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A continuous deformation algorithm on the product space of unit simplices

by

T.M. Doup and A.J.J. Talman

Abstract

A continuous deformation algorithm is introduced on $S \times [1,\infty)$, where S denotes the product space of unit simplices, with arbitrary grid refinement between two subsequent levels. The set $S \times [1,\infty)$ is triangulated in such a way that for each m, $m = 1,2,\ldots$, $S \times \{m\}$ is triangulated by the so-called V-triangulation. The algorithm starts by applying a variable dimension algorithm on S until an approximating simplex has been found on level 1. Then the algorithm follows a path of approximating simplices in $S \times [1,\infty)$, starting on level 1, until a certain level or a certain accuracy of a solution of the underlying problem has been reached. If the algorithm returns to level 1, then we again apply the variable dimension algorithm until a new approximating simplex is found on level 1, etc. We allow solutions to lie on the boundary of $S \times [1,\infty)$ in which case the algorithm, in general, will follow a path on the boundary of $S \times [1,\infty)$.

Keywords: triangulation, continuous deformation, homotopy, equilibrium

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1. Introduction

To compute equilibria or fixed points on the unit simplex $S^n = \{x \in R^{n+1} \mid \Sigma_{i=1}^{n+1} x_i = 1,\ x_i \geq 0,\ i = 1,\ldots,n+1\}$ several simplicial algorithms have been developed. In a simplicial subdivision of S^n into n-dimensional simplices such an algorithm searches for an n-simplex which yields an approximate solution. If the approximation is not good enough the simplicial subdivision is refined in the hope that the approximate solution found for the new subdivision is better, etc. The so-called variable dimension restart simplicial algorithms can start anywhere and find for a given simplicial subdivision within a finite number of steps an approximate solution by generating a sequence of adjacent simplices of varying dimension of the simplicial subdivision. If necessary these algorithms can be restarted in or close to the last found approximation for a finer subdivision to find a better one. The several variable dimension restart simplicial algorithms developed thusfar differ from each other by the underlying triangulation or simplicial subdivision of S^n and the number of rays along which the arbitrarily chosen starting point can be left. Simplicial algorithms with n+1 rays were introduced for the well known Q-triangulation of S^n in van der Laan and Talman [7], for the U-triangulation of the affine hull of S^n in van der Laan and Talman [8] and for the so-called V-triangulation of S^n in Doup and Talman [1]. Although the U-triangulation does not simplicially subdivide S^n itself this triangulation seems to be both in theory and in practice better than the Q-triangulation. The V-triangulation differs from both the U- and the Q-triangulation since it depends on the arbitrarily chosen starting point of the algorithm. In some way the V-triangulation is related to the K'triangulation of R^n originally proposed in Todd [13]. An algorithm with $2^{n+1}-2$ rays was recently proposed in Doup, van der Laan and Talman [2]. In this algorithm the V-triangulation underlies the algorithm. The other two triangulations do not seem to be appropriate for this algorithm with more than n+1 rays.

In van der Laan and Talman [10] the (n+1)-ray algorithm for both the Q- and U-triangulation has been generalized in order to compute equilibria or fixed points on the product space of several, say N, unit simplices S^{n_j}, $j = 1,\ldots,N$. This algorithm has $\Sigma_{j=1}^{N} (n_j+1)$ rays to leave the arbitrarily chosen starting point in S, one ray to each facet of S. Recently, Doup and Talman [1] introduced such an algorithm on S for the V-triangulation, generalized for S, with $\Pi_{j=1}^{N}(n_j+1)$ rays, one to each vertex of S. When applied for $N = 1$, $n_1 = n$, both algorithms simplify to the above mentioned algorithms on S^n with n+1 rays.

