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A CONSTRUCTION OF LATTICES IN SPLITTABLE SOLVABLE LIE GROUPS

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Abstract

In this paper, we consider a unified constructions of lattices in splittable solvable Lie groups.

1. Introduction

Let G be a connected solvable Lie group. A discrete co-compact subgroup of G is called a lattice in G. Auslander [2] has proven that a compact solvmanifold has a solvmanifold of the form G/Γ as a finite covering, where G is a simply connected solvable Lie group, and Γ is a lattice in G. It is well known that a nilpotent Lie group has a lattice if and only if its Lie algebra has a basis with respect to which the constants of structure are rational. Moreover, the de Rham cohomology groups of a compact nilmanifold N/Γ are isomorphic to the cohomology groups of the Lie algebra $\mathfrak{n}$ of N ([7]). In particular, the de Rham cohomology groups of a compact nilmanifold are independent of lattices.

In the case of non-nilpotent solvable Lie groups, it is not easy to check the existence of a lattice. The de Rham cohomology groups of a compact solvmanifold G/Γ are not isomorphic to the cohomology groups of the Lie algebra $\mathfrak{g}$ of G in general. Two solvmanifolds G_1/Γ_1 and G_2/Γ_2 with isomorphic fundamental groups are diffeomorphic (see [8, Theorem 3.6]). Auslander also have proven that a Wang group is pre-divible, then it is isomorphic to a lattice in some simply connected solvable Lie group (see [2] for details). On the other hand, it is also important to construct a lattice in a given simply connected solvable Lie group (see e.g., [4, Examples 2, 3]). In the papers [11], [9], Sawai and the author have constructed lattices in splittable solvable Lie groups. However, the constructions in [11], [9] seem somewhat technical.

In this paper, we consider a unified construction of lattices in splittable solvable Lie groups by using the following theorem.

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Main Theorem. *Let $G = N \rtimes_{\varphi} \mathbf{R}^s$ be a simply connected splittable solvable Lie group, where N is the nilradical of G. Then G has a splittable lattice $\Gamma = \Gamma_N \rtimes_{\varphi} L$, where Γ_N is a lattice in N and L is a lattices in $\mathbf{R}^s$, if and only if there exists a $\mathbf{Q}$-algebra $\mathfrak{n}_0$ of $\mathfrak{n}$, and a lattice L in $\mathbf{R}^s$ such that $d\varphi(L) \subset Aut(\mathfrak{n}_0)$ and $d\varphi(t)$ $(t \in L)$ acts as an integer unimodular matrix with respect to a basis of $\mathfrak{n}$ contained in $\mathfrak{n}_0$.*

The theorem can be considered as a weak version of Auslander's result [2, pp. 248–pp. 249]. However, it seems that a complete proof has not been published.

2. Necessary and sufficient conditions for the existence of splittable lattices

In this section, we consider a necessary and sufficient condition for the existence of splittable lattices in a splittable solvable Lie group.

There exists a necessary and sufficient condition of the existence for a lattice in a given nilpotent Lie group ([8, Theorem 2.12.]).

Theorem 2.1 ([8]). *Let N be a simply connected nilpotent Lie group, and $\mathfrak{n}$ its Lie algebra. Suppose that $\mathfrak{n}$ has a basis with respect to which the constants of structure are rational. Let $\mathfrak{n}_0$ be the vector space over $\mathbf{Q}$ spanned by this basis; if $\mathscr{L}$ is any lattice of maximal rank in $\mathfrak{n}$ contained in $\mathfrak{n}_0$, and $\exp : \mathfrak{n} \to N$ is the exponential map, then the group generated by $\exp \mathscr{L}$ is a lattice in N. Conversely, if Γ_N is a lattice in N, then the $\mathbf{Z}$-span of $\exp^{-1} \Gamma_N$ is a lattice $\mathscr{L}$ in the vector space $\mathfrak{n}$ such that the structural constants of $\mathfrak{n}$ with respect to any basis contained in $\mathscr{L}$ belong to $\mathbf{Q}$.*

Let Γ be a lattice in a connected solvable Lie group G, and N the nilradical of G. Then, the following theorem is well-known.

