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# A CONTINUED FRACTION EXPANSION FOR A $q$-TANGENT FUNCTION

MARKUS FULMEK

Abstract. We prove a continued fraction expansion for a certain $q$–tangent function that was conjectured by Prodinger.

## 1. Introduction

In [4], Prodinger defined the following $q$–trigonometric functions

$$\sin_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{[2n+1]_q!} q^{n^2},$$
$$\cos_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{[2n]_q!} q^{n^2}.$$

Here, we use standard $q$–notation:

$$[n]_q := \frac{1-q^n}{1-q},\ [n]_q! := [1]_q[2]_q \dots [n]_q,$$
$$(a;q)_n := (1-a)(1-aq)\dots(1-aq^{n-1}).$$

These $q$–functions are variations of Jackson's [2] $q$-sine and $q$-cosine functions.

For the $q$–tangent function $\tan_q = \frac{\sin_q}{\cos_q}$, Prodinger conjectured the following continued fraction expansion (see [4, Conjecture 10]):

$$-z\tan_q(z) = -\cfrac{z^2}{[1]_q q^0 - \cfrac{z^2}{[3]_q q^{-2} - \cfrac{z^2}{[5]_q q^1 - \cfrac{z^2}{[7]_q q^{-9} - \cdots}}}}. \qquad (1)$$

Here, the powers of $q$ are of the form $(-1)^{n-1} n(n-1)/2 - n + 1$.

The purpose of this note is to prove this statement. In our proof, we make use of the polynomials (see [3, §2, (11)]) $A_n(z)$ and $B_n(z)$, which

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are given recursively by

$$A_n(z) = b_n A_{n-1}(z) - z^2 A_{n-2}(z), \tag{2}$$

$$B_n(z) = b_n B_{n-1}(z) - z^2 B_{n-2}(z); \tag{3}$$

with initial conditions (see [3, §2, (12)])

$$A_{-1} = 1,\ B_{-1} = 0,\ A_0 = b_0,\ B_0 = 1,$$

where $b_0 = 0$, $b_n = [2n-1]_q q^{(-1)^{n-1} n(n-1)/2 - n + 1}$. As is well known (see [3, §2]), the continued fraction terminated after the term $b_n$ is equal to $\frac{A_n}{B_n}$, whence (1) follows from the assertion

$$A_n \cos_q + z B_n \sin_q = O(z^{2n+1}), \tag{4}$$

i.e., the leading $2n$ coefficients of $z$ vanish in (4).

In Section 2 we give a proof of (4) (and thus of (1)).

## 2. The proof

Both $A_n$ and $B_n$ are polynomials in $z^2$:

$$A_n(z) = \sum_j c_{n,j} z^{2j},\ B_n(z) = \sum_j d_{n,j} z^{2j}.$$

Observe that from the recursions (2) and (3) we obtain immediately

$$c_{n,k} = b_n c_{n-1,k} - c_{n-2,k-1} \text{ and } d_{n,k} = b_n d_{n-1,k} - d_{n-2,k-1}, \tag{5}$$

with initial conditions

$$c_{0,k} = d_{0,k} = c_{-1,k} = d_{-1,k} = c_{n,0} = d_{n,-1} = 0,\ c_{1,1} = -1,\ d_{0,0} = 1.$$

Given this notation, we have to prove the following assertion for the coefficients of $z^{2k}$ in (4): For $n \geq 1$, $0 \leq k \leq n$, there holds

$$\sum_{i=0}^{k} c_{n,i} \frac{(-1)^{k-i}}{[2k-2i]_q!} q^{(k-i)^2} + \sum_{i=0}^{k} d_{n,i} \frac{(-1)^{k-i-1}}{[2k-2i-1]_q!} q^{(k-i-1)^2} = 0. \tag{6}$$

In fact, we shall state and prove a slightly more general assertion:

**Lemma 1.** *Given the above definitions, we have for all $n \geq 1$, $k \geq 0$:*

$$\sum_{i=0}^{k-1} (-1)^i \frac{q^{(k-i-1)^2}}{[2k-2i-2]_q!} \left( c_{n,i+1} + \frac{d_{n,i}}{[2k-2i-1]_q} \right) = (-1)^n q^{(5+3(-1)^n - 12k - 4(-1)^n k + 8k^2 + 8n - 8kn + 4n^2 - 2(-1)^n n^2)/8} \times \frac{\prod_{s=k-n}^{k} [2s]_q}{[2k]_q!}. \tag{7}$$

Note that the left hand side of (7) is the same as in (6), and the right hand side of (7) vanishes for $0 \le k \le n$. Hence (6) (and thus Prodinger's conjecture) is an immediate consequence of Lemma 1.

*Proof.* We perform an induction on $k$ for arbitrary $n$.

The case $k = 0$ is immediate. For the case $k = 1$, observe that

$$-c_{n,1} = d_{n,0} = \prod_{s=1}^{n} b_n.$$

For the inductive step $(k-1) \to k$, we shall rewrite the recursions (5) in the following way:

$$c_{n,k} = -\sum_{i=0}^{n-2} \left( c_{i,k-1} \prod_{j=i+3}^{n} b_j \right), \; d_{n,k} = -\sum_{i=0}^{n-2} \left( d_{i,k-1} \prod_{j=i+3}^{n} b_j \right).$$

Substitution of these recursions into (7) and interchange of summations transform the identity into

$$\frac{q^{(k-1)^2}(1-[2k-1]_q)\prod_{s=1}^{n} b_s}{[2k-1]_q!} + \sum_{i=0}^{n-2} \left( \operatorname{rhs}(i,k-1) \prod_{j=i+3}^{n} b_j \right) = \operatorname{rhs}(n,k),$$

where $\operatorname{rhs}(n,k)$ denotes the right hand side of (7).

