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A Conservative Finite Element Method for the Korteweg-de Vries Equation

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Abstract. A finite element method for the 1-periodic Korteweg-de Vries equation

$$u_t + 2uu_x + u_{xxx} = 0$$

is analyzed. We consider first a semidiscrete method (i.e., discretization only in the space variable), and then we analyze some unconditionally stable fully discrete methods. In a special case, the fully discrete methods reduce to twelve point finite difference schemes (three time levels) which have second order accuracy both in the space and time variable.

1. Introduction. The purpose of this paper is to study a Galerkin-type method for the 1-periodic Korteweg-de Vries equation

$$\text{(1.1)} \qquad \begin{cases} u_t + 2uu_x + u_{xxx} = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

for $0 \leqslant t \leqslant T$, where $T > 0$ is a fixed real number. This equation arises for example as a model equation for unidirectional long waves in nonlinear dispersive media. For a discussion of this equation we refer the reader to Whitham [9] and references given there.

We derive the numerical method by writing the equation (1.1) in the conservative form

$$\text{(1.2)} \qquad u_t - w_x = 0,$$

where the flux w is given by

$$\text{(1.3)} \qquad w = -u_{xx} - u^2.$$

We insist that the conservation law (1.2) be satisfied pointwise, while the elliptic type relation (1.3) will be approximated by a Galerkin method.

We note that if we differentiate the relation (1.3) with respect to time, (1.2) and (1.3) imply the system

$$\text{(1.4)} \qquad \begin{cases} u_t - w_x = 0, \\ w_t + 2uw_x + w_{xxx} = 0. \end{cases}$$

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Our numerical method can be written either as an analog of the single equation (1.1) or of the system (1.4). In the first case, we only compute an approximation of the displacement u directly, while in the other case, the approximations of both u and w are simultaneously computed.

In Section 4 we derive error estimates for a semidiscrete version of this method, based on discretization in the space variable. If U and W denote the semidiscrete approximations of u and w, respectively, we prove estimates of the form ($I = (0, 1)$)

$$\|u - U\|_{L^2(I)} \leqslant ch^r,$$

and

$$\|w - W\|_{L^2(I)} \leqslant \begin{cases} ch^{r+1}, & \text{if } r \geqslant 3, \\ ch^2, & \text{if } r = 2, \end{cases}$$

where c is a constant depending on u, $r \geqslant 2$ is an integer and $h > 0$ is a small parameter indicating the space discretization. Here U and W are sought in a class of piecewise polynomials of degree less than r and $r + 1$, respectively.

In Section 5 we analyze some fully discrete schemes which have second order accuracy in time. For each time step a linear system of equations has to be solved. For one of the schemes considered, the coefficient matrix is a function of time, while another scheme has a coefficient matrix independent of time. All the schemes considered are unconditionally stable; i.e., no relation between h and the time step k need to be satisfied. This should be compared to the stability relation $kh^{-3} \leqslant c$, where c is a constant independent of k and h, which is usually required for explicit schemes for the equation (1.1) (see, for example, Fornberg and Whitham [5]).

The semidiscrete method is precisely formulated in Section 3 and in Section 6 we briefly discuss a closely related finite difference scheme.

Finally, we mention that a different semidiscrete method for the equation (1.1) was discussed by Wahlbin [8] and some numerical results for certain fully discrete versions of this scheme can be found in [2]. Spectral methods for the equation (1.1) have been proposed by Tappert [6] and Fornberg and Whitham [5]. Numerical results for some finite difference schemes are discussed by Vliegenthart [7].

Throughout this paper, c denotes a generic constant, not necessarily the same at different occurrences.

2. Notation and Preliminaries. On the space $L^2(I)$ let $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and norm, respectively. For any integer $m \geqslant 0$, H^m denotes the Sobolev space of 1-periodic functions on $\mathbf{R}$ with m derivatives in $L^2_{\text{loc}}(\mathbf{R})$, where the associated norm, $\|\cdot\|_m$, is given by

$$\|v\|_m = \left(\sum_{j=0}^{m} \left\| \left(\frac{\partial}{\partial x}\right)^j v \right\|^2 \right)^{1/2}.$$

If $m < 0$ is an integer, then H^m denotes the dual of H^{-m} with respect to the inner product $(\cdot, \cdot)$ with norm

$$\|v\|_m = \sup_{\substack{\varphi \in H^{-m} \\ \varphi \neq 0}} \frac{(v, \varphi)}{\|\varphi\|_{-m}}.$$

For any integer m take $\hat{H}^m$ to be the subspace of H^m consisting of functions of mean value zero; i.e.,

$$\hat{H}^m = \{v \in H^m \mid (v, 1) = 0\}.$$

If $v \in C(0, T; H^m)$, then let

$$|||v|||_m = \sup_{0 \leq t \leq T} \|v(t)\|_m.$$

Also, let $|\cdot|_\infty$ denote the norm in $L^\infty(I)$. All functions above are assumed to be real valued. We recall that $H^1 \subset L^\infty(I)$ with continuous injection; see for example [1]. This implies that there is a constant c such that

$$\|v\varphi\|_1 \leq c\|v\|_1 \|\varphi\|_1 \quad \text{for } v, \varphi \in H^1;$$

and hence,

$$\|v\varphi\|_{-1} \leq c\|v\|_{-1} \|\varphi\|_1 \quad \text{for } v \in H^{-1}, \varphi \in H^1.$$

We now recall that it was proved by Bona and Smith [3] that if the initial data u_0 of (1.1) is in H^m, $m \geq 2$, then there is a unique solution u of (1.1) and

$$\left(\frac{\partial}{\partial t}\right)^j u \in C(0, T; H^{m-3j}),$$

for each integer $j \geq 0$ such that $m - 3j \geq 0$. Furthermore, there is a constant $c(\|u_0\|_m)$ such that

$$\left|\left|\left|\left(\frac{\partial}{\partial t}\right)^j u\right|\right|\right|_{m-3j} \leq c(\|u_0\|_m). \tag{2.1}$$

3. The Semidiscrete Method. First, we introduce two classes of finite dimensional function spaces. For any $E \subset I$ let $P_j(E) = \{v: I \longrightarrow \mathbf{R} \mid v|_E \text{ is a polynomial of degree less than } j\}$. Now let Δ be a family of partitions of I; i.e., if $\delta \in \Delta$, then $\delta = \{x_i\}_{i=0}^M$, where $0 = x_0 < x_1 < \cdots < x_M = 1$. We shall use the notation $I_i = (x_{i-1}, x_i)$, $h = h(\delta) = \max_{1 \leq i \leq M}(x_i - x_{i-1})$ and $\widetilde{h} = \widetilde{h}(\delta) = \min_{1 \leq i \leq M}(x_i - x_{i-1})$. Throughout this paper we make the assumption that the family Δ is quasi-uniform in the sense that there is a constant c, independent of δ, such that

$$h \leq c\widetilde{h} \quad \text{for all } \delta \in \Delta. \tag{3.1}$$

For the rest of this paper we assume that r is a fixed integer ≥ 2. We now define two families of function spaces $\{S_\delta\}_{\delta \in \Delta}$ and $\{S_\delta^*\}_{\delta \in \Delta}$ by

$$S_\delta = \{\mu \in H^1 \mid \mu \in P_r(I_i);\ i = 1, 2, \ldots, M\},$$

and

$$S_\delta^* = \{\chi \in H^2 \mid \chi \in P_{r+1}(I_i);\ i = 1, 2, \ldots, M\}.$$

It is easy to see that $S_\delta \subset C(I)$, $S_\delta^* \subset C^1(I)$ and $\dim S_\delta = \dim S_\delta^* = M(r-1)$. We also observe that if $\chi \in S_\delta^*$, then $\chi_x \in S_\delta$.