Theorem 2.2 (Mostow [6]). *$N \cap \Gamma$ is a lattice in N.*

Let G be a simply connected solvable Lie group, and N the nilradical of G. Then G satisfies the exact sequence

$$1 \to N \to G \to \mathbf{R}^s \to 1.$$

We say that G is *splittable* if the short exact sequence splits. It is well-known that if G is *splittable*, then G is isomorphic to a semi-direct product $N \rtimes_{\varphi} \mathbf{R}^s$, where $\varphi : \mathbf{R}^s \to Aut(N)$ is a homomorphism.

Then, we have the following corollary.

Corollary 2.3. *Let $G = N \rtimes_{\varphi} \mathbf{R}^s$ be a simply connected splittable solvable Lie group, where N the nilradical of G, and $\Gamma = \Gamma_N \rtimes_{\varphi} L$ its splittable lattice.*

Then $C^k(N) \rtimes_\varphi \mathbf{R}^s$ *has a lattice, where* $\{C^k(N)\}$ *the descending central series for* N.

Proof. $(C^k(N) \cap \Gamma_N) \rtimes_\varphi L$ is a lattice in $C^k(N) \rtimes_\varphi \mathbf{R}^s$. □

THEOREM 2.4. *Let* $G = N \rtimes_\varphi \mathbf{R}^s$ *be a simply connected splittable solvable Lie group, where* N *is the nilradical of* G. *Then* G *has a splittable lattice* $\Gamma = \Gamma_N \rtimes_\varphi L$, *where* Γ_N *is a lattice in* N *and* L *is a lattices in* $\mathbf{R}^s$, *if and only if there exists a* $\mathbf{Q}$*-algebra* $\mathfrak{n}_0$ *of* $\mathfrak{n}$, *and a lattice* L *in* $\mathbf{R}^s$ *such that* $d\varphi(L) \subset Aut(\mathfrak{n}_0)$ *and* $d\varphi(t)$ $(t \in L)$ *acts as an integer unimodular matrix with respect to a basis of* $\mathfrak{n}$ *contained in* $\mathfrak{n}_0$.

Proof. Assume first that there exists a $\mathbf{Q}$-algebra $\mathfrak{n}_0$ of $\mathfrak{n}$, and lattice L in $\mathbf{R}^s$ such that $d\varphi(L) \subset Aut(\mathfrak{n}_0)$ and $d\varphi(t)$ $(t \in L)$ acts as an integer unimodular matrices with respect to a basis of $\mathfrak{n}$ contained in $\mathfrak{n}_0$. Let $\mathscr{L} \subset \mathfrak{n}_0$ be a lattice of maximal rank in $\mathfrak{n}$. By Theorem 2.1, the group Γ_N generated by $\exp \mathscr{L}$ is a lattice in N. Let $X \in \mathscr{L}$. Then $d\varphi(t)(X) \in \mathscr{L}$ for $t \in L$. Thus $\exp(d\varphi(t)X) = \varphi(t)(\exp X) \in \Gamma_N$. Since $(n,t)\cdot(n',t') = (n\varphi(t)(n'), tt')$, and $(n,t)^{-1} = (\varphi(t^{-1})(n^{-1}), t^{-1})$ for $(n,t), (n',t') \in N \rtimes_\varphi \mathbf{R}^s$, $\Gamma = \Gamma_N \rtimes_\varphi L$ is a discrete subgroup of $G = N \rtimes_\varphi \mathbf{R}^s$. Since a fiber bundle

$$N/N \cap \Gamma = N/\Gamma_N \to G/\Gamma \to G/N\Gamma = (G/N)/(N\Gamma/N)$$

has a compact base and a compact fiber, G/Γ is compact (note that $N\Gamma$ is closed by Theorem 1.13 in [8]). Thus, Γ is a lattice in G.

Conversely, assume that $\Gamma = \Gamma_N \rtimes_\varphi L$ is a lattice in G. Put

$$\mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{X \in \mathfrak{n} \mid X \in \exp^{-1} \Gamma_N\}.$$

Then

$$\mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{X_1, \dots, X_n\} \subset \mathfrak{n},$$

where $n = \dim \mathfrak{n}$ by Theorem 2.1. Moreover, $\mathscr{L}$ is a lattice in the vector space $\mathfrak{n}$ such that the structural constants of $\mathfrak{n}$ with respect to any basis contained in $\mathscr{L}$ belong to $\mathbf{Q}$ by Theorem 2.1. Since $\Gamma_N \rtimes_\varphi L$ is a subgroup, we see that for every $t \in L$ and $i = 1, \dots, n$, $(e,t)\cdot(\exp X_i, 0) = (\varphi(t)\exp X_i, t) \in \Gamma_N \rtimes_\varphi L$. Thus,

$$d\varphi(t)X_i \in \mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{X_1, \dots, X_n\}.$$

Thus, L acts as integer unimodular matrices with respect to $X_1, \dots, X_n$. □

The following lemma is obvious, however, it is useful to construct lattices (see Section 3).

