A CONJECTURE ON EXCEPTIONAL ORTHOGONAL POLYNOMIALS

DAVID GÓMEZ-ULLATE, NIKY KAMRAN, AND ROBERT MILSON

Abstract. Exceptional orthogonal polynomial systems (X-OPS) arise as eigenfunctions of Sturm-Liouville problems and generalize in this sense the classical families of Hermite, Laguerre and Jacobi. They also generalize the family of CPRS orthogonal polynomials introduced by Cariñena *et al.*, [3]. We formulate the following conjecture: *every exceptional orthogonal polynomial system is related to a classical system by a Darboux-Crum transformation*. We give a proof of this conjecture for codimension 2 exceptional orthogonal polynomials (X_2-OPs). As a by-product of this analysis, we prove a Bochner-type theorem classifying all possible X_2-OPS. The classification includes all cases known to date plus some new examples of X_2-Laguerre and X_2-Jacobi polynomials.

1. Introduction

The past several years have witnessed a considerable level of research activity in the area of exceptional orthogonal polynomials, which are new complete orthogonal polynomial systems arising as eigenfunctions of Sturm-Liouville operators, extending the classical families of Hermite, Laguerre and Jacobi. The first examples of exceptional orthogonal polynomial systems were discovered in [12] and [13] as a result of the development of a direct approach [8] to exact or quasi-exact solvability for spectral problems in quantum mechanics that would go beyond the classical Lie algebraic formulations [17, 22, 34].

The exceptional orthogonal polynomial systems and the Sturm-Liouville problems that define them have some key poperties that distinguish them from the classical orthogonal polynomial systems, and which we would like to highlight. The most apparent one is that they admit *gaps* in their degrees, in the sense that not all degrees are present in the sequence of polynomials that form a complete orthonormal set of the underlying weighted L^2 space, even though they are defined by a Sturm-Liouville problem. This means in particular that they are not covered by the hypotheses of Bochner's celebrated theorem on the characterization of orthogonal polynomial systems defined by Sturm-Liouville problems [2].

The number of gaps in the sequence of degrees of the polynomials appearing in a complete family will be referred to as the *codimension* and we will use the symbol X_m to denote the various complete orthogonal systems of codimension m. A second order differential operator is *exceptional* if it preserves some exceptional polynomial flag, but does not preserve the standard polynomial flag generated by the monomials. Thus, and in contrast with the classical families where the defining differential operator has only polynomial coefficients, the second order differential operators corresponding to the exceptional families have poles in their coefficients, although all their singular points happen to be regular.

The first explicit examples of families of exceptional orthogonal polynomials are the X_1-Jacobi and X_1-Laguerre polynomials, which are of codimension one, and were first introduced in [12] and [13]. In these papers, a characterization theorem was proved for these orthogonal polynomial families, realizing them as the unique complete codimension one families defined by a Sturm-Liouville problem. One of the key steps in the proof was the determination of normal forms for the flags of univariate polynomials of codimension one in the space of all such polynomials, and the determination of the second-order linear differential operators which preserve these flags [11, 16].

It is Quesne [25, 26] who first observed the presence of a relationship between exceptional orthogonal polynomials, the Darboux transformation and shape invariant potentials. This enabled her to obtain examples of potentials corresponding to orthogonal polynomial families of codimension two, as well as explicit families of X_2 polynomials. Higher-codimensional families were first obtained by Odake and Sasaki [28]. The same authors further showed the existence of two families of X_m-Laguerre and X_m-Jacobi polynomials, the existence of which was explained in [14] for X_m-Laguerre polynomials and in [16] for X_m-Jacobi polynomials, through the application of the isospectral algebraic Darboux transformation first introduced in [9, 10]. We also refer to [33] for similar results, and to [14, 16] for the proof of the completeness of the X_m-Laguerre families. We also note that some examples of exceptional Hermite polynomials were known to the quantum physics community in the early 90s, [4], and are actively studied today under the name of CPRS systems, [3, 7]. It should as well be noted that the exceptional Laguerre polynomials have already been used in a number of interesting physical contexts, for Dirac operators minimally coupled to external fields, [21], or in quantum information theory, [6].

The papers cited above contain many examples of orthogonal polynomial families of arbitrary codimension arising from the Laguerre and Jacobi system by the application of the Darboux transformation. However, as was shown in [15], the above list is not exhaustive: novel exceptional polynomials can be constructed by means of *multi-step* Darboux or Darboux-Crum transformations. The multi-step idea was further applied to exactly solvable and shape-invariant potentials up in [18, 27, 32]. However, an essential question that remains open is to know whether these families exhaust all the possibilities of higher-codimensional complete orthogonal polynomial systems, in other words whether *all* the higher-codimensional complete orthogonal polynomial systems are generated by the application of successive algebraic Darboux transformations. *We conjecture this result to be true.* In order to prove such a result, one approach would be to try to carry out for all codimensions an analysis similar to the one performed in [11–13] in codimension one, identify the complete orthogonal sets amongst the resulting families and show that all of these can be obtained from the classical codimension-zero families by iterating algebraic Darboux transformations (we will refer to these as *multi-step* Darboux transformations). This seems like a very challenging task in the absence of a general classification strategy that would lead to normal forms for flags of univariate polynomials for all codimensions. Even in the codimension two case, the question would be quite difficult to answer if we were only using the tools that were at our disposal in [11]. We are nevertheless able to give a complete answer to this question for codimension two families by suitably refining the approach taken in these earlier papers. In particular the possible pole structure of the coefficients of the operators that preserve the codimension two flags plays a key role in the analysis.

Since the main objects of our study are orthogonal polynomial systems that arise as eigenfunctions of a Sturm-Liouville problem, let us give a definition:

Definition 1.1. We define a Sturm-Liouville orthogonal polynomial system (SL-OPS) as a sequence of real polynomials $y_1(x), y_2(x), y_3(x), \ldots$, with $\deg y_i > \deg y_j$ if $i > j$, satisfying the following conditions:

(i) There exists a second order differential operator
$$T[y] = p(z)y'' + q(z)y' + r(z)y$$
with rational coefficients p, q, r such that $T[y_i] = \lambda_i y_i$ for all i, with the λ_i distinct.

(ii) There exists an interval $I = (a, b)$, $-\infty \le a < b \le \infty$ such that the weight function
$$W(x) = \frac{1}{p(x)} \exp\left(\int^x \frac{q}{p} dx\right)$$
is positive, that is $W(x) > 0$ for $x \in I$, such that all moments are finite,
$$\int_a^b x^i W(x) dx < \infty, \quad i = 0, 1, 2, 3 \ldots,$$
and such that $p(x)W(x) \to 0$ at the endpoints $x = a, b$.

(iii) The polynomial sequence is complete, meaning that $\mathrm{span}\{y_i\}_{i=1}^{\infty}$ is dense in $\mathrm{L}^2(W dx, I)$;

Remark 1.1. It follows from the above definition that the operator T is essentially self-adjoint on the weighted Hilbert space $L^2(I, W dx)$ and that the eigenpolynomials are orthogonal, meaning that
$$\int_a^b W y_i y_j dx = k_i \delta_{ij}, \quad k_i > 0,$$
for some constants k_i.

Remark 1.2. If $\deg y_i = i - 1$ for all i, we are dealing with one of the *classical* orthogonal polynomial systems of Hermite, Laguerre and Jacobi: the polynomials in question span the standard polynomial flag and p, q, r are polynomials of degrees $2, 1$ and 0 respectively, [2].

Remark 1.3. If the degree sequence $\{\deg y_i\}_{i=1}^{\infty}$ does not contain all non-negative integers, then we will have an exceptional polynomial system (X-OPS), and the coefficients of T will be purely rational (as opposed to polynomial) functions.

Remark 1.4. We shall see in Section 5.1 that the eigenvalue equation $T[y] = \lambda y$ can be put into Sturm-Liouville form.

Even though several families of X-OPS have now been described in the literature, the general question of classifying all such systems is still largely open. In particular, major porgress would be achieved in our understanding of the subject if we could obtain a classification or a characterization of all families of SL-OPS. (Recall the classification performed by Bochner [2] and Lesky [24] deals only with the classical OPS.) It seems clear by now that the Darboux transformation will be one of the key necessary tools in achieving such a goal. It should be noted that when referring to the Darboux transformation, we do not mean here the factorization of Jacobi matrices into upper triangular and lower triangular matrices [20]. Indeed, while such a transformation is defined for any OPS, the transformed OPS will in general not be a SL-OPS even if the original OPS was one. We will rather use *algebraic Darboux transformations*[1], also known as rational factorizations, which are defined only for SL-OPS. In these transformations, it is the second order operator T that needs to be factorized as the product of two first order operators $T = AB$, and the transformed operator $\hat{T}$ is obtained by reversing the order of the factors, $\hat{T} = BA$. We shall see that by construction, these algebraic Darboux transformations transform an SL-OPS into another SL-OPS.

We are now ready to state the main result of our paper:

Theorem 1.1. *Every* X_m *orthogonal polynomial system for* $m \leq 2$ *can be obtained by applying a sequence of at most* m *Darboux transformations to a classical orthogonal polynomial system.*

The proof of this theorem is done in several steps. The first step, carried out in Section 3, consists in the classification of X_2 flags and the determination of the corresponding pole structure for the coefficients of the second order linear differential operators that preserve them. This forms the substance of Theorem 3.2. It should be noted that in contrast with the codimension one case, the canonical codimension two flags contain free parameters (flag moduli). In Section 5 we provide the necessary background to select from the classification of X_2-flags those that give rise to a well defined SL-OPS. This selection is performed in Section 6, where Theorem 6.1 provides the classification of X_2 orthogonal polynomial systems. It is worth noting that this classification contains new families of X_2-Laguerre and X_2-Jacobi polynomials; for example the new Laguerre-type family of Section 6.5.6 with weight $e^{-x}x^{1/4}/(4x+3)^4$. The second step in the proof of Theorem 1.1 , which is carried out in Section 4, consists of the proof of the key property, stated in Theorem 4.2, that all X_1 and X_2 operators are related to a classical operator by a Darboux transformation or a sequence of two Darboux transformations.

Finally, we will conclude by stating our general, yet-to-be proved, conjecture, which extentends the result of Theorem 1.1 to arbitrary codimension.

Conjecture 1.1. *Every* X_m *orthogonal polynomial system for any codimension* m *can be obtained by applying a sequence of at most* m *Darboux transformations to a classical OPS.*

2. Preliminaries and definitions

2.1. **Polynomial flags.** Let $\mathcal{P}$ denote the infinite-dimensional space of real, univariate polynomials, and let $\mathcal{P}_n \subset \mathcal{P}$ be the $n+1$ dimensional subspace of polynomials having degree n or less. We define the degree of a finite dimensional polynomial subspace $U \subset \mathcal{P}$ to be

$$\deg U = \max\{\deg p : p \in \mathcal{U}\}. \tag{1}$$

Definition 2.1. A *polynomial flag* is an infinite sequence of polynomial subspaces $U_1 \subset U_2 \subset \ldots$, nested by inclusion, such that $\dim U_k = k$, and such that $\deg U_k < \deg U_{k+1}$ for all k. The *total space* of a polynomial flag is the infinite-dimensional polynomial subspace

$$\mathcal{U} = \bigcup_{k=1}^{\infty} U_k. \tag{2}$$

Definition 2.2. Let $\mathcal{U} \subset \mathcal{P}$ be an infinite dimensional polynomial subspace. A degree-regular basis of $\mathcal{U}$ is a sequence of polynomials $\{p_k\}_{k=1}^{\infty}$ such that $\deg p_k < \deg p_{k+1}$ and such that $\mathcal{U} = \operatorname{span}\{p_k\}$.

Proposition 2.1. *Let* $U_1 \subset U_2 \subset \cdots$ *be a polynomial flag,* $\mathcal{U}$ *the total space, and* $\{p_k\}_{k=1}^{\infty}$ *a degree regular basis. Then, for all* $k = 1, 2, \ldots,$ *we have*

$$U_k = \operatorname{span}\{p_1, \ldots, p_k\}.$$

[1] A wider class of these transformations has been extensively used in Quantum Mechanics to generate new exactly solvable problems from known ones. The subclass of interest to us in the context of OPS consists of the set of transformations that preserve the polynomial character of the eigenfunctions. This particular class of Darboux transformations was characterized in [9,10].

Proposition 2.2. *Let $\mathcal{U} \subset \mathcal{P}$ be an infinite dimensional polynomial subspace. Let $\hat{U}_k \subset \mathcal{U}$ be the unique k-dimensional subspace having minimal degree. Then $\hat{U}_1 \subset \hat{U}_2 \subset \cdots$ is a polynomial flag whose total space is $\mathcal{U}$.*

Proposition 2.3. *Let $U_1 \subset U_2 \subset \dots$ be a polynomial flag and $\mathcal{U}$ the corresponding total space. Let $\hat{U}_k$ be as above. Then, $\hat{U}_k = U_k$.*

The above propositions show that there is a natural bijection between the set of polynomial flags and the set of infinite-dimensional polynomial subspaces. Going forward it will sometimes be more conveninient to specify the total space rather than the actual flag. The identification of the flag and its total space will be implicitly assumed. We will use the complete notation $\mathcal{U} : U_1 \subset U_2 \subset \dots$ for the flag and its total space, but we will write only $\mathcal{U}$ to denote the flag $U_1 \subset U_2 \subset \dots$ where no confusion can arise.

Definition 2.3. Given a polynomial flag $\mathcal{U} : U_1 \subset U_2 \subset \dots$, we define the *degree sequence* $\{n_k\}_{k=1}^{\infty}$ and the *codimension sequence* $\{m_k\}_{k=1}^{\infty}$ as

$$n_k = \deg U_k, \quad m_k = n_k + 1 - k. \tag{3}$$

where m_k is the codimension of U_k in $\mathcal{P}_{n_k}$.

It is easy to see that $\{n_k\}$ is strictly increasing while $\{m_k\}$ is non-decreasing. In this paper we will focus on flags with *finite codimension*, which means that the total space $\mathcal{U}$ has finite codimension in $\mathcal{P}$, or equivalently, that the codimension sequence $\{m_k\}$ admits an upper bound $m = \max_k m_k$, which we call the *codimension of the flag*. As mentioned in the Introduction, one might also characterize m as the number of gaps in the degree sequence. We will say that a polynomial flag has *stable codimension* if $m_k = m$ for all k, or equivalently if the degree sequence satisfies $n_1 = m$ and $n_{k+1} = n_k + 1$ for all $k \geq 1$.

The simplest of all polynomial flags in the *standard flag* $\mathcal{U}_{\text{st}} : \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$. The total space for this flag is $\mathcal{P}$, its degree sequence is $\mathbb{N} \cup \{0\}$ and it has stable codimension zero.

Definition 2.4. We will say that a second order differential operator

$$T[y(z)] = p(z)y'' + q(z)y' + r(z)y, \tag{4}$$

is *rational*, if the coefficients p, q, r are rational functions of the independent variable z and the prime denotes derivation with respect to this variable, $y' = \frac{dy}{dz}$. The poles of a rational operator T are the poles of p, q and r. An operator T with no poles is said to be polynomial. If there is one or more poles, then we will refer to T as *non-polynomial*.

Definition 2.5. We say that a polynomial flag $\mathcal{U} : U_1 \subset U_2 \subset \dots$ is invariant under a rational operator $T[y]$ if $T(U_k) \subset U_k$ for all k. We let $\mathcal{D}_2(\mathcal{U})$ denote the vector space of all second order operators that preserve the flag $\mathcal{U}$.[2]

In the analysis of invariant polynomial flags, no generality is lost by considering only second order operators with rational coefficients, as evidenced by the following

Proposition 2.4. *Let $T[y] = py'' + qy' + ry$ be a differential operator such that*

$$T[y_i] = g_i, \ i = 1, 2, 3,$$

where y_i, g_i are polynomials with y_1, y_2, y_3 linearly independent. Then, p, q, r are rational functions.

Proof. It suffices to apply Cramer's rule to solve the linear system

$$\begin{pmatrix} y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \\ y_3'' & y_3' & y_3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

□

Definition 2.6. A polynomial flag is *imprimitive* if it admits a non-trivial common factor. Otherwise, the flag is said to be *primitive*.

[2]We stress that invariance of the whole flag $\mathcal{U} : U_1 \subset U_2 \subset \dots$ is a much stronger condition than the invariance of the total space $\mathcal{U}$. For the purpose of this study, we will always require invariance of the flag, since this condition leads to polynomial eigenfunctions of the operator.

Proposition 2.5. *Let $\mathcal{U}$ be a primitive flag, let μ be a polynomial of degree ≥ 1 and let*

$$\tilde{\mathcal{U}} = \mu\mathcal{U} = \{\mu p : p \in \mathcal{U}\}.$$

be the corresponding imprimitive flag. Suppose that $T[y]$ is a rational operator that preserves $\mathcal{U}$. Then, the gauge-equivalent rational operator $\tilde{T} = \mu T \mu^{-1}$ preserves $\tilde{\mathcal{U}}$.

Therefore, primitive flags can be regarded as canonical representatives for the equivalence relation modulo gauge transformations, and we can restrict our attention to primitive flags in the classification of invariant polynomial flags. The main object of our study is then the class defined in the following definition.

