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A CONSTRAINED H^∞ SMOOTH OPTIMIZATION TECHNIQUE

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SUMMARY

In H^∞ optimal control the cost function is the maximum singular value of a transfer function matrix over a frequency range. The optimization is over all stabilizing controllers. In constrained H^∞ control the controllers typically have a fixed structure, perhaps conveniently parametrized in terms of a parameter vector. Also, there may be functional constraints involving singular values representing, for example, robustness requirements. Such problems are usually cast as non-smooth optimization problems.

In this paper we consider a general class of constrained H^∞ optimization problems and show that these problems can be approximated by a sequence of smooth optimization problems. Thus each of the approximate problems is readily solvable by standard optimization software packages such as those available in the NAG or IMSL library. The proposed approach via smooth optimization is simple in terms of mathematical content, easy to implement and computationally efficient.

KEY WORDS Constrained H^∞ optimization Smoothing LQG design Multivariable linear control system design Optimal robust control

1. INTRODUCTION AND BACKGROUND

A method of linear multivariable control system design which is of current interest is H^∞ optimal control.[1,2] The simplest class of such problems, for which elegant solutions exist, are known as one-block problems, which can be formulated as

$$\min_{\mathbf{Q} \in RH^\infty} \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty \tag{1}$$

Here $\mathbf{T}_{ij} \in R_p$, the class of rational proper transfer function matrices, and RH^∞ denotes the class of such which are stable. Also, for continuous-time transfer functions $\mathbf{X}(s)$,

$$\|\mathbf{X}\|_\infty = \max_{i,\omega} \{\sigma_i[\mathbf{X}(s)]_{s=j\omega}\} = \max_{\omega} \bar{\sigma}[\mathbf{X}(\omega)] = \{\max_{\omega} \bar{\lambda}[\mathbf{X}^*(\omega)\mathbf{X}(\omega)]\}^{1/2} \tag{2}$$

where σ_i denotes the ith singular value, $\bar{\sigma}$ the maximum singular value, $\bar{\lambda}$ the maximum eigenvalue and the superscript asterisk the conjugate transpose. For discrete time transfer functions $\mathbf{X}(z)$, not dealt with explicitly here, there is an advantage that the frequency range $|z| = 1$ is finite.

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results in presenting a solution method for the constrained H^∞ optimization problem. The details are given in Section 3.

In Section 2 we review stabilizing controller theory background material and define more precisely the class of optimization problems of interest. In Section 3 the constrained H^∞ control problem under consideration is approximated by a sequence of smooth optimization problems. In Section 4 some aspects of the computational procedure are discussed and a design example is considered to assess the merits of the approach.

2. STABILIZING CONTROLLER THEORY

Consider the feedback control schemes of Figure 1. In Figure 1(a) there is a nominal plant with transfer function matrix $\mathbf{P} \in R_p$ and controller $\mathbf{K} \in R_p$. It is known that the control loop is well posed and the controller $\mathbf{K}$ is stabilizing for $\mathbf{G} = \mathbf{P}_{22}$ if and only if

$$\begin{bmatrix} \mathbf{I} & -\mathbf{K} \\ -\mathbf{G} & \mathbf{I} \end{bmatrix}^{-1} \tag{4}$$

exists and belongs to RH^∞. The theory for the class of all stabilizing controllers of Reference 2 allows the parametrization of all stabilizing controllers $\mathbf{K}$ for $\mathbf{G}$ as a linear fractional map

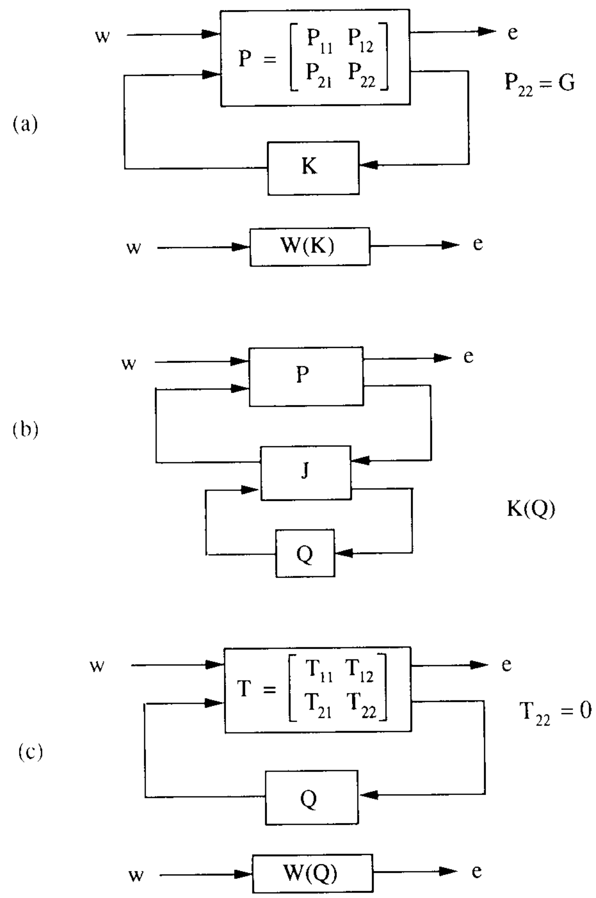

Figure 1. Stabilizing controller schemes

Elegant optimization techniques for (1) are available.[1–4] Also, these apply with modifications to the two-block and four-block problems, which are formulated as in (1) but with $\mathbf{Q}$ constrained as $[\mathbf{Q}_1\ \mathbf{0}]$, $[\mathbf{Q}_1^{\mathrm{T}}\ \mathbf{0}]$ or the $\mathbf{Q}$ block diagonal $\{\mathbf{Q}_1, \mathbf{0}\}$ respectively. Such methods fall short of dealing with other constraints, such as $\mathbf{Q}$ constrained as block diagonal $\{\mathbf{Q}_1, \mathbf{Q}_2\}$ or $\mathbf{Q}$ a constant.

In this paper we are concerned with the optimization task where additional constraints are included in the H^∞ optimization. These additional constraints may be in such a way that the controllers are of a fixed structure, perhaps conveniently parametrized in terms of a parameter vector. Also, they may be functional constraints involving singular values representing, for example, robustness requirements. Such problems cannot be solved by using the elegant techniques of References 1–4. It is well known that with these additional constraints, more complicated non-linear programming techniques must be employed and local minima are obtained rather global minima.