Note that by (3.1) there is a constant c such that

$$\|\mu\|_1 \leqslant ch^{-1}\|\mu\| \quad \text{for } \mu \in S_\delta. \tag{3.2}$$

This property, which is usually referred to as an inverse property, follows easily by a homogeneity argument. Finally, we observe that the spaces S_δ and S_δ^* have the approximation properties that for any $\varphi \in H^m$

$$\inf_{\mu \in S_\delta} \sum_{j=0}^{1} h^j \|\varphi - \mu\|_j \leqslant ch^m \|\varphi\|_m, \qquad 1 \leqslant m \leqslant r, \tag{3.3}$$

and

$$\inf_{\chi \in S_\delta^*} \sum_{j=0}^{2} h^j \|\varphi - \chi\|_j \leqslant ch^m \|\varphi\|_m, \qquad 2 \leqslant m \leqslant r+1, \tag{3.4}$$

where c is a constant independent of h and φ.

Now let $P\colon L^2(I) \longrightarrow S_\delta$ denote the L^2-projection on S_δ. For an arbitrary $\varphi \in H^m$, $m \geqslant 1$, choose $\mu \in S_\delta$ such that (3.3) is satisfied. Then we obtain from (3.2) that

$$\|\varphi - P\varphi\|_1 \leqslant \|\varphi - \mu\|_1 + \|\mu - P\varphi\|_1 \leqslant ch^{m-1}\|\varphi\|_m, \tag{3.5}$$

for $1 \leqslant m \leqslant r$. In particular, it follows that

$$\|P\varphi\|_1 \leqslant c\|\varphi\|_1 \quad \text{for } \varphi \in H^1, \tag{3.6}$$

where c is independent of h and φ.

The semidiscrete method for (1.1) is now derived by seeking $U\colon [0, T] \longrightarrow S_\delta$ and $W\colon [0, T] \longrightarrow S_\delta^*$ as approximations of u and w, respectively, such that

$$U_t - W_x = 0, \tag{3.7}$$

where U and W are related by

$$(U_x, \mu_x) - (U^2, \mu) = (W, \mu) \quad \text{for } \mu \in S_\delta. \tag{3.8}$$

Here (3.7) is a pointwise relation while (3.8) is a Galerkin approximation of the elliptic type relation (1.3). We note that if $\chi \in S_\delta^*$, then it follows from (3.7) and (3.8) that

$$(U_t, \chi) = -(W, \chi_x) = -(U_x, \chi_{xx}) + (U^2, \chi_x).$$

Hence, U satisfies

$$\begin{cases} (U_t, \chi) - (U^2, \chi_x) + (U_x, \chi_{xx}) = 0 & \text{for } \chi \in S_\delta^*, \\ U(0) = U_0, \end{cases} \tag{3.9}$$

where the initial data $U_0 \in S_\delta$ has to be specified.

Remark 3.1. We note that (3.9) is a nonlinear system of ordinary differential equations. We also observe that the coefficient matrix in front of the time-derivative can be singular, since it can occur that $P(S_\delta^*)$ is strictly contained in S_δ. However, in order to analyze (3.9), we shall in Section 4 assume that $P(S_\delta^*) = S_\delta$ for all $\delta \in \Delta$. This condition is for example always satisfied when $r = 2$ and δ is a uniform partition of I, which partitions I into an odd number of subintervals. The assumption will be removed in Section 5, where we consider an implicit time stepping scheme for (3.9).

Now note that if the initial value problem (3.9) is solved then, for any $t \in [0, T]$, $W(t) \in S_\delta^*$ can be determined from the facts that

$$W_x = U_t \quad \text{and} \quad (W, 1) = -(U^2, 1).$$

However, if we are interested in approximations of the flux w, then the method above can also be written as a semidiscrete analog of the system (1.4), where both the variables U and W are computed directly. In order to see this we differentiate (3.8) with respect to time, and use (3.7) to obtain the system

$$\text{(3.10)} \qquad \begin{cases} U_t - W_x = 0, \\ (W_t, \mu) + 2(UW_x, \mu) - (W_{xx}, \mu_x) = 0 \quad \text{for } \mu \in S_\delta, \\ U(0) = U_0, \quad W(0) = W_0, \end{cases}$$

where $U_0 \in S_\delta$ and $W_0 \in S_\delta^*$ have to be specified.

We remark that if U, W is any solution of (3.10), where U_0 and W_0 are chosen such that (3.8) is satisfied, then U satisfies (3.9). However, in general, the equations (3.9) and (3.10) are not equivalent. In fact, if U, W solves (3.10) for arbitrary U_0 and W_0, then

$$\text{(3.11)} \qquad (U_t, \chi) - (U^2, \chi_x) + (U_x, \chi_{xx}) = (\theta, \chi_x) \quad \text{for } \chi \in S_\delta^*,$$

where $\theta \in S_\delta$ is given by

$$\text{(3.12)} \qquad (\theta, \mu) = ((U_0)_x, \mu_x) - ((U_0)^2, \mu) - (W_0, \mu) \quad \text{for } \mu \in S_\delta.$$

We note particularly that the function θ is independent of t.

The main difference between (3.9) and (3.11) is that (3.11) allows us to specify initial values for both U and W. As we shall see in Section 4, this extra flexibility gives the equation (3.11) certain advantages over (3.9).

4. Error Estimates for the Semidiscrete Method. In this section we derive error estimates for the semidiscrete method (3.9). At the end of this section we also obtain certain variants of these results for the method (3.11) (or equivalently (3.10)). In order to avoid some technical difficulties we make the assumption throughout this section that $P(S_\delta^*) = S_\delta$ for all $\delta \in \Delta$ (cf. Remark 3.1). We remark that the need for this assumption does not indicate limited application of the method, but it is an assumption which simplifies the analysis of the semidiscrete method.

Define an operator Λ: $L^2(I) \longrightarrow H^1$ by

$$(\Lambda\varphi)_x = \varphi - (\varphi, 1) \quad \text{and} \quad (\Lambda\varphi, 1) = (\varphi, 1).$$

We note that if $\mu \in S_\delta$, then $\Lambda\mu \in S_\delta^*$ and, if we let $\hat{S}_\delta = S_\delta \cap \hat{H}^1$ and $\hat{S}_\delta^* = S_\delta^* \cap \hat{H}^2$, then $\Lambda(\hat{S}_\delta) = \hat{S}_\delta^*$.

Also let $P_1\colon H^1 \longrightarrow S_\delta$ be the H^1-projection onto S_δ; i.e.,

$$(\varphi - P_1\varphi, \mu) + (\varphi_x - (P_1\varphi)_x, \mu_x) = 0 \quad \text{for } \mu \in S_\delta.$$

We observe that it follows from (3.3) and standard theory for Galerkin methods for elliptic equations (see for example [4]), that for any $\varphi \in H^s$

$$\|\varphi - P_1\varphi\|_{-p} \leqslant ch^{s+p}\|\varphi\|_s, \qquad -1 \leqslant p \leqslant r-2,\ 1 \leqslant s \leqslant r, \tag{4.1}$$

where c is a constant independent of h and φ.

We now define $V\colon [0, T] \longrightarrow S_\delta$ by $V(t) = (P_1u)(t)$, and we let $\rho(t) = u(t) - V(t)$. We note that it follows from (2.1) and (4.1) that, for any integer $j \geqslant 0$, there is a constant $c = c(\|u_0\|_{s+3j})$ such that

$$\left|\!\left|\!\left|\left(\frac{\partial}{\partial t}\right)^j \rho\right|\!\right|\!\right|_{-p} \leqslant ch^{s+p}, \qquad -1 \leqslant p \leqslant r-2,\ 1 \leqslant s \leqslant r. \tag{4.2}$$

Also, observe that V satisfies the equation

$$(V_t, \chi) - (V^2, \chi_x) + (V_x, \chi_{xx}) = ((u + V)\rho + \rho, \chi_x) - (\rho_t, \chi), \tag{4.3}$$

for $\chi \in S_\delta^*$.

Now let $e = u - U$ and $\eta = V - U$, where U is the solution of (3.9). Then $e = \rho + \eta$, and we note that (4.2) implies that e can be estimated by estimating η which is a function in S_δ. This will be done by comparing (3.9) with (4.3).