Now we use the induction hypothesis. As it turns out, factorization of powers of $q$ from $\left(\operatorname{rhs}(i,k-1)\prod_{j=i+3}^{n} b_j\right)$ yields the same power for $2i$ and $2i+1$, whence we can group these terms together. After several steps of simplification we arrive at the following identity:

$$\left( \sum_{j=0}^{\lceil \frac{n-4}{2} \rceil} \frac{(q^{-2k+6};q^4)_j (q^{-2k+4};q^4)_j (q^{17/2-k};q^4)_j (-q^{17/2-k};q^4)_j}{(q^7;q^4)_j (q^9;q^4)_j (q^{9/2-k};q^4)_j (-q^{9/2-k};q^4)_j} q^{(2k-1)j} \right)$$
$$\times \frac{(q;q^2)_n q(1-q^{2k-1})(1-q^{2k-2})(1-q^{9-2k})}{(1-q)(1-q^3)(1-q^5)} - (1-q^{2k-1})(q^3;q^2)_{n-1}$$
$$- (-1)^n q^{(-1+(-1)^n+2k-2(-1)^n k+2n-4kn+4n^2)/4} (q^{2k-2n};q^2)_n$$
$$+ (q;q^2)_n + \chi(n)(1-q^{2k-1}) q^{n(2n-2k+1)/2} (q^{2k-2n+2};q^2)_{n-1} = 0, \quad (8)$$

where $\chi(n) = 1$ for $n$ even and 0 for $n$ odd.

The sum can be evaluated by means of the very–well–poised ${}_6\phi_5$ summation formula [1, (2.7.1); Appendix (II.20)]:

$$\sum_{j=0}^{\infty} \frac{(a;q)_j(\sqrt{a}q;q)_j(-\sqrt{a}q;q)_j(b;q)_j(c;q)_j(d;q)_j}{(q;q)_j(\sqrt{a};q)_j(-\sqrt{a};q)_j(\frac{aq}{b};q)_j(\frac{aq}{c};q)_j(\frac{aq}{d};q)_j}\left(\frac{aq}{bcd}\right)^j = \frac{(aq;q)_\infty(\frac{aq}{bc};q)_\infty(\frac{aq}{bd};q)_\infty(\frac{aq}{cd};q)_\infty}{(\frac{aq}{b};q)_\infty(\frac{aq}{c};q)_\infty(\frac{aq}{d};q)_\infty(\frac{aq}{bcd};q)_\infty}. \quad (9)$$

The sum we are actually interested in does not extend to infinity, so we rewrite is as follows:

$$\sum_{j=0}^{\lceil\frac{n-4}{2}\rceil} s(n,k,j) = \sum_{j=0}^{\infty} s(n,k,j) - \sum_{j=\lceil\frac{n-2}{2}\rceil}^{\infty} s(n,k,j) = \sum_{j=0}^{\infty} s(n,k,j) - s(n,k,\lceil\frac{n-2}{2}\rceil)\sum_{j=0}^{\infty}\frac{s(n,k,j+\frac{n-2}{2})}{s(n,k,\lceil\frac{n-2}{2}\rceil)},$$

where $s(n,k,j)$ denotes the summand in (8). Now, replacing $q$ by $q^4$, $a$ by $q^{-2k+8a+9}$, $b$ by $q^{-2k+4a+4}$, $c$ by $q^{-2k+4a+6}$ and $d$ by $q^4$ in the summand of (9) gives $\frac{s(n,k,j+a)}{s(n,k,a)}$ times the fraction $\frac{(q^{-2k+8a+9};q^4)_j(q^4;q^4)_j}{(q^{-2k+8a+9};q^4)_j(q^4;q^4)_j}$, which cancels. So we obtain after some simplification:

$$\sum_{j=0}^{x} \frac{(q^{-2k+6};q^4)_j(q^{-2k+4};q^4)_j(q^{17/2-k};q^4)_j(-q^{17/2-k};q^4)_j}{(q^7;q^4)_j(q^9;q^4)_j(q^{9/2-k};q^4)_j(-q^{9/2-k};q^4)_j} q^{(2k-1)j} = \frac{(1-q^3)(1-q^5)}{(1-q^{2k-1})(1-q^{9-2k})}\left(1-q^{(2k-1)(x+1)}\frac{(q^{-2k+4};q^2)_{2x+2}}{(q^3;q^2)_{2x+2}}\right).$$

Substitution of this evaluation in (8) and simplification yield for both cases $n$ even ($n=2N$) and odd ($n=2N-1$) the same equation

$$(q^{-2k+2};q^2)_{2N-1} = -q^{-2(k-N)(2N-1)}(q^{2k-4N+2};q^2)_{2N-1},$$

which, of course, is true. This finishes the proof. □

## References


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Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria
*E-mail address*: Markus.Fulmek@Univie.Ac.At
*WWW*: http://www.mat.univie.ac.at/~mfulmek