A computational technique has been developed for solving singular value inequalities over a continuum of frequencies.[5] The technique contains two parts: a master algorithm which constructs an infinite sequence of finite sets of inequalities, and a non-smooth subalgorithm which solves these finite sets of inequalities. When two singular values are close to being equal, it becomes difficult to compute the corresponding singular vectors with any precision. The case when two or more singular values are identical on an interval is excluded by assumption. In Reference 6 an optimization problem is considered where a differentiable cost functional is to be minimized subject to three kinds of constraints including singular value inequalities over a continuum of frequencies. An improved algorithm which overcomes the difficulties caused by singular vector computations is then proposed,[6] making use of outer approximations for problem decomposition and concepts of non-smooth optimization. Mathematically, these two papers are highly complex. Furthermore, they cannot make use of standard optimization software packages such as are available in the NAG or IMSL library.

The key contribution of Reference 7 is to give a simple yet efficient approach for solving functional inequality constrained optimization problems. The approach involves a new constraint transcription together with a local smoothing technique. On this basis a sequence of approximate optimization problems can be constructed. Each of these approximate problems can be viewed as a conventional optimization problem. The approach is very simple and can make use of existing optimization software packages. From numerical studies on simple problems the computational effort appears to be considerably less than using the non-smooth approach of References 8 and 9. It appears that gradient restoration algorithms[10,11] may also be used to solve problems of this type.

Our purpose in this paper is to demonstrate that, under reasonable assumptions, constrained H^∞ control problems and optimizations involving singular value constraints[5,6] can be tackled using existing smooth optimization software packages. This is achieved by using a well-known idea in functional analysis (see Reference 12 and the references cited therein) to approximate the objective functional (2) by

$$\| \bar{\sigma}[\mathbf{X}(\omega)] \|_{p,[0,c]} = \left\{ \int_0^c \{\bar{\lambda}[\mathbf{X}^*(\omega)\mathbf{X}(\omega)]\}^p \, \mathrm{d}\omega \right\}^{1/2p} \tag{3}$$

for an appropriate positive integer p and positive constant c. A simple proof is available for the convergence of $L_{p,[0,c]}$ to $L_{\infty,[0,c]}$ as $p \to \infty$.[12] In Reference 7 a technique has been developed to handle functional constraints in non-linear optimization. In this paper we use the above-mentioned convergence result, extend this technique[7] and develop a number of new

3. REFORMULATION AS A SMOOTH OPTIMIZATION PROBLEM

We reformulate the constrained H^∞ optimization problem as a smooth optimization problem in two steps. In step 1 we make a smooth approximation to the objective function using the L_p-norm. In step 2 the functional constraints are approximated by conventional constraints using a technique similar to that given in Reference 7. To begin, let us assume that the following conditions are satisfied.

(A1) $\mathbf{W}(\mathbf{x},s)$ is strictly proper (zero for $s=\infty$) with no poles on the jω-axis.
(A2) $\mathbf{W}(\mathbf{x},s)$ for each $\mathbf{x}$ has distinct singular values for almost all $s=\mathrm{j}\omega$ with no jω-axis poles.
(A3) $\mathbf{W}(\mathbf{x},s)$ is continuously differentiable with respect to $\mathbf{x}$ for almost all $s=\mathrm{j}\omega$.

Smooth approximation of the objective function

Consider the objective function given by (8a). By (2) it may be written as

$$\min_{\mathbf{x}\in\Xi}\|\mathbf{W}(\mathbf{x})\|_\infty=\left\{\min_{\mathbf{x}\in\Xi}\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x},\omega)\}\right\}^{1/2} \tag{9}$$

where $\bar{\lambda}(\mathbf{x},\omega)$ denotes the maximum eigenvalue of $\mathbf{W}^*(\mathbf{x},\omega)\mathbf{W}(\mathbf{x},\omega)$.

Lemma 1

Under assumption (A1), there exists an $\mathbf{x}^*\in\Xi$ and an $\omega^*\in[0,\infty)$ such that

$$\min_{\mathbf{x}\in\Xi}\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x},\omega)\}=\bar{\lambda}(\mathbf{x}^*,\omega^*)<\infty$$

Proof. By assumption (A1) it follows that for each $\mathbf{x}\in\Xi$, $\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x},\omega)\}$ is attained at some finite ω. Let $\{\mathbf{x}^i\}$ be a sequence in Ξ such that

$$f_i=\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x}^i,\omega)\}\rightarrow\bar{f}=\min_{\mathbf{x}\in\Xi}\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x},\omega)\}$$

Since Ξ is compact, $\{\mathbf{x}^i\}$ has a convergent subsequence, denoted again by the original sequence, which converges, say, to $\mathbf{x}^*\in\Xi$. By the above there exists some $\omega^*\in[0,\infty)$ such that

$$\bar{\lambda}(\mathbf{x}^*,\omega^*)=\max_{\omega\geqslant 0}\{\bar{\lambda}(\mathbf{x}^*,\omega)\}$$

Since for all $i=1,2,\ldots$ and for all $\omega\geqslant 0$

$$\bar{\lambda}(\mathbf{x}^i,\omega)\leqslant f_i$$

it follows that for all $i=1,2,\ldots$

$$\bar{\lambda}(\mathbf{x}^i,\omega^*)\leqslant f_i$$

Therefore, by virtue of the continuity of the function f,

$$\bar{\lambda}(\mathbf{x}^*,\omega^*)\leqslant\lim_{i\to\infty}f_i=\bar{f}$$

This in turn implies the conclusion of Lemma 1.

$\mathbf{K}(\mathbf{Q})$ in terms of arbitrary $\mathbf{Q} \in RH^{\infty}$ (see Figure 1(b)). The matrix $\mathbf{J} \in R_p$ (non-unique) is readily calculated from $\mathbf{G}$ and any stabilizing proper controller $\mathbf{K}$ for $\mathbf{G}$. From $\mathbf{P}$ and $\mathbf{J}$ of Figure 1(b) a $\mathbf{T} \in R_p$ can be constructed allowing Figure 1(b) to be re-organized as in Figure 1(c). A key property, namely $\mathbf{T}_{22} = \mathbf{0}$, ensures that closed transfer functions are affine in $\mathbf{Q}$. The stabilization theory tells us that

$$\{\mathbf{K} \text{ stabilizes } \mathbf{P}\} \Leftrightarrow \{\mathbf{Q} \text{ stabilizes } \mathbf{T}\} \tag{5}$$

Now, again referring to Figure 1, let us denote the transfer function matrix from the disturbance $\mathbf{w}$ to the disturbance response $\mathbf{e}$ as $\mathbf{W}$. Since this transfer function is $\mathbf{K}$- or $\mathbf{Q}$-dependent, we also use the notation $\mathbf{W}(\mathbf{K})$ or $\mathbf{W}(\mathbf{Q})$ as appropriate. A standard formulation of the H^{∞} optimization task is

$$\min_{\text{stabilizing } \mathbf{K} \text{ for } \mathbf{P}} \|\mathbf{W}(\mathbf{K})\|_{\infty} \tag{6}$$

