The uniform F-smoothness of S at H then follows from (29), (30), and the last theorem of the preceding section.

It remains only to show that the renormed version of l^2 lacks the A-property. For $0 \leqslant i < \infty$, let δ_i denote the point of l^2 such that $\delta_{ij} = 1$ or 0 according as $j = i$ or $j \neq i$; let δ_i^* denote the same point considered as a member of the conjugate space $(l^2)^*$. Note that for $i = 1, 2, \ldots$ and for $|\lambda| < 1$, any hyperplane in V parallel to the hyperplane $V_i = \{x \in V: x_i = 0\}$ is carried onto such a parallel hyperplane by the transformation $T_{\eta(\lambda)}$. Note also that $\tau_{\delta_i} = \eta_i$. Since η_i is concave, and since S_V is supported at δ_i in V by a translate of V_i, it follows that U is supported at the point $\lambda\delta_0 + \eta_i(\lambda)\delta_i$ by a hyperplane which contains a translate of V_i and also contains the tangent to η_i at this point. In particular (using (23) and (24)), with $x_i = (1-\varepsilon_i)\delta_0 + 2\varepsilon_i\delta_i \in S$ and $\{y_i\} = x_i^c$ relative to the new norm $\| \, \|$, we have

$$y_0 = (1-3\varepsilon_i)^{-1}(\delta_i^* - \delta_i^*).$$

As $\varepsilon_i \in]0, \frac{1}{6}[$ and as $\delta_1, \delta_2, \ldots \in S$ it follows that $\|y_i - y_j\| > \frac{1}{2}$ for $i \neq j$. But of course $x_1, x_2, \ldots \to \delta_0$, so the proof is complete.

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A construction of basis in $C^{(1)}(I^2)$

by

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The sequence $\{x_n, n = 1, 2, \ldots\}$ of elements of a given real Banach space $[X, \| \, \|]$ is called *basis* in X whenever each $x \in X$ has unique, convergent in the norm $\| \, \|$, expansion

$$x = \sum_{n=1}^{\infty} a_n x_n$$

with real coefficients $a_1, a_2, \ldots$ It is well known that the coefficients $a_n = a_n(x)$ are linear functionals over $[X, \| \, \|]$ and they are called *coefficient functionals* for the basis $\{x_n\}$.

There were two examples of seperable Banach spaces mentioned in the Banach monograph [1] (p. 238) for which it was not known how to construct bases. One of the examples is the space A of holomorphic functions in the interior and continuous on the boundary of the unit disc with uniform norm. The second example is the space $C^{(1)}(I^2)$, $I = \langle 0, 1 \rangle$, of all functions with continuous partial derivatives of the first order on I^2 with the norm

$$\|x\|^{(1)} = \|x\| + \|D_1 x\| + \|D_2 x\|$$

where

$$\|x\| = \max\{|x(s, t)|: s, t \in I^2\},$$

$$D_1 x(s, t) = \frac{\partial x}{\partial s}(s, t) \quad \text{and} \quad D_2 x(s, t) = \frac{\partial x}{\partial t}(s, t).$$

The aim of this paper is to give an effective construction of a basis in the Banach space $[C^{(1)}(I^2), \| \, \|^{(1)}]$. It follows immediately from the construction that this result can be extended to the case of $C^{(1)}(I^n)$ with arbitrary $n \geqslant 1$.

The construction depends heavily on the properties of the Franklin orthonormal system $\{f_n, n = 0, 1, \ldots\}$.

To define the orthonormal Franklin system we need to recall the definition of the Schauder functions: $s_0 = 1$, $s_1(t) = t$ for $t \in I$, and for

$m \geqslant 0, 1 \leqslant k \leqslant 2^m$, $s_{2^m+k}(t) = 1 - |2^{m+1}t - (2k-1)|$ if $|t - 2^{-(m+1)}(2k-1)| < 2^{-(m+1)}$ and $s_{2^m+k}(t) = 0$ elsewhere in I. The Schauder functions are linearly independent and the Schmidt orthonormalization procedure applied to $\{s_n, n = 0, 1, \ldots\}$ leads to the Franklin functions

$$f_n = \sum_{i=0}^{n} \mu_{in} s_i, \quad \mu_{nn} > 0, n = 0, 1, \ldots$$

The following result is due to Franklin (for a simple proof cf. [2]):

THEOREM A. *The orthonormal Franklin set $\{f_n, n = 0, 1, \ldots\}$ is a basis in $C(I)$ with uniform norm.*

Another property of the Franklin system was established in [3], Theorem 20,

THEOREM B. *The set of functions $\{1, \int_0^t f_n(u)\, du, n = 0, 1, \ldots\}$ is a basis in $C(I)$ and for each $x \in C(I)$*

$$x(t) = x(0) + \sum_{n=0}^{\infty} \Big[\int_0^1 f_n(s)\, dx(s)\Big] \int_0^t f_n(u)\, du, \quad t \in I,$$

and the convergence is uniform on I.