THEOREM 4.1. *Let s be an integer such that $1 \leqslant s \leqslant r$ and assume that $u_0 \in H^{s+6}$. Then there is a positive constant $c = c(\|u_0\|_{s+6})$ such that, if $\|\Lambda\eta(0)\|_2 \leqslant 1$ and $h \leqslant c^{-1}$, the initial value problem* (3.9) *has a unique solution U satisfying*

$$|\!|\!|e|\!|\!|_p \leqslant c\{h^{s-p} + \|\Lambda\eta(0)\|_2\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r). \tag{4.4}$$

Furthermore, if $u_0 \in H^{s+9}$, then there is a constant $c = c(\|u_0\|_{s+9})$ such that

$$|\!|\!|e_t|\!|\!|_p \leqslant c\{h^{s-p} + \|\Lambda\eta(0)\|_2 + \|\Lambda\eta_t(0)\|_2\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r). \tag{4.5}$$

Proof. We first prove the estimate (4.4). Observe that (3.9) is a system of ordinary differential equations which has a local solution for all initial values. Hence, the existence of the solution U on $[0, T]$ is established if we can show an a priori bound for $|\!|\!|U|\!|\!|_1$. Because of (2.1) it is, therefore, enough to show (4.4) under the assumption that U exists on $[0, T]$. Also, note that it suffices to show (4.4) under the assumption that there is a constant c, independent of h, such that

$$\sup_{0 \leqslant t \leqslant T} |U(t)|_\infty \leqslant |\!|\!|U|\!|\!|_0^{1/2}|\!|\!|U|\!|\!|_1^{1/2} \leqslant c. \tag{4.6}$$

Now let

$$q = \begin{cases} 0, & \text{if } r = 2, \\ 1, & \text{if } r \geqslant 3. \end{cases}$$

We note that in order to show (4.4), it is enough to show that

$$\|\|\eta\|\|_1 \leqslant c\{h^{s+q} + \|\Lambda\eta(0)\|_2\}. \tag{4.7}$$

Take $\xi = \Lambda\rho_t + \rho + (u + V)\rho$. Then, since $(\rho_t, 1) = 0$, it follows from (3.9) and (4.3) that η satisfies the equation

$$(\eta_t, \chi) - ((V + U)\eta, \chi_x) + (\eta_x, \chi_{xx}) = (\xi, \chi_x) \quad \text{for } \chi \in S_\delta^*. \tag{4.8}$$

The equation (4.8) will now be used to obtain error estimates for η. Take first $\chi = \Lambda\eta_t$ in (4.8). Note that $(\eta_t, 1) = 0$, which implies that $(\eta_t, \Lambda\eta_t) = 0$. Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\eta_x\|^2 = (\xi, \eta_t) + ((V + U)\eta, \eta_t).$$

By integration in time we, therefore, have for any $\bar{t} \in [0, T]$

$$\|\eta_x(\bar{t})\|^2 = \|\eta_x(0)\|^2 + 2\int_0^{\bar{t}} \{(\xi, \eta_t) + ((V + U)\eta, \eta_t)\}\, dt.$$

Since (4.2) implies that $\|(\partial/\partial t)^j \rho\|_{-1} \leqslant ch^{s+q}$, $j = 0, 1, 2$, we obtain by integration by parts in time that

$$\begin{aligned} 2\int_0^{\bar{t}} (\xi, \eta_t)\, dt &= 2(\xi, \eta)\Big|_0^{\bar{t}} - 2\int_0^{\bar{t}} (\xi_t, \eta)\, dt \\ &\leqslant \frac{1}{4}\|\eta_x(\bar{t})\|^2 + c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|_2^2 + \|\Lambda\eta(\bar{t})\|^2 + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\}. \end{aligned}$$

In order to estimate the second term above, observe that

$$((V + U)\eta, \eta_t) = 2(V\eta, \eta_t) - (\eta^2, \eta_t).$$

By (4.2), it follows that

$$\begin{aligned} 4\int_0^{\bar{t}} (V\eta, \eta_t)\, dt &= 2(V, \eta^2)\Big|_0^{\bar{t}} - 2\int_0^{\bar{t}} (V_t, \eta^2)\, dt \\ &\leqslant \frac{1}{4}\|\eta_x(\bar{t})\|^2 + c\left\{\|\Lambda\eta(0)\|_2^2 + \|\Lambda\eta(\bar{t})\|^2 + \int_0^{\bar{t}} \|\eta\|^2\, dt\right\}. \end{aligned}$$

Also, by (4.2) and (4.6) we have that

$$2\int_0^{t} (\eta^2, \eta_t)\, dt = \frac{2}{3}(\eta^2, 1)\Big|_0^{\bar{t}} \leqslant \frac{1}{4}\|\eta_x(\bar{t})\|^2 + c\{\|\Lambda\eta(0)\|_2^2 + \|\Lambda\eta(\bar{t})\|^2\}.$$

Hence, we have shown that there is a constant c such that, for any $\bar{t} \in [0, T]$

$$\|\eta_x(\bar{t})\|^2 - c\|\Lambda\eta(\bar{t})\|^2 \leqslant c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|_2^2 + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\}, \tag{4.9}$$

where c is independent of h and $\bar{t}$.

Next take $\chi = \Lambda P\Lambda\eta$ in (4.8). Then we have

$$-\frac{1}{2}\frac{d}{dt}\|P\Lambda\eta\|^2 - ((V + U)\eta, (\Lambda P\Lambda\eta)_x) + (\eta_x, (P\Lambda\eta)_x) = (\xi, (\Lambda P\Lambda\eta)_x). \tag{4.10}$$

First note that by (3.5)

$$(\eta_x, (P\Lambda\eta)_x) = (\eta_x, ((P-I)\Lambda\eta)_x) \leqslant ch\|\Lambda\eta\|_2^2.$$

By integrating (4.10) in time and using (3.6), (4.2), and (4.6) we, therefore, have

$$\|P\Lambda\eta(\bar{t})\|^2 \leqslant c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|^2 + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\},$$

for $0 \leqslant \bar{t} \leqslant T$. But now we observe that

$$\|P\Lambda\eta(\bar{t})\|^2 = \|\Lambda\eta(\bar{t})\|^2 - \|(I-P)\Lambda\eta(\bar{t})\|^2 \geqslant \|\Lambda\eta(\bar{t})\|^2 - ch^4\|\eta_x(\bar{t})\|^2;$$

and hence, we have

$$\|\Lambda\eta(\bar{t})\|^2 - ch^4\|\eta_x(\bar{t})\|^2 \leqslant c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\| + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\}. \tag{4.11}$$

By comparing this with (4.9) we now obtain that for h sufficiently small and $0 \leqslant \bar{t} \leqslant T$,

$$\|\Lambda\eta(\bar{t})\|_2^2 \leqslant c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|_2^2 + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\},$$

where c is independent of h and $\bar{t}$. Therefore, Gronwall's lemma implies that

$$|||\Lambda\eta|||_2 \leqslant c\{h^{s+q} + \|\Lambda\eta(0)\|_2\};$$

and this implies (4.7). Thus, (4.4) is established.

In order to prove (4.5), observe first that, in the same way as above, it is enough to show (4.5) under the assumption that

$$\sup_{0 \leqslant t \leqslant T} |U_t(t)|_\infty \leqslant c, \tag{4.12}$$

where c is independent of h.

