\gamma} \in \mathbb{R}$ such that $\hat{x} = \bar{d}^1 + \bar{\gamma} d_0^1$ for some $\bar{d}^1 \in \mathcal{E} \cap \mathcal{B}^= = \mathcal{C}_1 \cap \mathcal{B}^=$. Hence, $\hat{x} = \lambda \bar{d}^1 + (1-\lambda)\hat{d}^1$ where $\lambda = \hat{\gamma}/(\hat{\gamma} - \bar{\gamma}) \leq 1$, since $\hat{\gamma}$ and $\bar{\gamma}$ must have opposite signs. Additionally, given that the two cylinders are convex we get that $\bar{d}^1 \notin \mathcal{C}_2$. Then, $\mathcal{C}_1 \cap \mathcal{B}^= \neq \mathcal{C}_2 \cap \mathcal{B}^=$, which is a contradiction.

Let us assume now that $\hat{x} \in \mathcal{A}$ and $\hat{x} \notin \mathcal{B}$. By the separation theorem, there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n \mid h^\top x = \eta\}$ separating $\hat{x}$ from $\mathcal{C}_2$. By the definition of a cylinder, we have $h^\top d_0^2 = 0$. Now, let $\mathcal{H}$ be a supporting hyperplane of $\mathcal{C}_2$, which implies that $\mathcal{H} \cap \mathcal{C}_2$ is an exposed face of $\mathcal{C}_2$. Note that for any $\hat{y} \in \mathcal{H} \cap \mathcal{C}_2$ we have that $\{y \in \mathbb{R}^n \mid y = \hat{y} + \sigma d_0^2, \sigma \in \mathbb{R}\} \subseteq \mathcal{H} \cap \mathcal{C}_2$. Additionally, we know that the sets $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are bases for $\mathcal{C}_2$. Hence, there exists a point $w \in \mathcal{E} \cap \mathcal{B}^=$ such that $w \in \mathcal{H}$, and w is in an exposed face of $\mathcal{C}_2$.

Convexity of $\mathcal{C}_1$ implies that for any $\lambda \in [0,1]$, $\lambda\hat{x} + (1-\lambda)w \in \mathcal{C}_1$. On the other hand, since $w \in \mathcal{H}$ is a point in an exposed face of $\mathcal{C}_2$, $\lambda\hat{x} + (1-\lambda)w \notin \mathcal{C}_2$ for $0 < \lambda \leq 1$. Since $\hat{x} \in \mathcal{A} \cap \mathcal{C}_1$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}_1 = \emptyset$, we have that $a^\top \hat{x} \geq \alpha$ and $a^\top w < \alpha$. Hence, from the equation $a^\top(\lambda\hat{x} + (1-\lambda)w) = \lambda a^\top \hat{x} + (1-\lambda)a^\top w$, there exists a value $0 < \lambda \leq 1$ such that $a^\top(\lambda\hat{x} + (1-\lambda)w) = \alpha$. Therefore, there exists a point $\bar{x} = \lambda\hat{x} + (1-\lambda)w$ for some $0 < \lambda \leq 1$, such that $\bar{x} \in \mathcal{C}_1 \cap \mathcal{A}^=$, but $\bar{x} \notin \mathcal{C}_2$, which is a contradiction. An analogous argument can be used when $\hat{x} \in \mathcal{B}$ and $\hat{x} \notin \mathcal{A}$. □ □

As mentioned at the beginning of Section 1, Propositions 1 and 2 are rather general in that they apply to any convex set $\mathcal{E}$. However, their hypotheses, (3) and (4), are hard to satisfy and hence limit their applicability. To explore the full potential of this result remains the subject of future research. In this paper we demonstrate the power of this tool by exploring a class of MICO, the class of MISOCO problems, for which the assumptions are satisfied under mild conditions.

In the general setting, cone $\mathcal{K}$ or cylinder $\mathcal{C}$ of Propositions 1 and 2 can be used as a conic cut in MICO problems. For example, in Branch-and-Cut algorithms if either $\mathcal{K}$ or $\mathcal{C}$ exists for a disjunctive set, then $\mathcal{K}$ or $\mathcal{C}$ can be used to help tightening the description of a MICO problem. For practical use of this methodology, one needs to prove that a cone $\mathcal{K}$ or cylinder $\mathcal{C}$ exists that satisfies Definitions 2 or 5 respectively, and one needs to provide an easy to compute algebraic representation of the cone or cylinder. In the following section we analyze MISOCO problems, where the feasible set $\mathcal{E}$ comes from the intersection of a second order cone and an affine space. Given that for this case we can prove the existence of the cone and we can give a method to compute its algebraic representation, the resulting conic cut can be embedded in Branch-and-Cut algorithms to solve MISOCO problems.

4 The convex hull of the intersection of an ellipsoid and a disjunctive set

In the remainder of the paper, we turn our attention to the convex hull $\text{conv}(\mathcal{E}\cap(\mathcal{A}\cup\mathcal{B}))$ in a special case of (1) where $\mathcal{K}$ is the Cartesian product of Lorentz cones, i.e., $x = \left((x^1)^\top, (x^2)^\top, \ldots, (x^k)^\top\right)^\top$, $\mathbb{L}^{n_i} = \{x^i | x_1^i \geq \|x_{2:n_i}^i\|\}$, $i = 1, \ldots, k$ are Lorentz cones, and $\mathcal{K} = \mathbb{L}_1^{n_1} \times \cdots \times \mathbb{L}_k^{n_k}$. In this setting, we consider $\mathcal{K} = \mathbb{L}^n$, therefore $\mathcal{E}$ is an ellipsoid resulting from the intersection of a second order cone and an affine space. Consider, for example, the problem

$$\begin{array}{rlllll} \text{minimize:} & 3x_1 & +2x_2 & +2x_3 & +x_4 & \\ \text{subject to:} & 9x_1 & +x_2 & +x_3 & +x_4 & = 10 \\ & & (x_1, x_2, x_3, x_4) & & \in \mathbb{L}^4 & \\ & & & x_4 & \in \mathbb{Z}. & \end{array} \tag{5}$$

The feasible set of Problem (5) can be represented as an ellipsoid in $\mathbb{R}^3$ in terms of the variables x_2, x_3, x_4, as shown in Figure 5. In general, we consider the n-dimensional ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n | x^\top Q x + 2q^\top x + \rho \leq 0\}, \tag{6}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $x, q \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$.

The main goal of this section is to show the existence of the cone $\mathcal{K}$ or cylinder $\mathcal{C}$, as defined in Definitions 2 or 5, in order to use Proposition 1 or 2 for finding $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$. We are interested in two cases. In the first case, the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ are parallel (§4.1), while the two hyperplanes are in a general position in the second case (§4.2). In both cases, we are able to show that $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$ is obtained by intersecting $\mathcal{E}$ with a scaled second order cone $\mathcal{K}$ or a cylinder $\mathcal{C}$ and we show how to construct them.

4.1 Parallel disjunctions

In this section, we consider a disjunctive set $\mathcal{A} \cup \mathcal{B}$ such that $\mathcal{A} = \{x \in \mathbb{R}^n | a^\top x \geq \alpha\}$ and $\mathcal{B} = \{x \in \mathbb{R}^n | a^\top x \leq \beta\}$, i.e., the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ are parallel. We may assume w.l.o.g. that $\|a\| = 1$. We illustrate this case by using Problem (5), where one can use $\mathcal{A} = \{x \in \mathbb{R}^4 | x_4 \geq 0\}$ and $\mathcal{B} = \{x \in \mathbb{R}^4 | x_4 \leq -1\}$ to define a disjunctive set $\mathcal{A} \cup \mathcal{B}$. Figure 6(a) shows the hyperplanes defining this disjunctive set $\mathcal{A} \cup \mathcal{B}$ along with the feasible set of Problem (5).

4.1.1 Geometry of $\mathcal{E}$ and the hyperplanes $\mathcal{A}^=$, and $\mathcal{B}^=$

We begin this analysis by recalling some results from [8], where the authors study several properties of quadrics. A quadric is defined as

$$\mathcal{Q} = \left\{ x | x^\top Q x + 2q^\top x + \rho \leq 0 \right\}, \tag{7}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $q, x \in \mathbb{R}^n$ and $\rho \in \mathbb{R}$, and is denoted by the triplet $\mathcal{Q} = (Q, q, \rho)$. Note that under this definition, $\mathcal{E}$ is a quadric with a positive definite matrix Q. We first recall Theorem 3.2 of [8], which defines a uniparametric family of quadrics $\mathcal{Q}(\tau)$ parametrized by $\tau \in \mathbb{R}$ having the same intersection with two fixed parallel hyperplanes. This result is stated here as Theorem 1.

