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A CONSTRUCTION OF MAXIMAL COMMUTATIVE SUBALGEBRA OF MATRIX ALGEBRAS

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ABSTRACT. Let (B, m_B, k) be a maximal commutative k-subalgebra of $M_m(k)$. Then, for some element $z \in Soc(B)$, a k-algebra $R = B[X,Y]/I$, where $I = (m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$ will create an interesting maximal commutative k-subalgebra of a matrix algebra which is neither a C_1-construction nor a C_2-construction. This construction will also be useful to embed a maximal commutative k-subalgebra of matrix algebra to a maximal commutative k-subalgebra of a larger size matrix algebra.

1. Introduction

Let (B, m_B, k) be a maximal commutative k-subalgebra of $M_m(k)$. In this paper, we are interested in the following problem:

"How can we construct a maximal commutative k-subalgebra (R, m, k) of $M_n(k)$ for some n with $m < n$?"

In [2], Brown introduced a construction to produce a maximal commutative k-subalgebra (R, m, k) of $M_{m+1}(k)$ from (B, m_B, k). In this paper, we will present a construction to produce a maximal commutative k-subalgebra (R, m, k) of $M_{m+2}(k)$ from (B, m_B, k). In fact, the k-subalgebra R has dimension two more than the dimension of B. We will call this construction a C_2^2-construction.

Moreover, we will show the C_1-construction (C_2-construction) does not imply the C_2^2-construction in the next section and we can conclude that C_2^2-construction is another construction to produce maximal commutative k-subalgebras of matrix algebra.

Recall the C_1-construction and C_2-construction given by Brown and Call in [1] and [2].

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DEFINITION 1.1. Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity and N a finitely generated faithful B-module. Then $R = B \oplus N^\ell$ is a commutative k-algebra and $M = B^\ell \oplus N$ is a faithful R-module via the following multiplications:

$$\alpha(b, n_1, \ldots, n_\ell) = (\alpha b, \alpha n_1, \ldots, \alpha n_\ell)$$
$$(b, n_1, \ldots, n_\ell)(b', n'_1, \ldots, n_\ell) = (bb', n_1 b' + n'_1 b, \ldots, n_\ell b' + n'_\ell b)$$
$$(b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell) = (b_1 b, \ldots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i).$$

Moreover, it is known that $R \cong Hom_R(M, M)$ via the regular representation. Thus, R is isomorphic to a maximal commutative subalgebra of $M_n(k)$, where $n = dim_k(M)$. The k-algebra R of this form is called a C_1-construction.

The next theorem presents an equivalent condition to be a C_1-construction and the proof can be found in [1].

THEOREM 1.2. *Let (R, m, k) be a commutative local k-algebra. Then, R is a C_1-construction if and only if there is an ideal I satisfying the following conditions:*
(1) $Ann_R(I) = I$
(2) $0 \to I \to R \to R/I \to 0$ *splits as k-algebras.*

Throughout this paper, the socle of an algebra R will be denoted by $Soc(R)$ and the following theorem can be found in [2].

THEOREM 1.3. *Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity and N a finitely generated faithful B-module. Suppose $B \cong Hom_B(N, N)$ via the regular representation. Then there exists an element $z \in Soc(B)$ with $dim_k(Nz) = 1$.*

The following definition can be found in [3] and is a kind of generalization of the definition of C_2-construction in [2].

DEFINITION 1.4. Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity. If $R \cong B[X]/(m_B X, X^p - z)$ for some $z \in Soc(B) - \{0\}$ and a positive integer $p > 1$, then the k-algebra R of this form is called a C_2-construction.

Here is an equivalent condition to be a C_2-construction and can be found in [3].

THEOREM 1.5. *Let (R, m, k) be a commutative local k-algebra. Then, R is a C_2-construction if and only if R contains a commutative*

k-subalgebra (B, m_B, k) and an element $x \in m$ satisfying the following conditions:

(1) $0 \neq x^p \in Soc(B)$ *for some positive integer* $p > 1$,

(2) $m_B x = (0)$,

(3) $dim_k(R) = dim_k(B) + p - 1$.