Definition 2.7. A second order operator that preserves a primitive polynomial flag, but does not preserve the standard flag will be called *an exceptional operator*. An *exceptional flag* is the *maximal* primitive polynomial flag that is preserved by a second order exceptional operator. Exceptional flags and operators of finite codimension $m \geq 1$ will henceforth be called X_m flags and operators. By contrast, a second order differential operator that preserves the standard flag $\mathcal{P}$, will be referred to as a *classical operator*.

Theorem 2.1 (Bochner)**.** *A classical operator has the form*

$$T[y] = py'' + qy' + ry$$

where $p \in \mathcal{P}_2,\ q \in \mathcal{P}_1$ are polynomials of the indicated degree, and where r is a constant.

Proposition 2.6. *An exceptional operator is, necessarily, non-polynomial.*

Proof. See the proof of Lemma 3.1 in [16]. □

Thus, an exceptional operator has poles, but it also has an infinite number of polynomial eigenfunctions. When classifying exceptional flags by increasing codimension, each flag will give rise to new operators not considered at lower codimension, which justifies the definition above. Here are some examples to illustrate these definitions

Example 2.1. The flag with basis $\{1, z^2, z^3, \dots,\}$ is exceptional because the operator

$$T[y] = y'' - \frac{2y'}{z}$$

preserves the flag. The degree sequence is $\{0, 2, 3, \dots\}$ and the codimension sequence is $\{0, 1, 1, \dots\}$ so the flag has non-stable codimension 1.

Example 2.2. By contrast, the flag spanned by $\{z+1, z^2, z^3, \dots\}$ has a stable codimension $m = 1$. This flag is exceptional because it is preserved by the operator

$$T[y] = y'' - 2\left(1 + \frac{1}{z}\right)y' + \left(\frac{2}{z}\right)y.$$

Example 2.3. Let $H_k(z)$ denote the degree k Hermite polynomial. The codimension 1 flag spanned by $\{H_1, H_2, H_3, \dots\}$ is not exceptional. The flag is preserved by the operator $T[y] = y'' - zy'$. However, this operator also preserves the standard flag, which violates the maximality assumption.

Example 2.4. The codimension 1 flag spanned by $z, z^2, z^3, \dots$ is preserved by the operator

$$\tilde{T}[y] = y'' - \frac{2y'}{z} + \frac{2y}{z^2}.$$

This is not an exceptional flag because it is imprimitive as z is a non-trivial common factor. In fact, the operator $\tilde{T}$ is gauge equivalent $\tilde{T} = zTz^{-1}$ to the operator $T[y] = y''$ that preserves the standard flag.

Example 2.5. Let

$$\begin{cases} y_{2k-1} = z^{2k-1} - (2k-1)z, \\ y_{2k} = z^{2k} - kz^2, \end{cases} \qquad k = 2, 3, 4, \dots \tag{5}$$

Consider the flag spanned by $\{1, y_3, y_4, y_5, \dots\}$. The degree sequence of the flag is $0, 3, 4, 5, \dots$ so it is a non-stable codimension 2 flag. The flag is preserved by the following operators :

$$T_3[y] = (z^2 - 1)y'' - 2zy', \tag{6}$$

$$T_2[y] = zy'' - 2\left(1 + \frac{2}{z^2 - 1}\right)y', \tag{7}$$

$$T_1[y] = y'' + z\left(1 - \frac{4}{z^2 - 1}\right)y'. \tag{8}$$

The flag is exceptional, because T_1 and T_2 do not preserve the standard flag. Since T_2, T_1 have 2 distinct poles, they do not preserve a codimension 1 flag (see Lemma 3.3).

3. Classification of exceptional codimension 2 polynomial flags

In this Section we perform a classification of all X_2-flags up to affine transformations of the independent variable z. We exhibit degree-regular bases for each of them, and we determine the X_2-operators that preserve them. We begin by introducing the following flags:

$$\mathcal{E}^{(1)}(a;b) := \{p \in \mathcal{P} : p'(b) = ap(b)\} \tag{9}$$

$$\mathcal{E}^{(11)}(a_0, a_1; b_0, b_1) := \mathcal{E}^{(1)}(a_0; b_0) \cap \mathcal{E}^{(1)}(a_1; b_1) \tag{10}$$

$$\mathcal{E}^{(2)}(a_{01}, a_{03}, a_{23}; b) := \{p \in \mathcal{P} : p'(b) = a_{01}p(b),\ p'''(b) = 3a_{23}p''(b) + 6a_{03}p(b)\} \tag{11}$$

The first flag has codimension one and its associated X_1-operator will have a simple pole at $z = b$. The second flag has codimension two and its associated X_2 operator will have two simple poles at b_0 and b_1. The third flag has codimension two and its associated X_2 operator will have a simple pole at b. The notation in the superindices is connected to the the order of the poles of the weight for the exceptional orthogonal polynomial system based on the flag. This will become clear in Section 6. In any case, the sum of superindices must always coincide with the codimension of the flag.

Some, but not all of the parameters in the above flags can be normalized by means of an affine transformation. Thus, unlike the codimension one case, the X_2 flags contain free continuous parameters, which shall be refereed to as *flag moduli*. As explained before, the parameters b, b_0 and b_1 will be the positions of the poles of the operators. If there is one pole we will set $b = 0$ and if there are two poles we will normalize them as $b_0 = 0,\ b_1 = 1$. Note that any two poles in the complex plane can be transformed into 0 and 1 by a complex affine transformation, so there is no loss of generality involved in the above normalization.

Below, we describe each of the above flags in terms of a basis.

$$\mathcal{E}^{(1)}(a;0) = \operatorname{span}\{1 + az, z^2, z^3, z^4, \ldots\} \tag{12}$$

$$\begin{aligned}\mathcal{E}^{(11)}(a_0, a_1; 0, 1) = \operatorname{span}\{&z^2((a_1 - 2)(z - 1) + 1), (z - 1)^2((a_0 + 2)z + 1)\}\cup \\ &\{z^2(z - 1)^2 z^j\}_{j=0}^{\infty},\end{aligned} \tag{13}$$

$$\mathcal{E}^{(2)}(a_{01}, a_{03}, a_{23}; 0) = \operatorname{span}\{1 + a_{01}z + a_{03}z^3, z^2 + a_{23}z^3, z^4, z^5, \ldots\} \tag{14}$$

Let us first recall the main result of the classification of X_1-flags first proved in [12] (see [16] for a more recent and streamlined proof).

Theorem 3.1. *Every stable* X_1 *polynomial flag is affine-equivalent to*

$$\mathcal{E}^{(1)}(1;0) = \operatorname{span}\{1 + z, z^2, z^3, z^4, \ldots\}.$$

Every unstable X_1 *polynomial flag is affine-equivalent to the monomial flag*

$$\mathcal{E}^{(1)}(0;0) = \operatorname{span}\{1, z^2, z^3, z^4, \ldots\}.$$

Note that, as mentioned before, the most general X_1 flag up to affine transformations contains no flag moduli. The main result of this section is the following theorem that describes the situation for X_2 flags.

Theorem 3.2. *Up to an affine transformation every* X_2 *flag is equivalent to one of the following two flags:*

1. $\mathcal{E}^{(11)}(a_0, a_1; 0, 1)$
2. $\mathcal{E}^{(2)}(a_{01}, a_{03}, a_{23}; 0)$ *subject to the constraint*

$$a_{03}(a_{01} - a_{23})(6a_{03} + a_{01}a_{23}(a_{01} + a_{23})) = 0 \tag{15}$$

Before we can address the proof of the above theorem, we need to introduce more concepts and establish some key intermediate results.

For a polynomial $y(z)$ and a constant $b \in \mathbb{C}$, we define $\operatorname{ord}_b y \geq 0$ to be the order of b as a zero of $y(z)$. Let $\mathcal{U} \subset \mathcal{P}$ be a polynomial subspace. For $b \in \mathbb{C}$ define

$$I_b(\mathcal{U}) = \{\operatorname{ord}_b y : y \in \mathcal{U}\}. \tag{16}$$

Lemma 3.1. *Let T be a rational operator that preserves a primitive polynomial subspace $\mathcal{U} \subset \mathcal{P}$. Let*

$$T = \sum_{i=-d}^{\infty} T_i,$$

where

$$T_i[y] = z^i \left(p_i z^2 y'' + q_i z y' + r_i y\right)$$

for some constants p_i, q_i, r_i be the degree-homogeneous representation of T in terms of Laurent series. If T has a pole at $z = 0$, then $d = 2$, $r_{-2} = 0$, and there exists a positive integer $\alpha \geq 1$ such that

$$I_0(\mathcal{U}) = \mathbb{N}/\{1, 3, \ldots, 2\alpha - 1\} \tag{17}$$

Proof. Observe that T_i is degree-homogeneous, meaning that

$$T_i[z^j] = (p_i j(j-1) + q_i j + r_i) z^{i+j}.$$

So either T_i annihilates a given monomial z^j, or it shifts its degree by i. A non-zero T_i can annihilate at most two distinct monomials, whose exponents j satisfy the quadratic constraint

$$p_i j(j-1) + q_i j + r_i = 0.$$

By definition, $i \in I_0$ if and only if the flag contains a polynomial of the form $z^i +$ higher degree terms. Since T preserves $\mathcal{U}$ and since T_{-d} is the leading term of the operator, it follows that T_{-d} preserves the monomial subspace $\{z^i : i \in I_0\}$.

For T to have a pole at $z = 0$ we must have $d > 0$. Since $\mathcal{U}$ is primitive, $0 \in I_0$ and therefore T_{-d} must annihilate $z^0 = 1$. Observe that the leading order d must also be $d \geq 2$, since $d = 1$ would require that $T_{-1}[1] = 0 \Rightarrow r_{-1} = 0$, so operator T would be polynomial, contrary to the hypothesis. To conclude the proof, we will establish that d has to be precisely 2. Since the flag $\mathcal{U}$ has finite codimension, there are only a finite number of gaps (missing integers) in the set I_0. Let $i \notin I_0$ be one such gap, then either $i + d \notin I_0$, or T_{-d} annihilates z^{i+d}. Hence, $1 \notin I_0$ must be a gap. Otherwise, since $d \geq 2$, T_{-d} would need to annihilate three monomials: z^0, z^1 and at least one higher degree monomial, which is impossible. Thus, for some integer $\alpha \geq 1$, the gaps in the I_0 sequence are $1, 1+d, 1+2d, \ldots, 1+d(\alpha-1) \notin I_0$, with $T_{-d}[z^{d\alpha+1}] = 0$. Note that T_{-d} annihilates 1 and $z^{d\alpha+1}$ so it cannot annihilate any other monomial and therefore the above gaps are the only possible gaps in I_0. It follows that $2 \in I_0$ is *not* a gap. If the leading order was $d > 2$ then T_{-d} would be required to annihilate also z^2, which is impossible. We conclude then that $d = 2$ and since $T_{-2}[1] = 0$ we must have $r_{-2} = 0$. The assertions of the lemma are proved. □

The following lemma shows how to decompose a rational second order operator that preserves a primitive polynomial flag.

Lemma 3.2. *Let T be a second order rational operator with poles $b_1, \ldots, b_N \in \mathbb{C}$. If T preserves a primitive polynomial flag of finite codimension, then necessarily it has the form*

$$T[y] = p_{-2} y'' + (p_{-1} z y'' + q_{-1} y') + (p_0 z^2 y'' + q_0 y' + r_0 y) + \sum_{i=1}^{N} c_i \, \frac{y' - a_i y}{z - b_i},$$

where $p_i, q_i, r_i \in \mathbb{R}$ and $a_i, c_i \in \mathbb{C}$ are constants.

We see therefore that an exceptional operator must have rational coefficients that can only contain simple poles.

Proof. We decompose the given operator as

$$T = \sum_{i=0}^{N} T^{(i)}$$

where $T^{(0)}$ is a polynomial operator and where

$$T^{(i)}[y] = \frac{r_{-1}^{(i)} y}{z - b_i} + \frac{q_{-2}^{(i)} (z - b_i) y' + r_{-2}^{(i)} y}{(z - b_i)^2} + \sum_{j=3}^{d_i} \frac{p_{ij}(z - b_i)^2 y'' + q_{ij}(z - b_i) y' + r_{ij} y}{(z - b_i)^j}$$

for some positive integer $d_i \geq 1$ and constants p_{ij}, q_{ij}, r_{ij}.

Let $\mathcal{U}$ be the total space of the primitive flag preserved by T. Since $T(\mathcal{U}) \subset \mathcal{P}$, it follows that $T^{(i)}(\mathcal{U}) \subset \mathcal{P}$ for every $i = 0, 1, \dots, N$. By construction, the operators $T^{(1)}, \dots, T^{(N)}$ all lower degrees. Since T preserves an infinite flag, it cannot have a degree raising part. Therefore, $T^{(0)}$ has the form

$$T^{(0)}[y] = p_{-2}y'' + (p_{-1}zy'' + q_{-1}y') + (p_0 z^2 y'' + q_0 y' + r_0 y)$$

Expanding the operator coefficients as Laurent series in $z - b_i$, we apply Lemma 3.1 to conclude that $d_i = 2, r^{(i)}_{-2} = 0$ for all $i = 1 \dots, N$. The desired conclusion has been established. □

Note that if b_i is a real pole, then the constants a_i and c_i must also be real since the flag is real too. The next lemma shows that for every pole b_i of an exceptional differential operator, the elements of its invariant flag must satisfy a first order differential constraint at that pole.

Lemma 3.3. *Let $T[y]$ be a second order rational operator with poles $b_1, \dots, b_N$ that preserves a primitive flag $\mathcal{U}$ of finite codimension. Then, there exist constants $a_1, \dots, a_N$ such that the elements of $y \in \mathcal{U}$ obey 1st order differential constraints of the form*

$$y'(b_i) = a_i y(b_i), \quad i = 1, \dots, N.$$

Proof. By Lemma 3.1, for each $i = 1, \dots, N$ the total space $\mathcal{U}$ contains a polynomial of the form

$$y^{(i)}_0(z) = 1 + a_i(z - b_i) + O\big((z - b_i)^2\big),$$

but does not contain an element of the form

$$(z - b_i) + O\big((z - b_i)^2\big).$$

Therefore, every $y \in \mathcal{U}$ either starts as $y^{(i)}_0(z)$ or at degree 2 in $(z - b_i)$ so in any case it obeys the constraint $y'(b_i) = a_i y(b_i)$. □

At this point, it becomes necessary to describe and analyze certain degenerate subclasses of the $\mathcal{E}^{(11)}$ and $\mathcal{E}^{(2)}$ flags defined in (10)-(11). The distinguishing property of these subclasses is the first two elements of the degree sequence of the flag. Thus, when we write $\mathcal{E}_{ij}$ where $0 \le i < j \le 3$ we are referring to a codimension two flag whose degree sequence is $\{i, j, 4, 5, 6, \dots\}$. The generic case is the stable codimension two flag $\mathcal{E}_{23}$, which starts at degree 2 and has polynomials of all degrees $k \ge 2$. We analyze each of the above 3 families in more detail, and then give a proof of Theorem 3.2.

Proposition 3.1. *The $\mathcal{E}^{(11)}$ flags are classified into the following subclasses, according to their degree sequence:*

$$\begin{aligned}
\mathcal{E}^{(11)}_{23} &= \mathcal{E}^{(11)}(a_0, a_1; 0, 1), \quad \text{with} \quad a_1 a_0 + a_1 - a_0 \neq 0 && \text{(18a)}\\
&= \operatorname{span}\{(a_0 a_1 + a_1 - a_0)z^2 + (2 - a_1)(a_0 z + 1), (z-1)^2(1 + (2 + a_0)z)\} \cup \\
&\qquad \operatorname{span}\{z^2(z-1)^2 z^j\}_{j=0}^{\infty}\\
\mathcal{E}^{(11)}_{13} &= \mathcal{E}^{(11)}\left(a_0, \frac{a_0}{1 + a_0}; 0, 1\right), \ \text{with} \quad a_0 \neq -1, \ \text{and}\ (a_0, a_1) \notin \{(0,0), (-2,2)\} && \text{(18b)}\\
&= \operatorname{span}\{a_0 z + 1, (z-1)^2(1 + (2 + a_0)z)\} \cup \operatorname{span}\{z^2(z-1)^2 z^j\}_{j=0}^{\infty}\\
\mathcal{E}^{(11)}_{03} &= \mathcal{E}^{(11)}(0, 0; 0, 1) = \operatorname{span}\{1, (z-1)^2(1 + 2z)\} \cup \operatorname{span}\{z^2(z-1)^2 z^j\}_{j=0}^{\infty} && \text{(18c)}\\
\mathcal{E}^{(11)}_{12} &= \mathcal{E}^{(11)}(-2, 2; 0, 1) = \operatorname{span}\{2z - 1, z^2\} \cup \operatorname{span}\{z^2(z-1)^2 z^j\}_{j=0}^{\infty} && \text{(18d)}
\end{aligned}$$

Proof. This follows by direct inspection of (13). □

Also note that $\mathcal{E}^{(11)}_{12}$ can be obtained as a limit of $\mathcal{E}^{(11)}_{23}$ by setting $a_0 = t - 2$, $a_1 = t + 2$ and then sending $t = 0$. The flags in Proposition 3.1 are all X_2-flags whose operators have two simple poles at 0 and 1. In the following Proposition we provide a basis for the $\mathcal{D}_2$-spaces of operators that preserve them.