Theorem 1 ([8])**.** *Consider an ellipsoid $\mathcal{E} = (Q, q, \rho)$ and two parallel hyperplanes $\mathcal{A}^= = (a, \alpha)$ and $\mathcal{B}^= = (a, \beta)$. The uniparametric family of quadrics $\mathcal{Q}(\tau)$ parametrized by $\tau \in \mathbb{R}$ and having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$ as ellipsoid $\mathcal{E}$ is given by*

$$\begin{aligned} Q(\tau) &= Q + \tau a a^\top \\ q(\tau) &= q - \tau \frac{\alpha + \beta}{2}\ a \\ \rho(\tau) &= \rho + \tau \alpha \beta. \end{aligned}$$

From Theorem 1, for any $\tau \in \mathbb{R}$ the quadric $\mathcal{Q}(\tau)$ is such that $\mathcal{Q}(\tau) \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{Q}(\tau) \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=$. Hence, from Lemma 3 we need to investigate if there exists a value $\bar{\tau}$ such that $Q(\bar{\tau})$ is a two-sided cone one side of which, denoted $\mathcal{K}$, satisfies Definition 2 with a vertex $x^* \notin \bar{\mathcal{A}} \cap \bar{\mathcal{B}}$ or a convex cylinder $\mathcal{C}$. As a result, from Proposition 1 or 2 the intersections $\mathcal{K} \cap \mathcal{E}$ or $\mathcal{C} \cap \mathcal{E}$ would be the convex hull for $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$, respectively. Figures 6(b) and 6(c) illustrate the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$ and the four sets $\mathcal{E}, \mathcal{A}^=, \mathcal{B}^=, \mathcal{K}$ for Problem (5).

Now, let us assume first that we are given a τ such that $Q(\tau)$ is nonsingular. Under this assumption one can rewrite the quadric set $\mathcal{Q}(\tau)$ in (7) as

$$\left\{ x \,\middle|\, (x + Q(\tau)^{-1} q(\tau))^\top Q(\tau)(x + Q(\tau)^{-1} q(\tau)) \le q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) \right\}. \tag{8}$$

From (8), one can easily verify that the quadric $\mathcal{Q}(\tau)$ is empty if the matrix $Q(\tau)$ is positive definite and $q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) < 0$. Belotti et al. [8] prove that the quadric $\mathcal{Q}(\tau)$ defines a cone if $Q(\tau)$ is a non-singular symmetric matrix with exactly one negative eigenvalue and $q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) = 0$. They also prove that for any $\tau \in \mathbb{R}$, the matrix $Q(\tau)$ has at most one negative eigenvalue and at least $n-1$ positive eigenvalues. Therefore, we need to focus on $Q(\tau)$ to explore those τ values for which $q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) = 0$.

Let us define the vectors $u_a = Q^{-1/2} a$ and $u_q = Q^{-1/2} q$, where $Q^{-1/2}$ is the unique symmetric square root of Q^{-1}. Then, from Theorem 1 we can get the following expression, which is derived in Section 3.2.1 in [8]:

$$\begin{aligned} & q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) \\ = \ & \frac{(\alpha - \beta)^2 \|u_a\|^2}{4(1 + \tau \|u_a\|^2)} \tau^2 + \frac{\left(4 \|u_a\|^2 (\|u_q\|^2 - \rho) - (\alpha + \beta + 2 u_a^\top u_q)^2 + (\alpha - \beta)^2\right)}{4(1 + \tau \|u_a\|^2)} \tau \\ & + \frac{4(\|u_q\|^2 - \rho)}{4(1 + \tau \|u_a\|^2)}. \end{aligned} \tag{9}$$

Hence $q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau)$ is the ratio of two polynomials in τ. Two remarks are in order: first, note that at value $\hat{\tau} = -1/\|u_a\|^2$, the denominator of (9) becomes zero. Additionally, at $\hat{\tau}$, the matrix $Q(\tau)$ is positive semidefinite with one zero eigenvalue. Lemma 3.3 in [8] characterizes the behavior of $Q(\tau)$ at $\hat{\tau}$. There are two main ranges in this characterization. On one hand, for $\tau > \hat{\tau}$, the matrix $Q(\tau)$ is positive definite. On the other hand, for $\tau < \hat{\tau}$, the matrix $Q(\tau)$ is indefinite with one negative eigenvalue.

Second, for any $\tau \neq \hat{\tau}$, $q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau)$ becomes zero only at the roots $\bar{\tau}_1, \bar{\tau}_2$ of the numerator of (9). Let f be a function of τ that denotes the quadratic function in the numerator of (9). Hence, both roots $\bar{\tau}_1, \bar{\tau}_2$ of f are less than or equal to $\hat{\tau}$ [8]. Then, from Lemma 3.3 in

[8] the two roots $\bar{\tau}_1, \bar{\tau}_2$ correspond to the cones or the cylinders in the family of Theorem 1. A characterization of the family $\mathcal{Q}(\tau)$ for $\tau \in \mathbb{R}$ depending on the geometry of $\mathcal{E}$ and the hyperplanes $\mathcal{A}^=$, and $\mathcal{B}^=$ is presented in Theorem 3.4 of [8], which we recall here.

Theorem 2 ([8]). *Depending on the geometry of $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$, $\mathcal{Q}(\tau)$ can have the following shapes for $\tau \in \mathbb{R}$:*

- *$f(\tau)$ has two distinct roots $\bar{\tau}_1 < \bar{\tau}_2$ and $\bar{\tau}_2 < \hat{\tau}$: this is the general case, $\mathcal{Q}(\hat{\tau})$ is a paraboloid, and $\mathcal{Q}(\bar{\tau}_1), \mathcal{Q}(\bar{\tau}_2)$ are two cones.*
- *$f(\tau)$ has two distinct roots $\bar{\tau}_1 < \bar{\tau}_2$, and $\bar{\tau}_2 = \hat{\tau}$: the two hyperplanes are symmetric about the center of $\mathcal{E}$. $\mathcal{Q}(\bar{\tau}_1)$ is cone and $\mathcal{Q}(\bar{\tau}_2)$ is a cylinder.*
- *The two roots $\bar{\tau}_1, \bar{\tau}_2$ of $f(\tau)$ are equal, and $\bar{\tau}_2 < \hat{\tau}$: the discriminant of $f(\tau)$ is zero, which means that one of the hyperplanes intersect $\mathcal{E}$ in only one point. $\mathcal{Q}(\hat{\tau})$ is a paraboloid and $\mathcal{Q}(\bar{\tau}_2)$ is a cone.*
- *The two roots $\bar{\tau}_1, \bar{\tau}_2$ of $f(\tau)$ coincide with $\hat{\tau}$: this is the most degenerate case as both hyperplanes intersect $\mathcal{E}$ in only one point, and as such they are symmetric about the center of $\mathcal{E}$. In this case $\mathcal{Q}(\hat{\tau})$ is a line.*

4.1.2 Building a disjunctive conic cut

We can use the geometrical analysis of §4.1.1 to build a conic cut to convexify the intersection of a MISOCO problem with a parallel disjunction. To simplify the analysis, we separate the cylinder and conic cases.

Cylinders: First, we study the families $\mathcal{Q}(\tau)$, $\tau \in \mathbb{R}$ described in the second and fourth cases in Theorem 2, where there is a cylinder $\mathcal{C}$ at $\mathcal{Q}(\hat{\tau})$. In particular, $\mathcal{C}$ is given by $\mathcal{Q}(\bar{\tau}_2)$ in these cases. From equation (7), we have that

$$\mathcal{Q}(\bar{\tau}_2) = \left\{ x \in \mathbb{R}^n \;\middle|\; x^\top Q(\bar{\tau}_2)x + 2q(\bar{\tau}_2)^\top x + \rho(\bar{\tau}_2) \leq 0 \right\}, \tag{10}$$

where $Q(\bar{\tau}_2)$ is a positive semidefinite matrix. Hence, it follows from (10) that the quadric $\mathcal{Q}(\bar{\tau}_2)$ is a convex set and Proposition 2 proves that $\mathcal{C} \cap \mathcal{E}$ is the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. Finally, notice that the cylinder $\mathcal{C}$ described by (10) can be represented in terms of a second order cone, for that reason we classify $\mathcal{C}$ as a conic cut in this section.

Cones: Now we focus on the cones described in the first and third cases of Theorem 2. Our strategy is to show that the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ can be written as the union of two convex cones. Then, we derive a criterion to identify which cone gives the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$.