LEMMA 2.5. *Let* $G = N \rtimes_\varphi \mathbf{R}^s$ *be a simply connected splittable solvable Lie group, where* N *is the nilradical of* G. *We assume that*

(1) N *is* $(r+1)$*-step and there exists a* $\mathbf{Q}$*-algebra* $\mathfrak{n}_0$ *of* $\mathfrak{n}$.

(2) *There exists a lattice* $L = \mathbf{Z}\mathbf{t}_1 + \cdots + \mathbf{Z}\mathbf{t}_s$ *in* $\mathbf{R}^s$ *such that* $d\varphi(L) \subset Aut(\mathfrak{n}_0)$ *and* $d\varphi(t)$ $(t \in L)$ *acts as an integer unimodular matrix with respect to a basis of* $\mathfrak{n}$ *contained in* $\mathfrak{n}_0$.

(3) $d\varphi(\mathbf{t}_i)$ $(i = 1, \dots, s)$ *are semi-simple.*

Then there exist vector spaces $\mathfrak{m}_k$ $(k = 0, \dots, r)$ *which satisfy the following conditions*:

(1) $C^{k-1}(\mathfrak{n}) = \mathfrak{m}_{k-1} \oplus C^k(\mathfrak{n})$ *for* $k = 1, \dots, r+1$.

(2) $d\varphi(\mathbf{t}_i)(\mathfrak{m}_k) \subset \mathfrak{m}_k$ *for* $i = 1, \dots, s$.

(3) $d\varphi(\mathbf{t}_i) : \mathfrak{m}_k \to \mathfrak{m}_k$ *is unimodular for* $i = 1, \dots, s$.

Proof. We only prove the case of $s = 1$, because $d\varphi(\mathbf{t}_1), \dots, d\varphi(\mathbf{t}_s)$ are simultaneously diagonalizable. We write $L = t_0\mathbf{Z}$. Let $\mathfrak{m}_r = C^r(\mathfrak{n})$. Then, it is obvious that $\mathfrak{m}_r$ satisfies the conditions (2), (3) by Corollary 2.3. Assume that there exist $\mathfrak{m}_k, \dots, \mathfrak{m}_r$ which satisfy the conditions (2), (3). Since $d\varphi(t_0)$ is semi-simple, and $d\varphi(t_0)(C^{k-1}(\mathfrak{n})) \subset C^{k-1}(\mathfrak{n})$, there exist

$$X_1^{(k-1)}, \dots, X_{i_{k-1}}^{(k-1)} \in C^{k-1}(\mathfrak{n}) \backslash C^k(\mathfrak{n})$$

such that

$$C^{k-1}(\mathfrak{n}) = \operatorname{span}\{X_1^{(k-1)}, \dots, X_{i_{k-1}}^{(k-1)}\} \oplus C^k(\mathfrak{n}), \quad d\varphi(t_0)X_j^{(k-1)} = \lambda_j^{(k-1)}X_j^{(k-1)}$$

for $j = 1, \dots, i_{k-1}$, where $\lambda_j^{(k-1)} \in \mathbf{R}$. Put

$$\mathfrak{m}_{k-1} = \operatorname{span}\{X_1^{(k-1)}, \dots, X_{i_{k-1}}^{(k-1)}\}.$$

Then, $d\varphi(t_0)(\mathfrak{m}_{k-1}) \subset \mathfrak{m}_{k-1}$. Since $d\varphi(t)|_{C^{k-1}(\mathfrak{n})}$ and $d\varphi(t)|_{C^k(\mathfrak{n})}$ are unimodular, $d\varphi(t)|_{\mathfrak{m}_{k-1}}$ is also unimodular. □

3. Examples

In this section, we construct lattices in splittable solvable Lie groups, which are famous, by using Theorem 2.4. Similarly as in this section, we can construct lattices in other solvable Lie groups.