Proposition 3.2. *The generic flag $\mathcal{E}^{(11)}_{23}$ has a 2-dimensional $\mathcal{D}_2$ space. The non-stable flag $\mathcal{E}^{(11)}_{13}$ has a 3-dimensional $\mathcal{D}_2$, while $\mathcal{E}^{(11)}_{03}$ and $\mathcal{E}^{(11)}_{12}$ both have a 4-dimensional $\mathcal{D}_2$. The most general second order operator that preserves each of these flags is shown below (and therefore a basis of their $\mathcal{D}_2$-space). The* *symbols a_0, a_1 denote the flag moduli while the symbols $c, c_0, c_1, q_0, \lambda$ denote free constants appearing in the operator*

$$T^{(11)}_{23}[y] = c\left(-\frac{1}{2}z^2(a_0-a_1)(a_0-a_1+4) - z(a_0a_1 - a_0 - a_1^2 + 3a_1) - \frac{a_1^2}{2} + a_1\right)y''+ \tag{19a}$$
$$+ c\left(z((a_0-a_1)(a_0a_1 - 2a_0 + 2) + 2a_0^2) + (a_0-1)a_1^2 - (a_0-3)a_1 + a_0(a_0+1)\right)y'+$$
$$+ \frac{ca_0(a_0+2)}{z-1}(y' - a_1 y) + \frac{c(a_1-2)a_1}{z}(y' - a_0 y) + \lambda y$$

$$T^{(11)}_{13}[y] = \left(-(c_0+c_1)\frac{z^2}{2} + c_0\left(z - \frac{1}{2}\right)\right)y'' + \left((a_1c_1 - a_0c_0)z + (a_0-1)c_0 + c_1\right)y'+ \tag{19b}$$
$$\frac{c_0}{z}(y' - a_0 y) + \frac{c_1}{z-1}(y' - a_1 y) + \lambda y, \quad a_1 = \frac{a_0}{a_0+1}$$

$$T^{(11)}_{03}[y] = \left(-(q_0+c_0+c_1)\frac{z^2}{2} + \frac{q_0 z}{2} + c_0\left(z-\frac{1}{2}\right)\right)y'' + \left(q_0\left(z - \frac{1}{2}\right) - c_0 + c_1\right)y'+ \tag{19c}$$
$$+\left(\frac{c_0}{z} + \frac{c_1}{z-1}\right)y' + \lambda y$$

$$T^{(11)}_{12}[y] = \left((c_0+c_1-q_0)\frac{z^2}{2} + \left(\frac{q_0}{2} - c_1\right)z - \frac{c_0}{2}\right)y'' + \left(q_0\left(z-\frac{1}{2}\right) - 2c_0 + 2c_1\right)y'+$$
$$+ \frac{c_0}{z}(y' + 2y) + \frac{c_1}{z-1}(y' - 2y) + \lambda y \tag{19d}$$

Before we turn to the proof of this last Proposition, observe the duality between flag moduli and free parameters in the operator. In the general case $\mathcal{E}^{(11)}_{23}$ the flag has two moduli (a_0, a_1) and the $\mathcal{D}_2$-space has dimension two. In the case $\mathcal{E}^{(11)}_{13}$ the flag has one modulus a_0 and the operator has three free parameters, since $\dim \mathcal{D}_2\left(\mathcal{E}^{(11)}_{13}\right) = 3$. In the last two cases $\mathcal{E}^{(11)}_{03}$ and $\mathcal{E}^{(11)}_{12}$ the flag is completely specified (no flag moduli) but the operator contains four free parameters.

Proof. By Lemma 3.2, we must consider an operator of the form

$$T[y] := p_{-2}y'' + (p_{-1}zy'' + q_{-1}y') + (p_0 z^2 y'' + q_0 y') + \frac{c_0(y' - a_0 y)}{z} + \frac{c_1(y' - a_1 y)}{z-1},$$

where $p_{-2}, p_{-1}, q_{-1}, p_0, q_0, c_1, c_0$ are undetermined coefficients that need to be constrained so that T preserves the flag in question. Applying the relation

$$y'(0) = a_0 y(0), \quad y \in \mathcal{U}$$

to the constraint

$$T[y]'(0) - a_0 T[y](0) = 0, \quad y \in U \tag{20}$$

yields:

$$(c_0/2 + p_{-2})\,y'''(0) + (3a_0p_{-2} - (3a_0/2)c_0 - c_1 + p_{-1} + q_{-1})\,y''(0)+$$
$$+\left(a_0^3c_0 + (a_0^2 - a_0 + a_1)c_1 - a_0^2 q_{-1} + a_0 q_0\right)y(0) = 0$$

Since $y'''(0), y''(0), y(0)$ vary freely for $y \in \mathcal{U}$ the coefficients of all 3 terms must vanish in order for (20) to hold. An analogous constraint holds for

$$T[y]'(1) - a_1 T[y](1) = 0.$$

Since there are 7 parameters and only 6 linear, homogeneous constraints, there exists at least one non-trivial operator that preserves $\mathcal{E}^{(1)}$. The desired solution vector

$$[p_{-2}, p_{-1}, q_{-1}, p_0, q_0, c_1, c_0]^t$$

belongs to the null-space of the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ -a_0 & 1 & 1 & 0 & 0 & -1 & -3a_0/2 \\ 0 & 0 & -a_0^2 & 0 & a_0 & a_0^2 - a_0 + a_1 & a_0^3 \\ 1 & 1 & 0 & 1 & 0 & 1/2 & 0 \\ -a_1 & 1-a_1 & 1 & 2-a_1 & 1 & -3a_1/2 & 1 \\ 0 & 0 & -a_1^2 & 0 & -(a_1-1)a_1 & a_1^3 & a_0 - a_1(a_1+1) \end{pmatrix} \tag{21}$$

A direct calculation shows that all 6 minors of the above 6×7 have $a_0a_1 + a_1 - a_0$ as a factor, and that it is not possible for all the minors to vanish if $a_0a_1 + a_1 - a_0 \neq 0$. Hence, generically the above constraint matrix has rank 6, and there exists a unique, up to a scalar factor, solution, which after some calculation provides the operator $T^{(11)}_{23}$.

Setting $a_1 = a_0/(a_0+1)$ in the above matrix drops the rank of the matrix to 5, provided, $a_0 \notin \{0,-2\}$. Now the nullspace is 2-dimensional; this gives the form of $T^{(11)}_{13}$. Setting $a_0 = a_1 = 0$ in the constraint matrix gives a matrix of rank 4. The nullspace corresponds to the operator $T^{(11)}_{03}$. Similarly, $a_0 = -2, a_1 = 2$ also gives a rank 4 matrix, whose nullspace corresponds to the operator $T^{(11)}_{12}$. □

Proposition 3.3. *The flag $\mathcal{E}^{(11)}_{23}$ is an X_2 flag, provided $a_0 \notin \{0,-2\}$ and $a_1 \notin \{0,2\}$. The non-stable flags $\mathcal{E}^{(11)}_{13}, \mathcal{E}^{(11)}_{03}, \mathcal{E}^{(11)}_{12}$ are all X_2 flags.*

Proof. It is clear that all the operators preserve codimension two flags and since they have poles they do not preserve the standard flag. It only remains to prove the maximality assumption, i.e. that these operators do not preserve a flag of codimension one. By Lemma 3.3, an operator with two poles cannot preserve a codimension 1 flag. By inspection, if a_0, a_1 satisfy the conditions given above, the operator $T^{(11)}_{23}$ cannot preserve a codimension 1 flag. On the contrary, if $a_0 = 0$, a direct calculation shows that $T^{(11)}_{23}[1] = 0$ and hence $\mathcal{E}^{(1)}(0, a_1)$ is not the maximal flag preserved by $T^{(11)}_{23}$. Similarly, if $a_0 = -2$ then $2z - 1$ is again an eigenpolynomial. Similar remarks hold for the cases $a_1 = 0$ and $a_1 = 2$.

For the degenerate, non-stable flags, by taking $c_0, c_1 \neq 0$ we obtain operators that preserve these flags, but have 2 distinct poles. Therefore, by the same argument, these operators cannot preserve a flag of smaller codimension. □

We now turn to an analysis of the one-pole X_2 flag $\mathcal{E}^{(2)}$. In the language of Lemma 3.1, this flag is the most general codimension 2 flag with the order sequence $I_0 = \{0, 2, 4, 5, 6, \ldots\}$. The Lemma below derives the constraint (15) as the necessary and sufficient condition for such a flag to have a non-trivial $\mathcal{D}_2$.

Lemma 3.4. *Every X_2 flag that is preserved by an operator with a unique pole is translation-equivalent, to $\mathcal{E}^{(2)}(a_{01}, a_{03}, a_{23}; 0)$ where the parameters satisfy (15). Up to a multiplicative constant, a second order operator that preserves such a flag has the form*

$$T^{(2)}[y] = y'' + (p_{-1}zy'' + q_{-1}y') + (p_0z^2y'' + q_0y') - 4\frac{(y' - a_{01}y)}{z} + \lambda y \tag{22}$$

where

$$p_{-1} = 2a_{01} - 2a_{23} \tag{23}$$

$$q_{-1} = -7a_{01} + 5a_{23} \tag{24}$$

and where p_0, q_0 satisfy:

$$\begin{pmatrix} 0 & a_{01} & 3a_{01}^3 - 6a_{03} - 5a_{01}a_{23} \\ 2a_{03} & a_{03} & a_{03}(a_{01} - a_{23})(a_{01} + a_{23}) \\ 4a_{23} & a_{23} & 6a_{03} + 5a_{01}a_{23}^2 - 3a_{23}^3 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{25}$$

Proof. By Lemma 3.2 an operator with a unique pole at $z = 0$ that preserves a polynomial flag has the form

$$T[y] = p_{-2}y'' + (p_{-1}zy'' + q_{-1}y') + (p_0z^2y'' + q_0y') + c\frac{(y' - a_{01}y)}{z} + \lambda y$$

where $p_{-2}, p_{-1}, q_{-1}, p_0, q_0, \lambda, c$ are undetermined coefficients. If we demand that the flag has codimension 2, then the flag must be $\mathcal{E}^{(2)}$. By Lemma 3.1, it follows that we must also require $T_{-2}[z^5] = 0$, where

$$T_{-2}[y] = p_{-2}y'' + \frac{cy'}{z},$$

This imposes the condition $c = -4p_{-2}$ and since we require a non-trivial T_{-2}, we must have $p_{-2} \neq 0$. Hence, without loss of generality, we impose

$$c = -4, \quad p_{-2} = 1$$

from here on. The flag $\mathcal{E}^2$ in (11) is defined by the first and third order conditions

$$y'(0) = a_{01}y(0) \tag{26}$$

$$y'''(0) = 6a_{03}y(0) + 3a_{23}y''(0) \tag{27}$$

Imposing these conditions on $T[y]$ yields

$$
\begin{aligned}
&(5a_{01} - 3a_{23} + p_{-1} + q_{-1})\, y''(0) + \left(-4a_{01}^3 - 6a_{03} - a_{01}^2 q_{-1} + a_{01} q_0\right) y(0) = 0\\
&(a_{01} + a_{23} + 3p_{-1} + q_{-1})\, y^{(4)}(0) +\\
&\quad + 3\left(-4a_{01}a_{23}^2 + 6a_{03} - 6a_{23}^2 p_{-1} - 3a_{23}^2 q_{-1} + 4a_{23}p_0 + a_{23}q_0\right) y''(0) +\\
&\quad - 6a_{03}\left(4a_{01}^2 + 4a_{01}a_{23} + 6a_{23}p_{-1} + (3a_{23} + a_{01})q_{-1} - 6p_0 - 3q_0\right) y(0) = 0
\end{aligned}
$$

The values of $y^{(4)}(0), y^{(2)}(0), y(0)$ vary freely for $y \in \mathcal{E}^{(2)}$, and hence, invariance holds if and only if the coefficient of each of these expressions vanish. The conditions (23) (24) follow from the vanishing of the leading order coefficients. Once these values of p_{-1}, q_{-1} are imposed, the overdetermined constraint (25) expresses the vanishing of all the remaining coefficients. The vanishing of the determinant of the matrix in (25) is the compatibility condition for these constraints, and this is precisely condition (15). □

As we did before for the two-poles X_2 flags, the one-pole X_2 flags can be classified according to their degree sequence.

Proposition 3.4. *Every one-pole X_2 flag is affine-equivalent to one of the following:*

$$\mathcal{E}_{13}^{(2a)}(a) := \mathcal{E}^{(2)}(1,0,a;0) = \operatorname{span}\{1+z, z^2 + az^3, z^4, z^5, \dots\}, \quad a \neq 0, \tag{28a}$$

$$\mathcal{E}_{03}^{(2a)} := \mathcal{E}^{(2)}(0,0,1;0) = \operatorname{span}\{1, z^2 + z^3, z^4, z^5, \dots\}, \tag{28b}$$

$$\mathcal{E}_{12}^{(2a)} := \mathcal{E}^{(2)}(1,0,0;0) = \operatorname{span}\{1+z, z^2, z^4, z^5, z^6, \dots\}, \tag{28c}$$

$$\mathcal{E}_{02}^{(2a)} := \mathcal{E}^{(2)}(0,0,0;0) = \operatorname{span}\{1, z^2, z^4, z^5, z^6, \dots\}, \tag{28d}$$

$$\mathcal{E}_{23}^{(2b)}(a) := \mathcal{E}^{(2)}(a,a,a;0) = \operatorname{span}\{1 + az - z^2, z^2(1+az), z^4, z^5, \dots\}, \quad a \neq 0, \tag{28e}$$

$$
\begin{aligned}
\mathcal{E}_{23}^{(2c)}(a) &:= \mathcal{E}^{(2)}(a, -a(a+1)/6, 1), \quad a \neq 0\\
&= \operatorname{span}\{1 + az + a(a+1)z^2/6, z^2 + z^3, z^4, z^5, \dots\}
\end{aligned}
\tag{28f}
$$

Proof. The three types of flags labelled $(2a)$, $(2b)$ and $(2c)$ correspond to the three different ways of satisfying the defining constraint (15) on the three flag moduli. The type $(2a)$ flags are obtained by applying the constraint $a_{03} = 0$ to a general type $\mathcal{E}^{(2)}$ flag. By (14), the resulting degree regular basis is

$$1 + a_{01}z, z^2 + a_{23}z^3, z^4, z^5, \dots$$

If $a_{01}, a_{23} \neq 0$, then a scaling transformation can be used to send one (but not both) of the above parameters to 1. The various subclasses listed above arise if one or both of $a_{01}, a_{23} = 0$.

The type $(2b)$ flag is obtained by applying the constraint $a_{23} = a_{01}$. An examination of (25) shows that it is not possible for $a_{23} = a_{01} = 0, a_{03} \neq 0$. Therefore, for the type $(2b)$ subcase, we must have $a_{01} \neq 0$. Thus, in this case, transforming (14) to a degree regular basis gives

$$1 + a_{01}z - \frac{a_{03}}{a_{01}} z^2, z^2 + a_{01}z^3, z^4, z^5, \dots$$

Now a scaling transformation can be used to set $a_{03}/a_{01} = 1$.

The type $(2c)$ flags are obtained by imposing

$$a_{03} = -a_{01}a_{23}(a_{01} + a_{23})/6.$$

In this case, the degree regular basis is

$$1 + a_{01}z + a_{01}(a_{01} + a_{23})z^2/6, z^2(1 + a_{23}z), z^4, z^5, \dots$$

No generality is lost if we assume that $a_{01}, a_{23} \neq 0$, because otherwise we will obtain a flag of type $(2a)$. Finally, a scaling transformation is used to set $a_{23} = 1$. □

Note: as above the flag subscript indicates the degree sequence of the flag.