2. C_2^2-construction

In this section, we will introduce a method to produce a maximal commutative k-subalgebra (R, m, k) of $M_{m+2}(k)$ from a maximal commutative k-subalgebra (B, m_B, k) of $M_m(k)$.

THEOREM 2.1. *Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity and N a finitely generated faithful B-module. Suppose $B \cong Hom_B(N, N)$ via the regular representation. Let $R = B[X, Y]/(m_B X, m_B Y, X^2 - z, Y^2 - z, XY)$ and let $z \in Soc(B) - \{0\}$ with $dim_k(Nz) = 1$. If we let $M = N \oplus Nz \oplus Nz$, then the k-algebra R is isomorphic to $Hom_R(M, M)$ via the regular representation. In other words, R is isomorphic to a maximal commutative k-subalgebra of $M_n(k)$, where $n = dim_k(M)$.*

Proof. Obviously, $M = N \oplus Nz \oplus Nz$ is a $B[X, Y]$-module via the following operations:

$$(n, n_1 z, n_2 z)b = (nb, n_1 zb, n_2 zb)$$

$$(n, n_1 z, n_2 z)X = (n_1 z, nz, n_2 z^2) = (n_1 z, nz, 0)$$

$$(n, n_1 z, n_2 z)Y = (n_2 z, n_1 z^2, nz) = (n_2 z, 0, nz)$$

for all $n, n_1, n_2 \in N$ and $b \in B$. If we let x and y be the images of X and Y in R, then M is an R-module via the following operations:

$$(n, n_1 z, n_2 z)x = (n_1 z, nz, n_2 z^2) = (n_1 z, nz, 0)$$

$$(n, n_1 z, n_2 z)y = (n_2 z, n_1 z^2, nz) = (n_2 z, 0, nz).$$

Now let

$$(n, 0, 0)(b + \alpha x + \beta y) = (0, 0, 0)$$

for $n \in N$ and $\alpha, \beta \in k$. Then

$$(nb, n\alpha z, n\beta z) = (0, 0, 0)$$

and hence by the faithfulness of N, we obtain

$$b = 0, \quad \alpha = \beta = 0,$$

which implies M is a finitely generated faithful R-module.

Let $f \in Hom_R(M, M)$ and define $g : N \to M$ and $h : M \to N$ by

$$g(n) = (n, 0, 0), \quad h(n, n_1z, n_2z) = n$$

for $n, n_1, n_2 \in N$. Then, obviously g and h are B-module homomorphisms and hence the composition map $\pi = hfg$ is a B-module homomorphism.

Since $B \cong Hom_B(N, N)$ via the regular representation, $\pi = \mu_a$ for some $a \in B$. Thus,

$$h(f(n, 0, 0)) = \pi(n) = \mu_a(n) = na.$$

This implies that there are two functions

$$\phi_1 : N \to Nz, \ \ \phi_2 : N \to Nz$$

such that

$$f(n, 0, 0) = (na, \phi_1(n), \phi_2(n)).$$

Then, it is easy to show that ϕ_1 and ϕ_2 are B-module homomorphisms.

Since $dim_k(Nz) = 1$, there exists an element $n' \in N$ such that $\{n'z\}$ is a basis of k-vector space Nz. Thus, there exist $\gamma_1, \gamma_2 \in k$ such that

$$\phi_1(n') = \gamma_1 n'z, \quad \phi_2(n') = \gamma_2 n'z.$$

Then, $a + \gamma_1 x + \gamma_2 y \in R$ and we want to show

$$f = \mu_{a+\gamma_1 x+\gamma_2 y}.$$

To prove this, it is enough to show the following two identities:

(1) $f(0, n_1z, n_2z) = (\gamma_1 n_1 z + \gamma_2 n_2 z, n_1 az, n_2 az)$,

(2) $\phi_1(n) = \gamma_1 nz, \quad \phi_2(n) = \gamma_2 nz.$

It suffices to show the identity (1) for $n_1 = n', n_2 = n'$. In fact,

$$\begin{aligned} f(0, n_1z, n_2z) &= f((n_1, 0, 0)x + (n_2, 0, 0)y) \\ &= f((n_1, 0, 0))x + f((n_2, 0, 0))y \\ &= (n_1a, \phi_1(n_1), \phi_2(n_1))x + (n_2a, \phi_1(n_2), \phi_2(n_2))y \\ &= (\phi_1(n_1), n_1az, \phi_2(n_1)z) + (\phi_2(n_2), \phi_1(n_2)z, n_2az) \\ &= (\gamma_1 n_1 z, n_1az, 0) + (\gamma_2 n_2 z, 0, n_2az) \\ &= (\gamma_1 n_1 z + \gamma_2 n_2 z, n_1az, n_2az). \end{aligned}$$

Thus, identity (1) is satisfied.

For identity (2), note that

$$\begin{aligned} (\gamma_1 nz, naz, 0) &= f(0, nz, 0) = f((n, 0, 0)x) = f(n, 0, 0)x \\ &= (na, \phi_1(n), \phi_2(n))x = (\phi_1(n), naz, \phi_2(n)z) \end{aligned}$$

and from these identities, we obtain

$$\phi_1(n) = \gamma_1 nz.$$

Similarly, we have the following identities:

$$\begin{aligned}(\gamma_2 nz, 0, naz) &= f(0,0,nz) = f((n,0,0)y) = f(n,0,0)y \\ &= (na, \phi_1(n), \phi_2(n))y = (\phi_2(n), \phi_1(n)z, naz).\end{aligned}$$

Thus we have

$$\phi_2(n) = \gamma_2 nz.$$

The identity (2) is thus satisfied and finally we obtain

$$\begin{aligned} & f(n, n_1z, n_2z) \\ = \; & (na + n_1\gamma_1 z + n_2\gamma_2 z, n_1az + n\gamma_1 z, n_2az + n\gamma_2 z) \\ = \; & (na, n_1az, n_2az) + (n_1\gamma_1 z, n\gamma_1 z, 0) + (n_2\gamma_2 z, 0, n\gamma_2 z) \\ = \; & (na, n_1az, n_2az) + (n\gamma_1, n_1\gamma_1 z, n_2\gamma_1 z)x \\ & + (n\gamma_2, n_1\gamma_2 z, n_2\gamma_2 z)y \\ = \; & (n, n_1z, n_2z)(a + \gamma_1 x + \gamma_2 y) \\ = \; & \mu_{a+\gamma_1 x+\gamma_2 y}(n, n_1z, n_2z).\end{aligned}$$

Therefore, we have the following result:

$$f = \mu_{a+\gamma_1 x+\gamma_2 y}.$$

Since M is a faithful R-module, R is isomorphic to $Hom_R(M, M)$ via the regular representation and hence R is isomorphic to a maximal commutative k-subalgebra of $M_n(k)$, where $n = dim_k(M)$.

We will call the k-algebra R of the form in Theorem 2.1 a C_2^2-construction.

If (B_1, m_{B_1}, k) is a commutative k-algebra which is isomorphic to k-algebra (B, m_B, k). Then, B-module N is a B_1-module via $nb_1 = n\phi(b)$, where $\phi : B \to B_1$ is an isomorphism from B to B_1 and $\phi(b) = b_1$. Thus, the following corollary can be proved. □

COROLLARY 2.2. *Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity and N a finitely generated faithful B-module. Suppose B is isomorphic to $Hom_B(N, N)$ via the regular representation. Let (B_1, m_{B_1}, k) be a commutative k-algebra which is isomorphic to B. If we let $R = B_1[X, Y]/(m_{B_1}X, m_{B_1}Y, X^2 - z, Y^2 - z, XY)$, $z \in Soc(B_1) - \{0\}$ with $dim_k(Nz) = 1$, and $M = N \oplus Nz \oplus Nz$, then the k-algebra R is isomorphic to $Hom_R(M, M)$ via the regular representation. Thus, R is isomorphic to a maximal commutative k-subalgebra of $M_n(k)$, where $n = dim_k(M)$.*

REMARK 2.3. With the C_2^2-construction, a maximal commutative k-subalgebra B of $M_m(k)$ with $dim_k(B) = t$ can be embedded in a maximal commutative k-subalgebra R of $M_{m+2}(k)$ with $dim_k(R) = t{+}2$.