First, differentiate (4.8) with respect to time. We then obtain

$$(\eta_{tt}, \chi) - (((V+U)\eta)_t, \chi_x) + (\eta_{xt}, \chi_{xx}) = (\xi_t, \chi_x) \quad \text{for } \chi \in S_\delta^*. \tag{4.13}$$

Hence, if we take $\chi = \Lambda\eta_{tt}$ and integrate (4.13) in time, then we have

$$\|\eta_{xt}(\bar{t})\|^2 = \|\eta_{xt}(0)\|^2 + 2\int_0^{\bar{t}} \{(\xi_t, \eta_{tt}) + (((V+U)\eta)_t, \eta_{tt})\}\, dt.$$

By integration by parts in time, we obtain as above that

$$2\int_0^{\bar{t}} (\xi_t, \eta_{tt})\, dt = 2(\xi_t, \eta_t)\Big|_0^{\bar{t}} - 2\int_0^{\bar{t}} (\xi_{tt}, \eta_t)\, dt$$

$$\leqslant \frac{1}{4}\|\eta_{xt}(\bar{t})\|^2 + c\left\{h^{2(s+q)} + \|\Lambda\eta_t(0)\|_2^2 + \|\Lambda\eta(\bar{t})\|^2 + \int_0^{\bar{t}} \|\Lambda\eta\|_2^2\, dt\right\}.$$

In order to estimate the second term above we note that

$$((V+U)\eta)_t = (V+U)\eta_t + 2V_t\eta - \eta\eta_t.$$

By integration by parts in time, we have

$$\int_0^{\bar t}((V+U)\eta_t, \eta_{tt})\,dt = \frac{1}{2}((V+U), (\eta_t)^2)\Big|_0^{\bar t} - \frac{1}{2}\int_0^{\bar t}((V+U)_t, (\eta_t)^2)\,dt,$$

$$\int_0^{\bar t}(V_t, \eta\eta_{tt})\,dt = (V_t, \eta\eta_t)\Big|_0^{\bar t} - \int_0^{\bar t}\{(V_{tt}, \eta\eta_t) + (V_t, (\eta_t)^2)\}\,dt,$$

and

$$\int_0^{\bar t}(\eta\eta_t, \eta_{tt})\,dt = \frac{1}{2}\int_0^{\bar t}(\eta, ((\eta_t)^2)_t)\,dt = \frac{1}{2}(\eta, (\eta_t)^2)\Big|_0^{\bar t} - \frac{1}{2}\int_0^{\bar t}((\eta_t)^3, 1)\,dt.$$

Therefore, by applying (4.2), (4.4), and (4.12), we obtain

$$2\int_0^{\bar t}(((V+U)\eta)_t, \eta_{tt})\,dt \leq \frac{1}{4}\|\eta_{xt}(\bar t)\|^2 + c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|_2^2 + \|\Lambda\eta_t(0)\|_2^2 + \|\Lambda\eta_t(\bar t)\|^2 + \int_0^{\bar t}\|\Lambda\eta\|_2^2\,dt\right\}.$$

Hence, we have shown that there is a constant c, independent of $\bar t \in [0, T]$ and h, such that

$$\|\eta_{xt}(\bar t)\|^2 - c\|\eta_t(\bar t)\|^2 \leq c\left\{h^{2(s+q)} + \|\Lambda\eta(0)\|_2^2 + \|\Lambda\eta_t(0)\|_2^2 + \int_0^{\bar t}\|\Lambda\eta_t\|_2^2\,dt\right\}.$$

If we now take $\chi = \Lambda P\Lambda\eta_t$ in (4.13) and proceed exactly as we did when we derived (4.11), then the inequality above leads to the fact that

$$\text{(4.14)} \qquad |||\Lambda\eta_t|||_2 \leq c\{h^{s+q} + \|\Lambda\eta(0)\|_2 + \|\Lambda\eta_t(0)\|_2\};$$

and hence, (4.2) implies (4.5). Finally, we note that since (3.9) has a local (in time) unique solution, the estimate (4.4) implies the existence of a unique solution for $0 \leq t \leq T$. □

Remark 4.1. Assume for a moment that we wanted to solve the equation (1.1) on the interval $I_L = (0, L)$ instead of I. A careful examination of the proof of Theorem 4.1 would show, that in this case the constants in (4.4) and (4.5) are independent of L. This is desirable if the periodic version of (1.1) on I_L, with L large, is to be used to approximate the pure initial value problem for (1.1).

Remark 4.2. A careful study of the proof above would also show that the regularity assumptions stated in Theorem 4.1 can be weakened when $r \geq 4$. In this case we only need, for example, $u_0 \in H^{s+5}$ in order to prove (4.4), and if we are interested only in the H^1-part of (4.4), then $u_0 \in H^{s+3}$ is sufficient.

Now let U be the solution of (3.9), and let W: $[0, T] \longrightarrow S_\delta^*$ be the corresponding approximation of the flux w determined by (3.7) and (3.8). Then W is uniquely determined by the fact that

$$W_x = U_t \quad \text{and} \quad (W, 1) = -(U^2, 1)$$

or

$$W = \Lambda U_t - (U^2, 1).$$

Hence, at any time $t \in [0, T]$, $W(t)$ can be computed from $U(t)$ and $U_t(t)$. As above, it also follows from (1.2) and (1.3) that

$$w = \Lambda u_t - (u^2, 1).$$

Therefore, if we let $f = w - W$, then we have

$$f = \Lambda e_t - ((u + U)e, 1) = \Lambda \rho_t + \Lambda \eta_t - ((u + U)e, 1);$$

and hence, it follows from (4.2), (4.4) and (4.14) that

$$\text{(4.15)} \quad |||f|||_p \leqslant c\{h^{r+1-p} + \|\Lambda\eta(0)\|_2 + \|\Lambda\eta_t(0)\|\}, \qquad 2 \geqslant p \geqslant \max(0, 3 - r),$$

where $c = c(\|u_0\|_{r+9})$.

If we at any time $t \in [0, T]$ have computed $W(t)$ as above, then this can again be used to obtain a higher-order approximation U_a of u. For any $t \in [0, T]$ define $U_a(t) \in S_\delta^*$ by

$$\text{(4.16)} \qquad ((U_a)_x, \chi_x) + (U_a, \chi) = (W + U + U^2, \chi) \quad \text{for } \chi \in S_\delta^*.$$

Note that U_a is uniquely determined by (4.16), and we have the following error estimate.

THEOREM 4.2. *Assume that* $u_0 \in H^{r+9}$, *let* U *be the solution of* (3.9), *and let* W *be the corresponding approximation of the flux* w. *If* U_a: $[0, T] \longrightarrow S_\delta^*$ *is defined by* (4.16), *then there is a constant* $c = c(\|u_0\|_{r+9})$ *such that*

$$|||u - U_a|||_0 \leqslant c\{h^{r+q} + \|\Lambda\eta(0)\|_2 + \|\Lambda\eta_t(0)\|_2\},$$

where

$$q = \begin{cases} 0 & \text{if } r = 2, \\ 1 & \text{if } r \geqslant 3. \end{cases}$$

Proof. Note that (1.3) implies that u and w satisfy the relation

$$-u_{xx} + u = w + u + u^2.$$

By comparing this with (4.16), and by using standard theory for Galerkin methods for elliptic equations (see [4]), we obtain that

$$\begin{aligned} |||u - U_a|||_0 &\leqslant c\{h^{r+1} + |||e + f + (u + U)e|||_{-1}\} \\ &\leqslant c\{h^{r+1} + |||e|||_{-1} + |||f|||_{-1} + |||u + U|||_1 |||e|||_{-1}\}; \end{aligned}$$

and hence, the desired result follows from (4.4) and (4.15). □

We note that if we choose $U_0 = P_1 u_0$, then Theorem 4.1 implies that

$$|||e|||_p \leqslant ch^{r-p}, \qquad 1 \geqslant p \geqslant \max(-1, 2 - r).$$

However, if we are interested in the estimate for e_t (or the estimates for f or $u - U_a$), then this choice of U_0 might not lead to a good estimate for $\|\Lambda\eta_t(0)\|_2$. One way to construct U_0, such that both $\|\Lambda\eta(0)\|_2$ and $\|\Lambda\eta_t(0)\|_2$ are small, is to first choose $W_0 \in S_\delta^*$ close to $w(0)$ and then take $U_0 \in S_\delta$ to be a solution of (3.8) such that

$(U_0, 1) = (u_0, 1)$. This leads to a nonlinear equation for U_0. It can be shown, by a contractive mapping argument, that this equation has a unique solution U_0, in a neighborhood of u_0, if $\|u_0\|_1$ is sufficiently small. Furthermore, U_0 can be approximated by a linearly convergent iterative process.

An alternative way of overcoming the problem of choosing initial values described above, is to use the equation (3.11) instead of (3.9). In the rest of this section let U denote the solution of (3.11) and as above let $e = u - U$.