Proposition 3.5. *The flags $\mathcal{E}_{13}^{(2a)}, \mathcal{E}_{23}^{(2b)}, \mathcal{E}_{23}^{(2c)}$ have a 2-dimensional $\mathcal{D}_2$. The degenerate flags $\mathcal{E}_{03}^{(2a)}, \mathcal{E}_{12}^{(2a)}$ have a 3-dimensional $\mathcal{D}_2$, while $\mathcal{E}_{02}^{(2a)}$ has a 4-dimensional $\mathcal{D}_2$. The most general second order operator* *that preserves each of these flags is shown below. The symbol a represents the flag modulus while the symbols c, p_0, q_0, λ are free constants that appear in the operator.*

$$T_{13}^{(2a)}[y] = c\left((1-3a)(3-a)\frac{z^2}{4} + 2(1-a)z + 1\right)y'' + \tag{29a}$$

$$+ c((5a-3)az + 5a - 7)y' + \frac{4c(y-y')}{z} + \lambda y$$

$$T_{03}^{(2a)}[y] = \left((3c-q_0)\frac{z^2}{4} + c(1-2z)\right)y'' + (5c+q_0 z)y' - \frac{4cy'}{z} + \lambda y \tag{29b}$$

$$T_{12}^{(2a)}[y] = \left(p_0 z^2 + c(2z+1)\right)y'' - c(3z+7)y' + \frac{4c(y-y')}{z} + \lambda y \tag{29c}$$

$$T_{02}^{(2a)}[y] = \left(p_0 z^2 + c\right)y''(z) + q_0 z y'(z) - \frac{4cy'(z)}{z} + \lambda y \tag{29d}$$

$$T_{23}^{(2b)}[y] = c\left(1 - z^2\left(a^2+3\right)\right)y'' + c\left(2z\left(a^2+3\right) - 2a\right)y'(z) + \frac{4c(ay-y')}{z} + \lambda y \tag{29e}$$

$$T_{23}^{(3c)}[y] = c(1+(a-1)z)^2 y'' + c((a-1)(1-3a)z + 5 - 7a)y' + \frac{4c(ay-y')}{z} + \lambda y \tag{29f}$$

Proof. Each of the flags in question is a specialization of the $\mathcal{E}^{(2)}$ flag discsussed in Lemma 3.4, imposed in such a way so that (15) holds. The 3 factors in (15) give us the 3 possible cases: $\mathcal{E}^{(2a)}, \mathcal{E}^{(2b)}, \mathcal{E}^{(2c)}$. Imposing the respective constraints

$$a_{03} = 0, \quad a_{23} = a_{01}, \quad a_{03} = -a_{01}a_{23}(a_{01} + a_{23})/6$$

transforms (25) into a consistent, rank 2 system. We can further eliminate one more parameter by means of an appropriate scaling transformation. The form of the operators shown above follows from (23) (24) and the solution of the corresponding (25). □

Proposition 3.6. *The flags $\mathcal{E}_{13}^{(2a)}, \mathcal{E}_{03}^{(2a)}, \mathcal{E}_{12}^{(2a)}, \mathcal{E}_{02}^{(2a)}, \mathcal{E}_{23}^{(2b)}, \mathcal{E}_{23}^{(2c)}$ are all X_2 flags.*

Proof. For each of the above flags, we have exhibited a singular operator that preserves it. It remains to show that these operators cannot preserve a flag of smaller codimension. By Lemma 3.1 an X_1 flag preserved by an operator with a pole at $z = 0$, must have elements of order $0, 2, 3, 4, \ldots$. Therefore, it suffices to check that T_{-2} (see the Lemma for the explanation of the notation) does not annihilate z^3. For each of the operators shown in the preceding Proposition,

$$T_{-2}[y] = y'' - \frac{4y'}{z}.$$

Hence,

$$T_{-2}[z^3] = -6z.$$

Therefore, none of these operators can preserve an X_1 flag. □

Proof of Theorem 3.2. By the above Lemmas, an X_2 operator has either one or two poles. In the last case, the corresponding X_2 flag satisfies two distinct first order condtions

$$y'(b_i) = a_i y(b_i), \quad i = 1, 2$$

Applying an affine transformation, no generality is lost if we assume that the poles are at $z = 0$ and $z = 1$. This gives us flags of type $\mathcal{E}^{(11)}$. The corresponding X_2 operators are given in Proposition 3.2. The X_2 assertion is verified in Proposition 3.3.

In the case of one pole, without loss of generality the pole is at $z = 0$. In this case, the flag satisfies a first and a third order condition, which gives us a flag of type $\mathcal{E}^{(2)}$. As it was shown in Lemma 3.4, the moduli of the general flag must satisfy the constraint (15). This gives us the three cases: $\mathcal{E}^{(2a)}, \mathcal{E}^{(2b)}, \mathcal{E}^{(2c)}$. The corresponding operators for these flags are given in Proposition 3.5 and the X_2 condition is verified in Proposition 3.6 □

4. Factorization of exceptional operators

The results in this section are concerned with factorizations of the differential operators that preserve X_2 flags and their connection to the Darboux transformation. The usual Darboux transformation involves Schrödinger operators and square-integrable eigenfunctions but for our purposes it will be convenient to generalize it to second order operators with rational coefficients.

Definition 4.1. Let T be a second order differential operator that preserves a polynomial flag $\mathcal{U}$. Let

$$T = BA + \lambda_0 \tag{30}$$

be a factorization of T where A, B are first order operators with rational coefficients and λ_0 is a constant. If the partner operator defined by

$$\hat{T} = AB + \lambda_0. \tag{31}$$

also preserves a polynomial flag $\hat{\mathcal{U}}$ we will say that T and $\hat{T}$ are related by an *algebraic Darboux transformation.*

Definition 4.2. More generaly, we will say that two operators T and $\hat{T}$ are *Darboux-connected* if there exists a sequence of algebraic Darboux transformations that connect them.

The same notion can be defined for polynomial flags in the following manner:

Definition 4.3. Two polynomial flags $\mathcal{U} : U_1 \subset U_2 \subset \dots$ and $\hat{\mathcal{U}} : \hat{U}_1 \subset \hat{U}_2 \subset \dots$ are *Darboux-connected* if there exists two first order rational operators A and B such that one of the following three possibilities occur:

$$A[U_i] \subset \hat{U}_i, \qquad B[\hat{U}_i] \subset U_i, \quad i \geq 1; \tag{32}$$

$$A[U_{i+1}] \subset \hat{U}_i, \qquad B[\hat{U}_i] \subset U_{i+1}, \quad i \geq 1, \qquad A[U_1] = 0; \tag{33}$$

$$B[U_{i+1}] \subset \hat{U}_i, \qquad A[\hat{U}_i] \subset U_{i+1}, \quad i \geq 1, \qquad B[U_1] = 0. \tag{34}$$

In accordance with [14] we will refer to the above cases as formally isospectral, formally state-deleting and formally state-adding.

Note that this implies that the second order operators $T = BA$ and $\hat{T} = AB$ preserve the flags $\mathcal{U}$ and $\hat{\mathcal{U}}$ respectively, so Darboux-connected polynomial flags are always invariant. It is common to refer to the operators A, B as *intertwining operators*, or simply as *intertwiners.*

Definition 4.4. We say that a polynomial flag $\mathcal{U}$ is an m-step flag if there exists a sequence of m Darboux transformations that connect $\mathcal{U}$ to the standard flag.

Our main results in this section are summarized in the following two theorems:

Theorem 4.1. *Every* X_1 *flag is a 1-step flag. Every* X_2 *flag is either a 1-step or a 2-step flag.*

Theorem 4.2. *Every* X_1 *and* X_2 *operator is Darboux-connected to a classical operator. Furthermore, the intertwining operators that connect the classical operator to the* X*-operator also connect the standard flag to the exceptional flag.*

As we show in the next section, one consequence of Theorem 4.2 is that all X_2 and X_1 orthogonal polynomials can be expressed as certain Wronskians involving classical OPs.

Using the classification of X_1 and X_2 flags from the preceding section, the proof of Theorem 4.1 is broken up into a series of Lemmas. It turns out that Theorem 4.2 is a consequence of Theorem 4.1. Our proof strategy is to show that if two polynomial flags are Darboux-connected, then so are the operators that preserve them. This fact is established by Lemmas 4.2, 4.4 and 4.5. We complete the proof of Theorem 4.2 at the end of the present section.

Lemma 4.1. *Every* X_1 *polynomial flag is a 1-step flag.*

Proof. Let $\mathcal{U} = \mathcal{E}(a; b)$ be an X_1 flag as per Theorem 3.1. Without loss of generality, $b = 0$. Define the 1st order operators

$$A[y] := \frac{y' - ay}{z}, \quad B[y] := zy' - (az + 1)y \tag{35}$$

By inspection,

$$A[U_i] \subset \mathcal{P}_{i-1}, \quad i = 1, 2, \dots$$

Also,

$$B[y]'(0) - aB[y](0) = y'(0) - ay(0) - y'(0) + ay(0) = 0$$

Hence,

$$B[\mathcal{P}_{i-1}] \subset U_i, \quad i = 1, 2, \dots.$$

Therefore, AB preserves the standard flag, while BA leaves invariant U_i for every $i = 1, 2, \dots$. ☐

Lemma 4.2. *Every* X_1 *operator is Darboux-connected to a classical operator.*

Proof. Let $\mathcal{U} = \mathcal{E}(a; b)$ be an X_1 flag as per Theorem 3.1. Without loss of generality, $b = 0$. Let

$$A_{\alpha_1}[y] = A[y] + \alpha_1 y', \quad B_{\alpha_2}[y] = B[y] + \alpha_2 z^2 y',$$

where A, B are the operators defined in (35). Observe that

$$A_{\alpha_1}[U_i] \subset \mathcal{P}_{i-1}, \qquad B_{\alpha_2}[\mathcal{P}_{i-1}] \subset U_i, \quad i = 1, 2, \ldots. \tag{36}$$

and that

$$\dim\{cB_{\alpha_2} A_{\alpha_1} + \lambda : \alpha_1, \alpha_2, c, \lambda \in \mathbb{R}\} = 4. \tag{37}$$

It follows that every operator in the vector space in (37) preserves the X_1 flag. In [11, Proposition 4.10] it was shown that $\dim \mathcal{D}_2(\mathcal{U}) = 4$. Therefore, by dimensional exhaustion, every operator $T \in \mathcal{D}_2(\mathcal{U})$ admits a rational factorization of the form $T = cB_{\alpha_2} A_{\alpha_1} + \lambda$. To conclude, we observe that, by (36), the partner operator $\hat{T} = cA_{\alpha_1} B_{\alpha_2} + \lambda$ preserves the standard polynomial flag. □

Lemma 4.3. *Let $\mathcal{U}$ be a polynomial flag. If $\dim \mathcal{D}_2(\mathcal{U}) \geq 2$ then there exists a second order operator $T \in \mathcal{D}_2(\mathcal{U})$. If $\dim \mathcal{D}_2(\mathcal{U}) = 2$, exactly, then $\{1, T\}$ is a basis of $\mathcal{D}_2(\mathcal{U})$.*

Proof. It is clear that $1 \in \mathcal{D}_2(\mathcal{U})$. If there exists a first order operator $S \in \mathcal{D}_2(\mathcal{U})$, then $S^2 \in \mathcal{D}_2(\mathcal{U})$ is a second order operator, as was to be shown. It also follows that, if $\mathcal{D}_2(\mathcal{U})$ contains an operator of the 1st order, then $\dim \mathcal{D}_2(\mathcal{U}) \geq 3$. Hence, if $\dim \mathcal{D}_2(\mathcal{U}) = 2$, exactly, then every $T \in \mathcal{D}_2(\mathcal{U})$ is either a constant multiplication operator or an operator of the second order. □

Lemma 4.4. *Let $\mathcal{U} \subset \mathcal{P}$ be a polynomial flag and $A[y]$ a 1st order operator such that $\hat{\mathcal{U}} := A[\mathcal{U}] \subset \mathcal{P}$ is also a polynomial flag. Furthermore, suppose that $A[U_1] = \{0\}$ and that $\dim \mathcal{D}_2(\mathcal{U}) \geq 2$. Then, $\mathcal{U}, \hat{\mathcal{U}}$ are Darboux connected. Furthermore, every operator in $\mathcal{D}_2(\mathcal{U})$ is Darboux-connected to an operator in $\mathcal{D}_2(\hat{\mathcal{U}})$.*

Proof. Choose a non-zero $\phi \in U_1$. Let $T \in \mathcal{D}_2(\mathcal{U})$ be given. Since ϕ spans U_1 and since $T[U_1] \subset U_1$ we must have

$$(T - \lambda)[\phi] = 0$$

for some $\lambda \in \mathbb{R}$. Write

$$T[y] = py'' + qy' + ry$$
$$A[y] = b(y' - wy)$$

where $p(z), q(z), r(z), b(z)$ are rational functions and where $w(z) = \phi'(z)/\phi(z)$, because $A[\phi] = 0$, as per the above assumption. Next, set

$$B[y] = \hat{b}(y' - \hat{w}y)$$

where

$$\hat{w} = -w - q/p + b'/b, \quad \hat{b} = p/b$$

A direct calculation then shows that

$$T = BA + \lambda.$$

Since the kernel of $A|U_{i+1}$ is 1-dimensional we actually have

$$\hat{U}_i = A[U_{i+1}], \quad i = 1, 2, \ldots$$

Since

$$T[U_i] \subset U_i, \quad i = 1, 2, \ldots$$

it follows that

$$B[\hat{U}_i] \subset U_{i+1}, \quad i = 1, 2, \ldots$$

Therefore $AB \in \mathcal{D}_2(\mathcal{U})$ and $BA \in \mathcal{D}_2(\hat{\mathcal{U}})$. By Lemma 4.3, there exists a $T \in \mathcal{D}_2(\mathcal{U})$ such that $p(z) \neq 0$. This proves that $\mathcal{U}$ and $\hat{\mathcal{U}}$ are Darboux connected. □

Lemma 4.5. *Let $\mathcal{U}, \hat{\mathcal{U}}$ be Darboux-connected polynomial flags. If $\dim \mathcal{D}_2(\mathcal{U}) = 2$, then every operator in $\mathcal{D}_2(\mathcal{U})$ is Darboux-connected to an operator in $\mathcal{D}_2(\hat{\mathcal{U}})$.*

Proof. Let $A[y]$ and $B[y]$ be 1st order operators that connect the two flags. It is clear that $T = cBA + \lambda$ preserves $\mathcal{U}$ for all $c, \lambda \in \mathbb{R}$. By exhaustion every operator in $\mathcal{D}_2(\mathcal{U})$ has this form. By assumption, the partner operator $\hat{T} = cAB + \lambda$ preserves the partner flag $\hat{\mathcal{U}}$. □

Lemma 4.6. *The flag $\mathcal{E}_{23}^{(11)}$ is a 1-step flag.*

Proof. Recall that $\mathcal{E}^{(11)}_{23} = \mathcal{E}^{(1)}(a_0, a_1; 0, 1)$ where $a_0a_1 + a_1 - a_0 \neq 0$. Consider the 1st order operators

$$\begin{aligned} A[y] &= a_1 \frac{y' - a_0 y}{z} - a_0 \frac{y' - a_1 y}{z-1} \\ B[y] &= z(z-1)(2 - a_1 + (a_1 - a_0 - 4)z)\, y' + \\ &\quad + ((a_0a_1 + a_1 - a_0)z^2 + (2-a_1)a_0 z + 2 - a_1)\, y \end{aligned}$$

Let $U_1 \subset U_2 \subset \dots$ be the flag corresponding to the total space $\mathcal{E}^{(11)}_{23}$; see (18a) for a degree regular basis. A direct calculation shows that

$$B[y]'(0) = a_0 B[y](0), \quad B[y]'(1) = a_1 B[y][1]$$

Since B raises degree by 2, it follows that

$$B[\mathcal{P}_{j-1}] \subset U_j, \quad j = 1, 2, \dots$$

From the definition (10), we see that $A[\mathcal{E}^{(11)}_{23}] \subset \mathcal{P}$. Furthermore,

$$\begin{aligned} A[z^j] &= \frac{((a_1 - a_0)j + a_0a_1)z^{j-1} - z^{j-2} j a_1}{z-1} \\ &= \frac{(a_1 - j)a_0}{z-1} + (a_0a_1 + j(a_1 - a_0))z^{j-2} + \text{lower deg. terms} \end{aligned}$$

Since $\deg U_j = j + 1$, it follows that

$$A[U_j] \subset \mathcal{P}_{j-1}, \quad j = 1, 2, \dots$$

as was to be shown. □

Lemma 4.7. *The flag $\mathcal{E}^{(11)}_{13}$ is a 2-step flag.*

Proof. The degree regular basis is shown in (18b). In particular,

$$U_1 = \operatorname{span}\{1 + a_0 z\}.$$

Define

$$A[y] := \frac{a_1 \mathcal{W}[y, 1 + a_0 z]}{z(1-z)} = a_1 \frac{y' - a_0 y}{z} - a_0 \frac{y' - a_1 y}{z-1}, \quad a_1 = \frac{a_0}{1 + a_0}$$

A direct calculation shows that

$$A[y]'\,(-1/a_0) - a_1(2 + a_0) A[y]\,(-1/a_0) = 0, \quad a_1 = \frac{a_0}{1 + a_0}$$

Hence

$$A[\mathcal{E}^{(2)}_{13}] = \mathcal{E}^{(1)}\,(-a_1(2 + a_0); -1/a_0)$$

The latter is an X_1 flag, and X_1 flags are 1-step. Therefore, the desired conclusion follows by Lemma 4.4. □

Lemma 4.8. *The flag $\mathcal{E}^{(11)}_{03}$ is a 1-step flag.*

Proof. Define

$$A[y] := \frac{y'}{z(z-1)}$$

Using (18c), a direct calculation shows that

$$A[\mathcal{E}^{(11)}_{03}] = \mathcal{P}$$

where the last equality should be understood as an equality between polynomial flags. The desired conclusion follows by Lemma 4.4 □

Lemma 4.9. *The flag $\mathcal{E}^{(11)}_{12}$ is a 2-step flag.*

Proof. The degree regular basis is shown in (18d). In particular, note that

$$U_1 = \operatorname{span}\{2z - 1\}.$$

Define

$$A[y] := \frac{a_1 \mathcal{W}[y, 2z - 1]}{z(1-z)}$$

A direct calculation shows that

$$A[y]'(1/2) = 0.$$

Hence

$$A[\mathcal{E}^{(2)}_{12}] = \mathcal{E}^{(0)}(0,1/2)$$

The latter is an X_1 flag, and X_1 flags are 1-step. Therefore, the desired conclusion follows by 4.4. □