The equivalent task under (5) and the relationships

$$\mathbf{T}_{22} = \mathbf{0}, \qquad \mathbf{W}(\mathbf{K}) = \mathbf{W}(\mathbf{Q}) = \mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21} \tag{7}$$

is the derivative task (1), for which elegant solutions exist. It turns out that with $\mathbf{P}$ having McMillan degree n, the optimal $\mathbf{Q}$ leads, via the linear fractional map $\mathbf{K}(\mathbf{Q})$, to a controller of degree $n-1$.[4] Since $\mathbf{W}(\mathbf{Q})$ is affine in $\mathbf{Q}$ but $\mathbf{W}(\mathbf{K})$ is not, performing H^{∞} designs using constrained optimization techniques as proposed in this paper would be simplified by working with $\mathbf{W}(\mathbf{Q})$ rather than $\mathbf{W}(\mathbf{K})$ formulations. Certainly, this is the viewpoint taken in the $\mathbf{Q}$ design methods by Boyd *et al.*[13] and in the adaptive $\mathbf{Q}$ methods by Tay and Moore.[14,15] We stress, however, that the techniques of this paper apply equally well to $\mathbf{W}(\mathbf{K})$ formulations as to $\mathbf{W}(\mathbf{Q})$ formulations.

In practice the controller class may be specified with more restrictions than merely that $\mathbf{K}$ is stabilizing for $\mathbf{P}$. Let us consider the case when the structure of $\mathbf{K}$ is specified in terms of parameters $\mathbf{x} \in \Xi$ with Ξ a compact subset of $\mathbb{R}^l$. It may be that the parametrization is on $\mathbf{Q}$ and thereby on $\mathbf{K}$ since $\mathbf{K}(\mathbf{x}) = \mathbf{K}[\mathbf{Q}(\mathbf{x})]$. Also, in practice there may be frequency domain constraints on the closed-loop system behaviour other than mere stability. They may involve functional constraints on singular values. These motivate for us the following class of constrained H^{∞} optimization problems.

Constrained H^{∞} optimization problem

$$\min_{\mathbf{x} \in \Xi} \|\mathbf{W}(\mathbf{x})\|_{\infty}, \quad \mathbf{W}(\mathbf{x}) \in R^p \tag{8a}$$

subject to the constraints

(i) $$h_j(\mathbf{x}) \leqslant 0, \quad j = 1, 2, \ldots, N \tag{8b}$$

(ii) $$\max_{\omega \in \Omega} \phi_j(\mathbf{x}, \omega) \leqslant 0, \quad j = 1, \ldots, M \tag{8c}$$

where $\Omega = [0, \infty)$ and $h_j : \mathbb{R}^l \to \mathbb{R}$, $\phi_j : \mathbb{R}^l \times \mathbb{R} \to \mathbb{R}$. Notice that frequency domain constraints such as stability constraints, robustness constraints or frequency response constraints can be handled by (8c). These constraints can involve singular values in multivariable problems.[4] For convenience, let the constrained H^{∞} optimization problem (8) be referred to as Problem P.

Combining (12), (14) and (16) gives

$$|\omega_{ij}(\mathbf{x}, \omega)| \leqslant \frac{2\alpha(n+1)}{\beta\omega} \tag{17}$$

for all $\mathbf{x} \in \Xi$ and all $\omega \geqslant \bar{\omega} = 1 + 2\bar{\beta}$. The constants α, β and $\bar{\omega}$ all depend on the values of i and j. However, if we define

$$\alpha^* = \max_{i,j}\{\alpha\}, \qquad \beta^* = \min_{i,j}\{\beta\}, \qquad \bar{c} = \max_{i,j}\{\bar{\omega}\}$$

then from (11) and (17)

$$\operatorname{tr}\{\mathbf{W}^*(\mathbf{x}, \omega)\mathbf{W}(\mathbf{x}, \omega)\} \leqslant \frac{4q^2(\alpha^*)^2(n+1)^2}{\beta^2\omega^2} \tag{18}$$

for all $\mathbf{x} \in \Xi$ and all $\omega \geqslant \bar{c}$. This proves Lemma 2.

Theorem 1

Under assumptions (A1) and (A2) there exists a $c \in (0, \infty)$, independent of $\mathbf{x} \in \Xi$, such that

$$\max_{\omega \geqslant 0}\{\bar{\lambda}(\mathbf{x}, \omega)\} = \max_{0 \leqslant \omega \leqslant c}\{\bar{\lambda}(\mathbf{x}, \omega)\} \tag{19}$$

for each $\mathbf{x} \in \Xi$.

Proof. Clearly

$$\bar{\lambda}(\mathbf{x}, \omega) \leqslant \operatorname{tr}\{\mathbf{W}^*(\mathbf{x}, \omega)\mathbf{W}(\mathbf{x}, \omega)\}$$

By Lemma 1 and assumption (A2) we note that

$$\min_{\mathbf{x} \in \Xi} \max_{\omega \geqslant 0}\{\bar{\lambda}(\mathbf{x}, \omega)\} = \bar{\lambda}(\mathbf{x}^*, \omega^*) > 0$$

Let

$$\bar{\lambda}(\mathbf{x}^*, \omega^*) = \alpha \tag{20}$$

Hence by Lemma 2 there exist positive constants $\varkappa$ and $\bar{c}$ such that

$$\bar{\lambda}(\mathbf{x}, \omega) \leqslant \varkappa/\omega^2$$

for all $\omega > \bar{c}$ and $\mathbf{x} \in \Xi$. With c defined by

$$c = \max\{\bar{c}, \sqrt{(\varkappa/\alpha)}\} \tag{21}$$

we than have

$$\bar{\lambda}(\mathbf{x}, \omega) \leqslant \alpha \tag{22}$$

for all $\omega > c$ and $x \in \Xi$. Combining (20) and (22), we obtain the conclusion of Theorem 1.

Following from Theorem 1, we now consider the problem

$$\min_{\mathbf{x} \in \Xi} \|\bar{\sigma}(\mathbf{x}, \omega)\|_{p,[0,c]} = \min_{\mathbf{x} \in \Xi}\left(\int_0^c \{\bar{\lambda}(\mathbf{x}, \omega)\}^p \, \mathrm{d}\omega\right)^{1/2p} \tag{23a}$$

subject to the constraints

(i) $$h_j(\mathbf{x}) \leqslant 0, \quad j = 1, 2, \ldots, N \tag{23b}$$

Lemma 2

Under assumptions (A1) and (A2) there exist positive constants $\varkappa$ and $\bar{c}$ such that

$$\operatorname{tr}\{\mathbf{W}^*(\mathbf{x},\omega)\mathbf{W}(\mathbf{x},\omega)\} \leqslant \varkappa/\omega^2 \tag{10}$$

for all $\omega > \bar{c}$ and $\mathbf{x} \in \Xi$.