Example 3.1 ([10], Inoue surface of type S^0). Let $\mathfrak{g}$ be a solvable Lie algebra given by

$$\mathfrak{g} = \operatorname{span}\{T, X, Y, \overline{Y}\}$$

with nontrivial structure equations

$$[T, X] = aX, \quad [T, Y] = bY, \quad [T, \overline{Y}] = \bar{b}\overline{Y},$$

where $b \in \mathbf{C}$, and e^a, e^b, $e^{\bar{b}}$ are eigenvalues of $B \in SL(3, \mathbf{Z})$.

Let $\mathfrak{n} = \operatorname{span}\{X, Y, \overline{Y}\}$. Let G be the simply connected solvable Lie group corresponding to $\mathfrak{g}$, and $N \subset G$ the simply connected nilpotent Lie group corresponding to $\mathfrak{n}$.

Let ${}^t(a_1, a_2, a_3)$ be a real eigenvector of e^a, and ${}^t(b_1, b_2, b_3)$ an eigenvector of e^b. Let

$$P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{pmatrix}.$$

Put $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X, Y, \bar{Y})P$. Then $B \in SL(3, \mathbf{Z})$ is the matrix representation of $\exp ad(T)|_{\mathfrak{n}}$ with respect to $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$. Let $\mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$, and Γ_N the group generated by $\mathscr{L}$. By Theorem 2.4, G has a lattice $\Gamma = \Gamma_N \rtimes \mathbf{Z}$.

Example 3.2 ([1], [9], Inoue surface of type S^+). Let $\mathfrak{g}_1$ be a solvable Lie algebra given by

$$\mathfrak{g}_1 = \mathrm{span}\{T, X, Y, Z\}$$

with nontrivial structure equations

$$[T, X] = X, \quad [T, Y] = -Y, \quad [X, Y] = Z.$$

Let $\mathfrak{n} = \mathrm{span}\{X, Y, Z\}$. Let G_1 be the simply connected solvable Lie group corresponding to $\mathfrak{g}_1$, and $N \subset G_1$ the simply connected nilpotent Lie group corresponding to $\mathfrak{n}$.

Let $M \in SL(2, \mathbf{Z})$ be a unimodular matrix given by

$$M = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Then the characteristic polynomial of M is $f(x) = x^2 - nx + 1$. Let λ, λ^{-1} be the characteristic roots. Take $t_0 = \log \lambda$, i.e., $e^{t_0} = \lambda$. Let $P = \begin{pmatrix} 1 & \lambda \\ 1 & \lambda^{-1} \end{pmatrix}$. Then $PMP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Let $\mathfrak{m}_0 = \mathrm{span}\{X, Y\}$, and $\mathfrak{m}_1 = \mathrm{span}\{Z\}$ (cf. Lemma 2.5). Put $(\tilde{X}, \tilde{Y}) = (X, Y)P$. Then $M \in SL(2, \mathbf{Z})$ is the matrix representation of $\exp ad(t_0 T)|_{\mathfrak{m}_0}$ with respect to $\tilde{X}$, $\tilde{Y}$, and

$$[\tilde{X}, \tilde{Y}] = [X + Y, \lambda X + \lambda^{-1} Y] = (\lambda^{-1} - \lambda)Z = |P|Z.$$

Put $\tilde{Z} = |P|Z$. Then $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ is a basis of the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ with respect to which the constants of structure are rational, and the matrix representation of $\exp ad(t_0 T)|_{\mathfrak{n}}$ is an unimodular integer matrix. Let $\mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$, and Γ_N the group generated by $\mathscr{L}$. By Theorem 2.4, G_1 has a lattice $\Gamma = \Gamma_N \rtimes t_0\mathbf{Z}$.