Moreover, if $t < m$, then we can construct a maximal commutative k-subalgebra of matrix algebra whose dimension is not greater than the size of the matrix by applying C_2^2-construction.

For example, the k-algebra R in [4] by Courter is a maximal commutative k-subalgebra of $M_{14}(k)$ whose dimension is 13. Now, by applying C_2^2-construction successively, we can construct maximal commutative k-subalgebras $R_1, R_2, \ldots, R_t$ of $M_{16}(k), M_{18}(k), \ldots, M_{14+2t}(k)$ for $t \in \mathbb{N}$, respectively. Here, the dimension of R_t is obviously $13 + 2t$ for each $t \in \mathbb{N}$.

Here is an example of a C_2^2-construction. We will let E_{ij} be the (i, j)-th matrix unit.

EXAMPLE 2.4. Let B be a local k-subalgebra of $M_4(k)$ having maximal ideal consisting of the following matrices:

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix}$$

for $a, b, c \in k$. Then, B is a local maximal commutative k-subalgebra of $M_4(k)$.

Note that we can embed B into $M_6(k)$ via the following k-algebra homomorphism:

$$f\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix} + \alpha I_4\right] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ c & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \alpha I_6.$$

If $B_1 = f(B)$, then the map $f : B \longrightarrow B_1$ is a k-algebra isomorphism. Now let

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, the following identities hold:

$$m_{B_1} X = (0) = m_{B_1} Y, \;\; X^2 = E_{41} = Y^2, \;\; XY = 0$$

and hence

$$(m_{B_1}X, m_{B_1}Y, X^2 - E_{41}, Y^2 - E_{41}, XY) = (0).$$

Thus, the k-algebra R is given as follows:

$$R = B_1[X,Y] = B_1[X,Y]/(m_{B_1}X, m_{B_1}Y, X^2 - E_{41}, Y^2 - E_{41}, XY).$$

In fact,

$$R = \{r + \alpha I_6 \mid \alpha \in k\},$$

where r is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ c & b & a & 0 & d & e \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $a, b, c, d, e \in k$. Thus, the k-algebra R is a C_2^2-construction. Therefore, we construct a maximal commutative k-subalgebra R which is a C_2^2-construction and of dimension 6 from a maximal commutative k-subalgebra B of dimension 4.

REMARK 2.5. In the above Example, a maximal commutative k-subalgebra B of $M_4(k)$ is embedded to a maximal commutative k-subalgebra R of $M_6(k)$.

From the definition of C_2^2-construction, the following property can be obtained.

THEOREM 2.6. *Let (R, m, k) be a finite dimensional local commutative k-algebra. Then, R is a C_2^2-construction if and only if there exist a commutative k-subalgebra (B, m_B, k) and elements $x, y \in m$ satisfying the following properties:*

(1) $x^2 = y^2 \in Soc(B) - \{0\}$,
(2) $xy = 0$,
(3) $m_Bx = (0) = m_By$,
(4) $dim_k(R) = dim_k(B) + 2$.

Proof. Suppose R is a C_2^2-construction. Then, by the definition of C_2^2-construction, there exist a finite dimensional local commutative k-algebra (B, m_B, k) and a finitely generated faithful B-module N such that

$$R = B[X,Y]/(m_BX, m_BY, X^2 - z, Y^2 - z, XY)$$

for some $z \in Soc(B) - \{0\}$ with $dim_k(Nz) = 1$. Let x and y be the image of X and Y, respectively. Then, the conditions (1),(2),(3) and (4) can be shown by straightforward calculations. Conversely, suppose there exist a k-subalgebra B and elements x and y in m such that the given conditions are satisfied. Let $x^2 = y^2 = z \in Soc(B)$ and define a map

