A Continuity Property of the Entropy Density for Spin Lattice Systems

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Abstract. The entropy density of spin lattice systems is known to be a weak* upper semi-continuous functional on the set of the lattice invariant states. (It is even weak* discontinuous.) However we prove here that it is continuous with respect to the norm topology on those states.

I. Preliminaries

We consider a lattice \mathbb{Z}^d of N spin states per lattice site. By $\Lambda \subset \mathbb{Z}^d$ we will always mean a non-empty finite volume and by $V(\Lambda)$ the number of points in it.

To $\Lambda \subset \mathbb{Z}^d$ we associate the local algebra \mathcal{A}_{Λ} of observables:

$$\mathcal{A}_{\Lambda} = \mathcal{B}(\mathcal{H}_{\Lambda})$$
 where $\mathcal{H}_{\Lambda} = \bigotimes_{i \in \Lambda} \mathcal{H}_{i}$ and each \mathcal{H}_{i}

is an isomorphic copy of the *N*-dimensional Hilbert space \mathbb{C}^N . For $\Lambda_1 \subset \Lambda_2$ we trivially get an isometric embedding of \mathscr{A}_{Λ_1} in \mathscr{A}_{Λ_2} which maps A into $A \otimes 1_{\Lambda_2 \setminus \Lambda_1}$. This allows us to construct the C^* -algebra \mathscr{A} of quasilocal observables:

$$\mathscr{A} = \overline{\bigcup_{\Lambda \subset \mathbb{Z}^d} \mathscr{A}_{\Lambda}}^n$$
.

The natural translation mappings $\tau_x: \mathcal{A}_A \to \mathcal{A}_{A+x}, \ x \in \mathbb{Z}^d$ extend to a group $\{\tau_x | x \in \mathbb{Z}^d\}$ of automorphisms of \mathscr{A} . A state ω on \mathscr{A} is called lattice invariant if $\omega \circ \tau_x = \omega$ for all $x \in \mathbb{Z}^d$. Let \mathscr{E} denote the set of all lattice invariant states on \mathscr{A} .

For each state ω on \mathscr{A} and for each $\Lambda \subset \mathbb{Z}^d$ there exists a unique density matrix $\varrho_A \in \mathscr{B}(\mathscr{H}_A)$ such that $\forall A \in \mathscr{A}_A \ \omega(A) = \operatorname{Tr} \varrho_A A$.

The local entropy density of the state ω is given by

$$s_{\Lambda}(\omega) = -\frac{1}{V(\Lambda)} \operatorname{Tr} \varrho_{\Lambda} \log \varrho_{\Lambda}.$$

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The existence of $s(\omega) = \lim_{\Lambda \to \infty} s_{\Lambda}(\omega)$ has been proved for $\omega \in \mathscr{E}$ in the sense of growing cubes [1].

The continuity properties of the map $\omega \to s(\omega)$ are important with respect to the variational formulation of statistical mechanics [1] and explicit calculations of the entropy density [2]. It has been proved that the map is upper semi-continuous for the weak* topology on \mathscr{E} [1]; however it can easily be proved that it is not weak* continuous [2]. Here we prove the norm continuity.

II. A Continuity Property of the Entropy Density

Lemma 1 (Lidskii [3]). Let A, B be selfadjoint operators on the n-dimensional Hilbert space H. Denote by α_k , β_k and γ_k , k = 1, ..., n the repeated eigenvalues of A, B and A - B in ascending order. Then there exist numbers σ_{ki} , k, j = 1, ..., n such that:

1)
$$\sigma_{k,i} \geq 0$$
.

2)
$$\sum_{k}^{\infty} \sigma_{kj} = \sum_{i}^{\infty} \sigma_{kj} = 1$$
.

3)
$$\alpha_k - \beta_k = \sum_j \sigma_{kj} \gamma_j$$
.