Proof. Let $w_{ij}(\mathbf{x},\omega)$ be the (i,j) entry of the $q \times q$ matrix $\mathbf{W}(\mathbf{x},\omega)$. Then

$$\operatorname{tr}\{\mathbf{W}^*(\mathbf{x},\omega)\mathbf{W}(\mathbf{x},\omega)\} = \sum_{i=1}^{q}\sum_{j=1}^{q} |w_{ij}(\mathbf{x},\omega)|^2 \tag{11}$$

remembering that for each $\mathbf{x}$ and ω the $w_{ij}(\mathbf{x},\omega)$ are complex numbers.

Choose arbitrary i and j with $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant q$. By assumption (A1) it follows that there exists positive integers n and m, with $m \geqslant n+1$, and coefficients $\{a_i\}_{i=0}^{n}$ and $\{b_j\}_{j=0}^{m}$, with $b_m \neq 0$ and $b_0 \neq 0$, such that

$$w_{ij}(\mathbf{x},\omega) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0} \tag{12}$$

where $s = \mathrm{j}\omega$ and the coefficients $\{a_i\}_{i=0}^{n}$ and $\{b_j\}_{j=0}^{m}$ are functions of $\mathbf{x}$. Now

$$|a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0| \leqslant |a_n|\omega^n + |a_{n-1}|\omega^{n-1} + \cdots + |a_1|\omega + |a_0| \tag{13}$$

Since Ξ is a compact set, there exist positive constants α_k, $k = 0, 1, \ldots, n$, such that $|a_k| \leqslant \alpha_k$ for all $\mathbf{x} \in \Xi$. Let

$$\alpha = \max_{0 \leqslant k \leqslant n} \{\alpha_k\}$$

Then for $\omega \geqslant 1$ and all $\mathbf{x} \in \Xi$

$$|a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0| \leqslant \alpha(n+1)\omega^n \tag{14}$$

Also

$$\begin{aligned} |b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0| &\geqslant |b_m|\omega^m - |b_{m-1}|\omega^{m-1} - \cdots - |b_1|\omega - |b_0| \\ &= |b_m|\omega^m\left(1 - \frac{|b_{m-1}|}{|b_m|}\omega^{-1} - \cdots - \frac{|b_1|}{|b_m|}\omega^{-m+1} - \frac{|b_0|}{|b_m|}\omega^{-m}\right) \end{aligned} \tag{15}$$

Now, $b_m \neq 0$ and so there exist constants $\beta > 0$ and $\beta_l > 0$, $l = 0, 1, \ldots, m-1$, such that for all $\mathbf{x} \in \Xi$

$$|b_m| \geqslant \beta \quad \text{and} \quad \frac{|b_l|}{|b_m|} \leqslant \beta_l, \quad l = 0, 1, \ldots, m-1$$

Let

$$\bar{\beta} = \max_{0 \leqslant l \leqslant m-1} \{\beta_l\}$$

Then from the above

$$\begin{aligned} |b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0| &\geqslant \beta\omega^m[1 - \bar{\beta}(\omega^{-1} + \omega^{-2} + \cdots + \omega^{-m})] \\ &\geqslant \beta\omega^m\left(1 - \frac{\bar{\beta}}{\omega - 1}\right) \\ &\geqslant \tfrac{1}{2}\beta\omega^m \end{aligned} \tag{16}$$

for all $\mathbf{x} \in \Xi$ provided $\omega \geqslant 1 + 2\bar{\beta}$.

We shall establish conclusions (ii) and (iii) together. Recall that $\mathbf{x}^* \in \mathscr{F}$. Thus $\phi(\mathbf{x}^*) \geqslant m$. Furthermore, it is clear that $\phi_{p(i)}(\mathbf{x}^{p(i)}) \to \phi(\mathbf{x}^*)$ as $i \to \infty$. Hence

$$\lim_{i \to \infty} \phi_{p(i)}(\mathbf{x}^{p(i)}) \geqslant m$$

But

$$\lim_{i \to \infty} \phi_{p(i)}(\mathbf{x}^{p(i)}) \leqslant m$$

Thus we obtain

$$\lim_{i \to \infty} \phi_{p(i)}(\mathbf{x}^{p(i)}) = \phi(\mathbf{x}^*) = m$$

This in turn implies conclusions (ii) and (iii). Hence the proof of Theorem 2 is complete.

Let

$$f_p(\mathbf{x}) = \int_0^c \{\bar{\lambda}(\mathbf{x}, \omega)\}^p \, \mathrm{d}\omega \tag{25}$$

$f_p(\mathbf{x})$ can then be considered as the objective function for Problem P_p. The function $\bar{\lambda}(\mathbf{x}, \omega)$ is a non-differentiable function of $\mathbf{x}$ with the points of non-differentiability corresponding to two or more eigenvalues coming together. This has often led to the development of specialized non-smooth optimization algorithms for minimizing functions of this type. By assumption (A2) it follows that the eigenvalues of $\mathbf{W}^*(\mathbf{x}, \omega)\mathbf{W}(\mathbf{x}, \omega)$ are distinct except possibly at discrete values of ω. Therefore the functions $f_p(\mathbf{x})$ are differentiable functions for all $\mathbf{x} \in \Xi$. This means that standard optimization software can be used to minimize $f_p(\mathbf{x})$. The gradient of $f_p(x)$ is given by

$$\nabla_{\mathbf{x}} f_p = p \int_0^c \{\bar{\lambda}(\mathbf{x}, \omega)\}^{p-1} \nabla_{\mathbf{x}} \bar{\lambda}(\mathbf{x}, \omega) \, \mathrm{d}\omega \tag{26}$$

where

$$\nabla_{\mathbf{x}} \bar{\lambda} = 2\mathrm{Re}(\bar{\mathbf{v}}^* \mathbf{W}^* \nabla_{\mathbf{x}} \mathbf{W} \bar{\mathbf{v}}) \tag{27}$$

with $\bar{\mathbf{v}}$ being the right singular vector corresponding to the maximum singular value $\bar{\sigma}$ of $\mathbf{W}(\mathbf{x}, \omega)$.

For the non-generic case when assumption (A2) fails to hold, it appears that subgradient optimization techniques must be employed as in generalizing Reference 5 to Reference 6. Non-generic cases are of mathematical interest and for coping with possible ill-conditioning when singular values are close over a frequency range. These will be the subject of another paper.

Approximation of functional constraints

Consider the functional constraints (23c). Clearly they are equivalent to

$$G_j(\mathbf{x}) = \int_\Omega g_j(\mathbf{x}, \omega) \, \mathrm{d}\omega, \quad j = 1, \ldots, M$$

where

$$g_j(\mathbf{x}, \omega) = \max\{\phi_j(\mathbf{x}, \omega), 0\}$$

and $\Omega = [0, \infty)$. Since $G_j(\mathbf{x})$ is non-smooth in $\mathbf{x}$, standard routines would have difficulty coping

(ii) $$\max_{\omega \in \Omega} \phi_j(\mathbf{x}, \omega) \leqslant 0, \quad j = 1, 2, \ldots, M \tag{23c}$$

where p is a positive integer and c is a positive constant, both to be defined later.

