57. *A Construction for Idempotent Binary Relations*

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(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1970)

The problem of characterizing the idempotent elements of the semigroup $\mathcal{R}_A$ of all binary relations over a set A is of interest, since the semigroup $\mathcal{R}_A$ was a subject of numerous studies. This problem has been mentioned in [1]. Here we present a solution to this problem.

Let $\rho \in \mathcal{R}_A$. The set of all $a \in A$ such that $(a, a_1) \in \rho$ or $(a_1, a) \in \rho$ for some $a_1 \in A$ is called the *field* of ρ and is denoted as $pr\rho$. ρ may be considered as a binary relation over its field.

If $\mathfrak{a} \subset A$, then $\Delta_\mathfrak{a}$ denotes the binary relation over A defined as follows: $(a_1, a_2) \in \Delta_\mathfrak{a}$ iff $a_1 = a_2$ and $a_1 \in \mathfrak{a}$.

A reflexive and transitive binary relation is called a *quasi-order* relation. An antisymmetric quasi-order relation is called an *order* relation.

Let ρ be a binary relation over a set I and $(A_i)_{i \in I}$ be a family of pairwise disjoint nonempty sets. Then the binary relation $\bigcup_{(i,j) \in \rho} (A_i \times A_j)$ over the set $\bigcup_{i \in I} A_i$ is called an *inflation* of ρ. It is known that *inflations of order relations are quasi-order relations and every quasi-order relation is a uniquely determined inflation of a uniquely determined (up to isomorphism) order relation.* Thus, the structure of quasi-order relations may be considered as known modulo order relations.

Let ρ be a quasi-order relation over a set A. Then $\varepsilon_\rho = \rho \cap \rho^{-1}$ is the *symmetric part* of ρ (here ρ^{-1} is the *converse* of ρ). Clearly, ε_ρ is an equivalence relation over A. An element $a \in A$ is called *ρ-strict* if the ε_ρ-class containing a is a singleton, i.e., if $(a, a_0), (a_0, a) \in \rho$ imply $a = a_0$. An element $a_1 \in A$ *covers* an element $a_2 \in A$ if $(a_1, a), (a, a_2) \in \rho$ imply $a = a_1$ or $a = a_2$ for every $a \in A$, and $(a_1, a_2) \in \rho$. Two elements a_1 and a_2 are called *ρ-neighbors* if a_1 covers a_2 or a_2 covers a_1. A subset $\mathfrak{a} \subset A$ is called *ρ-permissible*, if all elements of $\mathfrak{a}$ are ρ-strict and $\mathfrak{a}$ does not contain ρ-neighbors.

A binary relation σ is called a *pseudo-order* relation if $\sigma = \rho \setminus \Delta_\mathfrak{a}$ where ρ is a quasi-order relation and $\mathfrak{a}$ is a ρ-permissible subset. Here $\setminus$ is the set-theoretical difference. In this case ρ is called the *completion* of σ, and $\mathfrak{a}$ is called the *defect* of σ. Since $\rho = \sigma \cup \Delta_A$ and $\mathfrak{a} = A \setminus pr(\sigma \cap \Delta_A)$, the completion and defect of σ are uniquely determined. Notice that $pr\sigma$ need not be equal to A.

Theorem. *A binary relation is idempotent if and only if it is a* *pseudo-order relation.*

Proof. Let σ be a pseudo-order relation and $(a, b), (b, c) \in \sigma$. Then $(a, b), (b, c) \in \rho$ and, by transitivity of ρ, $(a, c) \in \rho$. Here ρ and $\mathfrak{a}$ are the completion and defect of σ. If $(a, c) \in \Delta_{\mathfrak{a}}$, then $a = b$ (otherwise, a is not ρ-strict) and $(a, b) \in \Delta_{\mathfrak{a}}$— a contradiction. Hence, $(a, c) \notin \Delta_{\mathfrak{a}}$ and $(a, c) \in \sigma$, i.e., $\sigma \circ \sigma \subset \sigma$.

Now let $(a, b) \in \sigma$. If $a \notin \mathfrak{a}$ or $b \notin \mathfrak{a}$, then either $(a, a) \in \sigma$ or $(b, b) \in \sigma$. In both cases $(a, b) \in \sigma \circ \sigma$. Suppose $a, b \in \mathfrak{a}$. Then $(a, b) \in \rho = \rho \circ \rho$, i.e., $(a, c) \in \rho$ and $(c, b) \in \rho$ for some $c \in A$. Since a and b are not ρ-neighbors, one may suppose $c \neq a$ and $c \neq b$ in which case (a, c), $(c, b) \in \sigma$ and $(a, b) \in \sigma \circ \sigma$. Thus, $\sigma \subset \sigma \circ \sigma$. It follows that $\sigma \circ \sigma = \sigma$, i.e., σ is an idempotent.

Now let σ be an idempotent binary relation, $\rho = \sigma \cup \Delta_A$, $\mathfrak{a} = A \backslash pr(\sigma \cap \Delta_A)$. Then ρ is reflexive and $\rho \circ \rho = (\sigma \circ \sigma) \cup (\sigma \circ \Delta_A) \cup (\Delta_A \circ \sigma) \cup (\Delta_A \circ \Delta_A) = \sigma \cup \sigma \cup \sigma \cup \Delta_A = \rho$, i.e., ρ is transitive. Thus, ρ is a quasi-order relation. Clearly, $\sigma = \rho \backslash \Delta_{\mathfrak{a}}$. It remains to prove $\mathfrak{a}$ is ρ-permissible.

Let $a \in \mathfrak{a}$ and $(a, b), (b, a) \in \rho$. If $a \neq b$, then (a, b), $(b, a) \in \sigma$ and $(a, a) \in \sigma \circ \sigma = \sigma$— a contradiction. Hence, $a = b$, i.e., a is ρ-strict. Now let a and b be two distinct elements of $\mathfrak{a}$, $(a, b) \in \rho$. Then $(a, b) \in \sigma = \sigma \circ \sigma$, i.e., $(a, c), (c, b) \in \sigma \subset \rho$ for some $c \in A$. If $c = a$ or $c = b$, then $(a, c) \notin \sigma$ or $(c, b) \notin \sigma$. Therefore, $a \neq c \neq b$, i.e., a and b are not ρ-neighbors. Thus, σ is a pseudo-order relation.

The criterion of idempotence given in the Theorem seems to be the simplest possible, since the structure of idempotent binary relations is given modulo order relations. This criterion may be easily applied also in the case when the binary relation is given as a Boolean matrix or an oriented graph.

Reference

[1] S. Schwarz: The semigroup of binary relations on a finite set—presented at International Symposium on Semigroups, Smolenice, Czechoslovakia (1968).