$$\psi : B[X,Y]/(m_BX, m_BY, X^2 - z, Y^2 - z, XY) \longrightarrow R$$

by

$$\psi(b + I) = b,\ \psi(X + I) = x,\ \psi(Y + I) = y,$$

where $b \in B$ and $I = (m_BX, m_BY, X^2 - z, Y^2 - z, XY)$. Then, obviously ψ is a k-algebra homomorphism. Suppose $b + cX + dY + I \in ker\psi$. Then, we have

$$\psi(b + cX + dY + I) = b + cx + dy = 0.$$

Here, we may assume $c, d \in k$ since $m_Bx = m_By = (0)$. If $b \neq 0$, then $b \notin m_B$. For, if $b \in m_B$, then

$$cz = bx + cx^2 + dxy = 0,\ \ dz = by + cxy + dy^2 = 0$$

and $c = 0 = d$. This implies $b = 0$ which is impossible. Thus, $b \notin m_B$ and the element $b + cx + dy$ is a unit which is impossible. Thus, $b = 0$ and $cx + dy = 0$. If $c \neq 0$, then c^{-1} exists and hence $x + (c^{-1}d)y = 0$. By multiplying by x each side, we get

$$0 = x^2 + (c^{-1}d)xy = z.$$

This is also impossible and we should have $c = 0$. Finally we have $dy = 0$ and we can show $d = 0$ from the identity $dz = dy^2 = 0$. Thus,

$$b = c = d = 0$$

and this implies $b + cX + dY + I = I$ and ψ is a monomorphism. Since $dim_k(im(\psi)) = dim_k(B[x,y])$ and $dim_k(R) = dim_k(B) + 2$, the map ψ should be an isomorphism and we can conclude the k-algebra R is a C_2^2-construction. □

Now, in the rest of this paper, we want to prove C_i-construction does not imply C_2^2-construction for each $i = 1, 2$.

COROLLARY 2.7. *C_1-construction does not imply C_2^2-construction.*

Proof. Let (R, m, k) be a Schur algebra of size 4. That is, the element $r \in R$ is of the following form:

$$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha I_4$$

for some $a, b, c, d \in k$. It is known in [1] that the Schur algebra of size 4 is a C_1-construction. Note that the index of m is 2 and hence by Theorem 2.6, the k-algebra R can't be a C_2^2-construction. For, if R is a C_2^2-construction, then there exist elements x and y in m whose squares are not zero. But, this is impossible since the index of m is 2. □

COROLLARY 2.8. *C_2-construction does not imply C_2^2-construction.*

Proof. Let $R = m \oplus kI_4$ be a maximal commutative k-subalgebra of $M_4(k)$ such that each element $r \in m$ is of the following form:

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \end{pmatrix}$$

for some $a, b, c \in k$. If we let $B = k[E_{41}]$ and $s = E_{21} + E_{32} + E_{43}$, then

(1) $s^3 \in Soc(B) - \{0\}$
(2) $m_B s = (0)$
(3) $dim_k(R) = dim_k(B) + 2$.

This implies R is a C_2-construction.

Now, suppose R is a C_2^2-construction. Then R contains a k-subalgebra B and an element $x \in m$ satisfying the following conditions:

$$x^2 \in Soc(B) - \{0\}, \;\; m_B x = (0).$$

If we let

$$u = E_{21} + E_{32} + E_{43}, \;\; v = E_{31} + E_{42}, \;\; w = E_{41},$$

then $x = au + bv + cw$ for some $a, b, c \in k$. Note that

$$x^2 = a^2 v + abw \in Soc(B) \subseteq m_B.$$

Since $m_B x = (0)$, we obtain $a = 0$ from the identities

$$a^3 E_{41} = x^3 = 0.$$

Therefore, $x = bv + cw$ and $x^2 = 0$ which is impossible since $x^2 \in Soc(B) - \{0\}$. Now, we can conclude that the k-algebra R is not a C_2^2-construction. □

REMARK 2.9. Example 2.4 and the proof of Corollary 2.8 show that a C_2^2-construction can be constructed from a maximal commutative k-subalgebra that is not a C_2^2-construction.

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