Consider the roots $\bar{\tau}_i \neq \hat{\tau}$, $i = 1, 2$, and let $x(\bar{\tau}_i) = -Q(\bar{\tau}_i)^{-1}q(\bar{\tau}_i)$. Recall from §4.1.1 that $Q(\bar{\tau}_i)$ is a symmetric and non-singular matrix that has exactly one negative eigenvalue. Then, $Q(\bar{\tau}_i)$ can be diagonalized as $U(\bar{\tau}_i)D(\bar{\tau}_i)U(\bar{\tau}_i)^\top$, where $U(\bar{\tau}_i)$ is an orthogonal matrix and $D(\bar{\tau}_i)$ is a diagonal matrix having the eigenvalues of $Q(\bar{\tau}_i)$ in its diagonal. Let the index j_i be such that $D(\bar{\tau}_i)_{j_i,j_i} < 0$, and let $W(\bar{\tau}_i) = U(\bar{\tau}_i)\bar{D}(\bar{\tau}_i)^{1/2}$, where $\bar{D}(\bar{\tau}_i)_{l,k} = |D(\bar{\tau}_i)_{l,k}|$. Thus, we may write $\mathcal{Q}(\bar{\tau}_i)$ in terms of $W(\bar{\tau}_i)$ as follows

$$\left\{ x \in \mathbb{R}^n \middle| (x - x(\bar{\tau}_i))^\top W(\bar{\tau}_i)_{i \neq j_i} W(\bar{\tau}_i)^\top_{i \neq j_i} (x - x(\bar{\tau}_i)) \leq \left(W(\bar{\tau}_i)^\top_{j_i}(x - x(\bar{\tau}_i)) \right)^2 \right\},$$

where $W(\bar{\tau}_i)_{i\neq j_i}$ has the columns of $W(\bar{\tau}_i)$ that are different from j_i. Now, let us define the sets $\mathcal{Q}(\bar{\tau}_i)^+$, $\mathcal{Q}(\bar{\tau}_i)^-$ as follows

$$\mathcal{Q}(\bar{\tau}_i)^+ \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \left\| W(\bar{\tau}_i)_{i\neq j_i}^\top (x - x(\bar{\tau}_i)) \right\| \leq W(\bar{\tau}_i)_{j_i}^\top (x - x(\bar{\tau}_i)) \right\}, \tag{11}$$

$$\mathcal{Q}(\bar{\tau}_i)^- \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \left\| W(\bar{\tau}_i)_{i\neq j_i}^\top (x - x(\bar{\tau}_i)) \right\| \leq -W(\bar{\tau}_i)_{j_i}^\top (x - x(\bar{\tau}_i)) \right\}, \tag{12}$$

which are two second order cones. These two cones satisfy the general definition of a cone with the vertex at $x(\bar{\tau}_i)$ presented in Remark 1. It is easy to verify that $\mathcal{Q}(\bar{\tau}_i) = \mathcal{Q}(\bar{\tau}_i)^+ \cup \mathcal{Q}(\bar{\tau}_i)^-$. Also, it is clear from (11) and (12) that $\mathcal{Q}(\bar{\tau}_i)^+$ and $\mathcal{Q}(\bar{\tau}_i)^-$ are two convex sets. This shows that the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ can be written as the union of two convex cones.

Given the convex cones, we need a criterion to identify which cone gives the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. First, we choose one of the two quadrics $\mathcal{Q}(\bar{\tau}_i)$, $i = 1, 2$. For this purpose we can use Lemma 3, thus we need to verify if at least one of $\mathcal{Q}(\bar{\tau}_i)$, $i = 1, 2$ contains a cone with a vertex $x(\bar{\tau}_i) \notin \mathcal{A} \cap \mathcal{B}$. This criterion is presented in Lemma 9. The interested reader can review the proof in Appendix A.1.

Lemma 9. *The quadric $\mathcal{Q}(\bar{\tau}_2)$ found at the larger root $\bar{\tau}_2$ of $f(\tau)$ in the family $\mathcal{Q}(\tau)$ of the first and third case of Theorem 2 contains a cone that satisfies Definition 2.*

From Lemma 9, we reduce the choices to the cones $\mathcal{Q}(\bar{\tau}_2)^+$ and $\mathcal{Q}(\bar{\tau}_2)^-$. We now decide between the two cones using the sign of $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2))$. Thus, we choose $\mathcal{Q}(\bar{\tau}_2)^+$ if $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2)) > 0$, and we choose $\mathcal{Q}(\bar{\tau}_2)^-$ when $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2)) < 0$. Finally, it follows from Proposition 1 that the selected cone gives the convex hull for $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. Note that if $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2)) = 0$ the center of the ellipsoid $\mathcal{E}$ coincides with the vertex of the selected cone. In this case the feasible set is a single point, so by identifying this unique solution the problem is solved. This completes the procedure.

We have shown that for all the cases in Theorem 2, we can find a cone $\mathcal{K}$ or a cylinder $\mathcal{C}$ that satisfies Definitions 2 or 5 respectively. Hence, by combining Theorem 2 with Propositions 1 and 2 we can provide a procedure to find the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$, where the disjunctive set $\mathcal{A} \cup \mathcal{B}$ is such that the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$ are parallel. Thus, we have given easy to compute procedures to identify disjunctive conic cuts, and disjunctive cylindrical cuts in the respective cases of Theorem 2.

4.2 General disjunctions

Some of the results in §4.1 can be extended to general disjunctive sets $\mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = \{x \in \mathbb{R}^n | a^\top x \geq \alpha\}$ and $\mathcal{B} = \{x \in \mathbb{R}^n | b^\top x \leq \beta\}$ are defined such that there exists no $\kappa \in \mathbb{R}$ such that $b = \kappa a$.

An important example of general disjunction is given by *complementarity constraints*, usually described in the form $x_i x_j = 0$ and hence equivalent to the disjunction $x_i = 0 \vee x_j = 0$. An example of disjunctive cuts separated for problems with complementarity constraints is given by Júdice et al. [18], who study a problem where complementarity constraints are the only nonlinear ones, and whose relaxation yields an LP. Disjunctive cuts are separated using violated complementarity constraints by observing that both variables are basic and then applying a disjunctive procedure to the corresponding tableau rows.

We may assume w.l.o.g. that $\|a\| = \|b\| = 1$. These disjunctive sets are illustrated in Figure 7(a) for Problem (5) using $\mathcal{A} = \{x \in \mathbb{R}^4 \mid 0.45x_3 + 0.89x_4 \geq 0\}$ and $\mathcal{B} = \{x \in \mathbb{R}^4 \mid x_4 \leq -1\}$ to define the disjunctive set $\mathcal{A} \cup \mathcal{B}$.

4.2.1 Geometry of $\mathcal{E}$ and the hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$

We begin this analysis recalling Theorem 4.1 in [8]. This theorem defines a family of quadrics $\mathcal{Q}(\tau)$ for $\tau \in \mathbb{R}$ such that $\mathcal{Q}(\tau) \cap \mathcal{A}^= = \mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q}(\tau) \cap \mathcal{B}^= = \mathcal{Q} \cap \mathcal{B}^=$.

Theorem 3 ([8])**.** *Consider an ellipsoid $\mathcal{E} = (Q, q, q_0)$ and two nonparallel hyperplanes $\mathcal{A}^=$ and $\mathcal{B}^=$. The uniparametric family of quadrics $\mathcal{Q}(\tau)$ parametrized by $\tau \in \mathbb{R}$ and having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$ as the ellipsoid $\mathcal{E}$ is given by*

$$\begin{aligned} Q(\tau) &= Q + \tau \frac{ab^\top + ba^\top}{2} \\ q(\tau) &= q - \tau \frac{\beta a + \alpha b}{2} \\ \rho(\tau) &= \rho + \tau \alpha\beta. \end{aligned}$$

We need to investigate if there is a value $\bar{\tau}$ in the family of Theorem 3 for which $\mathcal{Q}(\bar{\tau})$ is a cone $\mathcal{K}$ or a cylinder $\mathcal{C}$. Thus, either $\mathcal{K} \cap \mathcal{E}$ or $\mathcal{C} \cap \mathcal{E}$ will give the convex hull for $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. Figure 7(b) and 7(c) illustrate this for the example in Problem (5).

Note that in Theorem 3, $Q(\tau)$ has a rank-2 update. This opens the possibility of having a matrix with two negative eigenvalues. However, it can be verified that under the assumption of Q being positive definite, $Q(\tau)$ can have at most one non-positive eigenvalue [8, § 4]. This property reduces the case of general disjunctive sets to the same set of geometrical objects that were considered in §4.1.1.

For any vector d, define $u_d = Q^{-1/2}d$. Using this notation, we get from Theorem 3 the following [8, § 4]:

$$q(\tau)^\top Q(\tau)^{-1} q(\tau) - \rho(\tau) = \frac{f(\tau)}{g(\tau)}, \tag{14}$$

where

$$g(\tau) = \tau^2 \left(\left(u_a^\top u_b \right)^2 - \|u_a\|^2 \|u_b\|^2 \right) + 4 u_a^\top u_b \tau + 4$$

and

$$\begin{aligned} f(\tau) =& \tau^2 \Big[\|u_a\|^2 (\beta + u_b^\top u_q)^2 \|u_b\|^2 (\alpha + u_a^\top u_q)^2 \\ &+ ((u_a^\top u_b)^2 - \|u_a\|^2 \|u_b\|^2)(\|u_q\|^2 - \rho) - 2u_a^\top u_b (u_a^\top u_q + \alpha)(u_b^\top u_q + \beta) \Big] \\ &+ 4\tau \Big[u_a^\top u_b (\|u_q\|^2 - \rho) - (\alpha + u_a^\top u_q)(\beta + u_b^\top u_q) \Big] + 4 \Big[\|u_q\|^2 - \rho \Big], \end{aligned} \tag{15}$$

which are two quadratic functions in τ. Let the two roots of $g(\tau)$ be denoted as $\hat{\tau}_1$ and $\hat{\tau}_2$, and we may assume w.l.o.g. that $\hat{\tau}_1 \leq \hat{\tau}_2$. It is proven in [8, §4] that at these two values $Q(\tau)$ is a positive semidefinite matrix with one zero eigenvalue. Now, let the roots of $f(\tau)$ be denoted as $\bar{\tau}_1$ and $\bar{\tau}_2$, and we may also assume w.l.o.g. that $\bar{\tau}_1 \leq \bar{\tau}_2$. It is easy to verify that (14) becomes zero for these two values when $Q(\bar{\tau}_i)$ is non-singular, $i = 1, 2$. Additionally, in [8, §4] it is shown that the situations $\hat{\tau}_1 < \bar{\tau}_1 < \hat{\tau}_2$, or $\hat{\tau}_1 < \bar{\tau}_2 < \hat{\tau}_2$, are only possible when the quadric $\mathcal{Q}$ is a single point, which is a trivial case. We use these observations in Theorem 4 to summarize the behavior of the family $\mathcal{Q}(\tau)$ when the quadric $\mathcal{Q}$ is not a single point, based on the values $\hat{\tau}_1, \hat{\tau}_2, \bar{\tau}_1, \bar{\tau}_2$. The interested reader can review the details of this theorem in [8, §4.2].