THEOREM 4.3. *Let s be an integer such that $1 \leqslant s \leqslant r$ and assume that $u_0 \in H^{s+6}$. Furthermore, let $U_0 = P_1 u_0$ and $W_0 = \Lambda P_1 w_x(0)$. Then there is a positive constant $c = c(\|u_0\|_{s+6})$ such that, if $h \leqslant c^{-1}$, the equation* (3.11) *has a unique solution U satisfying*

$$|||e|||_p \leqslant ch^{s-p}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r). \tag{4.17}$$

Also, if $u_0 \in H^{s+9}$, then there is a constant $c = c(\|u_0\|_{s+9})$ such that

$$|||e_t|||_p \leqslant ch^{s-p}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r). \tag{4.18}$$

Proof. First note that since the equation (3.11) is independent of the mean value of θ, it is enough to show the results above for $W_0 = \Lambda P_1 w_x(0) + (w(0), 1)$. In this case, the right-hand side θ of (3.11) is determined by

$$(\theta, \mu) = (\rho(0), \mu) + ((u_0 + V(0))\rho(0), \mu) + (\Lambda(I - P_1)w_x(0), \mu), \tag{4.19}$$

for $\mu \in S_\delta$. We also note that (3.6) implies that there is a constant c such that

$$\|\psi\|_{-1} \leqslant c \sup_{\substack{\mu \in S_\delta \\ \mu \neq 0}} \frac{(\psi, \mu)}{\|\mu\|_1} \quad \text{for any } \psi \in S_\delta.$$

Hence, it follows from (4.1) and (4.19) that

$$\|\theta\|_{-1} \leqslant ch^{s+q} \quad \text{where } q = \begin{cases} 0 & \text{if } r = 2, \\ 1 & \text{if } r \geqslant 3. \end{cases}$$

The estimate (4.17) follows now by a trivial modification of the argument that led to (4.4), by considering θ as an extra error term. The estimate (4.18) follows exactly in the same way as (4.5), since $\theta_t = 0$. □

We finally note that the estimates in Theorem 4.3 again can be used to prove estimates analogous to (4.15) and Theorem 4.2.

5. A Second Order Discretization in Time. The purpose of this section is to analyze examples of fully discrete versions of the semidiscrete methods discussed in Section 4. We shall consider two implicit methods which have second order accuracy in time. First, we consider a method where the associated system of linear equations has a coefficient matrix which is independent of time; and then, at the end of this section, we consider a method where a new matrix has to be inverted for each time step. The first method is of course desirable; but unfortunately, we need some extra regularity assumptions on the initial data in order to establish the convergence of this method. For the second method, however, we prove convergence with essentially the same assumptions as in Theorem 4.1. Also, recall that in this section we do not assume that $P(S_\delta^*) = S_\delta$.

Let N be a positive integer, and let $k = T/N$. For $t = 0, k, \ldots, Nk$, we seek an approximation of u in the space S_δ. In order to formulate the method, we first introduce some notation. For any v defined at $t = 0, k, \ldots, Nk$, we let $v^n = v(nk)$ and

$$v^{n+1/2} = \tfrac{1}{2}(v^n + v^{n+1}), \qquad n = 0, 1, \ldots, N-1,$$

$$\tilde{v}^n = \tfrac{1}{2}(v^{n-1} + v^{n+1}), \qquad n = 1, 2, \ldots, N-1,$$

$$(D_+v)^n = \frac{1}{k}(v^{n+1} - v^n), \qquad n = 0, 1, \ldots, N-1,$$

$$(D_0v)^n = (D_+v)^{n-1/2} \equiv (v^{n+1} - v^{n-1})/2k, \qquad n = 1, 2, \ldots, N-1.$$

We first consider the following fully discrete analog of (3.9):

$$(5.1) \qquad \begin{cases} ((D_0U)^n, \chi) - ((U^2)^n, \chi_x) + (\tilde{U}_x^n, \chi_{xx}) = 0 \quad \text{for } \chi \in S_\delta^*, \\ U^1 = U_1, \qquad U^0 = U_0, \end{cases}$$

where $\{U^n\}_{n=0}^N \subset S_\delta$ and where the initial values U_0 and U_1 have to be specified. We observe that $\{U^n\}$ is uniquely determined by (5.1).

Now let V: $[0, T] \longrightarrow S_\delta$ be as in Section 4; i.e., $V(t) = P_1u(t)$, and let $\rho(t) = u(t) - V(t)$. We note that it follows from (2.1) and (4.1) that for any $j \geq 0$,

$$(5.2) \qquad \max_{0 \leq n \leq N-j} \|(D_+^j\rho)^n\|_{-p} \leq ch^{s+p}, \qquad -1 \leq p \leq r-2,\ 1 \leq s \leq r,$$

where $c = c(\|u_0\|_{s+3j})$.

Also note that $\{V^n\}_{n=0}^N$ satisfies the equation

$$(5.3) \qquad \begin{cases} ((D_0V)^n, \chi) - ((V^2)^n, \chi_x) + (\tilde{V}_x^n, \chi_{xx}) \\ \quad = ((u^n + V^n)\rho^n + \tilde{\rho}^n + F_1(u)^n, \chi_x) - ((D_0\rho)^n + F_2(u)^n, \chi) \quad \text{for } \chi \in S_\delta^*, \end{cases}$$

where $F_1(u)^n = \widetilde{(u^2)}^n - (u^2)^n$ and $F_2(u)^n = \tilde{u}_t^n - (D_0u)^n$.

In analogy with Section 4, let $e^n = u^n - U^n$ and $\eta^n = V^n - U^n$. The following theorem will be derived by using arguments which are closely related to the ones given in the proof of Theorem 4.1. The main difference is that we here simultaneously prove error estimates for $\{e^n\}$ and $\{(D_+e)^n\}$.

THEOREM 5.1. *Let s be an integer such that $1 \leq s \leq r$ and assume that $u_0 \in H^{\max(s+6,10)}$. Then there are positive constants $c = c(\|u_0\|_{s+6})$ and $h_0 = h_0(\|u_0\|_{10})$ such that, if $\|\Lambda\eta^0\|_2$, $\|\Lambda\eta^1\|_2$, $\|\Lambda(D_+\eta)^0\|_2$, $\|\Lambda(D_+\eta)^1\|_2 \leq 1$ and $h \leq h_0$, the solution U of* (5.1) *satisfies*

$$(5.4) \qquad \max_{0 \leq n \leq N} \|e^n\|_p \leq c\{k^2 + h^{s-p} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2\}, \qquad 1 \geq p \geq \max(-1, 2-r).$$

Furthermore, if $u_0 \in H^{s+9}$, *then there is a constant* $c = c(\|u_0\|_{s+9})$ *such that*

$$\max_{0 \leq n \leq N-1} \|(D_+e)^n\|_p \leq c\{k^2 + h^{s-p} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2 \tag{5.5}$$
$$+ \|\Lambda(D_+\eta)^0\|_2 + \|\Lambda(D_+\eta)^1\|_2\},$$
$$1 \geq p \geq \max(-1, 2 - r).$$

Proof. We first note that it is enough to prove the estimates above under the assumption that

$$\max_{0 \leq n \leq N} |U^n|_\infty, \max_{0 \leq n \leq N-1} |(D_+U)^n|_\infty \leq c, \tag{5.6}$$

for some constant c, independent of k and h. Now let

$$q = \begin{cases} 0 & \text{if } r = 2, \\ 1 & \text{if } r \geq 3. \end{cases}$$

Observe that in order to show (5.4), it is enough to show that

$$\max_{0 \leq n \leq N} \|\eta^n\|_1 \leq c\{k^2 + h^{s+q} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2\}. \tag{5.7}$$

We also observe that it follows from (5.1) and (5.3) that $\{\eta^n\}$ satisfies the difference equation

$$((D_0\eta)^n, \chi) - ((V^n + U^n)\eta^n, \chi_x) - (\tilde{\eta}_x^n, \chi_{xx}) = (\xi^n, \chi_x) \quad \text{for } \chi \in S_\delta^*, \tag{5.8}$$

where $\xi^n = \Lambda((D_0\rho)^n + F_2(u)^n) + (u^n + V^n)\rho^n + \tilde{\rho}^n + F_1(u)^n$.