For brevity, let $\mathscr{F}$ be the subset of Ξ such that the constraints (23b) and (23c) are satisfied. With this abbreviation the problem (23) may be restated as: find a parameter vector $\mathbf{x} \in \mathscr{F}$ such that the objective function (23a) is minimized over $\mathscr{F}$. This restated problem will be referred to as Problem P_p.

Theorem 2

Let assumptions (A1) and (A2) be satisfied and let the constant c in Problem P_p be defined as in (21). Define

$$m_p = \min_{\mathbf{x} \in \mathscr{F}} \phi_p(\mathbf{x})$$

where

$$\phi_p(\mathbf{x}) = \left\{ \int_0^c \{\bar{\lambda}(\mathbf{x}, \omega)\}^p \, d\omega \right\}^{1/2p} \tag{24}$$

Furthermore, for each positive integer p let $\mathbf{x}^p \in \mathscr{F}$ be such that $m_p = \phi_p(\mathbf{x}^p)$. Then there exists a subsequence $\{p(i) : i = 1, 2, \ldots\}$ of the sequence $\{p : p = 1, 2, \ldots\}$ such that

(i) $\mathbf{x}^{p(i)} \to \mathbf{x}^*$ as $i \to \infty$
(ii) $\mathbf{x}^*$ is an optimal parameter of Problem P
(iii) $m_{p(i)} \downarrow m$ as $i \to \infty$, where

$$m = \min_{\mathbf{x} \in \mathscr{F}} \phi(\mathbf{x})$$

with

$$\phi(\mathbf{x}) = \max_{\omega \geqslant 0} \{\bar{\lambda}(\mathbf{x}, \omega)\} = \max_{0 \leqslant \omega \leqslant c} \{\bar{\lambda}(\mathbf{x}, \omega)\}$$

Proof. By Theorem 1 we have

$$\max_{\omega \geqslant 0} \{\bar{\lambda}(\mathbf{x}, \omega)\} = \max_{0 \leqslant \omega \leqslant c} \{\bar{\lambda}(\mathbf{x}, \omega)\}$$

for each $\mathbf{x} \in \mathscr{F}$. Now we note that $\phi_p(\mathbf{x}) \uparrow \phi(\mathbf{x})$ as $p \to \infty$ for each $\mathbf{x} \in \mathscr{F}$, where $\mathscr{F}$ is a compact subset of $\mathbb{R}^l$. Thus by Dini's theorem we have $\phi_p(\mathbf{x}) \uparrow \phi(\mathbf{x})$ as $p \to \infty$ uniformly with respect to $\mathbf{x} \in \mathscr{F}$. Furthermore, for all $\mathbf{x} \in \mathscr{F}$ and for all positive integers p, $\phi_p(\mathbf{x}) \leqslant \phi(\mathbf{x})$.

For each p let $\mathbf{x}^p \in \mathscr{F}$ be such that

$$\min_{\mathbf{x} \in \mathscr{F}} \phi_p(\mathbf{x}) = \phi_p(\mathbf{x}^p)$$

Then $\phi_p(\mathbf{x}^p) \leqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \mathscr{F}$ and for all positive integers p. Therefore $\phi_p(\mathbf{x}^p) \leqslant m$. Next we recall that $\mathscr{F}$ is compact. Thus there exists a subsequence $\{p(i) : i = 1, 2, \ldots\}$ of the sequence $\{p : p = 1, 2, \ldots\}$ such that

$$\lim_{i \to \infty} \mathbf{x}^{p(i)} = \mathbf{x}^*$$

with $\mathbf{x}^* \in \mathscr{F}$. This implies conclusion (i).

Lemma 3

Let $\phi_j(\mathbf{x}, \omega)$ represent singular values. Then the relaxed constraint conditions (C)–(E) hold if and only if assumptions (A1)–(A3) are satisfied.

Proof. It is straightforward to show that (C) $\Leftrightarrow$ (A1), (D) $\Leftrightarrow$ (A2) and [(D) and (E)] $\Leftrightarrow$ [(A2) and (A3)]. For the nongeneric case when (E) fails, as for the function $f_p(\mathbf{x})$ defined in (25), it appears that subgradient-based optimization techniques must be employed.[5,6]

4. COMPUTATIONAL ASPECTS AND AN EXAMPLE

Consider initially Problem P in which there are no functional constraints. As described in the previous section, we choose a sequence of positive integers, p, and solve (approximate) Problems P_p which are

$$\min_{\mathbf{x}} \|\bar{\sigma}(\mathbf{x}, \omega)\|_{p,[0,c]} = \min_{\mathbf{x}} \phi_p(\mathbf{x}) = \left\{\min_{\mathbf{x}} f_p(\mathbf{x})\right\}^{1/2p} \tag{30}$$

with $f_p(\mathbf{x})$ defined by (25) as

$$f_p(\mathbf{x}) = \int_0^c \{\bar{\lambda}(\mathbf{x}, \omega)\}^p \, d\omega \tag{31}$$

The point at which each minimum is attained will be denoted by $\mathbf{x}^p$ and so the minimum value of the corresponding objective function will be $\phi_p(\mathbf{x}^p)$. The solution procedure for Problem P consists of the following steps.

(I) Choose a positive integer $p(1)$ (a suggested value is $p(1) = 5$) and a value of c. Set $i = 1$.

(II) Use a standard optimization algorithm to solve Problem $P_{p(i)}$ and obtain $\mathbf{x}^{p(i)}$. For each value of $\mathbf{x}$ the objective function $f_p(\mathbf{x})$ and its derivative with respect to $\mathbf{x}$ can be calculated by the following procedure.
- (a) For each $\omega \in [0, c]$ calculate the value of $\bar{\lambda}(\mathbf{x}, \omega)$ and $\nabla_{\mathbf{x}}\bar{\lambda}(\mathbf{x}, \omega)$ by
  - (i) using a complex singular value decomposition to obtain the singular values and right singular vectors of $\mathbf{W}(\mathbf{x}, \omega)$
  - (ii) calculating $\bar{\lambda}(\mathbf{x}, \omega) = \{\bar{\sigma}(\mathbf{x}, \omega)\}^2$
  - (iii) calculating $\nabla_x\bar{\lambda}(\mathbf{x}, \omega)$ from equation (27).
- (b) Use numerical integration to calculate $f_p(\mathbf{x})$ and $\nabla_{\mathbf{x}} f_p(\mathbf{x})$ from (25) and (26).

(III) Choose a value of $p(i + 1) > p(i)$ and use $\mathbf{x}^{p(i)}$ as the initial point in the next optimization. Set $i = i + 1$ and go to step (II).