Theorem 4 ([8])**.** *Depending on the geometry of $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{B}$, $\mathcal{Q}(\tau)$ can have the following shapes for $\tau \in \mathbb{R}$:*

- *$f(\tau)$ has two distinct roots $\bar{\tau}_1 < \bar{\tau}_2$ such that $\hat{\tau}_2 < \bar{\tau}_1$, or $\bar{\tau}_2 < \hat{\tau}_1$, or $\bar{\tau}_1 < \hat{\tau}_1 \leq \hat{\tau}_2 < \bar{\tau}_2$: this is the general case, $\mathcal{Q}(\hat{\tau}_1)$ and $\mathcal{Q}(\hat{\tau}_2)$ are paraboloids, and $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ are cones.*

- *$f(\tau)$ has two distinct roots $\bar{\tau}_1 < \bar{\tau}_2$, and exactly one of them coincides with either $\hat{\tau}_1$ or $\hat{\tau}_2$: this case has two possibilities. First, $\mathcal{Q}(\hat{\tau}_1)$ is a cylinder and $\mathcal{Q}(\hat{\tau}_2)$ is a paraboloid. Second, $\mathcal{Q}(\hat{\tau}_2)$ is a cylinder and $\mathcal{Q}(\hat{\tau}_1)$ is a paraboloid. In both situations we have that either $\mathcal{Q}(\bar{\tau}_1)$ is a cylinder and $\mathcal{Q}(\bar{\tau}_2)$ is a cone or that $\mathcal{Q}(\bar{\tau}_2)$ is a cylinder and $\mathcal{Q}(\bar{\tau}_1)$ is a cone.*

- *$f(\tau)$ has two distinct roots $\bar{\tau}_1 < \bar{\tau}_2$ such that $\bar{\tau}_1 = \hat{\tau}_1$ and $\bar{\tau}_2 = \hat{\tau}_2$: in this case both hyperplanes contain the center $-Q^{-1}q$ of the ellipsoid $\mathcal{E}$. Both quadrics $\mathcal{Q}(\hat{\tau}_1)$ and $\mathcal{Q}(\hat{\tau}_2)$ are cylinders.*

- *The two roots of $f(\tau)$ coincide, and either $\bar{\tau}_1 = \bar{\tau}_2 < \hat{\tau}_1$ or $\hat{\tau}_2 < \bar{\tau}_1 = \bar{\tau}_2$: in this case the discriminant of $f(\tau)$ is zero, which implies that one of the hyperplanes intersects $\mathcal{E}$ in only one point. We have that $\mathcal{Q}(\bar{\tau}_1)$ is a cone and the quadrics $\mathcal{Q}(\hat{\tau}_1)$, $\mathcal{Q}(\hat{\tau}_2)$ are two paraboloids.*

- *The two roots of $f(\tau)$ coincide and either $\bar{\tau}_1 = \bar{\tau}_2 = \hat{\tau}_1$ or $\hat{\tau}_2 = \bar{\tau}_1 = \bar{\tau}_2$: in this case both hyperplanes intersect $\mathcal{E}$ in only one point. Then, either $\mathcal{Q}(\hat{\tau}_1)$ is a line and $\mathcal{Q}(\hat{\tau}_2)$ is a paraboloid or $\mathcal{Q}(\hat{\tau}_2)$ is a line and $\mathcal{Q}(\hat{\tau}_1)$ is a paraboloid.*

4.2.2 Building a disjunctive conic cut

Using the results of the geometrical analysis of §4.2.1 we give now the guidelines to build a conic cut to convexify the intersection of a MISOCO feasible set with a general disjunction.

First of all, observe that from Assumption 1 the third case in Theorem 4 cannot occur. Hence, this case is not considered for building a cut for general disjunctions. We classify the remaining cases as cylinders and cones.

Cylinders: We look at the cylinders $\mathcal{C}$ in the families $\mathcal{Q}(\tau)$ described in the second and fifth cases of Theorem 4 of [8]. Observe that in general, $\mathcal{C}$ can be found at either $\hat{\tau}_2$ or $\hat{\tau}_1$. This can be decided by comparing $\hat{\tau}_2$ or $\hat{\tau}_1$ with the roots $\bar{\tau}_1$ and $\bar{\tau}_2$ using the criteria described in Theorem 4. Let $\hat{\tau}$ be a value such that $\mathcal{Q}(\hat{\tau})$ is a cylinder. From equation (7) it is easy to verify that $\mathcal{Q}(\hat{\tau})$ is a convex set. Consequently, from Proposition 2 we get that $\mathcal{C} \cap \mathcal{E}$ is the convex hull of $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$. Finally, note that the cylinder $\mathcal{C}$ can be represented in terms of a second order cone. For that reason, we classify $\mathcal{C}$ as a conic cut in this section too.

Cones: We need to focus now on the cones described in the first and fourth cases of Theorem 4. Let $\bar{\tau}_i \neq \hat{\tau}_1, \hat{\tau}_2$, $i = 1, 2$. In this two cases $Q(\bar{\tau}_i)$ is symmetric and non-singular matrix with exactly one negative eigenvalue. This is a similar situation as the first and third cases of Theorem 2. From the analysis in §4.1.2 it follows that $\mathcal{Q}(\bar{\tau}_i) = \mathcal{Q}(\bar{\tau}_i)^+ \cup \mathcal{Q}(\bar{\tau}_i)^-$, where $\mathcal{Q}(\bar{\tau}_i)^+, \mathcal{Q}(\bar{\tau}_i)^-$ are the second order cones (11) and (12). Observe that $x(\bar{\tau}_i) = -q(\bar{\tau}_i)^\top Q(\bar{\tau}_i)$ is the vertex of $\mathcal{Q}(\bar{\tau}_i)^+$ and $\mathcal{Q}(\bar{\tau}_i)^-$. Then, using Lemma 3 we can verify if there is a cone in $\mathcal{Q}(\bar{\tau}_i)^+, \mathcal{Q}(\bar{\tau}_i)^-$, $i = 1, 2$, that satisfies Definition 2. In particular, we need to prove that there is one $x(\bar{\tau}_i)$, $i = 1, 2$, that is either in $\mathcal{A}$ or $\mathcal{B}$. This criteria is stated in Lemma 10.

Lemma 10. *Let the two roots $\bar{\tau}_i$, $i = 1, 2$ of $f(\bar{\tau})$ be different from $\hat{\tau}_1$, and $\hat{\tau}_2$. Then, in the first and fourth cases of Theorem 4, the cone $\mathcal{Q}(\bar{\tau}_2)$ contains a convex cone that satisfies Definition 2.*

The proof of Lemma 10 is presented in Appendix A.2. Now we can define a procedure to identify a conic cut. We need to identify which of the cones $\mathcal{Q}(\bar{\tau}_2)^+, \mathcal{Q}(\bar{\tau}_2)^-$ gives the conic cut. For this purpose we use the sign of $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2))$. Hence, we choose $\mathcal{Q}(\bar{\tau}_2)^+$ if $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2)) > 0$, and we choose $\mathcal{Q}(\bar{\tau}_2)^-$ when $W(\bar{\tau}_2)_1^\top(-Q^{-1}q - x(\bar{\tau}_2)) < 0$. This completes the procedure.

In summary, excluding the third case in Theorem 4, we have shown that it is possible to find a cone $\mathcal{K}$ or cylinder $\mathcal{C}$ satisfying Definitions 2 or 5 for all the relevant cases in Theorem 4. Hence, combining Theorem 4 with Propositions 1 and 2 we provided a procedure to find a disjunctive conic cut for $\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})$ for a general disjunctive set $\mathcal{A} \cup \mathcal{B}$.

5 Disjunctive conic cut vs nonlinear conic mixed integer rounding inequality

Atamtürk and Narayanan [2] present a procedure for generating a *nonlinear conic mixed integer rounding* cut. Since this is a conic cut, we examine how it compares to the disjunctive conic cut introduced here. For this purpose, let us consider the following example

$$\begin{array}{rl} \text{minimize:} & -x \quad -y \\ \text{subject to:} & \left\| \begin{matrix} x - \frac{4}{3} \\ y - 1 \end{matrix} \right\| \le \frac{4}{3} - \frac{x}{2} - \frac{y}{2} \\ & x \in \mathbb{Z},\ y \in \mathbb{R}. \end{array} \tag{16}$$

First, notice that the example in (16) is in the format used in [2], which is different from the one in (1). The main difference is the way we write the conic constraint. Despite this difference we can still construct a disjunctive conic cut, because the feasible region of this problem is an ellipsoid in the (x, y) space.