Lemma 4.10. *The flag $\mathcal{E}^{(2b)}_{23}(a)$ is a 2-step flag.*

Proof. Define the operator

$$A[y] := (y' - ay)/z + Ky', \quad K = \sqrt{a^2 \pm 3}$$

Applying A to the degree regular basis shown in (28e) gives a flag with a stable degree sequence of $1, 2, \ldots$. Imposing

$$y'(0) = ay(0), \quad y'''(0) = 3ay''(0) \pm 6ay(0),$$

a direct calculation shows that

$$A[y]'(0) = (a+K)A[y][0].$$

Since the former conditions defines $\mathcal{E}^{(2b)}$ and since the latter conditions defines $\mathcal{E}^{(1)}(a+K;0)$ (see (9) for the definition), it follows that

$$A[\mathcal{E}^{(2b)}] \subset \mathcal{E}^{(1)}(a+K;0)$$

Next, define

$$B[y] := z(1-Kz)y' - (3 + (a-2K)z)y$$

If we suppose that

$$y'(0) = (a+K)y(0)$$

then by direct calculation,

$$B[y]'(0) = aB[y](0), \quad B[y]'''(0) = 3ay''(0) + 6ay(0)$$

Therefore,

$$B[\mathcal{E}^{(1)}(a+K;0)] \subset \mathcal{E}^{(2b)}.$$

Next, observe that A lowers degree by 1 and that B raises degree by 1. Hence BA and AB do not raise degree and they preserve their respective flags. Since $\mathcal{E}(a+K;0)$ is a 1-step flag (Theorem 3.1 and Lemma 4.1) it follows that $\mathcal{E}^{(3b)}_{23}$ is a 2-step flag. □

Lemma 4.11. *The flag $\mathcal{E}^{(2c)}_{23}(a)$ is a 2-step flag.*

Proof. The argument is the same as for the proof of Lemma 4.10, but with the following operators:

$$A[y] := \frac{y' - ay}{z} + \frac{a-1}{2}y'$$
$$B[y] := z(1 + (a-1)z)y' - (3 + (2a-1)z)y.$$

We then have

$$A[\mathcal{E}^{(2c)}] \subset \mathcal{E}^{(0)}(1;0), \quad B[\mathcal{E}^{(0)}(1;0)] \subset \mathcal{E}^{(2c)}.$$

□

Lemma 4.12. *The flags $\mathcal{E}^{(2a)}_{13}, \mathcal{E}^{(2a)}_{03}, \mathcal{E}^{(2a)}_{12}, \mathcal{E}^{(2a)}_{02}$ are all 2-step flags.*

Proof. By Proposition 3.4, all of the above flags are various specializations of

$$\mathcal{E}^{(2)}(a_{01}, 0, a_{23}; 0) = \mathrm{span}\{1 + a_{01}z, z^2 + a_{23}z^3, z^4, z^5, \ldots\}.$$

Hence, it suffices to prove the assertion for this general case. Equivalently, the above flag consists of polynomials satisfying

$$y'(0) = a_{01}y(0), \quad y'''(0) = 3a_{23}y''(0) \tag{38}$$

Consider the operator

$$A[y] := \frac{y' - a_{01}y}{z} + a_{01}y'$$

and note that

$$A[a_{01}z + 1] = 0.$$

Next, observe that

$$A[y]'(z) - \frac{1}{2}(a_{01} + 3a_{23})A[y](z) = (a_{01}y(0) - y'(0))\left(\frac{1}{z^2} + \frac{(a_{01}+3a_{23})/2}{z}\right) + \\ + \quad \frac{1}{2}(y'''(0) - 3a_{23}y''(0)) + O(z)$$

Hence, if $y(z)$ satisfies (38), then $A[y] \in \mathcal{E}^{(1)}((a_{01} + 3a_{23})/2; 0)$.

At this point, let us suppose that $a_{01} \neq 0$ and note that

$$A[y]'(z) - a_{01}A[y](z) = (1 + a_{01}z)\left(\frac{a_{01}}{z^2}y - \frac{1 + a_{01}z}{z^2}y' + \frac{1}{z}y''\right)$$

Hence $A[y] \in \mathcal{E}^{(1)}(a_{01}; -1/a_{01})$ for all polynomials $y(z)$. Together, the above calculations demonstrate that if $a_{01} \neq 0$, then

$$A[\mathcal{E}^{(2a)}] \subset \mathcal{E}^{(11)}((a_{01} + 3a_{23})/2, a_{01}; 0, -1/a_{01}).$$

Hence, by Lemma 4.4, the flags $\mathcal{E}^{(2a)}_{13}, \mathcal{E}^{(2a)}_{12}$ are Darboux connected to the flag above. We already showed that $\mathcal{E}^{(11)}$ is a 1-step flag, so this concludes the proof for the case $a_{01} \neq 0$.

Finally, let us consider the case $a_{01} = 0$. In this case,

$$\begin{aligned} A[y] &= \frac{y'}{z}, \\ \mathcal{E}^{(2)}(0, 0, a_{23}; 0) &= \operatorname{span}\{1, z^2 + a_{23}z^3, z^4, z^5, \ldots\} \\ A[\mathcal{E}^{(2)}(0, 0, a_{23}; 0)] &= \operatorname{span}\{2 + 3a_{23}z, z^2, z^3, \ldots\} \\ &= \mathcal{E}^{(1)}(3a_{23}/2; 0) \end{aligned}$$

By Lemma 4.1, the latter is a 1-step flag. Since $A[1] = 0$, applying Lemma 4.4 shows that $\mathcal{E}^{(2a)}_{03}, \mathcal{E}^{(2a)}_{02}$ are both 2-step flags. □

Proof of Theorem 4.2. There are two basic mechanisms which we use to give the proof of the conjecture for X_2 and X_1 operators. The first mechanism is that of dimensional exhaustion, and is utilized in Lemma 4.2 and in Lemma 4.5. This mechanism is used to prove the conjecture for X_1 flags (Lemma 4.2) and also used in the proof of Lemmas 4.6, 4.10 and 4.11. All these cases require that we exhibit both an A operator, which relates the given flag $\mathcal{U}$ to a “simpler” flag $\hat{\mathcal{U}}$, and a B operator that relates $\hat{\mathcal{U}}$ back to $\mathcal{U}$.

The other basic argument is conceptually related to state-deleting transformations in quantum mechanics. Here it suffices to show that a 1st order operator that annihilates U_1 maps the given flag $\mathcal{U}$ to a simpler flag $\hat{\mathcal{U}}$ and to have in hand a second order operator that preserves the given $\mathcal{U}$. This is the argument of Lemma 4.4. This argument is utilized in Lemmas 4.7, 4.8 and 4.9 4.12. Taken together, these Lemmas cover the cases of all possible X_1 and X_2 flags and the operators that preserve them. □

5. Polynomial Sturm-Liouville problems and Darboux transformations

Our main goal is to complete the classification of X_2 OPS and what remains to do is to select from all the X_2 operators given in Section 3 for each X_2 flag, those that give rise to a well defined Sturm Liouville problem. For this reason, in this Section we need to review some preliminary results from the theory of Sturm Liouville problems. We will also provide the main definitions and properties of algebraic Darboux transformations for second order differential operators. We emphasize that by construction these transformations will map an SL-OPS into an SL-OPS.

5.1. **Orthogonal polynomials on the real line defined by a Sturm-Liouville problem.** Every second-order eigenvalue equation

$$T[y] := p(z)y'' + q(z)y' + r(z)y = \lambda y$$

can be put into formal Sturm-Liouville form

$$-(Py')' + Ry = -\lambda W y$$

where

$$P(z) = \exp\left(\int^z q/p\,dz\right), \tag{39}$$

$$W(z) = (P/p)(z), \tag{40}$$

$$R(z) = -(rW)(z), \tag{41}$$

With the above definitions, the operator $T[y]$ is formally self-adjoint with respect to the weight $W(z)dz$ in the sense that Green's formula, below, holds:

$$\int T[f]g\,W dz - \int T[g]f\,W dz = P(f'g - fg') \tag{42}$$

If the operator $T[y]$ has infinitely many polynomial eigenfunctions, and if an interval of orthogonality can be appropriately chosen so that $W(z)dz$ has finite moments and the right-hand side of (42) vanishes for polynomials $f(z), g(z)$, then the eigenpolynomials of $T[y]$ constitute an SL-OPS.

By direct inspection, every X_2 operator listed in Propositions 3.2 and 3.5 has the form

$$T[y] := p(z)(y'' - 2(\log\xi)') + q(z)y' + r(z)y$$

where $p(z)$ is a quadratic polynomial, $q(z)$ is a linear form, $\xi(z)$ is either $z(z-1)$ or z and $r(z)$ is a rational function with $\xi(z)$ in the denominator. Applying an affine change of variable,

$$z = ax + b$$

the coefficients $p(z)$ and $q(z)$ can be put into a normal form. There are five classes of these normal forms, which we display in Table 1 together with the interval of orthogonality and the weight defined by (39) -(40).

TABLE 1.

$p(x)$	$q(x)$	$W(x)$	I	OPS family
1	$-2x$	$\frac{e^{-x^2}}{\xi(x)^2}$	$(-\infty,\infty)$	Hermite
x	$\alpha+1-x$	$\frac{e^{-x}x^\alpha}{\xi(x)^2}$	$(0,\infty)$	Laguerre
$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$	$\frac{(1-x)^\alpha(1+x)^\beta}{\xi(x)^2}$	$(-1,1)$	Jacobi
x^2	$2(x\pm 1)$	$\frac{e^{\mp 2/x}}{\xi(x)^2}$	n/a	Bessel
$1+x^2$	$\alpha+2(\beta+1)x$	$\frac{(1+x^2)^\beta e^{a\tan^{-1}x}}{\xi(x)^2}$	n/a	twisted Jacobi

Just as in the analysis of classical orthogonal polynomial systems [24] the Bessel and twisted Jacobi cases can be excluded because it is not possible to choose a interval of orthogonality that satisfies the finite-moment condition. Therefore the search for X_2 orthogonal polynomial systems narrows to the first 3 cases. In each case, the requirement is that $\xi(z)$ have no zeros on the corresponding interval of orthogonality. For the Laguerre subcase, there is the additional constraint that $\alpha > -1$. For the Jacobi subcase, the constraint is that $\alpha, \beta > -1$.

5.2. **Factorization and orthogonal polynomials.** Consider two differential operators:

$$T[y] = py'' + qy' + ry, \tag{43}$$

$$\hat{T}[y] = py'' + \hat{q}y' + \hat{r}y, \tag{44}$$

related by a factorization (30) (31). Let us write

$$A[y] = b(y' - wy), \tag{45}$$

$$B[y] = \hat{b}(y' - \hat{w}y), \tag{46}$$

where $p(z), q(z), r(z), b(z), w(z), \hat{b}(z), \hat{w}(z)$ are all rational functions. We will refer to

$$\phi(z) = \exp\int^z w\,dz, \quad w = \phi'/\phi \tag{47}$$

as a *quasi-rational factorization eigenfunction* and to $b(z)$ as the *factorization gauge*. The reason for the above terminology is as follows. By (30),

$$T[\phi] = \lambda_0 \phi; \tag{48}$$

hence the term factorization eigenfunction. Next, consider two factorization gauges $b_1(z), b_2(z)$ and let $\hat{T}_1[y], \hat{T}_2[y]$ be the corresponding partner operators. Then,

$$\hat{T}_2 = \mu^{-1} \hat{T}_1 \mu, \quad \text{where } \mu(z) = b_1(z)/b_2(z).$$

Therefore, the choice of $b(z)$ determines the gauge of the partner operator. This is why we refer to $b(z)$ as the factorization gauge.

Proposition 5.1. *Let $T[y]$ be a second order rational operator that preserves a polynomial flag. Let $\phi(z)$ be a quasi-rational factorization eigenfunction with eigenvalue λ_0. Then, there exists a rational factorization* (30) *such that the partner operator preserves a primitive polynomial flag.*

Proof. Let $w(z) = \phi'(z)/\phi(z)$ and let $b(z)$ be an as yet unspecified rational function. Set

$$\hat{w} = -w - q/p + b'/b, \tag{49}$$

$$\hat{b} = p/b, \tag{50}$$

and let $A[y], B[y]$ be as shown in in (45) (46). An elementary calculation shows that (30) holds. Let $y_1, y_2, \dots$ be a degree-regular basis of the flag preserved by T. We require that the flag spanned by $A[y_j]$ be polynomial and primitive (no common factors). Observe that if we take $b(z)$ to be the reduced denominator of $w(z)$, then $A[y_j]$ is a polynomial for all j. However, this does not guarantee that $A[y_j]$ is free of a common factor. That is indeed a stronger condition which in fact fixes the gauge $b(z)$ up to a choice of scalar multiple. Finally, the intertwining relation

$$\hat{T} A = AT \tag{51}$$

implies that $A[y_j]$ are eigenpolynomials of the partner $\hat{T}$. □

In the preceding subsection, we showed that a second-order operator $T[y]$ is formally self-adjoint relative to a weight W defined by (39) (40). The following Proposition describes the effect of a factorization transformation on the corresponding factorization function and the weight.

Proposition 5.2. *Suppose that rational operators*

$$T[y] = py'' + qy' + ry, \quad \hat{T}[y] = py'' + \hat{q}y' + \hat{r}y$$

are related by a rational factorization with factorization eigenfunction $\phi(z)$ and factorization gauge $b(z)$, Then the dual factorization gauge, factorization eigenfunction and weight function are given by

$$b\hat{b} = p \tag{52}$$

$$\hat{W}/\hat{b} = W/b, \tag{53}$$

$$\hat{b}\hat{\phi} = 1/(W\phi) \tag{54}$$

Proof. Equation (52) follows immediately from (45) (46) (30). From there, equation (31) implies that

$$w + \hat{w} = -q/p + b'/b = -\hat{q}/p + \hat{b}'/\hat{b}. \tag{55}$$

Hence,

$$\hat{q} = q + p' - 2pb'/b. \tag{56}$$

From here, (53) follows by equations (39) (40). Equation (54) follows from (47). □

The dual weights $W, \hat{W}$ allow us to interpret the intertwining operators $A[y], B[y]$ in terms of a formally adjoint relation

$$\int A[f]g\,\hat{W}dx + \int B[g]f\,W dx = (P/b)fg \tag{57}$$

If the right hand side vanishes on an appropriately chosen interval of orthogonality, and if the partner operators $T, \hat{T}$ both admit an infinite sequence of eigenpolynomials, then the operators T and $\hat{T}$ and their corresponding eigenfunctions are related by a 1-step Darboux transformation.

The dual factorization functions $\phi, \hat{\phi}$ allow us to express the adjoint intertwiners as Wronskians:

$$A[y] = b\phi^{-1}\mathcal{W}[\phi, y]$$
$$B[y] = \hat{b}\hat{\phi}^{-1}\mathcal{W}[\hat{\phi}, y].$$

In Theorem 4.2 of Section 4, we established that every X_2-operator is Darboux-connected to a classical operator and that the requisite intertwiners also connect the corresponding exceptional flag with the standard polynomial flag. Theorem 1.1 follows as an immediate corollary. In light of the above remarks, it is convenient to give the connecting intertwiners as Wronskians of factorizing functions of the classical operators. Therefore, before turning to the exhaustive classification, we must review the possible quasi-rational factorizing functions for the classical operators.

5.3. **The X_2 Hermite polynomials.** The classical Hermite orthogonal polynomials are orthogonal relative to the weight

$$W(x) = e^{-x^2}.$$

The nth Hermite polynomial $H_n(x)$ satisfies the differential equation

$$\mathcal{H}[H_n] = -2nH_n$$

where

$$\mathcal{H}[y] = y'' - 2xy'$$

The exhaustive classification of the X_2 polynomials confirms the factorization conjecture. This means that all X_2 Hermite polynomials are given as Wronskians of the classical polynomials together with fixed quasi-rational factorization eigenfunction of the classical Hermite operator $\mathcal{H}[y]$. These quasi-rational eigenfunctions are listed below:

$$\psi_n^{(1)}(x) = H_n(x), \qquad \mathcal{H}[\psi^{(1)}] = -2n\psi^{(1)} \tag{58}$$
$$\psi_n^{(2)}(x) = e^{x^2}H_n(ix), \qquad \mathcal{H}[\psi^{(2)}] = 2(n+1)\psi^{(2)}. \tag{59}$$

We will use $\hat{H}_n(x)$ to denote the X_2 Hermite polynomials, where the degree index n skips exactly two values. These exceptional Hermite polynomials are orthogonal relative to a weight of the form

$$\hat{W}(x;\alpha,\beta) = \frac{e^{-x^2}}{\xi(x)^2}$$

where the denominator $\xi(x)$ is a quadratic polynomial. Consequently, the $\hat{H}_n(x)$ are eigenpolynomials of an operator of the form

$$\hat{\mathcal{H}}[y] := \mathcal{H}[y] - 2(\log\xi)'y' + r(x)y,$$

where $r(x)$ is rational in x and where the prime denotes a derivative with respect to x. In order for the weight to be non-singualr, the quadratic $\xi(x)$ must have imaginary roots. Also, as we show below, the rational term $r(x) = 0$ always vanishes. This is established on a case-by-case basis, and has no apriori explanation.