Next, we express the above lattice in G_1 explicitly. The solvable Lie group G_1 can be written as

$$G_1 = \left\{ \left. \begin{pmatrix} 1 & -\frac{1}{2}ye^t & \frac{1}{2}xe^{-t} & 0 & z \\ 0 & e^t & 0 & 0 & x \\ 0 & 0 & e^{-t} & 0 & y \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x, y, z \in \mathbf{R} \right\} \cong N \ltimes \mathbf{R}^1.$$

Thus, the solvable Lie group G_1 is isomorphic to $(\mathbf{R}^2 \times \mathbf{R}^1 \times \mathbf{R}^1, *)$, where

$$\begin{aligned}(\mathbf{x}_1, z_1, t_1) &* (\mathbf{x}_2, z_2, t_2)\\ &= \left(\mathbf{x}_1 + \begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}\mathbf{x}_2, z_1 + \frac{1}{2}{}^t\mathbf{x}_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^{t_1} & 0 \\ 0 & e^{-t_1} \end{pmatrix}\mathbf{x}_2 + z_2, t_1 + t_2\right).\end{aligned}$$

Then, a subgroup

$$\Gamma = \left\{\left(P\mathbf{m}, \frac{1}{2}|P|n\right) \,\middle|\, \mathbf{m} \in \mathbf{Z}^2, n \in \mathbf{Z}\right\} \rtimes t_0\mathbf{Z}$$

is a lattice in $(\mathbf{R}^2 \times \mathbf{R}^1 \times \mathbf{R}^1, *) \cong G_1$. Indeed, note that

$$\begin{aligned}&\exp(a\tilde{X} + b\tilde{Y} + c\tilde{Z})\exp(a'\tilde{X} + b'\tilde{Y} + c'\tilde{Z})\\ &\quad= \exp\left((a + a')\tilde{X} + (b + b')\tilde{Y} + \left(c + \frac{1}{2}ab' - \frac{1}{2}a'b + c'\right)\tilde{Z}\right),\end{aligned}$$

and this product coincides with the product of $N \subset G_1$. By the first canonical coordinates, $\exp(a\tilde{X} + b\tilde{Y} + c\tilde{Z}) \mapsto \left(\begin{pmatrix} a \\ b \end{pmatrix}, c\right)$, we have

$$\Gamma_N \to \left\{\left(\mathbf{m}, \frac{1}{2}n\right) \,\middle|\, \mathbf{m} \in \mathbf{Z}^2, n \in \mathbf{Z}\right\}.$$

Since $(\tilde{X}, \tilde{Y}) = (X, Y)P$, and $\tilde{Z} = |P|Z$, we see that Γ is a lattice in G_1.

Example 3.3 ([4], [9]). Let $\mathfrak{g}_2$ be a solvable Lie algebra given by

$$\mathfrak{g}_2 = \mathrm{span}\{T, X_1, X_2, X_3, Z_1, Z_2, Z_3\}$$

with nontrivial equations

$$\begin{aligned}&[X_1, X_2] = Z_3, && [X_2, X_3] = Z_1, && [X_3, X_1] = Z_2,\\ &[T, X_1] = -a_1X_1, && [T, X_2] = -a_2X_2, && [T, X_3] = -a_3X_3,\\ &[T, Z_1] = a_1Z_1, && [T, Z_2] = a_2Z_2, && [T, Z_3] = a_3Z_3,\end{aligned}$$

where a_1, a_2, a_3 are distinct real numbers such that $a_1 + a_2 + a_3 = 0$, and $t_0 \in \mathbf{R}$ and $m, n \in \mathbf{N}$ satisfy that $e^{a_1t_0}$, $e^{a_2t_0}$, $e^{a_3t_0}$ are distinct roots of the polynomial $f(x) = x^3 - mx^2 + nx - 1$ (cf. [9, Thereom 1]). Let $\mathfrak{n} = \mathrm{span}\{X_1, X_2, X_3, Z_1, Z_2, Z_3\}$. Let G_2 be the simply connected solvable Lie group corresponding to $\mathfrak{g}_2$, and $N \subset G_2$ the simply connected nilpotent Lie group corresponding to $\mathfrak{n}$.

Let B be a unimodular matrix given by

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -n \\ 0 & 1 & m \end{pmatrix}.$$

Then, the characteristic polynomial of B is $f(x) = x^3 - mx^2 + nx - 1$. Put $\lambda_i = e^{a_i t_0}$ $(i = 1, 2, 3)$, and $P = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$. Then $PBP^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$. Let $\mathfrak{m}_0 = \mathrm{span}\{X_1, X_2, X_3\}$, and $\mathfrak{m}_1 = \mathrm{span}\{Z_1, Z_2, Z_3\}$. Put

$$(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) = (X_1, X_2, X_3)P.$$

Then $B^{-1} \in SL(3, \mathbf{Z})$ is the matrix representation of $\exp ad(t_0 T)|_{\mathfrak{m}_0}$ with respect to $\tilde{X}_1$, $\tilde{X}_1$, $\tilde{X}_3$. Moreover, let

$$\tilde{Z}_3 = [\tilde{X}_1, \tilde{X}_2], \quad \tilde{Z}_1 = [\tilde{X}_2, \tilde{X}_3], \quad \tilde{Z}_2 = [\tilde{X}_3, \tilde{X}_1].$$