Relaxing the integrality constraint, the resulting relaxation from problem (16) can be solved easily (the KKT conditions give a 2×2 linear system). First, notice that this relaxation is just a problem of minimizing a linear function over an ellipsoid. Particularly, we can rewrite the relaxation of problem (16) as follows,

$$\begin{array}{rl} \text{minimize:} & -x - y \\ \text{subject to:} & \frac{3}{4}x^2 + \frac{3}{4}y^2 - \frac{1}{2}xy - \frac{4}{3}x - \frac{2}{3}y + 1 \le 0 \\ & x,\ y \in \mathbb{R}. \end{array} \tag{17}$$

The feasible set of this problem is presented in Figure 8, which is an ellipsoid. The optimal objective function value is -2.471, and the relaxed optimal solution for the example in problem (16) is $(x^*, y^*) = (1.402, 1.069)$.

We can rewrite problem (16) in the following form:

$$\begin{array}{rl} \text{minimize:} & -x \quad -y \\ \text{subject to:} & x \quad +y \quad +2t \qquad = \frac{8}{3} \\ & \sqrt{(x - \frac{4}{3})^2 + (y - 1)^2} \le t \\ & x \in \mathbb{Z},\ y \in \mathbb{R},\ t \in \mathbb{R}. \end{array} \tag{18}$$

Figure 9 presents the feasible region of this equivalent problem. Using a branch-and-bound procedure one can easily solve the mixed integer problem in (18), and get that the optimal solution is $(t^*, x^*, y^*) = (1/3, 1, 1)$ with the optimal cost of -2.

The problem reformulation (18) presents a case similar to the one studied in Example 1 in [2], which shows how to obtain a nonlinear conic mixed integer rounding inequality for the set

$$T_0 = \left\{ (x, y, t) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} : \sqrt{(x - \frac{4}{3})^2 + (y-1)^2} \leq t \right\}, \tag{19}$$

which is the set of solutions satisfying the last constraint in (18). In general, the procedure discussed by Atamtürk and Narayanan [2] focuses on generating the convex hull for each *polyhedral second-order conic constraint* in the problem. Then, by adding those new cuts they tighten the original formulation. In particular, applying that procedure to the set in (19) they obtain the cut

$$\sqrt{\left(\frac{x}{3}\right)^2 + (y-1)^2} \leq t, \tag{20}$$

which is a valid cut for the problem in (18).

Analyzing the relaxed solution showed in Figure 8, we can see that the solution is not feasible for the integer problem. First, observe that if we use the disjunction $x \leq 1 \vee x \geq 2$ it is not possible to apply the disjunctive conic cut here, because the line $x = 2$ does not intersect the set of feasible solutions that is an ellipsoid, violating one of the assumptions in §3. However, we can still use the nonlinear conic mixed integer rounding inequality procedure. Figure 10 shows the result of applying the nonlinear conic cut (19) to the problem in (18). The point $(t^*, x^*, y^*) = (1/3, 1, 1)$ is the new optimal solution for the continuous relaxation of the resulting problem with the cut added, which turns out to be optimal for the mixed integer problem. The optimal objective value is -2.

Now, let us modify the first constraint in (18) as follows

$$x + y + 2t = \frac{14}{3}.$$

Figure 11 shows the new feasible region. With this modification, the relaxed optimal solution is $(t^*, x^*, y^*) = (0.68, 1.81, 1.48)$, which is not feasible for the integer problem. Now, for this example we can use the disjunction $x \leq 1 \vee x \geq 2$ and obtain a disjunctive conic cut that can be represented in the (x, y) space as follows:

$$\sqrt{(y - 0.33x + 0.22)^2} \leq 2.67 - 0.93x. \tag{21}$$

Observe that the nonlinear conic mixed integer rounding inequality (20) stays the same, since we have not modified the conic constraint. Figure 11 shows these two cuts and highlights the difference between applying the nonlinear conic mixed integer rounding inequality and the disjunctive conic cut to the modified problem. More specifically, the disjunctive conic cut gives the convex hull of the intersection between the disjunction $x \leq 1 \vee x \geq 2$ and the feasible set of problem (18). This is not the case for the nonlinear conic mixed integer rounding inequality (20). The new optimal solution for the relaxed problem when either of the cuts is applied is $(t^*, x^*, y^*) = (0.71, 2.0, 1.25)$. In particular, we can see that any of the cuts is enough to find the optimal solution. The optimal value for the objective function is -3.25.

Finally, we perform an additional test modifying the first constraint in (18) as follows:

$$x + y + 2t = 8.$$

In this case we use the disjunction $x \leq 2 \vee x \geq 3$. Then, we can obtain a disjunctive conic cut that can be represented in the (x, y) space as follows:

$$\sqrt{(y - 0.33x + 1.33)^2} \leq 6.04 - 1.21x.$$

For this example the nonlinear conic mixed integer rounding inequality (20) fails to eliminate the continuous optimal solution found for the relaxed problem, as illustrated in Figure 12. Thus, there is no gain in adding this cut to the problem. However, the disjunctive conic cut is violated by the current fractional solution, and the addition of the disjunctive conic cut is enough to find the integer solution for the problem.

6 Concluding remarks

In this paper, we analyzed the convex hull of the intersection of a convex set $\mathcal{E}$ and a linear disjunctive set $\mathcal{A} \cup \mathcal{B}$. This analysis is done for general convex sets. We assume the existence of a convex cone $\mathcal{K}$ (resp. a convex cylinder $\mathcal{C}$) that has the same intersection with the boundary $\mathcal{A}^=$, $\mathcal{B}^=$ of the disjunction as $\mathcal{E}$. Given the cone $\mathcal{K}$ (resp. cylinder $\mathcal{C}$), we proved that the convex hull $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$ is $\mathcal{E} \cap \mathcal{K}$ (resp. $\mathcal{E} \cap \mathcal{C}$). Additionally, we were able to prove that if $\mathcal{K}$ (resp. a cylinder $\mathcal{C}$) exists, then it is unique.

We then showed the existence of such a cone $\mathcal{K}$ (resp. a cylinder $\mathcal{C}$) for MISOCO problems. We consider the feasible set of the continuous relaxation of a MISOCO problem, assumed to be an ellipsoid, intersected with a general linear disjunction. We showed that in this case $\mathcal{K}$ is a second order cone, and provided a closed formula to describe $\mathcal{K}$ (resp. a cylinder $\mathcal{C}$) for MISOCO problems. This cone provides a novel conic cut for MISOCO and because it gives the convex hull of the disjunction, it is the strongest possible cut for MISOCO problems. Having a closed form for this disjunctive conic cut makes them ready to use. The development of an efficient Branch-and-Cut software package for MISOCO problems is the subject of ongoing research.

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A The proofs of Lemmas 9 and 10

For the sake of simplifying the algebra of these proofs we use the following observation. If $Q \succ 0$ and the quadric $\mathcal{Q}$ is not single point, $\mathcal{Q}$ can be transformed to a unit hypersphere $\{y \in \mathbb{R}^n \mid \|y\|^2 \leq 1\}$ using the affine transformation

$$y = \frac{Q^{1/2}(x + Q^{-1}q)}{\sqrt{\|u_q\|^2 - \rho}}. \tag{22}$$

Observe that this transformation preserves the inertia of Q, hence the classification of the quadric is not changed. Additionally, observe that if we apply the same transformation to two parallel hyperplanes, the resulting hyperplanes are still parallel. Hence, throughout this proof, if $Q \succ 0$, we assume w.l.o.g. that the quadric $\mathcal{Q}$ is a unit hypersphere centered at the origin. In this case, we have that the positive definite matrix Q of Section 4 is the identity matrix, the vector q is the zero vector, $\rho = -1$. Additionally, given Assumption 2 and the assumption that $\|a\| = \|b\| = 1$, we have that $|\alpha| \leq 1$, and $|\beta| \leq 1$.

A.1 Proof of Lemma 9

From Section 4.1 we have that $\hat{\tau} = -1$, and the numerator of the right hand side of (9) reduces to

$$f(\tau) = \tau^2 \frac{(\alpha - \beta)^2}{4} + \tau(1 - \alpha\beta) + 1.$$

Recall from Section 4.1.1 that the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_1)$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$, are computed using the roots $\bar{\tau}_1$ and $\bar{\tau}_2$ of the function $f(\tau)$. Particularly, we have that

$$\bar{\tau}_1 = 2\left(\frac{\alpha\beta - 1 - \sqrt{(1-\alpha^2)(1-\beta^2)}}{(\alpha - \beta)^2}\right),$$
$$\bar{\tau}_2 = 2\left(\frac{\alpha\beta - 1 + \sqrt{(1-\alpha^2)(1-\beta^2)}}{(\alpha - \beta)^2}\right),$$

where $\bar{\tau}_1 \leq \bar{\tau}_2$. Note that if $\alpha = \beta$, then $f(\tau)$ is a linear function. In this case we would have that $\mathcal{A} = \mathcal{B}$ and is easy to verify that $\operatorname{conv}(\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}$. However, recall that our assumption is $\beta \neq \alpha$. Hence, for the rest of this proof we assume that $\alpha \neq \beta$.