5.4. **X_2-Laguerre polynomials.** The classical Laguerre weight is

$$W_\alpha(x) = e^{-x}x^\alpha$$

The classical Laguerre operator is

$$\mathcal{L}_\alpha[y] := xy'' + (\alpha + 1 - x)y'$$

The quasi-rational eigenfunctions of this operator are

$$\phi_n^{(1)}(x;\alpha) = L_n^{(\alpha)}(x) \qquad \mathcal{L}_\alpha[\phi_n^{(1)}] = -n\phi_n^{(1)} \tag{60}$$
$$\phi_n^{(2)}(x;\alpha) = x^{-\alpha}L_n^{(-\alpha)}(x) \qquad \mathcal{L}_\alpha[\phi_n^{(2)}] = (\alpha - n)\phi_n^{(2)} \tag{61}$$
$$\phi_n^{(3)}(x;\alpha) = e^x L_n^{(\alpha)}(-x) \qquad \mathcal{L}_\alpha[\phi_n^{(3)}] = (\alpha + n + 1)\phi_n^{(3)} \tag{62}$$
$$\phi_n^{(4)}(x;\alpha) = e^x x^{-\alpha}L_n^{(-\alpha)}(-x) \qquad \mathcal{L}_\alpha[\phi_n^{(4)}] = (n+1)\phi_n^{(4)} \tag{63}$$

In confirmation of the factorization conjecture, all X_2 Laguerre polynomials are given as first and second-order Wronskians of the classical Laguerres and the above factorization functions. The X_2 polynomials themselves will be denoted by $\hat{L}_n^{(\alpha)}$ the range of n omits exactly two degrees. In all cases, the $\hat{L}_n^{(\alpha)}$ are orthogonal relative to a weight of the form

$$\hat{W}(x;\alpha) := \frac{e^{-x}x^{\alpha}}{\xi(x;\alpha)^2}, \tag{64}$$

where the denominator $\xi(x;\alpha)$ is a quadratic polynomial in x. The parameter α has to be restricted so that $\xi(x;\alpha)$ has no zeros in the interval of orthogonality $x \in (0,\infty)$. The exceptional polynomials $\hat{L}_n^{(\alpha)}$ arise as eigenpolynomials of a second order operator

$$\hat{\mathcal{L}}_\alpha[y] = xy'' + (1+\alpha-x)y' - 2(\log\xi)'y' + r(x;\alpha)y$$

where $r(x;\alpha)$ is a rational function in x which will be adjusted so that, in all cases,

$$\hat{\mathcal{L}}_\alpha[\hat{L}_n^{(\alpha)}] = -n\hat{L}_n^{(\alpha)}$$

5.5. **The X_2 Jacobi polynomials.** The classical Jacobi OPs are orthogonal relative to the weight

$$W(x;\alpha,\beta) = (1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta > -1.$$

The nth Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$\mathcal{T}_{\alpha,\beta}[P_n^{(\alpha,\beta)}] = -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}$$

where

$$\mathcal{T}_{\alpha,\beta}[y] = (1-x^2)y'' + (\beta-\alpha-(2+\alpha+\beta)x)y'$$

The exhaustive classification of the X_2 polynomials confirms the factorization conjecture. This means that all X_2 Jacobi polynomials are given as Wronskians of the classical polynomials together with fixed quasi-rational factorization eigenfunction of the classical Jacobi operator $\mathcal{T}_{\alpha,\beta}[y]$. These quasi-rational eigenfunctions are listed below:

$$\phi_n^{(1)}(x;\alpha,\beta) = P^{(\alpha,\beta)}(x), \qquad \mathcal{T}[\phi^{(1)}] = -n(n+\alpha+\beta+1)\phi^{(1)} \tag{65}$$

$$\phi_n^{(2)}(x;\alpha,\beta) = (1+x)^{-\beta}P^{(\alpha,-\beta)}(x), \qquad \mathcal{T}[\phi^{(2)}] = (\beta-n)(n+\alpha+1)\phi^{(2)} \tag{66}$$

$$\phi_n^{(3)}(x;\alpha,\beta) = (1-x)^{-\alpha}P^{(-\alpha,\beta)}(x), \qquad \mathcal{T}[\phi^{(3)}] = (\alpha-n)(n+\beta+1)\phi^{(3)} \tag{67}$$

$$\phi_n^{(4)}(x;\alpha,\beta) = (1-x)^{-\alpha}(1+x)^{-\beta}P^{(-\alpha,-\beta)}(x), \qquad \mathcal{T}[\phi^{(4)}] = (n+1)(\alpha+\beta-n)\phi^{(4)} \tag{68}$$

We will use $\hat{P}_n^{(\alpha,\beta)}(x)$ to denote the X_2 Jacobi polynomials, where the degree index n skips exactly two values. These exceptional Jacobi polynomials are orthogonal relative to a weight of the form

$$\hat{W}(x;\alpha,\beta) = \frac{(1-x)^\alpha(1+x)^\beta}{\xi(x;\alpha,\beta)^2}$$

where the denominator $\xi(x;\alpha,\beta)$ is a quadratic polynomial. Consequently, the $\hat{P}_n^{(\alpha,\beta)}(x)$ are eigenpolynomials of an operator of the form

$$\hat{\mathcal{T}}_{\alpha,\beta}[y] := \mathcal{T}_{\alpha,\beta}[y] - 2(1-x^2)(\log\xi)'y' + r(x;\alpha,\beta)y,$$

where $r(x;\alpha,\beta)$ is rational in x and where the prime denotes a derivative with respect to x. The parameters $\alpha,\beta > -1$ are so restricted in order to have finite moments of all orders. Additional restrictions must be imposed on α,β to ensure that $\xi(x;\alpha,\beta)$ has no zeros in the interval of orthogonality $x \in (-1,1)$.

6. Classification of codimension 2 XOPs

The main result of this section is a complete list of X_2 orthogonal polynomial systems together with the intertwining operators that connect them to the classical families of Hermite, Laguerre and Jacobi. The classification is summarized in the following.

Theorem 6.1. *Up to a real affine transformation of the independent variable, all* X_2 *orthogonal polynomial systems are gathered in the following table:*

TABLE 2. Classification of X_2 orthogonal polynomial systems

	$\mathcal{E}^{(11)}_{23}$	$\mathcal{E}^{(11)}_{13}$	$\mathcal{E}^{(11)}_{03}$	$\mathcal{E}^{(2a)}_{13}$	$\mathcal{E}^{(2a)}_{03}$
Hermite			***1-step*** §***6.1.3***		
Laguerre	***1-step*** §***6.2.1***	*2-step* §*6.2.2*	*1-step* §*6.2.3*	*2-step* §*6.2.5*	*2-step* §*6.2.6*
Jacobi	***1-step*** §***6.3.1***	*2-step* §*6.3.2*	*1-step* §*6.3.3*	*2-step* §*6.3.5*	*2-step* §*6.3.6*

In Table 2 we find the classification of X_2 orthogonal polynomial systems. In each cell we give the number of iterated Darboux transformations to obtain these families from a classical OPS, and we specify the subsection where each family is described. Empty cells mean that an OPS of that type does not exist for the given flag, and the same is true for all the other X_2 flags not included in the table. The cells marked in bold correspond to X_2-OPS previously known in the literature, while all other cases are new.

In the rest of this section we will select the X_2 operators for each of the X_2 flags in Section 3 that can be transformed into a well defined Sturm Liouville problem of Hermite, Laguerre or Jacobi type. We allow affine changes of variables and basically we need to transform the leading order of the X_2 operator into 1, x or $1-x^2$ and verify that the weight is non-singular in the corresponding interval and it has well defined moments of all orders. This will exclude many cases and it will impose constraints on the remaining free parameters for the cases that survive.

6.1. X_2-Hermite OPS.

6.1.1. *No Hermite polynomials for the 2-pole flag* $\mathcal{E}^{(11)}_{23}$. The leading order coefficient in (19a), is

$$-\frac{1}{2}z^2(a_0-a_1)(a_0-a_1+4)-z(a_0a_1-a_0-a_1^2+3a_1)-\frac{a_1^2}{2}+a_1$$

We require the coefficient of z^2 to vanish. Setting $a_1=a_0$ transforms the above into

$$-a_0(2z+a_0/2-1)$$

Setting $a_1=a_0+4$ gives

$$(a_0+2)(2z-a_0/2+2)$$

In other case, it is impossible to obtain a Hermite-like operator.

6.1.2. *No Hermite polynomials for the 2-pole flag* $\mathcal{E}^{(11)}_{13}$. The leading order coefficient in (19b), is

$$-(c_0+c_1)\frac{z^2}{2}+c_0\left(z-\frac{1}{2}\right)$$

It is not possible to specialize c_0, c_1 so that the above polynomial reduces to a constant.

6.1.3. *1-step Hermite polynomials that span the 2-pole flag* $\mathcal{E}^{(11)}_{03}$. Setting $\alpha_0, \alpha_1=-1/2, q_0=1$ in (19c) and applying the change of varibles

$$z=i/\sqrt{2}x+1/2$$

gives a Hermite-type operator

$$\hat{\mathcal{H}}[y]:=y''-2xy-2(\log\xi)'y'$$

where

$$\xi(x)=1+2x^2=-\frac{1}{2}H_2(ix).$$

The adjoint intertwiners and the exceptional polynomials are shown below:

$$B[y] = e^{-x^2} W[\psi^{(2)}, y] \tag{69}$$

$$A[y] = \frac{y'}{\xi(x)} \tag{70}$$

$$\hat{H}_0 = 1 \tag{71}$$

$$\hat{H}_n = B[H_{n-3}], \quad n = 3, 4, 5, \ldots \tag{72}$$

$$\hat{\mathcal{H}}[\hat{H}_n] = -2n\hat{H}_n \tag{73}$$

$$A[\hat{H}_n] = 4nH_{n-3}, \quad n = 0, 3, 4, 5, \ldots \tag{74}$$

The above polynomials are related to the CPRS exactly-solvable potential [3, 7] and constitute the codimension-2 instance of the modified Hermite polynomials introduced in [4]. This family was also described independently in [5] for arbitrary codimension.

6.1.4. *No Hermite polynomials for the 2-pole flag* $\mathcal{E}^{(11)}_{12}$. By inspection of (19d), a Hermite-type operator requires

$$\alpha_0 = \alpha_1 = \frac{1}{2} q_0 \neq 0$$

Applying a change of variable

$$z = ax$$

yields the weight

$$W(x) = e^{-2a^2x^2}(1 - 4a^2x^2)^2$$

To have a real weight requires a to be either real, or purely imaginary. In the first, case, the weight is singular; in the latter case there are no singularities but the finite-moment condition is violated.

6.1.5. *No Hermite-type polynomials for the 1-pole flags* $\mathcal{E}^{(2a)}, \mathcal{E}^{(2b)}$ *and* $\mathcal{E}^{(2c)}$. A real valued operator and weight requires the unique pole to be real. However, a Hermite-type weight requires the entire real line as the interval of orthogonality. Therefore, even if Hermite type weights of the form

$$W(x) = \frac{e^{-x^2}}{(x-b)^4}$$

do exist, since b is real, the resulting weight is singular.

6.2. X_2-Laguerre OPS.

6.2.1. *1-step Laguerre polynomials that span the 2-pole flag* $\mathcal{E}^{(11)}_{23}$. By direct inspection of (19a) a Laguerre-type operator requires either $a_1 = a_0 + 4$ or $a_1 = a_0$. We consider these two cases in turn

(I) Imposing $a_1 = a_0 + 4$ in (19a), making an affine change of variable

$$x = (a_0 + 2)(z - a_0/4 - 1),$$

and setting

$$\alpha = a_0(4 + a_0)/4$$

gives the operator

$$\hat{\mathcal{L}}_\alpha[y] := xy'' + (1 + \alpha - x)y' - 2(\log \xi)'(xy' + \alpha y)$$

where

$$\xi(x;\alpha) = L^{(\alpha-1)}_2(-x) = (x^2 + 2(\alpha+1)x + \alpha(\alpha+1))/2$$

and the prime symbol denotes the derivative with respect to x. We impose $\alpha > 0$ in order to avoid positive zeros of $\xi(x;\alpha)$. The resulting orthogonal polynomials are codimension-2 instances of the type I exceptional Laguerre polynomials [14, 28]. The corresponding polynomials and the adjoint intertwining relation are shown below:

$$A[y] := x^{\alpha+1}\mathcal{W}[x^{-\alpha}, y]/\xi(x;\alpha) \tag{75}$$

$$B[y] := e^{-x}\mathcal{W}[\phi^{(3)}_2(x;\alpha-1), y] \tag{76}$$

$$\hat{L}^{(\alpha)}_n(x) = B[L^{(\alpha-1)}_{n-2}], \qquad n = 2, 3, 4, \ldots \tag{77}$$

$$A[\hat{L}^{(\alpha)}_n] = (\alpha + n)L^{(\alpha-1)}_{n-2}. \tag{78}$$

(II) Imposing $a_1 = a_0$ in (19a), making an affine change of variable

$$x = a_0(4z - 2 + a_0)/4,$$

and setting

$$\alpha = a_0^2/4 - 1$$

gives the operator

$$\hat{\mathcal{L}}_\alpha[y] := xy'' + (1+\alpha - x)y' - 2x\left(\log \xi\right)'(y' - y)$$

where

$$\xi(x;\alpha) = L_2^{(-\alpha-1)}(x) = (x^2 + 2(\alpha-1)x + \alpha^2 - \alpha)/2, \quad \alpha > 1$$

The resulting orthogonal polynomials are codimension-2 instances of the type II exceptional Laguerre polynomials [14,28]. The definition of these polynomials and the adjoint differential relation are shown below

$$A[y] := \frac{e^{-x}}{\xi(x;\alpha)} \mathcal{W}[e^x, y] \tag{79}$$

$$B[y] := x^{\alpha+2}\mathcal{W}[\phi_2^{(2)}(x;\alpha+1), y] \tag{80}$$

$$\hat{L}_n^{(\alpha)} = B[L_{n-2}^{(\alpha+1)}], \quad n = 2,3,4,5,\ldots \tag{81}$$

$$A[\hat{L}_n^{(\alpha)}] = (3-\alpha-n)L_{n-2}^{(1+\alpha)} \tag{82}$$

6.2.2. *2-step Laguerre polynomials that span the 2-poles flag* $\mathcal{E}_{13}^{(11)}$. By direct inspection of (19b), a Laguerre-type operator requires $c_0 = 1, c_1 = 0$. Applying the affine transformation

$$x = (z - 1/2)\frac{a_0(2+a_0)}{a_0+1}$$

and setting

$$\alpha = \frac{a_0^2 + 2a_0 + 2}{2(a_0+1)}$$

gives the operator

$$\hat{\mathcal{L}}_\alpha[y] := xy'' + (1+\alpha-x)y' - 2x(\log\xi)'y' + \frac{2(\alpha-1)(\alpha+1-x)}{\xi(x;\alpha)}y$$

where

$$\xi(x;\alpha) = x^2 + 1 - \alpha^2 = e^{-2x}x^{1+\alpha}\mathcal{W}[\phi_1^{(4)}(x;\alpha), \phi_1^{(3)}(x;\alpha)], \quad |\alpha| < 1$$

The adjoint intertwiners and the exceptional polynomials are:

$$B_\alpha[y] := \frac{1}{\alpha}e^{-2x}x^{2+\alpha}\mathcal{W}[\phi_1^{(3)}(x;\alpha), \phi_1^{(4)}(x;\alpha), y] \tag{83}$$

$$\hat{L}_1^{(\alpha)} := L_1^{(\alpha)}(-x) = x + \alpha + 1 \tag{84}$$

$$\hat{L}_n^{(\alpha)} := B_\alpha[L_{n-3}^{(\alpha)}], \quad n = 3,4,5,\ldots \tag{85}$$

$$A_\alpha[y] := \frac{x^{2+\alpha}}{\alpha\xi(x;\alpha)^2}\mathcal{W}[x^{-\alpha}(x-\alpha+1), x+\alpha+1, y] \tag{86}$$

$$A_\alpha[\hat{L}_n^{(\alpha)}] = -(n-1)(\alpha+n-1)L_{n-3}^{(\alpha)}, \quad n = 1,3,4,5,\ldots \tag{87}$$

Note: for $\alpha = 0$, the above definitions have to be treated as a limit process. A straightforward calculation shows that

$$B_0[y] = -x(1+x^2)y'' + (2x^3 + x^2 + 2x - 1)y' - (x^3 + x^2 + 2x - 2)y \tag{88}$$

6.2.3. *1-step Laguerre polynomials that span the 2-poles flag* $\mathcal{E}_{03}^{(11)}$. Inspection of (19c) reveals that a Laguerre-type operator requires

$$q_0 + c_0 + c_1 = 1$$

Since we are free to scale the operator, no generality is lost by imposing $c_0 - c_1 = 1$, which gives us

$$c_0 = (1 - q_0)/2, \quad c_1 = -(1 + q_0)/2$$

Applying the affine change of variables

$$x = q_0(1 - q_0 - 2z)$$

and setting

$$\alpha = 1 - q_0^2$$

gives the operator

$$\hat{\mathcal{L}}_k[y] := xy'' + (1 + \alpha - x)y' - 2x(\log \xi)'y' \tag{89}$$

where

$$\xi(x; \alpha) = L_2^{(-\alpha-1)}(-x) = (x^2 + 2(1 - \alpha)x + \alpha^2 - \alpha)/2, \tag{90}$$

and where

$$\alpha \in (-1, 0) \cup (1, \infty)$$

in order to avoid positive zeros in $\xi(x; \alpha)$ and to have finite moments. The corresponding exceptional polynomials and intertwiners are shown below:

$$B[y] := e^{-x}x^{2+\alpha}\mathcal{W}[\phi_2^{(4)}(x; 1 + \alpha), y] \tag{91}$$

$$A[y] = \frac{y'}{\xi(x; \alpha)} \tag{92}$$

$$\hat{L}_0^{(\alpha)}(x) = 1 \tag{93}$$

$$\hat{L}_n^{(\alpha)}(x) = B[L_{n-3}^{(\alpha+1)}], \quad n = 3, 4, 5, \dots \tag{94}$$

$$A[\hat{L}_n^{(\alpha)}] = nL_{n-3}^{(\alpha+1)}, \quad n = 0, 3, 4, 5, \dots \tag{95}$$

6.2.4. *No Laguerre polynomials for the 2-poles flag* $\mathcal{E}_{12}^{(11)}$. By inspection of (19d), $q_0 = c_0 + c_1$. Without loss of generality,

$$c_0 - c_1 = 1, \quad c_0 + c_1 = a$$

where a is a new operator parameter. Making the affine change of variables

$$x = a((1 + a) - 2z)$$

gives the weight

$$\hat{W}_\alpha(x) = \frac{e^{-x}x^{a^2-1}}{(x - a^2 - a)^2(x - a^2 + a)^2}$$

In order to have a real weight we need a to be either real or pure imaginary. In the first case, the denominator will have a positive zero; the weight is singular. In the former case, the finite moment condition is violated. Therefore, there are no X_2 polynomials that span this flag.