Then we can easily see that

$$(\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3) = (Z_1, Z_2, Z_3)|P|{}^t P^{-1},$$

and ${}^t B$ is the matrix representation of $\exp ad(t_0 T)|_{\mathfrak{m}_1}$ with respect to $\tilde{Z}_1$, $\tilde{Z}_2$, $\tilde{Z}_3$. Then $\tilde{X}_1$, $\tilde{X}_1$, $\tilde{X}_3$, $\tilde{Z}_1$, $\tilde{Z}_1$, $\tilde{Z}_2$, $\tilde{Z}_3$ is a basis of the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ with respect to which the constants of structure are rational, and the matrix representation of $\exp ad(t_0 T)|_{\mathfrak{n}}$ is an unimodular integer matrix. Let $\mathscr{L} = \mathrm{span}_{\mathbf{Z}}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3\}$, and Γ_N the group generated by $\mathscr{L}$. By Theorem 2.4, G_2 has a lattice $\Gamma = \Gamma_N \rtimes t_0\mathbf{Z}$.

Next, we express the above lattice in G_2 explicitly. The solvable Lie group G_2 corresponding to $\mathfrak{g}_2$ can be written as

$$G_2 = \left\{ \left. \begin{pmatrix} e^{a_1 t} & 0 & 0 & 0 & -\frac{1}{2}x_3 e^{-a_2 t} & \frac{1}{2}x_2 e^{-a_3 t} & 0 & z_1 \\ 0 & e^{a_2 t} & 0 & \frac{1}{2}x_3 e^{-a_1 t} & 0 & -\frac{1}{2}x_1 e^{-a_3 t} & 0 & z_2 \\ 0 & 0 & e^{a_3 t} & -\frac{1}{2}x_2 e^{-a_1 t} & \frac{1}{2}x_1 e^{-a_2 t} & 0 & 0 & z_3 \\ 0 & 0 & 0 & e^{-a_1 t} & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & e^{-a_2 t} & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & e^{-a_3 t} & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x_i, z_i \in \mathbf{R} \right\}$$

$$\cong N \ltimes \mathbf{R}.$$

Thus, the solvable Lie group G_2 is isomorphic to $(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^1, *)$, where

$$\begin{aligned} &(\mathbf{x}_1, \mathbf{z}_1, t_1) * (\mathbf{x}_2, \mathbf{z}_2, t_2) \\ &\quad = \left(\mathbf{x}_1 + \begin{pmatrix} e^{-a_1 t_1} & 0 & 0 \\ 0 & e^{-a_2 t_1} & 0 \\ 0 & 0 & e^{-a_3 t_1} \end{pmatrix} \mathbf{x}_2, \mathbf{z}_1 + A(\mathbf{x}_1) \begin{pmatrix} e^{-a_1 t_1} & 0 & 0 \\ 0 & e^{-a_2 t_1} & 0 \\ 0 & 0 & e^{-a_3 t_1} \end{pmatrix} \mathbf{x}_2 \right. \\ &\qquad \left. + \begin{pmatrix} e^{a_1 t_1} & 0 & 0 \\ 0 & e^{a_2 t_1} & 0 \\ 0 & 0 & e^{a_3 t_1} \end{pmatrix} \mathbf{z}_2, t_1 + t_2 \right), \end{aligned}$$

where $A(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$ for $\mathbf{x} = {}^t(x_1, x_2, x_3)$. Then, a subgroup

$$\Gamma = \left\{ \left(P\mathbf{m}, \frac{1}{2}|P|{}^tP^{-1}\mathbf{n} \right) \middle| \mathbf{m}, \mathbf{n} \in \mathbf{Z}^3 \right\} \rtimes t_0\mathbf{Z}$$

is a lattice in G_2.