In order to simplify the writing we now introduce the following notation:

$$\begin{cases} \alpha^n = \tfrac{1}{2}(\|\Lambda\eta^n\|^2 + \|\Lambda\eta^{n+1}\|^2), \\ \beta^n = \tfrac{1}{2}(\|\eta_x^n\|^2 + \|\eta_x^{n+1}\|^2), \\ \gamma^n = \tfrac{1}{2}(\|\Lambda\eta^n\|_2^2 + \|\Lambda\eta^{n+1}\|_2^2). \end{cases} \tag{5.9}$$

Now take $\chi = \Lambda(D_0\eta)^n$ in (5.8). We observe that (5.8) implies that $((D_0\eta)^n, 1) = 0$ and, hence, $((D_0\eta)^n, \Lambda(D_0\eta)^n) = 0$. We, therefore, obtain

$$\frac{1}{4k}(\|\eta_x^{n+1}\|^2 - \|\eta_x^{n-1}\|^2) = (\xi^n, (D_0\eta)^n) + ((V^n + U^n)\eta^n, (D_0\eta)^n);$$

and hence, by summing from $n = 1$ to $n = m$ $(1 \leq m \leq N - 1)$,

$$\beta^m - \beta^0 = 2k \sum_{n=1}^{m} \{(\xi^n, (D_0\eta)^n) + ((V^n + U^n)\eta^n, (D_0\eta)^n)\}.$$

By summation by parts we now have

$$2k \sum_{n=1}^{m} (\xi^n, (D_0\eta)^n) = (\xi^m, \eta^{m+1}) + (\xi^{m-1}, \eta^m) - (\xi^2, \eta^1) - (\xi^1, \eta^0)$$
$$- 2k \sum_{n=2}^{m-1} ((D_0\xi)^n, \eta^n).$$

Therefore, we obtain from (5.2) and Taylor's Theorem that

$$2k \sum_{n=1}^{m} (\xi^n, (D_0\eta)^n) \leqslant \frac{1}{4}\beta^m + c\left\{k^4 + h^{2(s+q)} + \gamma^0 + \alpha^m + k\sum_{n=2}^{m-2} \gamma^n\right\}.$$

Similarly, we also obtain from (5.6) that

$$2k \sum_{n=1}^{m} ((V^n + U^n)\eta^n, (D_0\eta)^n) = \sum_{n=1}^{m} ((V^n + U^n), \eta^{n+1}\eta^n - \eta^n\eta^{n-1})$$

$$= ((V^m + U^m), \eta^{m+1}\eta^m) - ((V^1 + U^1), \eta^1\eta^0)$$

$$- k\sum_{n=1}^{m-1} ((D_+(V + U))^n, \eta^{n+1}\eta^n)$$

$$\leqslant \frac{1}{4}\beta^m + c\left\{\gamma^0 + \alpha^m + k\sum_{n=2}^{m-2} \gamma^n\right\}.$$

(Note that we here particularly have used that $|(D_+U)^n|_\infty$ is uniformly bounded.) The estimates above imply that there is a constant c, independent of k, h, and m, such that

(5.10) $$\beta^m - c\alpha^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + k\sum_{n=2}^{m-2} \gamma^n\right\} \quad \text{for } 0 \leqslant m \leqslant N - 1.$$

Now observe that if we take $\chi = P\Lambda P\widetilde{\eta}^n$ in (5.8), then an argument analogous to the one that led to (4.11) implies that for $0 \leqslant m \leqslant N - 1$,

$$\alpha^m - ch^4\beta^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + k\sum_{n=2}^{m-2} \gamma^n\right\}.$$

Therefore, we obtain from (5.10) that, for h sufficiently small,

$$\gamma^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + k\sum_{n=2}^{m-2} \gamma^n\right\},$$

where c is independent of k, h, and m ($0 \leqslant m \leqslant N - 1$). Hence, by the discrete version of Gronwall's Lemma we have

$$\max_{0 \leqslant n \leqslant N-1} \gamma^m \leqslant c\{k^4 + h^{2(s+q)} + \gamma^0\}$$

or

(5.11) $$\max_{0 \leqslant n \leqslant N} \|\Lambda\eta^n\|_2 \leqslant c\{k^2 + h^{s+q} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2\};$$

and this implies (5.7). Thus, (5.4) is established.

In order to show (5.5), we note that if we apply the difference operator D_+ to Eq. (5.8), then we have

(5.12) $$((D_0D_+\eta)^n, \chi) - ((D_+(V^2 - U^2))^n, \chi_x) + ((D_+\widetilde{\eta}_x)^n, \chi_{xx}) = (D_+\xi^n, \chi_x) \quad \text{for } \chi \in S_\delta^*.$$

In analogy with the notation above, we now let α_t^n, β_t^n, and γ_t^n be defined by an analog of (5.9), where η^n is replaced by $(D_+\eta)^n$. We also note that

$$((D_+\tilde{\eta}_x)^n, (D_0D_+\eta_x)^n) = \frac{1}{4k}(\|(D_+\eta)^{n+1}\|^2 - \|(D_+\eta)^{n-1}\|^2).$$

Therefore, if we take $\chi = \Lambda(D_0D_+\eta)^n$ in (5.12) and sum from $n = 1$ to $n = m$ $(1 \leqslant m \leqslant N-2)$, then we have

$$\beta_t^m - \beta_t^0 = 2k \sum_{n=1}^{m} \{((D_+\xi)^n + (D_+(V^2 - U^2))^n, (D_0D_+\eta)^n)\}. \tag{5.13}$$

By summation by parts we now obtain in the same way as above that

$$2k \sum_{n=1}^{m} ((D_+\xi)^n, (D_0D_+\eta)^n)$$

$$\leqslant \frac{1}{4}\beta_t^m + c\left\{k^4 + h^{2(s+q)} + \gamma_t^0 + \alpha_t^m + k\sum_{n=2}^{m-2}\gamma_t^n\right\}.$$

In order to estimate the second term above we use the identity

$$(D_+(V^2 - U^2))^n = (V^{n+1/2} + U^{n+1/2})(D_+\eta)^n$$

$$+ 2(D_+V)^n\eta^{n+1/2} - \eta^{n+1/2}(D_+\eta)^n.$$

By summation by parts we now have

$$2k \sum_{n=1}^{m} ((V^{n+1/2} + U^{n+1/2})(D_+\eta)^n, (D_0D_+\eta)^n)$$

$$= (V^{m+1/2} + U^{m+1/2}, (D_+\eta)^{m+1}(D_+\eta)^m)$$

$$- (V^{3/2} + U^{3/2}, (D_+\eta)^1(D_+\eta)^0)$$

$$- 2k\sum_{n=2}^{m} ((D_0(V + U))^n, (D_+\eta)^n(D_+\eta)^{n-1}).$$

Similarly, we also have

$$2k \sum_{n=1}^{m} ((D_+V)^n\eta^{n+1/2}, (D_0D_+\eta)^n)$$

$$= ((D_+V)^m\eta^{m+1/2}, (D_+\eta)^{m+1}) + ((D_+V)^{m-1}\eta^{m-1/2}, (D_+\eta)^m)$$

$$- ((D_+V)^1\eta^{3/2}, (D_+\eta)^0) - ((D_+V)^2\eta^{5/2}, (D_+\eta)^1)$$

$$- k\sum_{n=2}^{m-1} (2(D_0D_+V)^n\eta^{n+1/2} + (D_+V)^{n+1}(D_0\eta)^{n+1}$$

$$+ (D_+V)^{n-1}(D_0\eta)^n, (D_+\eta)^n).$$

Finally, we have

$$2k \sum_{n=1}^{m} (\eta^{n+1/2}(D_+\eta)^n, (D_0D_+\eta)^n)$$

$$= (\eta^{m+1/2}, (D_+\eta)^m(D_+\eta)^{m+1}) - (\eta^{3/2}, (D_+\eta)^0(D_+\eta)^1)$$

$$- 2k \sum_{n=2}^{m} ((D_0\eta)^n, (D_+\eta)^{n-1}(D_+\eta)^n).$$

Therefore, it follows from (5.2), (5.6) and (5.11) that

$$2k \sum_{n=1}^{m} ((D_+(V^2 - U^2))^n, (D_0D_+\eta)^n)$$

$$\leqslant \frac{1}{4}\beta_t^m + c\left\{k^4 + h^{2(s+q)} + \gamma^0 + \gamma_t^0 + \alpha_t^m + k\sum_{n=2}^{m-2} \gamma_t^m\right\}.$$

Hence, we obtain from (5.13) that there is a constant c, independent of k, h, and m, such that

$$\beta_t^m - c\alpha_t^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + \gamma_t^0 + k\sum_{n=2}^{m-2} \gamma_t^n\right\}, \tag{5.14}$$

for $0 \leqslant m \leqslant N-2$.