6.2.5. *1-step Laguerre polynomials for the 1-pole flag* $\mathcal{E}_{13}^{(2a)}$. We refer to the $\mathcal{E}^{(2)}$ flags and the corresponding OPS as 1-pole because the weight function has one pole, unlike the 2-poles present in the weight functions of the $\mathcal{E}^{(11)}$ families. This pole in the weight has higher multiplicity.

By direct inspection of (29a), a Laguerre-type operator requires either $a = 1/3$, or $a = 3$. Setting $a = 1/3$, making the change of variables $x = z + 3/4$ yields a singular weight, namely

$$\frac{e^{-x}x^{-1/4}}{(4x - 3)^4}$$

Setting $a = 3$ and making the change of variables

$$x = 3z - 3/4$$

gives the operator

$$\hat{\mathcal{L}}[y] := xy'' + (5/4 - x)y' - \frac{4xy' + y}{x + 3/4}$$

and the weight

$$\hat{W}(x) = \frac{e^{-x}x^{1/4}}{(4x+3)^4},$$

which is both non-singular and has finite moments of all orders. The remarkable feature of this weight is that it has a fourth order pole, unlike the two second order poles of the previously discussed X_2 families. The adjoint intertwiners and the exceptional polynomials for this weight are shown below:

$$B[y] := \frac{e^{-2x}x^{9/4}}{(x+3/4)}\mathcal{W}[\phi_1^{(4)}(x;1/4), \phi_2^{(3)}(x;1/4), y] \tag{96}$$

$$\hat{L}_1(x) := x + 15/4 \tag{97}$$

$$\hat{L}_n(x) := B[L_{n-3}^{(1/4)}], \quad n = 3,4,5,\dots \tag{98}$$

$$A[y] := \frac{x^{9/4}}{(x+3/4)^3}\mathcal{W}[x^{-1/4}, x+15/4, y] \tag{99}$$

$$A[\hat{L}_n] = \frac{25}{128}(n-1)(4n+1)L_{n-3}^{(1/4)}, \quad n = 1,3,4,5,\dots \tag{100}$$

6.2.6. *2-step Laguerre polynomials for the 1-pole flag $\mathcal{E}_{03}^{(2a)}$.* By inspection of (29b), a Laguerre-type operator requires $q_0 = 3$. Making the affine change of variable

$$x = \frac{3}{4}(2z-1)$$

gives the operator

$$\hat{\mathcal{L}}[y] := xy'' + (3/4 - x)\,y' - \frac{4xy'}{x+3/4}$$

and the weight

$$\hat{W}(x) := \frac{e^{-x}x^{-1/4}}{(4x+3)^4}$$

The adjoint intertwiners and the exceptional polynomials are shown below:

$$B[y] := \frac{e^{-2x}x^{7/4}}{x+3/4}\mathcal{W}[\phi_2^{(4)}(x;-1/4), \phi_1^{(3)}(x;-1/4), y] \tag{101}$$

$$\hat{L}_0 = 1 \tag{102}$$

$$\hat{L}_n := B[L_{n-3}^{(-1/4)}], \quad n = 3,4,5,\dots \tag{103}$$

$$A[y] := \frac{x^{7/4}}{(x+3/4)^3}\mathcal{W}[1, x^{1/4}(x+15/4), y] \tag{104}$$

$$A[\hat{L}_n] = \frac{25}{128}n(5-4n)L_{n-3}^{(-1/4)}, \quad n = 0,3,4,5,\dots \tag{105}$$

6.2.7. *No Laguerre polynomials for the 1-pole flags $\mathcal{E}_{02}^{(2a)}, \mathcal{E}_{12}^{(2a)}, \mathcal{E}_{23}^{(2b)}$ and $\mathcal{E}_{23}^{(2c)}$.* Setting $p_0 = 0$ and applying an affine transformation, the operator (29c) yields a singular Laguerre-type weight

$$\hat{W}(x) = \frac{e^{-x}x^{1/4}}{(4x-3)^4}$$

By direct inspection of (29d), (29e) (29f), the operators in question do not admit a Laguerre form.

6.3. X_2 **-Jacobi OPS.**

6.3.1. *1-step Jacobi polynomials that span the 2-pole flag $\mathcal{E}_{23}^{(11)}$.* The quadratic coefficient of y'' in (19a) factors as

$$-\frac{1}{2}(a_1-a_0)(a_1-a_0-4)(z-z_1)(z-z_2) \tag{106}$$

where

$$z_1 = \frac{a_1}{a_1-a_0-4}, \quad z_2 = \frac{a_1-2}{a_1-a_0-4} \tag{107}$$

We seek an affine change of variable that transforms this quadratic into $1-x^2$. There are two possibilities according to which root is sent to $+1$ or -1. However, since the two resulting families are related by an affine change of variable, it suffices to consider just one such transformation. Employing the transformation

$$z = \frac{z_2}{2}(x+1) - \frac{z_1}{2}(x-1)$$

setting

$$\alpha = \frac{2(z_1-1)z_1(2z_2-1)}{z_1-z_2}, \quad \beta = \frac{2(2z_1-1)z_2(z_2-1)}{z_1-z_2}$$

and adding a constant term, transforms $T^{(11)}_{23}[y]$ into the operator

$$\hat{\mathcal{T}}_{\alpha,\beta}[y] = \mathcal{T}_{\alpha,\beta}[y] - 2(\log\xi)'((1-x^2)y' + \beta(1-x)y) + 2(\alpha-\beta-1)y$$

where

$$\xi(x;\alpha,\beta) = P_2^{(-\alpha-1,\beta-1)}(x) \tag{108}$$

$$= \frac{1}{4}\binom{\beta-\alpha+2}{2}(x-1)^2 + \frac{1}{2}(\beta-\alpha+1)(1-\alpha)(x-1) + \binom{\alpha}{2} \tag{109}$$

In this way, we have arrived at the codimension-2 instance of the exceptional Jacobi-type polynomials introduced by Odake and Sasaki [16, 28].

We require that $\xi(x;\alpha,\beta)$ have no zeros in the interval of orthogonality $x \in [-1,1]$. The above affine transformation maps $-1, 1$ to the roots of $\xi(x)$ and maps

$$z_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{-a(1+a+b)/b}, \quad a = \alpha-1,\ b = -\beta-1 \tag{110}$$

$$z_2 = \frac{1}{2} \mp \frac{1}{2}\sqrt{-b(1+a+b)/a} \tag{111}$$

to ± 1. Therefore, an equivalent condition is that z_1, z_2 are either complex-valued or lie in the interval $(0,1)$. The solutions to this constraint in the (a,b) plane are the disjoint union of the following regions: (i) $a, b > 0$; (ii) $a > 0, b < -1$; (iii) $a < -1, b > 0$; (iv) $-1 < a, b < 0$. Finite moments require $\alpha, \beta > -1$. Therefore, in the final analysis, we have two classes orthogonal polynomials with a non-singular weight and finite moments: $\alpha > -1, \beta > 0$ and $0 < \alpha < 1, -1 < \beta < 0$; c.f., Proposition 4.5 of [16].

The exceptional polynomials and the adjoint intertwiners are shown below:

$$A[y] := \frac{(1+x)^{\beta+1}}{\xi(x;\alpha,\beta)}\mathcal{W}[(1+x)^{-\beta}, y] \tag{112}$$

$$B[y] := (1-x)^{\alpha+2}\mathcal{W}[\phi_2^{(2)}(x;\alpha+1,\beta-1), y] \tag{113}$$

$$\hat{P}_n^{(\alpha,\beta)} = B[P_{n-2}^{(\alpha+1,\beta-1)}] \tag{114}$$

$$\hat{\mathcal{T}}_{\alpha,\beta} = BA + (2+\beta)(\alpha-1) \tag{115}$$

$$\mathcal{T}_{\alpha+1,\beta-1} = AB + (2+\beta)(\alpha-1) \tag{116}$$

$$\hat{\mathcal{T}}[\hat{P}_n] = -(n-2)(n-1+\alpha+\beta)\hat{P}_n \tag{117}$$

$$A[\hat{P}_n^{(\alpha,\beta)}] = -(\alpha+n-3)(\beta+n)P_{n-2}^{(\alpha+1,\beta-1)} \tag{118}$$

6.3.2. *2-step Jacobi polynomials that span the 2-pole flag* $\mathcal{E}^{(11)}_{13}$. The quadratic coefficient of y'' in (19b) factors as

$$\frac{c_0}{2}((R+1)z-1)((R-1)z+1), \quad \text{where } R = \sqrt{-\frac{c_1}{c_0}}$$

Employing the affine transformation

$$z = \frac{Rx+1}{1-R^2}$$

and setting

$$\alpha = \frac{1}{1-R} + \frac{a_0}{1-R} - \frac{R}{(1+a_0)(1-R)} \tag{119}$$

$$\beta = \frac{1}{1+R} + \frac{a_0}{1+R} + \frac{R}{(1+a_0)(1+R)} \tag{120}$$

transforms the operator $T_{13}^{(11)}$ into

$$\hat{\mathcal{T}}_{\alpha,\beta}[y] = \mathcal{T}_{\alpha,\beta}[y] - 2(1-x^2)(\log\xi)'y' - \frac{8(\alpha-1)(\beta-1)P_1^{(\alpha,\beta)}(x)}{\xi(x;\alpha,\beta)}y$$

where

$$\xi(x;\alpha,\beta) = (x^2+1)(\alpha^2-\beta^2) + 2x(\alpha^2+\beta^2-2) \tag{121}$$

$$= \frac{4a_0(2+a_0)(1+a_0-R)(1+a_0+R)}{(1+a_0)^2(R^2-1)^2}(x+R)(Rx+1) \tag{122}$$

For a real, non-singular weight, we require $R = e^{it},\ t\in\mathbb{R}$ to be a unit-length complex number. A direct calculation shows that

$$R = \frac{\alpha^2+\beta^2-2}{\alpha^2-\beta^2} \pm \frac{2\sqrt{(\alpha^2-1)(\beta^2-1)}}{\alpha^2-\beta^2}$$

$$\frac{1}{R} = \frac{\alpha^2+\beta^2-2}{\alpha^2-\beta^2} \mp \frac{2\sqrt{(\alpha^2-1)(\beta^2-1)}}{\alpha^2-\beta^2}$$

Therefore, the parameters α,β must satisfy

$$-1<\alpha<1,\ \beta>1, \quad \text{or} \quad \alpha>1,\ -1<\beta<1$$

The corresponding exceptional polynomial and the adoint intertwiners are shown below

$$A[y] := \frac{(1+x)^{\beta+2}}{\beta\xi(x;\alpha,\beta)}\mathcal{W}[(1+x)^{-\beta}P_1^{(\alpha,\beta-2)}, P_1^{(-\alpha-2,\beta)}, y] \tag{123}$$

$$B[y] := \frac{(1-x)^{6+2\alpha}(1+x)^{2+\beta}}{\beta}\mathcal{W}[\phi_1^{(2)}(x;\alpha+2,\beta), \phi_1^{(4)}(x;\alpha+2,\beta), y] \tag{124}$$

$$\hat{P}_1^{(\alpha,\beta)} = P_1^{(-\alpha-2,\beta)} \tag{125}$$

$$\hat{P}_n^{(\alpha,\beta)} = B\left[P_{n-3}^{(\alpha+2,\beta)}\right], \quad n=3,4,5,\dots \tag{126}$$

$$\hat{\mathcal{T}}[\hat{P}_n] = -n(n-3+\alpha+\beta)\hat{P}_n \tag{127}$$

$$A[\hat{P}_n^{(\alpha,\beta)}] = \frac{1}{16}(n-1)(n+\alpha-2)(n+\beta-1)(n+\alpha+\beta-2)P_{n-3}^{(\alpha+2,\beta)}, \quad n=1,3,4,5,\dots \tag{128}$$

As above, for the case of $\beta=0$, the definitions above must be treated as a limit.

6.3.3. *1-step Jacobi polynomials that span the 2-pole flag* $\mathcal{E}_{03}^{(11)}$. The quadratic coefficient of y'' in (19c) factors as

$$-(q_0+c_0+c_1)\frac{z^2}{2} + \frac{q_0 z}{2} + c_0\left(z-\frac{1}{2}\right) = -\frac{1}{2z_1z_2}(z-z_1)(z-z_2) \tag{129}$$

where,

$$c_0=1,\ c_1 = \left(1-\frac{1}{z_1}\right)\left(1-\frac{1}{z_2}\right), \quad q_0 = -2+\frac{1}{z_1}+\frac{1}{z_2} \tag{130}$$

Note that no generality is lost by scaling $c_0=1$ because, if $c_0=0$, then the operator does not have a pole at $z=0$. Employing the affine transformation

$$z = \frac{z_1(x+1)-z_2(1-x)}{2}$$

and setting

$$\alpha = \frac{2(z_1-1)z_1(2z_2-1)}{z_1-z_2}, \quad \beta = -\frac{2(2z_1-1)z_2(z_2-1)}{z_1-z_2}$$

and adding a constant term, transforms $2z_1z_2T_{23}^{(11)}$ into the operator

$$\hat{\mathcal{T}}_{\alpha,\beta}[y] = \mathcal{T}_{\alpha,\beta}[y] - 2(\log\xi)'(1-x^2)y'$$

where

$$\xi(x;\alpha,\beta) = P_2^{(-\alpha-1,-\beta-1)}(x) \tag{131}$$

$$= \frac{1}{4}\binom{2-\beta-\alpha}{2}(x-1)^2 + \frac{1}{2}(1-\beta-\alpha)(1-\alpha)(x-1) + \binom{\alpha}{2} \tag{132}$$

We require that $\xi(x;\alpha,\beta)$ have no zeros in the interval of orthogonality $x \in [-1,1]$. The above affine transformation maps $-1, 1$ to the roots of $\xi(x)$ and maps

$$z_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{-a(1+a+b)/b}, \quad a = \alpha - 1,\ b = \beta - 1 \tag{133}$$

$$z_2 = \frac{1}{2} \mp \frac{1}{2}\sqrt{-b(1+a+b)/a} \tag{134}$$

to ± 1. Therefore, an equivalent condition is that z_1, z_2 are either complex-valued or lie in the interval $(0,1)$. This constraint, toghether with the finite moments constraint, gives us 4 disjoint classes of acceptable parameter values:

(i) $\alpha, \beta > 1$;
(ii) $1 < \alpha < 3,\ -1 < \beta < 0,\ \alpha + \beta < 2$;
(iii) $1 < \beta < 3,\ -1 < \alpha < 0,\ \alpha + \beta < 2$;
(iv) $0 < \alpha, \beta < 1$

The exceptional polynomials and the adjoint intertwiners are shown below:

$$A[y] := \frac{y'}{P_2^{(-\alpha-1,-\beta-1)}(x)} \tag{135}$$

$$B[y] := (1-x)^{2+\alpha}(1+x)^{2+\beta}\mathcal{W}[\phi_2^{(4)}(x;\alpha+1,\beta+1), y] \tag{136}$$

$$\hat{P}_0^{(\alpha,\beta)} = 1 \tag{137}$$

$$\hat{P}_n^{(\alpha,\beta)} = B[P_{n-3}^{(\alpha+1,\beta+1)}], \quad n = 3,4,5,\ldots \tag{138}$$

$$\hat{\mathcal{T}}[\hat{P}_n] = -(n-2)(n-1+\alpha+\beta)\hat{P}_n \tag{139}$$

$$A[\hat{P}_n^{(\alpha,\beta)}] = -n(\alpha+n-3)P_{n-3}^{(\alpha+1,\beta+1)}, \quad n = 0,3,4,5,\ldots \tag{140}$$

6.3.4. *No Jacobi polynomials for the 2-pole flag* $\mathcal{E}_{12}^{(11)}$. The quadratic coefficient of y'' in (19d) factors as

$$(c_0 + c_1 - q_0)\frac{z^2}{2} + \left(\frac{q_0}{2} - c_1\right) - \frac{c_0}{2} = -\frac{1}{2z_1z_2}(z-z_1)(z-z_2) \tag{141}$$

where,

$$c_0 = 1,\ c_1 = \left(1 - \frac{1}{z_1}\right)\left(1 - \frac{1}{z_2}\right), \quad q_0 = 2 - \frac{1}{z_1} - \frac{1}{z_2} + \frac{2}{z_1z_2} \tag{142}$$

Note that no generality is lost by scaling $c_0 = 1$ because, if $c_0 = 0$, then the operator does not have a pole at $z = 0$. Employing the affine transformation

$$z = \frac{z_1(x+1) - z_2(1-x)}{2}$$

and setting

$$\alpha = -\frac{2(z_1-1)z_1(2z_2-1)}{z_1 - z_2}, \quad \beta = \frac{2(2z_1-1)z_2(z_2-1)}{z_1-z_2}$$

gives a weight of the form

$$\hat{W}(x;\alpha,\beta) = \frac{(1-x)^\alpha(1+x)^\beta}{\left(P_2^{(\alpha-1,\beta-1)}(x)\right)^2}$$

Since

$$z_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{a(1+a+b)/b}, \quad a = \alpha + 1,\ b = \beta + 1 \tag{143}$$

$$z_2 = \frac{1}{2} \mp \frac{1}{2}\sqrt{b(1+a+b)/a} \tag{144}$$

and since $\alpha, \beta > -1$ is required for finite moments, the roots z_1, z_2 are real, and one of them lies outside the interval $(0,1)$. Therefore, if $\alpha, \beta > -1$, the above weight must be singular on $x \in (-1,1)$.