The vertices of the cones $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_2)$ are $x(\bar{\tau}_i) = -Q(\bar{\tau}_i)^{-1}p(\bar{\tau}_i)$, $i = 1, 2$. We can express $x(\bar{\tau}_i)$ in terms of a, α, and β as follows

$$\begin{aligned} x(\bar{\tau}_i) = -Q(\bar{\tau}_i)^{-1}q(\bar{\tau}_i) &= -\left(I - \frac{\bar{\tau}_i}{(1+\bar{\tau}_i)}aa^\top\right)\left(-\bar{\tau}_i\frac{(\alpha+\beta)}{2}a\right) \\ &= \bar{\tau}_i\frac{(\alpha+\beta)}{2}\left(1 - \frac{\bar{\tau}_i}{(1+\bar{\tau}_i)}\right)a \\ &= \bar{\tau}_i\frac{(\alpha+\beta)}{2(1+\bar{\tau}_i)}a. \end{aligned}$$

Consider the inner product

$$a^\top x(\bar{\tau}_i) = -a^\top Q(\bar{\tau}_i)^{-1}q(\bar{\tau}_i) = \bar{\tau}_i\frac{(\alpha+\beta)}{2(1+\bar{\tau}_i)}a^\top a = \bar{\tau}_i\frac{(\alpha+\beta)}{2(1+\bar{\tau}_i)}.$$

Note that if $\alpha = -\beta$ then $a^\top x(\bar{\tau}_i) = 0$. Recall from Theorem 2 that in that case $Q(\bar{\tau}_1)$ is a cylinder. For that reason, we assume that $\alpha \neq -\beta$ for the rest of this proof.

Next, note that since $\mathcal{A}^=$ and $\mathcal{B}^=$ are parallel, then $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then, we need to show that in the first and third cases of Theorem 2 the vertex $x(\bar{\tau}_2)$ cannot be in the set $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Assume to the contrary that $x(\bar{\tau}_2) \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Now, since we are analyzing the first and fourth cases of Theorem 4 we know that $\bar{\tau}_2 < -1$. Thus, if $a^\top x(\bar{\tau}_2) \leq \alpha$ and $a^\top x(\bar{\tau}_2) \geq \beta$, then

$$\bar{\tau}_2(\beta - \alpha) \leq 2\alpha \quad \text{and} \quad \bar{\tau}_2(\alpha - \beta) \geq 2\beta. \tag{23}$$

Substituting $\bar{\tau}_2$ in (23) we obtain that $\sqrt{\frac{(1-\alpha^2)}{(1-\beta^2)}} = 1$. The las equality is possible only if either $\alpha = -\beta$ or $\alpha = \beta$. In the first case we obtain that $\bar{\tau}_2 = -1$, which is not in the cases considered. In the second case we obtain that $\overline{\mathcal{A}} \cap \overline{\mathcal{B}} = \emptyset$. Hence, in the first and third cases of Theorem 2 the vertex $x(\bar{\tau}_2)$ cannot be in the set $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

Thus, since the intersections $\mathcal{Q}(\bar{\tau}_2) \cap \mathcal{A}^=$ and $\mathcal{Q}(\bar{\tau}_2) \cap \mathcal{B}^=$ are bounded, then one of the following two cases holds:

- Case 1: $\mathcal{Q}^+(\bar{\tau}_2) \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{Q}^+(\bar{\tau}_2) \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=$;
- Case 2: $\mathcal{Q}^-(\bar{\tau}_2) \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{Q}^-(\bar{\tau}_2) \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=$.

Consequently, we have that one of the cones $\mathcal{Q}^+(\bar{\tau}_2)$ and $\mathcal{Q}^-(\bar{\tau}_2)$ found at the root $\bar{\tau}_2$ satisfy Proposition 1.

□

A.2 Proof of Lemma 10

Recall from Section 4.2.1 that the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_1)$ in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ of Theorem 4, are computed using the roots of the function (15), which in this case simplifies to

$$f(\tau) = \left((\alpha\beta - a^\top b)^2 - (1-\alpha^2)(1-\beta^2)\right)\tau^2 + 4(a^\top b - \alpha\beta)\tau + 4.$$

The roots of $f(\tau)$ are

$$\bar{\tau}_1 = 2\left(\frac{\alpha\beta - a^\top b - \sqrt{(1-\alpha^2)(1-\beta^2)}}{(\alpha\beta - a^\top b)^2 - (1-\alpha^2)(1-\beta^2)}\right) = \frac{2}{\alpha\beta - a^\top b + \sqrt{(1-\alpha^2)(1-\beta^2)}},$$
$$\bar{\tau}_2 = 2\left(\frac{\alpha\beta - a^\top b + \sqrt{(1-\alpha^2)(1-\beta^2)}}{(\alpha\beta - a^\top b)^2 - (1-\alpha^2)(1-\beta^2)}\right) = \frac{2}{\alpha\beta - a^\top b - \sqrt{(1-\alpha^2)(1-\beta^2)}},$$

where $\bar{\tau}_1 \leq \bar{\tau}_2$.

Also, recall that the classification of the the quadrics $\mathcal{Q}(\bar{\tau}_1)$ and $\mathcal{Q}(\bar{\tau}_1)$ is done based on the ratio $f(\tau)/g(\tau)$, where $g(\tau)$ simplifies in this case to

$$g(\tau) = ((a^\top b)^2 - 1)\tau^2 + 4a^\top b\tau + 4.$$

The roots of $g(\tau)$ are

$$\hat{\tau}_1 = -\frac{2}{a^\top b + 1} < 0 \qquad \text{and} \qquad \hat{\tau}_2 = -\frac{2}{a^\top b - 1} > 0.$$

Note that if $a^\top b - 1 = 0$, then we obtain that $\hat{\tau}_2$ is given by a division by zero. However, since $\|a\| = \|b\| = 1$, in this case we obtain that $a^\top b = \cos(0)$, which implies that $a = b$. This is the case when we have parallel hyperplanes, which was already analyzed in A.1. For that reason, for the rest of this proof we assume that $a \neq b$.

The vertex of the cone $\mathcal{Q}(\bar{\tau}_2)$ is $x(\bar{\tau}_2) = -Q(\bar{\tau}_2)^{-1}q(\bar{\tau}_2)$. We can express $x(\bar{\tau}_2)$ in terms of a, b, α, and β as follows

$$\begin{aligned}
x(\bar{\tau}_2) &= -Q(\bar{\tau}_2)^{-1}q(\bar{\tau}_2) \\
&= -\left(I - \frac{(aa^\top + bb^\top)\bar{\tau}_2^2 - (a^\top b\bar{\tau}_2^2 + 2\bar{\tau}_2)(ba^\top + ab^\top)}{(1-(a^\top b)^2)\bar{\tau}_2^2 - 4a^\top b\bar{\tau}_2 - 4}\right)\left(-\bar{\tau}_2\frac{\beta a + \alpha b}{2}\right) \\
&= \frac{\bar{\tau}_2\left(\left((\alpha - a^\top b\beta)\bar{\tau}_2 - 2\beta\right)a + \left((\beta - a^\top b\alpha)\bar{\tau}_2 - 2\alpha\right)b\right)}{(1-(a^\top b)^2)\bar{\tau}_2^2 - 4a^\top b\bar{\tau}_2 - 4}.
\end{aligned}$$

Consider the inner products

$$a^\top x(\bar{\tau}_2) = \frac{\bar{\tau}_2\left((1-(a^\top b)^2)\alpha\bar{\tau}_2 - 2(a^\top b\alpha + \beta)\right)}{(1-(a^\top b)^2)\bar{\tau}_2^2 - 4a^\top b\bar{\tau}_2 - 4}$$

and

$$b^\top x(\bar{\tau}_2) = \frac{\bar{\tau}_2\left((1-(a^\top b)^2)\beta\bar{\tau}_2 - 2(a^\top b\beta + \alpha)\right)}{(1-(a^\top b)^2)\bar{\tau}_2^2 - 4a^\top b\bar{\tau}_2 - 4}.$$

Next, we show that in the first and fourth cases of Theorem 4 the vertex $x(\bar{\tau}_2)$ cannot be in the set $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Assume to the contrary that $x(\bar{\tau}_2) \in \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Note that $\hat{\tau}_1$ and $\hat{\tau}_2$ are the roots of $(1-(a^\top b)^2)\tau^2 - 4a^\top b\tau - 4 = -g(\tau)$. Now, since we are analyzing the first and fourth cases of Theorem 4 we know that $\hat{\tau}_2 < \bar{\tau}_1$, or $\bar{\tau}_2 < \hat{\tau}_1$, or $\bar{\tau}_1 < \hat{\tau}_1 < \hat{\tau}_2 < \bar{\tau}_2$. Even more, since $1-(a^\top b)^2 \geq 0$ we have that $(1-(a^\top b)^2)\bar{\tau}_2^2 - 4a^\top b\bar{\tau}_2 - 4 \geq 0$. Thus, if $a^\top x(\bar{\tau}_2) \leq \alpha$ and $b^\top x(\bar{\tau}_2) \geq \beta$, then

$$(a^\top b\alpha - \beta)\bar{\tau}_2 \leq -2\alpha \quad \text{and} \quad (a^\top b\beta - \alpha)\bar{\tau}_2 \geq -2\beta. \tag{24}$$