Remark 3.4. The assumptions with respect to M, P, and B are not essential. Indeed, let $M \in SL(2, \mathbf{Z})$ be a unimodular matrix with distinct real positive eigenvalues, say, λ, $1/\lambda$. Take $t_0 = \log \lambda$. Let $P \in GL(2, \mathbf{R})$ be a matrix which satisfies

$$PMP^{-1} = \begin{pmatrix} e^{t_0} & 0 \\ 0 & e^{-t_0} \end{pmatrix}.$$

Then,

$$\Gamma = \left\{ \left(P\mathbf{m}, \frac{1}{2}|P|n \right) \middle| \mathbf{m} \in \mathbf{Z}^2, n \in \mathbf{Z} \right\} \rtimes t_0\mathbf{Z}$$

is a lattice in G_1. Similarly, we have the same argument in Example 3.3.

Remark 3.5. We can explain the isomorphisms of Lie groups in [9, pp. 3127, pp. 3132]. Let V^n be an n dimensional real vector space, and $B : V \to \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R}^m)$ a linear mapping. We define a multiplication on the set $N(B) = V \times \mathbf{R}^m$ by

$$(v_1, z_1) * (v_2, z_2) = (v_1 + v_2, z_1 + z_2 + (B(v_1))(v_2)), \quad v_i \in V,\ z_i \in \mathbf{R}^m \ (i = 1, 2).$$

It is straightforward to verify that $(v, z)^{-1} = (-v, -z + (B(v))(v))$. Since $Z = \{0\} \times \mathbf{R}^m$ is a normal subgroup and $N(B)/Z \cong V$ is abelian, $N(B)$ is a 2-step nilpotent Lie group. If $v = {}^t(x_1, \ldots, x_n) = \mathbf{x}$ and $B(\mathbf{x}) = (a_{ij}(\mathbf{x}))$, $i, j = 1, \ldots, n$ relative to a basis of V, then

$$(\mathbf{x}, z) \mapsto \begin{pmatrix} I_m & B(\mathbf{x}) & z \\ 0 & I_n & \mathbf{x} \\ 0 & 0 & 1 \end{pmatrix},$$

where I_n is the $n \times n$ unit matrix, is a faithful representation of $N(B)$. Let $m = n$. Then we can write $B(\mathbf{x}) = A(\mathbf{x}) + S(\mathbf{x})$, where $A(\mathbf{x})$ is the alternate matrix and $S(\mathbf{x})$ is the symmetric matrix corresponding to $B(\mathbf{x})$, respectively. Note that we can consider a 2-step nilpotent Lie group $N(A)$. Let a subscript 0 denote that the element is in $N(A)$ and a subscript 1 denote that the element is in $N(B) = N(A + S)$. Let

$$\pi_S((\mathbf{x}, z)_0) = \left(\mathbf{x}, z + \frac{1}{2}(S(\mathbf{x}))(\mathbf{x}) \right)_1.$$

If $(S(\mathbf{x}_1))(\mathbf{x}_2) = (S(\mathbf{x}_2))(\mathbf{x}_1)$ for each $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbf{R}^n$, then π_S is an isomorphism.

In the case of $m = 1$, B can be considered as a bilinear form. Hence, we write $(B(v_1))(v_2) = B(v_1, v_2)$. If $v = {}^t(x_1, \dots, x_n) = \mathbf{x}$ and $B = (a_{ij})$, $i, j = 1, \dots, n$ relative to a basis of V, then we can write the above representation as

$$(\mathbf{x}, z) \mapsto \begin{pmatrix} 1 & {}^t\mathbf{x}B & z \\ 0 & I_n & \mathbf{x} \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $B = A + S$, where A is the alternating bilinear form and S is the symmetric bilinear form corresponding to $B(\mathbf{x})$, respectively. Then we see that π_S is always an isomorphism (see [3, pp. 1–pp. 2] for $m = n = 1$).

For example, let G be the following solvable Lie group.

$$G = N(B) \rtimes \mathbf{R} = \left\{ \left. \begin{pmatrix} 1 & 0 & xe^{-t} & 0 & z \\ 0 & e^t & 0 & 0 & x \\ 0 & 0 & e^{-t} & 0 & y \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x, y, z \in \mathbf{R} \right\},$$

where

$$N(B) = \left\{ \left. \begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbf{R} \right\}.$$

Hence, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, the alternate part A of B is $\dfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, we have

$$G \cong N(A) \rtimes \mathbf{R} = \left\{ \left. \begin{pmatrix} 1 & -\frac{1}{2}ye^t & \frac{1}{2}xe^{-t} & 0 & z \\ 0 & e^t & 0 & 0 & x \\ 0 & 0 & e^{-t} & 0 & y \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| t, x, y, z \in \mathbf{R} \right\}$$

because $\exp ad(tT) \circ \pi_S = \pi_S \circ \exp ad(tT)$ for any $t \in \mathbf{R}$.