6.3.5. *2-step Jacobi polynomials that span the 1-pole flag $\mathcal{E}_{13}^{(2a)}$.* The quadratic coefficient of y'' in (29a) factors as

$$\left((1-3a)\,(3-a)\,\frac{z^2}{4}+2\,(1-a)\,z+1\right)=\frac{1}{4}((a-3)z-2)((3a-1)z-2)$$

In order to have a Jacobi-type operator, we require $a\neq 3,1/3,-1$; in the latter case we obtain a perfect square. Applying the affine transformation

$$z=\frac{(x+1)}{a-3}-\frac{x-1}{3a-1}$$

yields the operator

$$\hat{\mathcal{T}}_a[y]:=\mathcal{T}_{\alpha,\beta}[y]-4(1-x^2)(\log\xi)'y'-\frac{8}{\xi(x;a)}y$$

where

$$\xi(x;a)=(1+a)x+2(a-1)$$

and where

$$\alpha=2+\frac{6}{a-3},\quad \beta=\frac{2}{3a-1}$$

Just as for the Laguerre-type polynomials, the corresponding weight involves a 4th order pole:

$$\hat{W}(x;a)=\frac{(1-x)^{\alpha}(1+x)^{\beta}}{\xi(x;a)^4}$$

In order to obtain a non-singular weight we must have $a>3$ or $a<1/3$. However, in order to have $\alpha,\beta>-1$ (finite moments), we must restrict the latter condition to $a<-1/3,\ a\neq-1$. The corresponding values of α,β range from $\alpha>2,\ 0<\beta<2$ in the former case, and $1/5<\alpha<2,\ -1<\beta<0,\ (\alpha,\beta)\neq(1/2,-1/2)$ in the latter case. Of course α,β are not independent, but rather are linked by the relation

$$4\alpha\beta+\beta-\alpha+2=0$$

The adjoint intertwiners and the exceptional polynomials for this flag and weight are shown below:

$$B[y]:=\frac{(1-x)^{2\alpha+6}(1+x)^{\beta+2}}{a(a-1)(1+3a)\xi(x;a)}\mathcal{W}[\phi_1^{(4)}(x;\alpha+2,\beta),\phi_2^{(2)}(x;\alpha+2,\beta),y] \tag{145}$$

$$A[y]:=\frac{(3a-1)^5(a-3)^3}{36(1+3a)\xi(x;a)^3}\mathcal{W}[(1+x)^{-\beta},2(1+a)(x-1)+(a-1)(3a-1),y] \tag{146}$$

$$\hat{P}_1(x;a)=2(1+a)(x-1)+(a-1)(3a-1) \tag{147}$$

$$\hat{P}_n(x;a):=B[P_{n-3}^{(\alpha,\beta)}],\quad n=3,4,5,\dots, \tag{148}$$

$$A[\hat{P}_n1]=(n-1)(n-3+\alpha)(n+\beta)(n-2+\alpha+\beta)P_{n-3}^{(\alpha,\beta)},\quad n=1,3,4,5,\dots \tag{149}$$

6.3.6. *2-step Jacobi polynomials that span the 1-pole flag $\mathcal{E}_{03}^{(2a)}$.* The quadratic coefficient of y'' in (29b) factors as

$$\left((3-q_0)\frac{z^2}{4}-2z+1\right)=\frac{(z-z_1)(z-z_2)}{z_1^2}$$

where

$$z_1,z_2=\frac{-4\pm2\sqrt{1+q_0}}{q_0-3},\quad z_2=\frac{z_1}{2z_1-1}.$$

Applying the affine transformation

$$z=\frac{z_1(x+1)}{2}-\frac{(x-1)z_2}{2},\quad z_1\neq z_2$$

yields the operator

$$\hat{\mathcal{T}}[y]:=\mathcal{T}_{\alpha,\beta}[y]-4(1-x^2)(\log\xi)'y'$$

where

$$\xi(x;z_1)=(z_1-1)x+z_1$$

and where

$$\alpha=\frac{3}{2}z_1-1,\quad \beta=\frac{3}{2}z_2-1,\quad 4\alpha\beta+\alpha+\beta-2=0$$

Just as for the Laguerre-type polynomials, the corresponding weight involves a 4th order pole:

$$\hat{W}(x;z_1) = \frac{(1-x)^\alpha(1+x)^\beta}{\xi(x;z_1)^4}$$

In order to obtain a non-singular weight we require $z_1 \neq z_2$ to have the same sign. This implies that $z_1 > 1/2,\ z_1 \neq 1$, which in turn implies that $\alpha, \beta > -1/4,\ \alpha, \beta \neq 1/2$ but are subject to the relation

$$4\alpha\beta + \alpha + \beta - 2 = 0$$

The finite moment condition is therefore automatically satisfied. The adjoint intertwiners and the exceptional polynomials for this flag and weight are shown below:

$$B[y] := \frac{(1-x)^{2\alpha+6}(1+x)^{\beta+2}}{P_1^{(-\alpha-2,\beta)}(x)} \mathcal{W}[\phi_2^{(4)}(x;\alpha+2,\beta), \phi_1^{(2)}(x;\alpha+2,\beta), y] \tag{150}$$

$$A[y] := \frac{2(1+\alpha)^3(1+x)^{2+\beta}}{(\beta-1)^2\alpha(\alpha-2)^2} \mathcal{W}[1, (1+x)^{-\beta}(1+\alpha+(x-1)\beta(1-2\alpha), y] \tag{151}$$

$$\hat{P}_0(x;z_1) = 1 \tag{152}$$

$$\hat{P}_n(x;z_1) := B[P_{n-3}^{(\alpha+2,\beta)}], \quad n = 3,4,5,\dots, \tag{153}$$

$$A[\hat{P}_n] = n(n-2+\alpha)(n-1+\beta)(n-3+\alpha+\beta)P_{n-3}^{(2+\alpha,\beta)}, \quad n = 0,3,4,5,\dots \tag{154}$$

6.3.7. *No Jacobi polynomials for the 1-pole flags* $\mathcal{E}_{02}^{(2a)}, \mathcal{E}_{12}^{(2a)}, \mathcal{E}_{23}^{(2b)}, \mathcal{E}_{23}^{(2c)}$. Setting

$$z_1, z_2 = \frac{-1 \pm \sqrt{1-p_0}}{p_0}$$

and applying the affine change of variables

$$z = \frac{z_1(x+1)}{2} - \frac{(x-1)z_2}{2}, \quad z_1 \neq z_2$$

transforms the operator in (29c) into Jacobi form. The corresponding weight is

$$\hat{W}(x;z_1) = \frac{(1-x)^\alpha(1+x)^\beta}{(x(z_1+1)+z_1)^4}$$

where

$$\alpha = -1 + \frac{3}{2}z_1, \quad \beta = -1 + \frac{3}{2}z_2$$

A non-singular weight requires that z_1, z_2 be real and have the same sign. Since

$$z_2 = \frac{-z_1}{2z_1+1}$$

the only possibility is that $z_1, z_2 < -1/2$. However, this means that $\alpha, \beta < -1$, which violates the finite moments condition.

By direct inspection of (29d) (29e), a Jacobi-type operator must have a singularity at $x = 0$. The coefficient of y'' in (29f) is a perfect square, which does not permit a Jacobi-type operator.

7. Summary and outlook

In the present paper we have given a classification of exceptional orthogonal polynomial systems of codimension two (X_2-OPS). The classification includes all the cases previously known in codimension two plus some new examples of exceptional polynomials. Among the new families, the one-pole flags are clearly special. Generically, the weight of a X_m-OPS is a rational modification of a classical weight with m double poles, and this is the case for all the families known to date. The Jacobi and Laguerre OPS that span the $\mathcal{E}^{(2a)}$ flag have codimension two but only one pole in their weight, with quadruple multiplicity. They also have one less free parameter than the usual Laguerre and Jacobi families, i.e. no free parameters for the $\mathcal{E}^{(2a)}$-Laguerre and just one free parameter for the $\mathcal{E}^{(2a)}$-Jacobi. The explanation for the presence of these exotic families is that generically they would belong to a higher-codimensional family, but that a careful tuning of the parameters can make the codimension drop by one and have two of the poles of the weight coalesce. Thus, the generic weight of an X_m-OPS is a classical weight divided by the square of a certain degree m polynomial $\xi(x)$ with simple roots that lie outside the interval of orthogonality, but we know that degenerate cases are also possible.

We have also shown that every X_2-OPS can be obtained from a classical OPS by a sequence of at most two Darboux transformations, and we conjecture this result to be true *mutatis mutandis* for any codimension m. Even if the conjecture could be proved to be true, the scheme of multiple step Darboux transformations is still very rich: there are four quasi-rational factorizing functions for the Laguerre and Jacobi families and two for the Hermite. The SL-OPS obtained by 1-step Darboux transformations have been studied in all cases, but multi-step Darboux transformations might mix factorizing functions of different kinds and all the possibilities have not yet been explored. It could also happen that even if the intermediate weights in a multi-step Darboux transformation are singular, the final weight will be regular. All cases when this happens have been studied for multi-step state-deleting Darboux transfomations in a more general Sturm-Liouville context (not necessarily polynomial) by Krein and Adler [1, 23]. A generalization of Krein-Adler's Theorem to multi-step isospectral transformations has been performed by Grandati [18], but the full characterization of SL-OPS obtainable via multi-step Darboux transformations of mixed type remains an open problem.

Another consequence of the conjecture is that all exceptional polynomials could be written as Wronskian determinants involving essentially classical orthogonal polynomials (more specifically, involving one classical polynomial and many quasi-rational factorizing functions).

Acknowledgements. The research of DGU was supported in part by MICINN-FEDER grant MTM2009-06973 and CUR-DIUE grant 2009SGR859. The research of NK was supported in part by NSERC grant RGPIN 105490-2011. The research of RM was supported in part by NSERC grant RGPIN-228057-2009.

References

[1] V. E. Adler, A modification of Crum's method, *Theor. Math. Phys.* **101** (1994) 1381–1386.

[2] S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, *Math. Z.* **29** (1929), 730-736.

[3] JF Cariñena, A. M. Perelomov, M. F. Rañada and M. Santander, A quantum exactly solvable nonlinear oscillator related to the isotonic oscillator, *J. Phys. A* **41** (2008) 085301.

[4] S. Y. Dubov, V. M. Eleonskii, and N. E. Kulagin, Equidistant spectra of anharmonic oscillators, *Sov. Phys. JETP* **75** (1992) 446–451 *Chaos* **4** (1994) 47–53.

[5] D. Dutta and P. Roy, Conditionally exactly solvable potentials and exceptional orthogonal polynomials, *J. Math. Phys.* **51** (2010) 042101.

[6] D. Dutta and P. Roy, Information entropy of conditionally exactly solvable potentials, *J. Math. Phys* **52** (2011) 032104.

[7] J. M. Fellows and R. A. Smith, Factorization solution of a family of quantum nonlinear oscillators, *J. Phys. A* **42** (2009) 335303.

[8] D. Gómez-Ullate, N. Kamran and R. Milson, Quasi-exact solvability and the direct approach to invariant subspaces. *J. Phys. A* 38(9) (2005) 2005–2019.

[9] D. Gómez-Ullate, N. Kamran and R. Milson, The Darboux transformation and algebraic deformations of shape-invariant potentials, *J. Phys. A* **37** (2004), 1789–1804.

[10] D. Gómez-Ullate, N. Kamran and R. Milson, Supersymmetry and algebraic Darboux transformations, *J. Phys. A* **37** (2004), 10065–10078.

[11] D. Gómez-Ullate, N. Kamran and R. Milson, Quasi-exact solvability in a general polynomial setting, *Inverse Problems*, **23** (2007) 1915–1942.

[12] D. Gómez-Ullate, N. Kamran, and R. Milson, An extension of Bochner's problem: exceptional invariant subspaces *J. Approx. Theory* **162** (2010) 987–1006.

[13] D. Gómez-Ullate, N. Kamran, and R. Milson, An extended class of orthogonal polynomials defined by a Sturm-Liouville problem , *J. Math. Anal. Appl.* **359** (2009) 352–367.

[14] D. Gómez-Ullate, N. Kamran, and R. Milson, Exceptional orthogonal polynomials and the Darboux transformation, *J. Phys. A* **43** (2010) 434016.

[15] D. Gómez-Ullate, N. Kamran, and R. Milson, Two-step Darboux transformations and exceptional Laguerre polynomials, *J. Math. Anal. Appl.*, **387** (2012), 410–418.

[16] D. Gómez-Ullate, N. Kamran, and R. Milson, On orthogonal polynomials spanning a non-standard flag, in *Algebraic aspects of Darboux transformations, quantum integrable systems and supersymmetric quantum mechanics*. P. Acosta-Humánez et al. Eds. *Contemp. Math.* **563** (2012) 51–72.

[17] A. González-López, N. Kamran and P.J. Olver, Normalizability of one-dimensional quasi-exactly solvable Schrdinger operators, *Comm. Math. Phys.* **153** (1993), no. 1, 117146.

[18] Y. Grandati, Multistep DBT and regular rational extensions of the isotonic oscillator, arXiv:1108.4503

[19] Y. Grandati, Solvable rational extensions of the isotonic oscillator , *Ann Phys.* **326** (2011) 2074–2090.

[20] F.A. Grünbaum and L. Haine, Orthogonal polynomials satisfying differential equations: the role of the Darboux transformation, *Symmetries an Integrability of Differential Equations*, CRM Proc. Lecture Notes, **9**, Amer. Math. Soc. Providence, RI, 1996, 143–154.

[21] C.-L Ho, Dirac(-Pauli), Fokker-Planck equations and exceptional Laguerre polynomials, *Annals of Physics*, **326**,(2011), 797-807.

[22] N. Kamran, and P.J. Olver, Lie algebras of differential operators and Lie-algebraic potentials, *J. Math. Anal. Appl.* **1**45 (1990), no. 2, 342356

[23] M.G. Krein, *Doklady acad. Nauk. CCCP*, **113** (1957) 970-973.
[24] P. Lesky, Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouvillesche Differentialgleichungen, *Arch. Rat. Mech. Anal.* **10** (1962), 341-352.
[25] C. Quesne, Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry, *J. Phys. A* **41** (2008) 392001–392007
[26] C. Quesne, Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics, SIGMA **5** (2009), 084, 24 pages.
[27] C. Quesne, Higher-order SUSY, exactly solvable potentials, and exceptional orthogonal polynomials, *Mod. Phys. Lett.* **A 26** (2011) 1843-1852.
[28] S. Odake and R. Sasaki, Infinitely many shape invariant potentials and new orthogonal polynomials, *Phys. Lett. B* **679** (2009) 414–417.
[29] S. Odake and R. Sasaki, Infinitely many shape invariant discrete quantum mechanical systems and new exceptional orthogonal polynomials related to the Wilson and Askey-Wilson polynomials, *Phys. Lett. B* **682** (2009), 130–136.
[30] S. Odake and R. Sasaki, Infinitely many shape invariant potentials and cubic identities of the Laguerre and Jacobi polynomials, *J. Math. Phys.* **51** (2010), 053513.
[31] S. Odake and R. Sasaki, Another set of infinitely many exceptional (X_m) Laguerre polynomials, *Phys. Lett. B* **684** (2010), 173–176.
[32] S. Odake and R. Sasaki, Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials, *Phys. Lett. B* **702** (2011), 164–170.
[33] R. Sasaki, S. Tsujimoto, and A. Zhedanov, Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations, *J. Phys. A* **43** (2010), 315204.
[34] A. Turbiner, Quasi-exactly-solvable problems and sl(2) algebra, *Comm. Math. Phys.* **118** (1988), no. 3, 467-474.

Departamento de Física Teórica II, Universidad Complutense de Madrid, 28040 Madrid, Spain

Department of Mathematics and Statistics, McGill University Montreal, QC, H3A 2K6, Canada

Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, B3H 3J5, Canada