Lemma 2. The function $f: [0, 1] \rightarrow [0, 1/e]: x \rightarrow f(x) = -x \log x$ satisfies the inequality:

$$|f(x) - f(y)| \le 2|y - x| + f(|y - x|).$$

Proof. For $0 \le x \le y \le 1$ we have:

$$|f(x) - f(y)|$$

$$= |-x(\log x - 1) + y(\log y - 1) + y - x|$$

$$\leq |-x(\log x - 1) + y(\log y - 1)| + y - x$$

$$= \left| \int_{x}^{y} dt \log t \right| + y - x$$

$$= -\int_{x}^{x + (y - x)} dt \log t + y - x$$

$$\leq -\int_{0}^{y - x} dt \log t + y - x = 2(y - x) + f(y - x). \quad \Box$$

Theorem. The function $s : \mathscr{E} \to [0, \log N] : \omega \to s(\omega)$ is continuous with respect to the norm topology on \mathscr{E} .

Proof. Let $\omega_1, \omega_2 \in \mathscr{E}$ and $\varrho_A^1, \varrho_A^2, A \in \mathbb{Z}^d$ the corresponding density matrices.

Denote by $\lambda_k^1, \lambda_k^2, \mu_k, k = 1, ..., N^{V(A)}$ the repeated eigenvalues of ϱ_A^1, ϱ_A^2 and $\varrho_A^1 - \varrho_A^2$ in ascending order. By Lemma 1

$$\begin{split} \sum_{k} |\lambda_{k}^{1} - \lambda_{k}^{2}| &= \sum_{k} |\sum_{j} \sigma_{kj} \mu_{j}| \leq \sum_{k} \sum_{j} \sigma_{kj} |\mu_{j}| \\ &= \sum_{j} |\mu_{j}| = \operatorname{Tr} |\varrho_{A}^{1} - \varrho_{A}^{2}| = \sup_{\substack{\|A\| \leq 1 \\ A \in \mathscr{A}_{A}}} |\operatorname{Tr} (\varrho_{A}^{1} - \varrho_{A}^{2}) A| \\ &= \sup_{\substack{\|A\| \leq 1 \\ A \in \mathscr{A}_{A}}} |(\omega_{1} - \omega_{2}) (A)| \leq \sup_{\substack{\|A\| \leq 1 \\ A \in \mathscr{A}}} |(\omega_{1} - \omega_{2}) (A)| \\ &= \|\omega_{1} - \omega_{2}\|. \end{split} \tag{1}$$

Using Lemma 2

$$\begin{split} |s_A(\omega_1) - s_A(\omega_2)| &= \frac{1}{|V(A)|} \left| \sum_k f(\lambda_k^1) - f(\lambda_k^2) \right| \\ &\leq \frac{1}{|V(A)|} \sum_k |f(\lambda_k^1) - f(\lambda_k^2)| \\ &\leq \frac{1}{|V(A)|} \sum_k 2|\lambda_k^1 - \lambda_k^2| + f(|\lambda_k^1 - \lambda_k^2|) \,. \end{split}$$

Let $\varepsilon_k = |\lambda_k^1 - \lambda_k^2|$, then

$$|s_{A}(\omega_{1}) - s_{A}(\omega_{2})| \leq \frac{1}{V(A)} \sum_{k} 2 \varepsilon_{k} + f(\varepsilon_{k})$$
 and by (1)
$$\sum_{k} \varepsilon_{k} = a \leq \|\omega_{1} - \omega_{2}\|.$$

The case a = 0 implies $\varepsilon_k = 0$ for all k and hence

$$|s_A(\omega_1) - s_A(\omega_2)| = 0.$$

If a > 0 then by the convexity of the logarithm

$$\frac{1}{V(\Lambda)} \sum_{k} 2 \, \varepsilon_{k} + f(\varepsilon_{k}) = \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \sum_{\varepsilon_{k} \neq 0} \frac{\varepsilon_{k}}{a} \log \frac{1}{\varepsilon_{k}}$$

$$\leq \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \log \sum_{\varepsilon_{k} \neq 0} \frac{1}{a} \leq \frac{2a}{V(\Lambda)} + \frac{a}{V(\Lambda)} \log \left(\frac{N^{V(\Lambda)}}{a}\right)$$

$$= \frac{2a - a \log a}{V(\Lambda)} + a \log N.$$

Hence

$$\begin{split} |s(\omega_1) - s(\omega_2)| &= \lim_{A \to \infty} |s_A(\omega_1) - s_A(\omega_2)| \\ &\leq \|\omega_1 - \omega_2\| \log N \,. \quad \Box \end{split}$$

References

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