By taking $\chi = \Lambda P\Lambda(D_+\widetilde{\eta})^n$ in (5.12) we obtain in the same way as above that

$$\alpha_t^m - ch^4\beta_t^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + \gamma_t^0 + k\sum_{n=2}^{m-2} \gamma_t^n\right\}.$$

By comparing this with (5.14) we have that, for h sufficiently small,

$$\gamma_t^m \leqslant c\left\{k^4 + h^{2(s+q)} + \gamma^0 + \gamma_t^0 + k\sum_{n=2}^{m-2} \gamma_t^n\right\}.$$

Therefore, the discrete version of Gronwall's Lemma again implies that

$$\max_{0\leqslant n\leqslant N-2} \gamma_t^n \leqslant c\{k^4 + h^{2(s+q)} + \gamma^0 + \gamma_t^0\}$$

or

$$\max_{0\leqslant n\leqslant N-1} \|\Lambda(D_+\eta)^n\|_2 \leqslant c\{k^2 + h^{s+q} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2 + \|\Lambda(D_+\eta)^0\|_2 + \|\Lambda(D_+\eta)^1\|_2\}, \tag{5.15}$$

where c is a constant independent of k and h. Since (5.2) and (5.15) imply (5.5), this completes the proof of the theorem. □

In the same way as in the semidiscrete case we can now use the computed approximation $\{U^n\}_{n=0}^N$ of the displacement u, in order to obtain an approximation of the flux w. Define $\{W^n\}_{n=1}^{N-1} \subset S_\delta^*$ by

$$W_x^n = (D_0 U)^n \quad \text{and} \quad (W^n, 1) = -((U^2)^n, 1)$$

or

$$W^n = \Lambda(D_0 U)^n - ((U^2)^n, 1).$$

Since it follows from (1.2) and (1.3) that $w^n = \Lambda u_t^n - ((u^2)^n, 1)$, we immediately obtain from (5.2), (5.4), (5.15) and Taylor's Theorem that

$$\max_{1 \leqslant n \leqslant N-1} \|f^n\| = c\{k^2 + h^{r+1-p} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2\} + \|\Lambda(D_+\eta)^0\|_2 + \|\Lambda(D_+\eta)^1\|_2, \tag{5.16}$$

for $2 \geqslant p \geqslant \max(0, 3 - r)$, where $f^n = w^n - W^n$ and $c = c(\|u_0\|_{r+9})$.

If we for any n, $1 \leqslant n \leqslant N - 1$, define $U_a^n \in S_\delta^*$ by

$$((U_a^n)_x, \chi_x) + (U_a, \chi) = (W^n + U^n + (U^2)^n, \chi) \quad \text{for } \chi \in S_\delta^*,$$

then it also follows, in the same way as we proved Theorem 4.2, that (with $c = c\|u_0\|_{r+9}$)

$$\|u^n - U_a^n\| \leqslant c\{k^2 + h^{r+q} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2 + \|\Lambda(D_+\eta)^0\|_2 + \|\Lambda(D_+\eta)^1\|_2\},$$

where

$$q = \begin{cases} 0 & \text{if } r = 2, \\ 1 & \text{if } r \geqslant 3. \end{cases}$$

We shall now discuss how to choose the initial values U_0 and U_1 in (5.1). Note that even if we are interested only in the estimate (5.4), Theorem 5.1 applies only if U_0 and U_1 are chosen such that $\|\Lambda(D_+\eta)^0\|_2$ and $\|\Lambda(D_+\eta)^1\|_2$ are uniformly bounded for all $\delta \in \Delta$. In the same way as it was indicated in Section 4, such initial values can be found by an iterative process if $\|u_0\|_1$ is sufficiently small. Here we shall instead consider an analog of Theorem 4.3; i.e., we shall consider a fully discrete version of (3.11).

First let, for any $\epsilon > 0$, $\bar{w}(\epsilon) = w(0) + \epsilon w_t(0)$. Now define $W_1 \in S_\delta^*$ by $W_1 = \Lambda P_1 \bar{w}_x(k)$ and $U_0, U_1 \in S_\delta$ by $U_0 = P_1 u_0$ and $U_1 = P_1 u_0 + kP_1 \bar{w}_x(k/2)$. Furthermore, let $\theta \in S_\delta$ be given by the following analog of (3.12):

$$(\theta, \mu) = ((U_0)_x + k(W_1)_{xx}, \mu_x) - (U_1^2, \mu) - (W_1, \mu) \quad \text{for } \mu \in S_\delta. \tag{5.17}$$

We shall consider the following fully discrete analog of (3.11):

$$\begin{cases} ((D_0 U)^n, \chi) - ((U^2)^n, \chi_x) + (\widetilde{U}_x^n, \chi_{xx}) = (\theta, \chi_x) & \text{for } \chi \in S_\delta^*, \\ U^1 = U_1, \quad U^0 = U_0. \end{cases} \tag{5.18}$$

We note that (5.17) and (5.18) imply that $(D_0 U)^1 = (W_1)_x$; and therefore,

$$\|(D_0\eta)^1\|_1 = \|P_1((D_0u)^1 - \bar{w}_x(k))\|_1$$
$$\leqslant \|(D_0u)^1 - u_t(k)\|_1 + \|w_x(k) - \bar{w}_x(k)\|_1 \leqslant ck^2.$$

Similarly, since $(D_+U)^0 = P_1\bar{w}_x(k/2)$, we also obtain

$$\|(D_+\eta)^0\|_1 \leqslant ck^2;$$

and therefore, there is a constant c, independent of k and h, such that

$$\|\Lambda(D_+\eta)^0\|_2,\ \|\Lambda(D_+\eta)^1\|_2 \leqslant ck^2.$$

Note also that $\eta^0 = 0$ and $\eta^1 = P_1\{\int_0^k w_x(s)\,ds - k\bar{w}_x(k/2)\}$; and hence,

$$\|\Lambda\eta^0\|_2,\ \|\Lambda\eta^1\|_2 \leqslant ck^2. \tag{5.19}$$

Finally, we observe that, if $1 \leqslant s \leqslant r$, then it follows in the same way as the corresponding result proved in Theorem 4.3 that

$$\|\theta\|_{-1} \leqslant c\{k^2 + h^{s+q}\}, \quad \text{where } q = \begin{cases} 0 & \text{if } r = 2, \\ 1 & \text{if } r \geqslant 3. \end{cases}$$

Therefore, the following theorem follows by a simple modification of the proof of Theorem 5.1.

THEOREM 5.2. *Let s be an integer such that $1 \leqslant s \leqslant r$, and assume that $u_0 \in H^{\max(s+6,10)}$. Furthermore, let $W_1 = \Lambda P_1\bar{w}_x(k)$, $U_0 = P_1u_0$ and $U_1 = P_1u_0 + kP_1\bar{w}_x(k/2)$. Then there are positive constants $c = c(\|u_0\|_{s+6})$ and $h_0 = h_0(\|u_0\|_{10})$ such that, if $h \leqslant h_0$, the solution U of* (5.18) *satisfies*

$$\max_{0 \leqslant n \leqslant N} \|e^n\|_p \leqslant c\{k^2 + h^{s-p}\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r).$$

Also, if $u_0 \in H^{s+9}$, then there is a constant $c = c(\|u_0\|_{s+9})$ such that

$$\max_{0 \leqslant n \leqslant N-1} \|(D_+e)^n\|_p \leqslant c\{k^2 + h^{s-p}\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r).$$

Finally, we consider a modification of the method (5.1), where a new matrix has to be inverted for each time step. In this case, the error estimate for $\{e^n\}$ can be obtained with the same regularity assumptions as in Theorem 4.1. The reason for this improved result is that the error estimate for $\{e^n\}$ can be proven directly without using results for $\{(D_+e)^n\}$.