Similarly, let $B(\mathbf{x}) = \begin{pmatrix} 0 & 0 & x_2 \\ x_3 & 0 & 0 \\ 0 & x_1 & 0 \end{pmatrix}$. Then, $A(\mathbf{x}) = \dfrac{1}{2}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$.
Thus, we have an isomorphism in [9, pp. 3127].

4. Lattice of a family of solvable Lie groups

Let $B_1, \dots, B_{n-1} \in SL(n, \mathbf{Z})$ be matrices which satisfy that $B_iB_j = B_jB_i$ for each i, j, and each eigenvalue is positive. Let $P \in GL(n, \mathbf{R})$ be a matrix such that

$$PB_iP^{-1} = \begin{pmatrix} \alpha_1^i & & 0 \\ & \ddots & \\ 0 & & \alpha_n^i \end{pmatrix} = \begin{pmatrix} e^{\log \alpha_1^i} & & 0 \\ & \ddots & \\ 0 & & e^{\log \alpha_n^i} \end{pmatrix}$$

for each i (note that $\sum_{k=1}^{n} \log \alpha_k^i = 0$). If $\begin{pmatrix} \log \alpha_1^i \\ \vdots \\ \log \alpha_{n-1}^i \end{pmatrix} \in \mathbf{R}^{n-1}$ $(i = 1, \dots, n-1)$ are linearly independent, then a solvable Lie group

$$G = \left\{ \left. \begin{pmatrix} e^{t_1} & 0 & \cdots & 0 & 0 & x_1 \\ 0 & \ddots & & \vdots & 0 & \vdots \\ \vdots & & e^{t_{n-1}} & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & e^{-(t_1+\cdots+t_{n-1})} & 0 & x_n \\ 0 & \cdots & \cdots & 0 & I_{n-1} & \mathbf{t} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} \right| \mathbf{t} \in \mathbf{R}^{n-1}, x_1, \dots, x_n \in \mathbf{R} \right\},$$

where $\mathbf{t} = {}^t(t_1, \dots, t_{n-1})$, has a lattice. Indeed, note that $G \cong \mathbf{R}^n \rtimes_\varphi \mathbf{R}^{n-1}$, where $\varphi(t_1, \dots, t_{n-1}) = \mathrm{diag}(e^{t_1}, \dots, e^{t_{n-1}}, e^{-(t_1+\cdots+t_{n-1})})$. Then,

$$\Gamma = \left\{ P \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \middle| m_i \in \mathbf{Z} \right\} \rtimes_\varphi \left(\mathbf{Z} \begin{pmatrix} \log \alpha_1^1 \\ \vdots \\ \log \alpha_{n-1}^1 \end{pmatrix} \times \cdots \times \mathbf{Z} \begin{pmatrix} \log \alpha_1^{n-1} \\ \vdots \\ \log \alpha_{n-1}^{n-1} \end{pmatrix} \right)$$

is a lattice in $G \cong \mathbf{R}^n \rtimes_\varphi \mathbf{R}^{n-1}$ by Thereom 2.4.

Remark 4.1. The solvable Lie group has a left invariant contact form.

In the case of $n = 3$, if $\dfrac{\log \alpha_1^1}{\log \alpha_2^1} \neq \dfrac{\log \alpha_1^2}{\log \alpha_2^2}$, then the solvable Lie group G has a lattice.

Let

$$B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -5 \\ 0 & 1 & 6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -4 & -4 & -3 \\ 21 & 16 & 11 \\ -4 & -3 & -2 \end{pmatrix}.$$

Then, $B_1, B_2 \in SL(3, \mathbf{Z})$ satisfy $B_1B_2 = B_2B_1$, and $\dfrac{\log \alpha_1^1}{\log \alpha_2^1} \neq \dfrac{\log \alpha_1^2}{\log \alpha_2^2}$ (see [5, Proposition 4.4]).

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