Substituting $\bar{\tau}_2$ in (24) we obtain that $\frac{\alpha}{\sqrt{1-\alpha^2}} = -\frac{\beta}{\sqrt{1-\beta^2}}$, which implies that $\alpha = -\beta$. This is possible if $\bar{\tau}_2 = \hat{\tau}_1$, which is not in the cases being considered. Hence, in the first and fourth cases of Theorem 4 $x(\bar{\tau}_2)$ cannot be in the set $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

Similarly, we can show that in the first and fourth cases of Theorem 4 the vertex $x(\bar{\tau}_2)$ cannot be in the set $\mathcal{A} \cap \mathcal{B}$. In particular, if $a^\top x(\bar{\tau}_2) \geq \alpha$ and $b^\top x(\bar{\tau}_2) \leq \beta$, then

$$(a^\top b\alpha - \beta)\bar{\tau}_2 \geq -2\alpha \quad \text{and} \quad (a^\top b\beta - \alpha)\bar{\tau}_2 \leq -2\beta. \tag{25}$$

Substituting $\bar{\tau}_2$ in (25) we obtain that $\frac{\alpha}{\sqrt{1-\alpha^2}} = -\frac{\beta}{\sqrt{1-\beta^2}}$. This implies that $\bar{\tau}_2 = \hat{\tau}_1$, which is not in the cases being considered. Hence, the vertex $x(\bar{\tau}_2)$ cannot be in the set $\mathcal{A} \cap \mathcal{B}$.

Thus, since the intersections $\mathcal{Q}(\bar{\tau}_2) \cap \mathcal{A}^=$ and $\mathcal{Q}(\bar{\tau}_2) \cap \mathcal{B}^=$ are bounded, then one of the following two cases is true:

- Case 1: $\mathcal{Q}^+(\bar{\tau}_2) \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{Q}^+(\bar{\tau}_2) \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=$.
- Case 2: $\mathcal{Q}^-(\bar{\tau}_2) \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{Q}^-(\bar{\tau}_2) \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=$.

Consequently, we have that one of the cones $\mathcal{Q}^+(\bar{\tau}_2)$, $\mathcal{Q}^-(\bar{\tau}_2)$ found at the root $\bar{\tau}_2$ satisfies Proposition 1. □

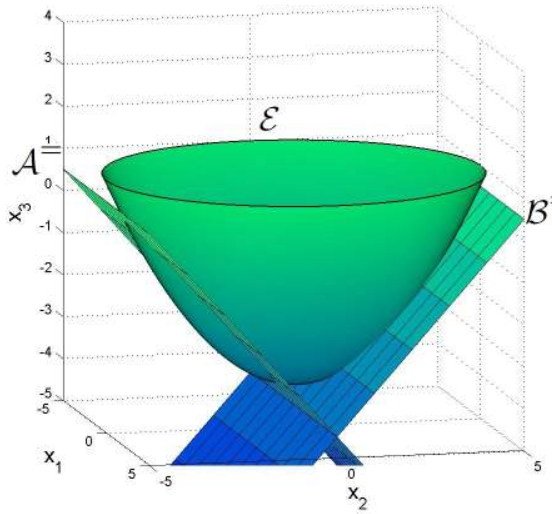
B Additional lemma

Lemma 11. *Let $\mathcal{C} \subset \mathbb{R}^n$ be a cylinder with a compact base. Then $\mathcal{C}$ is closed.*

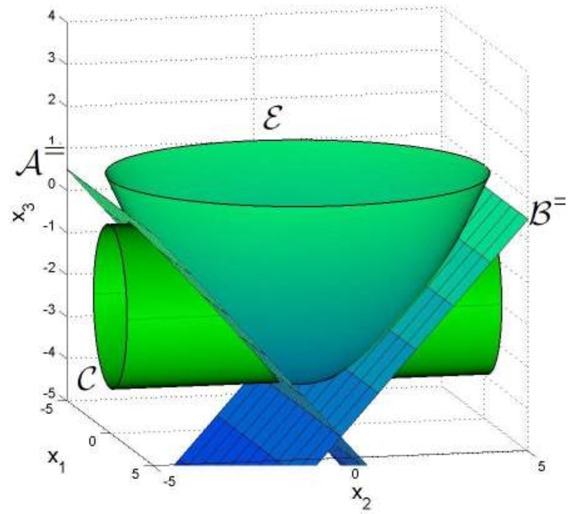
Proof. Let $\mathcal{D}$ be a compact base for $\mathcal{C} = \{x \in \mathbb{R}^n | x = d + \sigma d_0, d \in \mathcal{D}, \sigma \in \mathbb{R}\}$ and let $u \in \mathbb{R}^n$ be a vector such that $u \notin \mathcal{C}$. Our goal is to show that there is a neighborhood $\mathcal{U}$ of u such that $\mathcal{U} \cap \mathcal{C} = \emptyset$.

Let $\delta = \max\{\|u - x\| \mid x \in \mathcal{D}\} > 0$ be the maximum distance from a point $x \in \mathcal{D}$ to u. Let us choose $\sigma_o = (\delta + 1)/\|d_0\|$ and let $\mathcal{B}$ be the open ball of radius 1 centered at u. Define the set $\mathcal{C}_1 = \{x \in \mathbb{R}^n \mid x = d + \sigma d_o, d \in \mathcal{D}, \sigma \leq -\sigma_o\} \cup \{x \in \mathbb{R}^n \mid x = d + \sigma d_0, d \in \mathcal{D}, \sigma \geq \sigma_o\}$. Then, we have that $\mathcal{B} \cap \mathcal{C}_1 = \emptyset$.

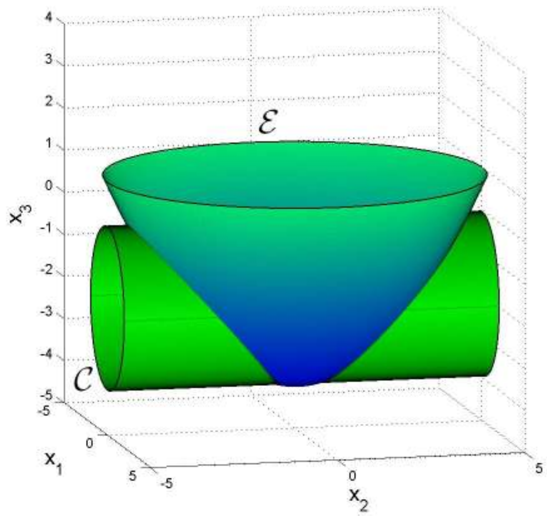
Let $\mathcal{X} = \mathcal{D} \times [-\sigma_o, \sigma_o]$, and consider the map $h : \mathcal{X} \mapsto \mathbb{R}^n$, defined by $h(\sigma, x) = x + \sigma d_o$. Since $\mathcal{D}$ and $[-\sigma_o, \sigma_o]$ are compact we have that $\mathcal{X}$ is compact. Since h is continuous in $\mathcal{X}$ we have that the image $h(\mathcal{X})$ is a compact set as well, and hence closed in $\mathbb{R}^n$. Furthermore, note that $h(\mathcal{X}) \subset \mathcal{C}$, thus $u \notin h(\mathcal{X})$. Hence, there is a neighborhood $\mathcal{N}$ of u such that $\mathcal{N} \cap h(\mathcal{X}) = \emptyset$. Let $\mathcal{U} = \mathcal{B} \cap \mathcal{N}$, then for any $\sigma \in \mathbb{R}$ we have that $\mathcal{U} \cap (\sigma d_0 + \mathcal{D}) = \emptyset$. This proves that the complement of $\mathcal{C}$ is open, thus $\mathcal{C}$ is closed. □ □



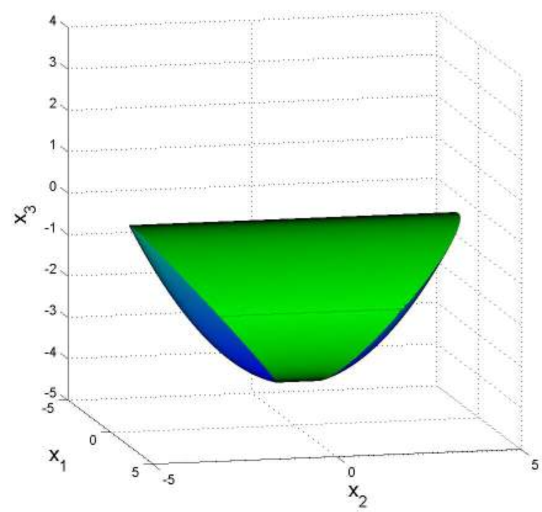
(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and $\mathcal{E}$



(b) The cylinder $\mathcal{C}$ yielding $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$



(c) $\mathcal{E} \cap \mathcal{C}$



(d) $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

Figure 4: Illustration of a disjunctive cylindrical cut as specified in Proposition 2