For further purposes it is convenient to introduce the following notation. Let $N = \{1, 2, \ldots\}$ and let $\nu_i: N \times N \to N$ $(i = 0, 1, 2)$ be functions defined as follows:

(1) $$\nu_0(n, m) = \begin{cases} p^2 + k \text{ for } n = p+1 \text{ and } m = k, 1 \leqslant k \leqslant p, \\ p^2 + p + k \text{ for } n = k \text{ and } m = p+1, 1 \leqslant k \leqslant p+1; \end{cases}$$

(2) $$\nu_1(n, m) = \begin{cases} p^2 + k \text{ for } n = k \text{ and } m = p+1, 1 \leqslant k \leqslant p, \\ p^2 + p + k \text{ for } n = p+1 \text{ and } m = k, 1 \leqslant k \leqslant p+1; \end{cases}$$

(3) $$\nu_2(n, m) = \begin{cases} p(p+1) + k \text{ for } n = p+2 \text{ and } m = k, 1 \leqslant k \leqslant p, \\ p(p+2) + k \text{ for } n = k \text{ and } m = p+1, 1 \leqslant k \leqslant p+2. \end{cases}$$

It is easily seen that the mappings ν_i are one-to-one and onto.

The following result can be found in [4] and [5]:

THEOREM C. *Let $\{\varphi_n, n = 1, 2, \ldots\}$ and $\{\psi_n, n = 1, 2, \ldots\}$ be two bases in $C(I)$ and let for $i = 0, 1, 2$*

$$\lambda_k^{(i)}(s, t) = \varphi_n(s)\psi_m(t) \quad \textit{whenever } k = \nu_i(n, m).$$

Then, for each i, $\{\lambda_k^{(i)}, k = 1, 2, \ldots\}$ is a basis in $[C(I^2), \| \, \|]$.

This is a good place to mention how the coefficient functionals $\{\zeta_k^{(i)}\}$ of the basis $\{\lambda_k^{(i)}\}$ are constructed in terms of the coefficient functionals $\{\xi_n\}, \{\eta_m\}$ corresponding to the bases $\{\varphi_n\}, \{\psi_m\}$, respectively. For functions $z \in C(I^2)$ of the form $z(s, t) = x(s)y(t)$, $x, y \in C(I)$, we have

(4) $$\zeta_k^{(i)}(z) = \xi_n(x)\eta_m(y) \quad \text{whenever } k = \nu_i(n, m).$$

To state our result we need additional notation:

$$F_1(t) = 1, \quad F_n(t) = \int_0^t f_{n-2}(u)\, du, \quad n = 2, 3, \ldots; t \in I,$$

and

$$H_k^{(0)}(s, t) = F_n(s)F_m(t) \quad \text{whenever } k = \nu_0(n, m), \quad s, t \in I.$$

THEOREM. *The set of functions $\{H_k^{(0)}, k = 1, 2, \ldots\}$ is a basis for the Banach space $[C^{(1)}(I^2), \| \, \|^{(1)}]$.*

For the proof we need to consider three bases in $C(I^2)$.

According to Theorems B and C $\{H_k^{(0)}, k = 1, 2, \ldots\}$ is a basis in $[C(I^2), \| \, \|]$.

Now let us define

$$H_k^{(1)}(s, t) = f_{n-1}(s)F_m(t) \quad \text{whenever } k = \nu_1(n, m).$$

Applying Theorems A, B and C we find that $\{H_k^{(1)}\}$ is again a basis in $[C(I^2), \| \, \|]$.

The third basis $\{H_k^{(2)}, k = 1, 2, \ldots\}$ is defined as follows:

$$H_k^{(2)}(s, t) = F_n(s)f_{m-1}(t) \quad \text{whenever } k = \nu_2(n, m).$$

Consequently, for each $i = 0, 1, 2$ and for each $x \in C(I^2)$ we have

(5) $$x = \sum_{k=1}^{\infty} a_k^{(i)}(x) H_k^{(i)},$$

and the series converges in the maximum norm. Let

(6) $$S_N^{(i)} = \sum_{k=1}^{N} a_k^{(i)} H_k^{(i)}.$$

It is easy to see from the definitions that $D_1 H^{(0)}_{\nu_0(1,m)} = D_1 H^{(0)}_{\nu_0(n,1)} = 0$ and

$$D_1 H^{(0)}_{\nu_0(n,m)} = H^{(1)}_{\nu_1(n-1,m)} \quad \text{for } n > 1, m \geqslant 1,$$
$$D_2 H^{(0)}_{\nu_0(n,m)} = H^{(2)}_{\nu_2(n,m-1)} \quad \text{for } n \geqslant 1, m > 1.$$

These formulas and (6) give

(7) $$D_1 S_N^{(0)}(x) = \sum_{E_1} a^{(0)}_{\nu_0(n,m)}(x) H^{(1)}_{\nu_1(n-1,m)}, \qquad D_2 S_N^{(0)}(x) = \sum_{E_2} a^{(0)}_{\nu_0(n,m)}(x) H^{(2)}_{\nu_2(n,m-1)},$$

where $E_1 = \{(n, m):\ n > 1,\ \nu_0(n, m) \leqslant N\}$, $E_2 = \{(n, m):\ m > 1,\ \nu_0(n, m) \leqslant N\}$.