Instead of restarting a variable dimension simplicial algorithm on S^n, as soon as an approximating solution has been found, one can also continue the algorithm with the simplex yielding the approximating solution by embedding S^n into the set $S^n \times [1,\infty)$. This set is triangulated in such a way that for each m, $m = 1,2,\ldots$, S^n is triangulated on level m with mesh tending to zero if m goes to infinity. In this way a path of adjacent (n+1)-simplices of the triangulation of $S^n \times [1,\infty)$ is generated such that each generated simplex yields an approximate solution. Under some boundary condition, guaranteeing that the algorithm can not terminate in the boundary of $S^n \times [1,\infty)$, such an algorithm will exceed each level m, $m = 1,2,\ldots$, within a finite number of steps. As soon as some accuracy for the approximation is reached the algorithm can be stopped. Such algorithms are called homotopy or continuous deformation algorithms and were initiated in Eaves [3] for problems on S^n. However, the triangulation used in the latter algorithm only allows for a grid refinement between two subsequent levels of at most two. Arbitrary grid refinement algorithms were developed in van der Laan and Talman [9] and Shamir [12] for the Q- and U-triangulation. Continuous deformation algorithms on the product space of more than one unit simplex are thusfar unknown although both the Q- and the U-triangulation of S allow us to construct triangulations of $S \times [1,\infty)$. However the system of the $\Sigma_{j=1}^{N}(n_j+1)$-ray algorithm is not appropriate for continuation in $S \times [1,\infty)$ when on level 1 an approximation has been found.

In this paper we will show how the recently developed variable dimension restart algorithm on S described in [1] can be adapted to a continuous deformation algorithm on $S \times [1,\infty)$ with arbitrary grid refinement between two subsequent levels. The triangulation of $S \times [1,\infty)$ which underlies the algorithm is based on the V-triangulation of S itself whereas the system of equations in which the l.p. pivot steps are made coincides with the system of equations for the restart algorithm. To start the algorithm the variable dimension restart algorithm of [1] is applied in order to find an approximating simplex in S on level 1. Then the algorithm generates a path of adjacent approximating simplices in $S \times [1,\infty)$ by alternating l.p. pivot steps and replacement steps in the triangulation. As soon as the algorithm returns to $S \times \{1\}$, the restart algorithm is again applied in order to find a new approximating simplex in S on level 1. Then the algorithm continues with this simplex in $S \times \{1\}$ generating again a path of adjacent approximating simplices in $S \times [1,\infty)$, etc. Since the number of simplices in $S \times [1,m]$ is finite for each m, $m = 1,2,\ldots$, the algorithm must exceed each level m within a finite number of steps. The algorithm can be terminated when some accuracy is reached or a simplex on some specific level has been generated. Since we will not assume that the boundary condition holds for the underlying equilibrium or fixed point problem, we allow the algorithm to generate lower dimensional approximating simplices on the boundary of $S \times [1,\infty)$ so that the algorithm will generate in general a path of adjacent simplices of variable dimension. Restart algorithms on S which allow for these general type of problems were developed for the Q-triangulation in Freund [4] and in van der Laan, Talman and Van der Heyden [11], and for the V-triangulation in [1].

The advantage of a continuous deformation algorithm seems to be that as soon as an approximating simplex on say level m, $m \geq 1$, is found more information is used to find an approximating simplex on level m+1 when compared to a restart algorithm. More precisely a restart algorithm only uses the information of the approximating solution whereas a continuation algorithm uses the information of the whole approximating simplex which includes the function values of the vertices of this simplex. Although this information might be of little value when the grid size of the triangulation is large, it could accelerate the algorithm considerable when the mesh becomes smaller, especially when the underlying problem is smooth so that a grid refinement factor of more than two can be taken between two subsequent levels.

The algorithm presented in this paper can be used to approximate Nash equilibria strategy vectors in an N-person noncooperative game. Then $S = \Pi_{j=1}^{N} S^{n_j}$ is the strategy space of the game and S^{n_j}, $j = 1,\ldots,N$ is the strategy space of player j if n_j+1 is the number of pure strategies of player j. Another application is the international trade model (see e.g. van der Laan [6]). Furthermore the homotopy parameter t, $t \geq 1$, in $S \times [1,\infty)$ can be considered as a time parameter. For example, the excess demand functions for the different goods in the international trade model might change continuously over time and we are interested in the path of equilibria considered as a function of time (see John [5]). By triangulating $S \times [1,\infty)$ as described in this paper this path of solutions can be followed. For this application one in general does not need to refine the grid size on a new level. Although we have described the triangulation for a sequence of decreasing grid sizes on the subsequent levels, it will appear that the description of the triangulation is still valid if we take the same grid size on each level.

The paper is organized as follows. In section 2 we give a short description of the variable dimension restart algorithm on S. Section 3 describes the triangulation of $S \times [1,\infty)$ and in section 4 we show that this triangulation induces a triangulation of the boundary of $S \times [1,\infty)$. Finally, section 5 describes the steps of the algorithm.