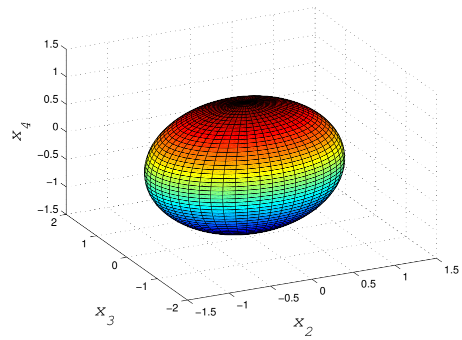
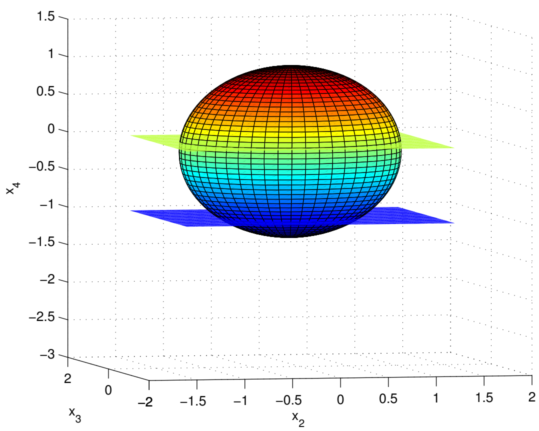
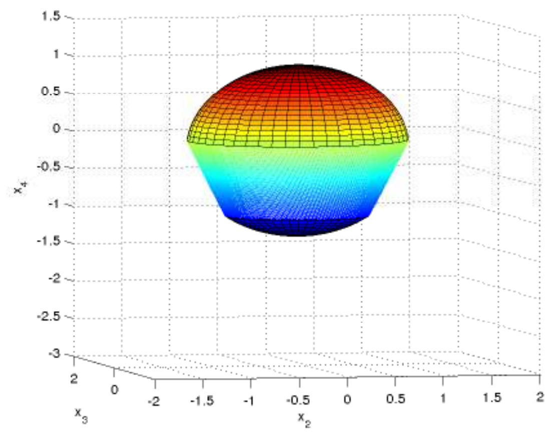


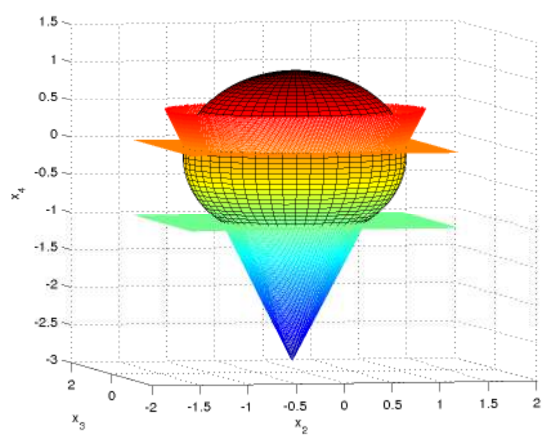
Figure 5: The feasible region of Problem (5)



(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and $\mathcal{E}$

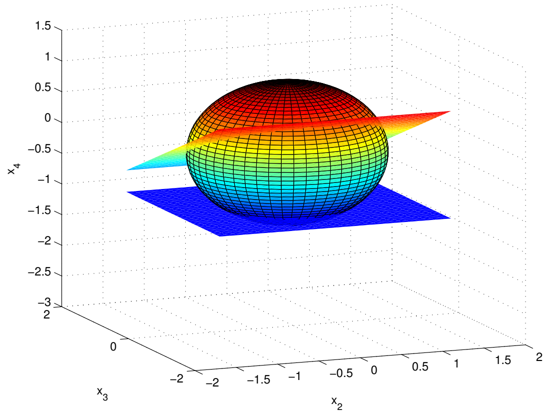


(b) $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

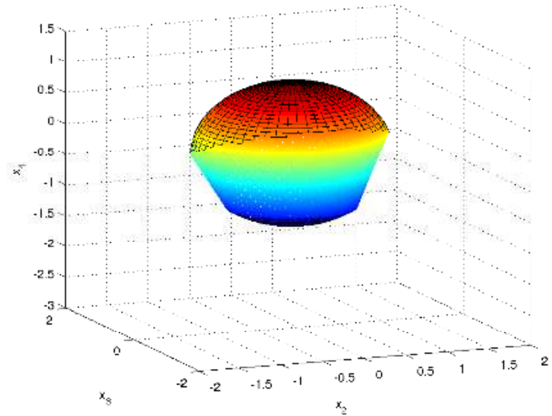


(c) The cone yielding $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

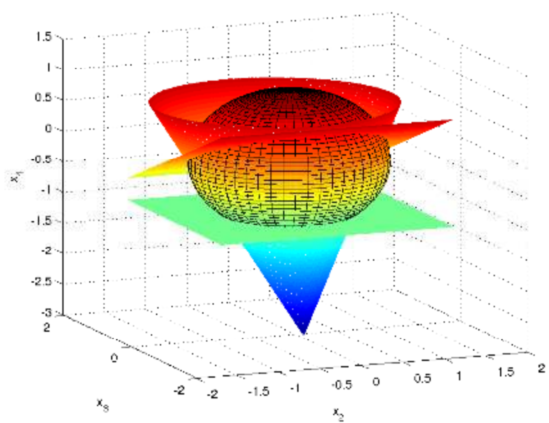
Figure 6: The convex hull of the intersection of a parallel disjunction and an ellipsoid.



(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and $\mathcal{E}$



(b) $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$



(c) The cone yielding $\operatorname{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$

Figure 7: Convex hull of the intersection between a non-parallel disjunction and an ellipsoid.

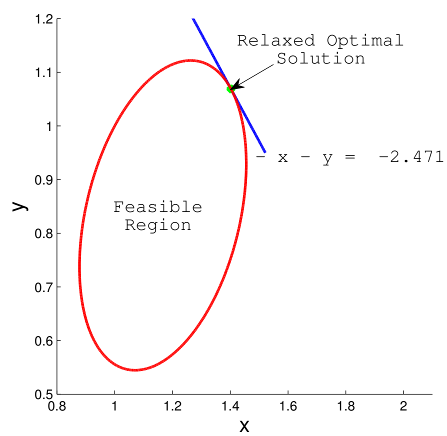


Figure 8: An optimal solution of problem (17).

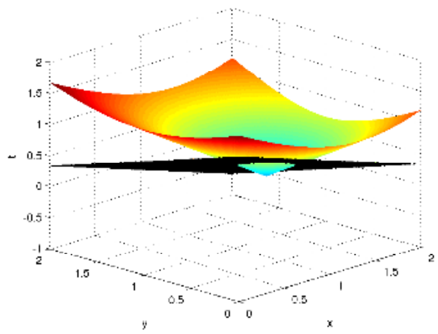


Figure 9: The feasible region of the continuous relaxation of problem (18).

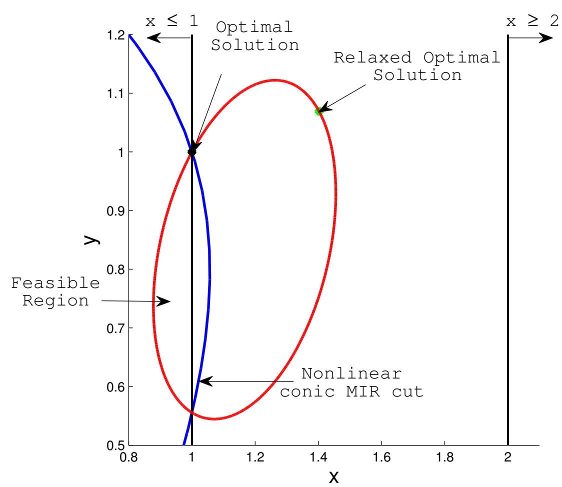


Figure 10: Nonlinear conic mixed integer rounding inequality

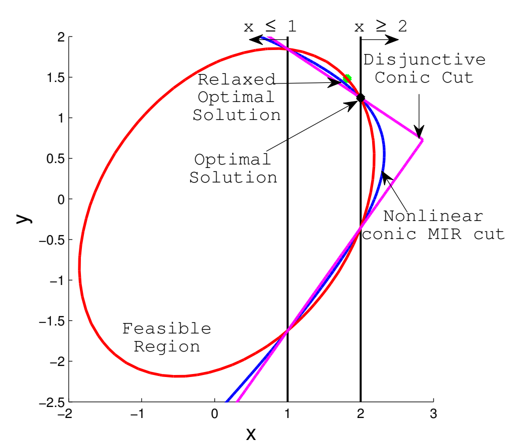


Figure 11: The Disjunctive conic cut and the Nonlinear conic mixed integer rounding inequality cutting off the relaxed optimal solution.

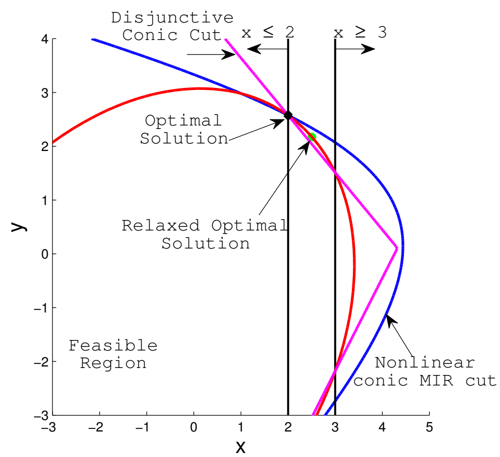


Figure 12: The nonlinear conic mixed integer rounding inequality fails to cut off the optimal solution fo the relaxed problem