Remarks

(i) If a complex singular-valued decomposition is not available, then an alternative procedure is to form $\mathbf{W}^*(\mathbf{x}, \omega)\mathbf{W}(\mathbf{x}, \omega)$ and compute its eigenvalues to obtain $\bar{\lambda}(\mathbf{x}, \omega)$. The eigenvectors of $\mathbf{W}^*(\mathbf{x}, \omega)\mathbf{W}(\mathbf{x}, \omega)$ are then the right singular vectors of $\mathbf{W}(\mathbf{x}, \omega)$.

(ii) In practice we usually have a fairly good idea what to chose for the value of c in step (I). For example, if $\bar{\lambda}(\mathbf{0}, j\omega)$ is computed, then since $\bar{\lambda}(\mathbf{0}, j\omega) \to 0$ as $\omega \to \infty$, the likely interval of interest should be apparent. Thus we should not need to iterate on the finite approximation to the infinite frequency range. Normally a value of $c = 10$ will suffice, although this does depend on the particular problem. In the discrete-time case the frequency range $|z| = 1$ is finite so that this issue does not arise. In fact, as pointed out with these constraints. However, by using the smoothing technique suggested in Reference 7, we define

$$g_{j,\varepsilon}(\mathbf{x},\omega)=\begin{cases}0 & \text{if } \phi_j(\mathbf{x},\omega)\leqslant -\varepsilon\\ [\phi_j(\mathbf{x},\omega)+\varepsilon]^2/4\varepsilon, & \text{if } |\phi_j(\mathbf{x},\omega)|<\varepsilon\\ \phi_j(\mathbf{x},\omega), & \text{if } \phi_j(\mathbf{x},\omega)\geqslant\varepsilon\end{cases}$$

and

$$G_{j,\varepsilon}(\mathbf{x})=\int_\Omega g_{j,\varepsilon}(\mathbf{x},\omega)\,\mathrm{d}\omega \tag{28}$$

Notice that

$$\lim_{\varepsilon\to 0} g_{j,\varepsilon}(\mathbf{x},\omega)=g_j(\mathbf{x},\omega) \qquad \lim_{\varepsilon\to 0} G_{j,\varepsilon}(\mathbf{x})=G_j(\mathbf{x})$$

Let $\mathscr{F}_{\varepsilon,\tau}$ be the subset of Ξ defined by the constraint (23b) together with the constraint

$$G_{j,\varepsilon}(\mathbf{x})\leqslant\tau \tag{29}$$

With these definitions a further approximation to Problem P_p can be defined as: find a parameter vector $\mathbf{x}\in\mathscr{F}$ such that the objective function (23a) is minimized over $\mathscr{F}_{\varepsilon,\tau}$. For each $\varepsilon>0$ and $\tau>0$ let the corresponding approximate problem be denoted by Problem $P_{p,\varepsilon,\tau}$.

In reference 7 it is assumed that the following conditions are satisfied.

(A) Ω is a compact interval in $\mathbb{R}$.
(B) $\phi_j(\mathbf{x},\omega)$ is continuously differentiable in $\mathbf{x}$ and ω for all j.

In applying the techniques of Reference 7 to achieve a solution to the constrained H^∞ optimization task, it is important to note that the proofs used there can be trivially generalized to allow a relaxation of conditions (A) and (B) to the following.

(C) $\int_\Omega g_{j,\varepsilon}(\mathbf{x},\omega)\,\mathrm{d}\omega$ exists for all $\mathbf{x}$ and for each j.
(D) $\partial\phi_j(\mathbf{x},\omega)/\partial\omega$ is piecewise continuous in $\omega\in\Omega$ for each $\mathbf{x}$ and j.
(E) $\phi_j(\mathbf{x},\omega)$ is continuously differentiable with respect to $\mathbf{x}$ for almost all ω and all j.

Under the constraint conditions (A) and (B) it is proposed in Reference 7 that approximate problems parametrized in terms of ε and τ be solved for decreasing ε and τ until a suitable approximation to the optimal $\mathbf{x}$ is found. The following theorem shows that the key theoretical result in Reference 7 remains valid under the relaxed constraint conditions (C)–(E).

Theorem 3

Consider Problem P_p for a particular positive integer p. Then under the relaxed constraint conditions (C)–(E) an arbitrarily close approximation to the optimal $\mathbf{x}^p$ for Problem P_p is achieved by solving Problem $P_{p,\varepsilon,\tau}$ for ε and τ suitably small. Moreover, for each $\varepsilon>0$ the approximate solution satisfies the constraints $G_{j,\varepsilon}(\mathbf{x})\leqslant\tau$ for $\tau>0$ suitably small. (There exists a $\tau(\varepsilon)>0$ such that for all $0<\tau<\tau(\varepsilon)$ the approximate solution is feasible.)

Proof. See the relevant parts of Reference 7.

Note that if $\phi_j(\mathbf{x},\omega)$ represents singular values, then $\phi_j(\mathbf{x},\omega)$ is not continuously differentiable in $\mathbf{x}$ and ω when multiple singular values occur, so that the constraint condition (B) can fail.

the steady state estimator gain matrix for the plant is given by

$$\mathbf{H} = \begin{bmatrix} -3\cdot 0843 & -2\cdot 7837 \\ 0\cdot 9002 & -0\cdot 8184 \\ 1\cdot 3711 & -0\cdot 3416 \end{bmatrix}$$

Thus the LQG regulator for the plant is given as

$$\mathbf{K} = \mathbf{F}[s\mathbf{I} - (\mathbf{A} + \mathbf{BF} + \mathbf{HC} + \mathbf{HDF})]^{-1}(-\mathbf{H})$$

i.e.

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} + \mathbf{BF} + \mathbf{HC} + \mathbf{HDF} & \vdots & -\mathbf{H} \\ \cdots & \cdots & \cdots \\ \mathbf{F} & \vdots & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2\cdot 2435 & 0\cdot 4476 & 2\cdot 4335 \\ -3\cdot 4051 & 1\cdot 3717 & -3\cdot 6719 \\ -12\cdot 4143 & 2\cdot 9412 & -14\cdot 2656 \end{bmatrix} & \vdots & \begin{bmatrix} 3\cdot 0843 & 2\cdot 7837 \\ -0\cdot 9002 & 0\cdot 8184 \\ -1\cdot 3711 & 0\cdot 3416 \end{bmatrix} \\ \cdots & \cdots & \cdots \\ \begin{bmatrix} -4\cdot 1399 & 3\cdot 2740 & -1\cdot 9588 \\ 0\cdot 3212 & -3\cdot 2691 & -2\cdot 4968 \end{bmatrix} & \vdots & \mathbf{0} \end{bmatrix}$$