2. The variable dimension algorithm on S revisited

In this section we will give a short description of the variable dimension algorithm in S described in Doup and Talman [1]. This algorithm, starting with a 0-dimensional simplex traces a sequence of adjacent simplices of varying dimension of the V-triangulation of S until a complete simplex, say τ^0, is found. We will adapt the algorithm in such a way that it can also start with a complete simplex, say τ^1, different from τ^0, from which it traces a sequence of adjacent simplices of varying dimension to a complete simplex different from τ^0 and τ^1.

Let $S = \Pi_{j=1}^N S^{n_j}$ be the product space of unit simplices, i.e. $x = (x_1,\ldots,x_N) \in S$ iff $x_j \in S^{n_j}$, $j = 1,\ldots,N$, let $I(j)$ be the index set $\{(j,1),\ldots,(j,n_j+1)\}$, $j = 1,\ldots,N$, let n denote $\Sigma_{j=1}^N n_j$ and let $I = \cup_{j=1}^N I(j)$. Let T be a proper subset of I with $z(j) = |T_j| - 1 \geq 0$, for all j, where T_j is defined by $T_j = T \cap I(j)$. Furthermore let $\gamma_j = ((j,k_0^j),\ldots,(j,k_{z(j)}^j))$ be a permutation of the $z(j)+1$ elements in T_j, $j = 1,\ldots,N$, let Z_j^0 be given by $Z_j^0 = \{(j,k_0^j)\}$, $Z^0 = \bigcup_{j=1}^N Z_j^0$, $Z_j = T_j \setminus Z_j^0$, $Z = \bigcup_{j=1}^N Z_j$, and let $\gamma = (\gamma_1,\ldots,\gamma_N)$. Finally, let v be an arbitrary point in the interior of S.

<u>Definition 2.1</u>. Let T be a proper subset of I and let the γ_j, Z_j^0, Z_j's and γ, Z^0, Z be as defined above, then the set $A(\gamma)$ is given by

$$A(\gamma) = \{x \in S \,|\, x = v + bq^\gamma(Z^0) + \sum_{(i,h) \in Z} a(i,h)q^\gamma(i,h),$$

$$0 \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_1^j) \leq b \leq 1,\ j \in I_N\}$$

where the (N+n)-vector $q^\gamma(Z^0)$ is given by

$$q_j^\gamma(Z^0) = p_j(Z_j^0) - v_j\ ,\ j \in I_n$$

and where for $i = 1,\ldots,z(j)$, $j \in I_N$, $q_h^\gamma(j,k_i^j) = 0$, $h \neq j$, and

$$q_j^\gamma(j,k_i^j) = p_j(\{(j,k_0^j),\ldots,(j,k_i^j)\}) - p_j(\{(j,k_0^j),\ldots,(j,k_{i-1}^j)\}),$$

where for $j \in I_N$ the (n_j+1)-vector $p_j(K_j)$, $K_j \subset I(j)$, $K_j \neq \emptyset$ is given by

$$p_{j,k}(K_j) = \begin{cases} v_{j,k}\left(\sum_{(j,h) \in K_j} v_{j,h}\right)^{-1} & , (j,k) \in K_j \\ 0 & , (j,k) \notin K_j. \end{cases}$$

The vector $p_j(K_j)$ is a relative projection of v_j on the boundary of S^{n_j}. The set $A(T)$ is now given by the union of $A(\gamma)$ over all permutation vectors γ of T. In fact $A(T)$ is the convex hull of v and the vertices $e(j,h)$, $(j,h) \in T$. The t-dimensional set $A(\gamma)$, $t = |T| - N + 1$, is triangulated by the collection $G(\gamma)$ of t-simplices $\tau(w^1, \omega)$ with vertices $w^1, \ldots, w^{t+1}$, where

i) $$w^1 = v + bd_1^{-1}q^{\gamma}(Z^0) + \sum_{(i,h) \in Z} a(i,h)d_1^{-1}q^{\gamma}(i,h),$$

for nonnegative integers b and $a(i,h), (i,h) \in Z$, such that for all $j \in I_N$, $0 \leq a(j,k^j_{z(j)}) \leq \ldots \leq a(j,k^j_1) \leq b \leq d_1 - 1$;

ii) $\omega = (\omega_1, \ldots, \omega_t)$ is a permutation of the t elements consisting of Z^0 and the t-1 elements of Z such that for all $i = 1, \ldots, z(j)$: $s > s'$ if $a(j,k^j_i) = a(j,k^j_{i-1})$ where $\omega_s = (j,k^j_i)$ and $\omega_{s'} = (j,k^j_{i-1})$, $j \in I_N$. In the case $i = 1$, $a(j,k^j_0) = b$ and $\omega_{s'} = Z^0$;

iii) $w^{i+1} = w^i + d_1^{-1}q^{\gamma}(\omega_i)$, $i = 1, \ldots t$

where $q^{\gamma}(Z^0)$ and $q^{\gamma}(i,h), (i,h) \in Z$, are defined as before.