Now, let $y \in C(I^2)$, $z_i \in C(I^2)$, $i = 1, 2$, and let z_i be constant in the i-th variable, i.e. $z_1(s, t) = 1 \cdot z_1(t)$, $z_2(s, t) = z_2(s) \cdot 1$.

Let

$$I_1 y(s, t) = \int_0^s y(u, t)\,du, \qquad I_2 y(s, t) = \int_0^t y(s, u)\,du.$$

It is clear that

$$a_k^{(0)}(I_i y + z_i) = a_k^{(0)}(I_i y) + a_k^{(0)}(z_i).$$

Now, by Theorems B and C and by (4)

$$a_{\nu_0(n,m)}^{(0)}(z_1) = 0 \quad \text{for } n > 1, m \geqslant 1;$$

$$a_{\nu_0(n,m)}^{(0)}(z_2) = 0 \quad \text{for } n \geqslant 1, m > 1.$$

Thus,

$$\begin{aligned} a_k^{(0)}(I_1 y + z_1) &= a_k^{(0)}(I_1 y), \quad k = \nu_0(n, m), n > 1, \\ a_k^{(0)}(I_2 y + z_2) &= a_k^{(0)}(I_2 y), \quad k = \nu_0(n, m), m > 1. \end{aligned} \tag{8}$$

Assuming that $y(s, t) = z(s)x(t)$, $z, x \in C(I)$ we find by Theorems A, B, C and by (4) that

$$\begin{aligned} a_{\nu_0(n,m)}^{(0)}(I_1 y) &= a_{\nu_1(n-1,m)}^{(1)}(y) \quad \text{for } n > 1, m \geqslant 1, \\ a_{\nu_0(n,m)}^{(0)}(I_2 y) &= a_{\nu_2(n,m-1)}^{(2)}(y) \quad \text{for } n \geqslant 1, m > 1. \end{aligned} \tag{9}$$

Since the I_i's are continuous linear operators in $C(I^2)$ and the $a_k^{(i)}$'s are continuous linear functionals on $C(I^2)$ it follows that formulas (9) can be extended immediately to arbitrary $y \in C(I^2)$. In particular, if for given $x \in C^{(1)}(I^2)$ we put $y = D_i x$ and $z_1(s, t) = x(0, t)$, $z_2(s, t) = x(s, 0)$, then formulas (8) and (9) give

$$a_{\nu_0(n,m)}^{(0)}(x) = a_{\nu_1(n-1,m)}^{(1)}(D_1 x) \quad \text{for } n > 1, m \geqslant 1,$$

$$a_{\nu_0(n,m)}^{(0)}(x) = a_{\nu_2(n,m-1)}^{(2)}(D_2 x) \quad \text{for } n \geqslant 1, m > 1.$$

Substituting this into (7) we obtain

$$D_1 S_N^{(0)}(x) = \sum_{E_1} a_{\nu_1(n-1,m)}^{(1)}(D_1 x) H_{\nu_1(n-1,m)}^{(1)},$$

$$D_2 S_N^{(0)}(x) = \sum_{E_2} a_{\nu_2(n,m-1)}^{(2)}(D_2 x) H_{\nu_2(n,m-1)}^{(2)}.$$

These identities and definitions (1), (2) and (3) give for $x \in C^{(1)}(I^2)$

$$D_i S_N^{(0)}(x) = S_{N_i}^{(i)}(D_i x), \quad i = 1, 2,$$

where the N_i's are uniquely determined by N and N_i for each i goes to infinity with N. Since $\{H_k^{(i)}\}$ is a basis in $C(I^2)$ and for $x \in C^{(1)}(I^2)$ we have that $D_i x \in C(I^2)$ it follows therefore that

$$\|D_i x - D_i S_N^{(0)}(x)\| = \|D_i x - S_{N_i}^{(i)}(D_i x)\| \to 0 \quad \text{as } N \to \infty,$$

and this completes the proof.

Added in proof. While the paper was in print the author learned that the same basis was earlier constructed by S. Schonefeld and his paper is going to appear in the Bulletin of the American Mathematical Society.

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