The problem of achieving maximally input sensitivity recovery via a $\mathbf{Q} \in RH^\infty$ feeding back the estimation residuals can be formulated as the following standard H^∞ optimization problem:[16]

$$\min_{\mathbf{Q} \in RH^\infty} \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\|_\infty$$

where

$$\mathbf{T}_{11} = \begin{bmatrix} \dfrac{5\cdot 0s^5 + 174\cdot 4s^4 + 670\cdot 7s^3 - 2222\cdot 3s^2 - 612\cdot 9s + 36\cdot 6}{s^6 + 23\cdot 3s^5 + 173\cdot 7^4 + 588\cdot 4s^3 + 1012\cdot 7s^2 + 836\cdot 1s + 225\cdot 5} & \dfrac{4\cdot 8s^5 + 131\cdot 7s^4 + 413\cdot 7s^3 - 1697\cdot 3s^2 + 232\cdot 2s + 214\cdot 3}{s^6 + 23\cdot 3s^5 + 173\cdot 7s^4 + 588\cdot 4s^3 + 1012\cdot 7s^2 + 836\cdot 1s + 225\cdot 5} \\ \dfrac{-12\cdot 0s^5 - 148\cdot 8s^4 - 511\cdot 5s^3 + 1224\cdot 0s^2 + 831\cdot 4s - 152\cdot 3}{s^6 + 23\cdot 3s^5 + 173\cdot 7s^4 + 588\cdot 4s^3 + 1012\cdot 7s^2 + 836\cdot 1s + 225\cdot 5} & \dfrac{-8\cdot 0s^5 - 100\cdot 4s^4 - 298\cdot 1s^3 + 956\cdot 4s^2 - 995\cdot 3s + 41\cdot 0}{s^6 + 23\cdot 3s^5 + 173\cdot 7s^4 + 588\cdot 4s^3 + 1012\cdot 7s^2 + 836\cdot 1s + 225\cdot 5} \end{bmatrix}$$

$$\mathbf{T}_{12} = -\begin{bmatrix} \dfrac{s^3 + 7\cdot 5989s^2 - 14\cdot 8430s - 3\cdot 0578}{s^3 + 15\cdot 6564s^2 + 35\cdot 4320s + 13\cdot 6191} & \dfrac{-4\cdot 6724s^2 + 15\cdot 3487s + 5\cdot 6400}{s^3 + 15\cdot 6564s^2 + 35\cdot 4320s + 13\cdot 6191} \\ \dfrac{-4\cdot 6724s^2 + 5\cdot 8158s - 9\cdot 4676}{s^3 + 15\cdot 6564s^2 + 35\cdot 4320s + 13\cdot 6191} & \dfrac{s^3 + 5\cdot 0576s^2 - 12\cdot 2951s - 0\cdot 3528}{s^3 + 15\cdot 6564s^2 + 35\cdot 4320s + 13\cdot 6191} \end{bmatrix}$$

$$\mathbf{T}_{21} = \begin{bmatrix} \dfrac{s^3 + 4\cdot 6741s^2 - 11\cdot 3723s + 8\cdot 1171}{s^3 + 7\cdot 6586s^2 + 18\cdot 3219s + 16\cdot 5548} & \dfrac{-2\cdot 4653s^2 - 20\cdot 9025s - 0\cdot 1737}{s^3 + 7\cdot 6586s^2 + 18\cdot 3219s + 16\cdot 5548} \\ \dfrac{6\cdot 0347s^2 + 6\cdot 1714s - 4\cdot 9359}{s^3 + 7\cdot 6586s^2 + 18\cdot 3219s + 16\cdot 5548} & \dfrac{s^3 + 11\cdot 0845s^2 + 18\cdot 1915s + 7\cdot 8557}{s^3 + 7\cdot 6586s^2 + 18\cdot 3219s + 16\cdot 5548} \end{bmatrix}$$

The singular values of $\mathbf{T}_{11}(j\omega)$ are shown in Figure 2(a).

by a reviewer, it might be useful to use a bilinear transformation to convert the infinite frequency range in continuous time to the finite one associated with discrete time. Details on this are omitted.

(iii) Typically only two or three different values of p will be required. For example, $\{p(i)\} = \{5, 20, 100\}$ will usually give a good result. It is a relatively simple task to compute the ∞-norm for a particular value of $\mathbf{x}^p$, i.e. $\phi(\mathbf{x}^p)$, and compare it with the value of the corresponding p-norm, $\phi_p(\mathbf{x}^p)$. By Theorem 2 the value of $\phi_p(\mathbf{x}^p)$ and hence $\phi(\mathbf{x}^p)$ converges to the solution of Problem P as $p \to \infty$.

(iv) For larger values of p the integrand in (31) will need to be scaled so as to avoid numerical overflow. One means of accomplishing this is to scale the integrand by the square of $\phi(\mathbf{x}^p)$, where $\mathbf{x}^p$ is the previous solution point. Then

$$\phi_{p(i)}(\mathbf{x}) = \phi(\mathbf{x}^{p(i-1)}) \left\{ \int_0^c \left[\frac{\bar{\lambda}(\mathbf{x}, \omega)}{\{\phi(\mathbf{x}^{p(i-1)})\}^2} \right]^p \mathrm{d}\omega \right\}^{1/2p} \tag{32}$$

where $\phi(\mathbf{x}^{p(0)})$ is set to unity. The integrand in (32) now has a maximum of approximately unity and so overflow is avoided.

In the several examples we have tried the previously described computational procedure works extremely well. Admittedly we have yet to solve any particularly complicated problems, but we do not anticipate any severe difficulties. The following example shows the application of the procedure to control system design. The details of the LQG design can be found in Reference 16.

Example: H^∞ sensitivity recovery based on LQG design

Consider the following state-space description of a linear plant:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where

$$\mathbf{A} = \begin{bmatrix} 3 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0\cdot5 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0\cdot1 & 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which quadratic performance index

$$\int_0^\infty (\mathbf{x}^{\mathrm{T}}\mathbf{Q}_c\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}_c\mathbf{u})\, \mathrm{d}t$$

Given that

$$\mathbf{Q}_c = \mathbf{C}^{\mathrm{T}}\mathbf{C} = \begin{bmatrix} 5\cdot00 & -0\cdot80 & 6.00 \\ -0\cdot80 & 1\cdot01 & 0.30 \\ 6\cdot00 & 0\cdot30 & 9\cdot00 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_c = \mathbf{I}$$

the steady state feedback gain matrix obtained from the LQ design is

$$\mathbf{F} = \begin{bmatrix} -4\cdot1399 & 3\cdot2740 & -1\cdot9588 \\ 0\cdot3212 & -3\cdot2691 & -2\cdot4968 \end{bmatrix}$$