We consider the following method:

$$\begin{cases} ((D_0U)^n, \chi) - \left(\frac{1}{3}(U^2)^n + \frac{2}{3}\tilde{U}^nU^n, \chi_x\right) + (\tilde{U}_x^n, \chi_{xx}) = 0 \quad \text{for } \chi \in S_\delta^*, \\ U^1 = U_1, \qquad U^0 = U_0, \end{cases} \tag{5.20}$$

where $\{U^n\}_{n=0}^N \subset S_\delta$. In the same way as above let $e^n = u^n - U^n$ and $\eta^n = V^n - U^n$. We then have the following convergence result.

THEOREM 5.3. *Let s be an integer such that $1 \leqslant s \leqslant r$, and assume that $u_0 \in H^{s+6}$. Then there is a positive constant $c = c(\|u_0\|_{s+6})$ such that, if $\|\Lambda\eta^0\|_2$, $\|\Lambda\eta^1\|_2 \leqslant 1$ and $k, h \leqslant c^{-1}$, the equation* (5.20) *has a unique solution such that*

$$\max_{0 \leqslant n \leqslant N} \|e^n\|_p \leqslant c\{k^2 + h^{s-p} + \|\Lambda\eta^0\|_2 + \|\Lambda\eta^1\|_2\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r).$$

Proof. As in the proof of Theorem 4.1, it is enough to show the estimate under the assumption that

$$\max_{0 \leqslant n \leqslant N} |U^n|_\infty \leqslant c, \tag{5.21}$$

where c is independent of k and h. It is also easy to see that (5.21) implies that the linear equations which define (5.20) have a unique solution for k sufficiently small. By comparing (5.20) with Eq. (1.1), we now obtain the following difference equation for $\{\eta^n\}$:

$$\begin{cases} ((D_0\eta)^n, \chi) + \frac{2}{3}((V^n + \widetilde{V}^n)\eta^n + V^n\widetilde{\eta}^n, \chi_x) \\ \qquad - \left(\frac{1}{3}(\eta^2)^n + \frac{2}{3}\eta^n\widetilde{\eta}^n, \chi_x\right) + (\widetilde{\eta}^n_x, \chi_{xx}) = (\xi^n, \chi_x), \end{cases}$$

for $\chi \in S_\delta^*$. Here ξ^n depends only on u and V. As in the proof of Theorem 5.1, we now obtain estimates for $\{\eta^n\}$ by energy methods. The only difference from the proof of the estimate (5.4) is that the term

$$k \sum_{n=1}^{m} \left(\frac{1}{3}(\eta^2)^n + \frac{2}{3}\eta^n\widetilde{\eta}^n, D_0\eta^n\right), \qquad 1 \leqslant m \leqslant N-1,$$

can be estimated without using results for $\{(D_+\eta)^n\}$. This follows since

$$\begin{aligned} k \sum_{n=1}^{m} &\left(\frac{1}{3}(\eta^2)^n + \frac{2}{3}\eta^n\widetilde{\eta}^n, (D_0\eta)^n\right) \\ &= \frac{1}{6}\sum_{n=1}^{m} ((\eta^2)^n\eta^{n+1} + (\eta^2)^{n+1}\eta^n - (\eta^2)^{n-1}\eta^n - (\eta^2)^n\eta^{n-1}, 1) \\ &= \frac{1}{6}((\eta^2)^m\eta^{m+1} + (\eta^2)^{m+1}\eta^m, 1) - \frac{1}{6}((\eta^2)^0\eta^1 + (\eta^2)^1\eta^0, 1). \end{aligned}$$

Therefore, we obtain from (5.21) that

$$k \sum_{n=1}^{m} \left(\frac{1}{3}(\eta^2)^n + \frac{2}{3}\eta^n\widetilde{\eta}^n, (D_0\eta)^n\right) \leqslant \frac{1}{4}\beta^m + c\{\gamma^0 + \alpha^m\},$$

where c is independent of k, h, and m and where we have adopted the notation from the proof of Theorem 5.1. The desired result now follows by simple modifications of the arguments given in that proof. □

We note that if we take $U_0 = P_1u_0$ and $U_1 = P_1u_0 + kP_1\bar{w}_x(k/2)$, then it follows from (5.19) and Theorem 5.3 that

$$\max_{0 \leqslant n \leqslant N} \|e^n\|_p \leqslant c\{k^2 + h^{s-p}\}, \qquad 1 \geqslant p \geqslant \max(-1, 2-r).$$

6. Finite Difference Methods. In this section we shall briefly discuss some finite difference methods for the equation (1.1), closely related to the method (5.1). Throughout this section we shall consider (5.1) when $r = 2$ and when δ is a uniform partition of I; i.e., $\delta = \{x_j\}_{j=0}^{M}$, where $x_j = jh$ and $h = 1/M$. In this case we let $U_j^n = U^n(jh)$, where $\{U^n\}_{n=0}^{N}$ is the solution of (5.1). In analogy with the notation in Section 5, we also let

$$\widetilde{U}_j^n = \frac{1}{2}(U_j^{n-1} + U_j^{n+1}) \quad \text{and} \quad D_0 U_j^n = \frac{1}{2k}(U_j^{n+1} - U_j^{n-1}).$$

We observe that since U^n is a piecewise linear function, U^n is determined by the nodal values U_j^n, $j = 1, 2, \ldots, M$.

The method (5.1) can be shown to be equivalent to the following finite difference scheme:

$$\text{(6.1)} \qquad \left\{ \begin{aligned} &\frac{1}{24}D_0 U_{j+2}^n + \frac{11}{24}D_0 U_{j+1}^n + \frac{11}{24}D_0 U_j^n + \frac{1}{24}D_0 U_{j-1}^n \\ &\quad + \frac{5}{12}\,\frac{U_{j+1}^n U_{j+1}^n - U_j^n U_j^n}{h} + \frac{1}{3}\,\frac{U_{j+2}^n U_{j+1}^n - U_j^n U_{j-1}^n}{2h} \\ &\quad + \frac{1}{4}\,\frac{U_{j+2}^n U_{j+2}^n - U_{j-1}^n U_{j-1}^n}{3h} + \frac{\widetilde{U}_{j+2}^n - 3\widetilde{U}_{j+1}^n + 3\widetilde{U}_j^n - \widetilde{U}_{j-1}^n}{h^3} = 0, \end{aligned} \right.$$

where U_j^1, U_j^0, $j = 1, 2, \ldots, M$, have to be specified and where U^n is extended periodically for $j \notin \{1, 2, \ldots, M\}$.

It now follows directly from Theorem 5.1, that the solution $\{U_j^n\}$ of (6.1) satisfies the convergence estimate

$$\text{(6.2)} \qquad \left(h \sum_{j=1}^{M} |u(jh, hk) - U_j^n|^2 \right)^{1/2} \leqslant c(k^2 + h^2), \qquad 0 \leqslant nk \leqslant T,$$

if the initial approximations are sufficiently accurate in the sense of Theorem 5.1.

By a finite difference argument, it can also be shown that the estimate (6.2) holds for the following slightly simpler difference scheme:

$$\frac{1}{2}(D_0 U_{j+1}^n + D_0 U_j^n) + \frac{(U_{j+1}^n)^2 - (U_j^n)^2}{h} + \frac{\widetilde{U}_{j+2}^n - 3\widetilde{U}_{j+1}^n + 3\widetilde{U}_j^n - \widetilde{U}_{j-1}^n}{h^3} = 0.$$

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