Associated with

$$\mathbf{Q}_e = \mathbf{I} \quad \text{and} \quad \mathbf{R}_e = \mathbf{I}$$

If the controller $\mathbf{K}[\mathbf{Q}(\mathbf{x})]$ is further required to be able to stabilize internally the perturbed plant $(1+\mu)\mathbf{P}$ for all scalar unimodular $\mu \in RH^\infty$ satisfying $\|\mu\| < \alpha$, we need to impose the following additional constraint on $\mathbf{Q}(\mathbf{x}) \in RH^\infty$:

$$\|\mathbf{T}_1 + \mathbf{Q}(\mathbf{x})\mathbf{T}_2\|_\infty < 1/\alpha \tag{34}$$

where

$$\mathbf{T}_1 = \begin{bmatrix} \dfrac{13\cdot030s^5 + 207\cdot55s^4 + 860\cdot72s^3 + 407\cdot54s^2 + 82\cdot558s + 281\cdot94}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} & \dfrac{9\cdot5139s^5 + 79\cdot365s^4 - 110\cdot40s^3 + 203\cdot16s^2 + 383\cdot42s - 3\cdot7723}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} \\ \dfrac{-7\cdot3569s^5 - 88\cdot996s^4 - 119\cdot33s^3 + 848\cdot86s^2 + 612\cdot06s - 219\cdot69}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} & \dfrac{2\cdot6342s^5 + 107\cdot33s^4 + 634\cdot93s^3 + 665\cdot76s^2 + 493\cdot27s + 303\cdot07}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} \end{bmatrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} \dfrac{20\cdot242s^5 + 141\cdot72s^4 + 380\cdot54s^3 + 131\cdot88s^2 - 33\cdot997s + 179\cdot91}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} & \dfrac{-3\cdot2070s^5 - 27\cdot069s^4 - 149\cdot96s^3 - 14\cdot159s^2 + 140\cdot05s - 40\cdot002}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} \\ \dfrac{5\cdot2929s^5 + 104\cdot35s^4 + 645\cdot93s^3 + 1315\cdot3s^2 + 895\cdot18s - 86\cdot059}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} & \dfrac{29\cdot490s^5 + 256\cdot57s^4 + 747\cdot83s^3 + 1097\cdot0s^2 + 894\cdot73s + 387\cdot17}{s^6 + 23\cdot315s^5 + 173\cdot66s^4 + 588\cdot38s^3 + 1012\cdot7s^2 + 836\cdot09s + 225\cdot46} \end{bmatrix}$$

Note that if the constraint (34) is satisfied, then the closed-loop system composed of $\mathbf{G}$ (the plant transfer function) and $\mathbf{K}[\mathbf{Q}(\mathbf{x})]$ has a gain margin of at least $(1+\alpha)/(1-\alpha)$.

If we use the optimal $\mathbf{Q}$ of the form (33) for the unconstrained H^∞ optimization problem, then we obtain

$$\|\mathbf{T}_1 + \mathbf{Q}(\mathbf{x})\mathbf{T}_2\|_\infty = 5\cdot043$$

which corresponds to $\alpha = 0\cdot198$ in (34) and a gain margin of at least $1\cdot493$. We will now improve the robustness of the controller by solving the constrained H^∞ optimization problem with $\alpha = 0\cdot5$ in the constraint (34). This would correspond to a gain margin of at least 3.

The values of the L_p- and L_∞-norms of $\mathbf{W}(\mathbf{x}^p)$ resulting from the optimization are displayed in Table II along with the values of $\mathbf{x}^p$ for $p = 5$, 20, 100 and 200. Figure 3 shows the singular

Table II. Values of the L_p- and L_∞-norms of $\mathbf{W}(\mathbf{x}, j\omega)$ for the constrained case

p	$\mathbf{x}^p$				$\phi_p(\mathbf{x}^p)$	$\phi(\mathbf{x}^p)$
5	−18·61	2·38	14·65	−11·05	5·21	4·09
20	−20·04	2·02	15·95	−10·87	4·19	4·04
100	−20·41	1·95	16·24	−10·77	4·05	4·04
200	−20·79	1·86	15·91	−10·86	4·04	4·04

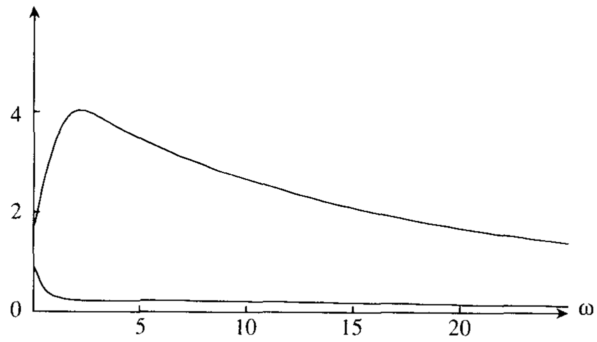

Figure 3. Singular values of $\mathbf{W}(\mathbf{x}, j\omega)$ for optimal $\mathbf{x} = [-20\cdot79, 1\cdot86, 15\cdot91, -10\cdot86]^T$ for the constrained problem

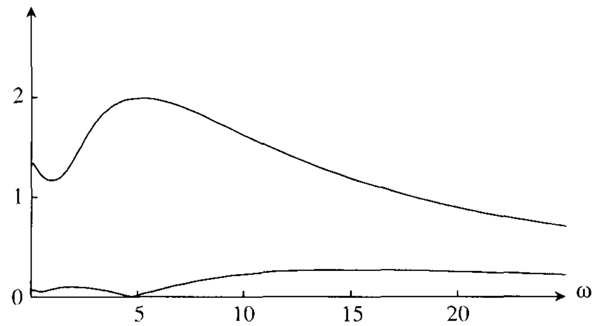

Figure 4. Singular values of $\mathbf{T}_1(j\omega) + \mathbf{Q}(\mathbf{x}, j\omega)\mathbf{T}_2(j\omega)$ for optimal $\mathbf{x}$ for the constrained problem

values of $\mathbf{W}(\mathbf{x}^p, j\omega)$ corresponding to $p = 200$. The optimal value obtained for $\|\mathbf{W}(\mathbf{x})\|_\infty$ is 4·03, which is only marginally worse than the value of 3·28 obtained in the unconstrained case. The singular values of $\mathbf{T}_1(j\omega) + \mathbf{Q}(\mathbf{x}, j\omega)\mathbf{T}_2(j\omega)$ are also illustrated in Figure 4.

5. CONCLUSIONS

We have demonstrated a practical successive approximation method for performing constrained H^∞ optimization and an associated convergence theory. The method is attractive because it is based on smooth optimization theory and standard software optimization routines. Theory has shown that the results of the method approximate arbitrarily closely optimal results. The proposed method is easy to implement and, in our limited numerical experience, appears to be highly reliable and efficient. Since the constraints permitted can be of functional form, robustness measures can be treated in an H^∞-based control system design. The application of this method has been illustrated in a non-trivial design example.

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