The number d_1^{-1}, d_1 a positive integer, denotes the grid size of the triangulation. In fact we consider the so-called V-triangulation of S with relative projection (see [1]). In the following we define $a(Z^0) = a(j,k^j_0) = b$, $j \in I_N$.

Let the (N+n)-vector $\ell(x)$, $x \in S$, be a labelling function from S into R^{N+n}.

Definition 2.2. Let T be a subset of I with $|T_j| \geq 1$, $j \in I_N$. For $k = t-1, t$, where $t = |T| - N+1$, a k-simplex $\tau(w^1,\ldots,w^{k+1})$ is T-complete if the system of linear equations

$$\Sigma_{i=1}^{k+1}\lambda_i\binom{\ell(w^i)}{1} + \Sigma_{(i,h)\notin T}\mu_{i,h}\binom{e(i,h)}{0} - \Sigma_{j=1}^{N}\beta_j\binom{\bar{e}(j)}{0} = \binom{\underline{0}}{1}, \qquad (2.1)$$

where $e(i,h)$ denotes the $(\Sigma_{j=1}^{i-1}(n_j+1)+h)$ - th unit vector in R^{N+n}, $\bar{e}(j) = \Sigma_{h=1}^{n_j+1} e(j,h)$, $j \in I_N$, and $\underline{0}$ is the (N+n)-dimensional zero vector in R^{N+n}, has a solution $\lambda_i^* \geq 0$, $i = 1,\ldots,k+1$, $\mu_{i,h}^* \geq 0$, $(i,h) \notin T$ and β_j^*, $j \in I_N$.

A solution will be denoted by $(\lambda^*, \mu^*, \beta^*)$. For a T-complete k-simplex with $k = t-1$ we assume that the system (2.1) has a unique solution $(\lambda^*, \mu^*, \beta^*)$, $\lambda_i^* > 0$, $i = 1,\ldots,t$ and $\mu_{i,h}^* > 0$ $(i,h) \notin T$, and that for $k = t$ the system has a line segment of solutions with at most one variable of (λ^*, μ^*) equal to zero (Nondegeneracy assumption) so that each T-complete (t-1)-simplex in A(T) is a facet of either two T-complete simplices in A(T) or of one in which case it lies on the boundary of A(T).

Definition 2.3. A T-complete (t-1)-simplex $\tau(w^1,\ldots,w^t)$ is complete if for all x in τ: $x_{i,h} = 0$, $(i,h) \notin T$.

Observe that we allow T to be equal to I.

As described in Doup and Talman [1] the T-complete t-simplices in A(T), $T \subset I$, determine paths of adjacent simplices of varying dimension such that each path is either a loop or has two end points. Exactly one end point is the zero-dimensional simplex $\tau(v)$, whereas all other end points are complete simplices. Exactly one path connects $\tau(v)$ with a complete simplex whereas all other paths with two end points connect two complete simplices. We will now give the replacement steps occuring in the algorithm which follows such a path.

Let $\tau(w^1,\omega)$ be a T-complete t-simplex in $G(\gamma)$ such that the T-complete facet of τ opposite vertex w^s, $1 \leq s \leq t+1$, is a facet of another T-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$ in $G(\gamma)$, then $\bar{\tau}$ can be obtained from τ as given in table 1, where $e(Z^0) = \Sigma_{j=1}^{N} e(j,k_0^j)$. The vertex w^s is replaced by the new vertex of $\bar{\tau}$.

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s = 1$	$w^1+d_1^{-1}q^{\gamma}(\omega_1)$	$(\omega_2,\dots,\omega_t,\omega_1)$	$a+e(\omega_1)$
$1 < s < t+1$	w^1	$(\omega_1,\dots,\omega_{s-2},\omega_s,\omega_{s-1},\dots,\omega_t)$	a
$s = t+1$	$w^1-d_1^{-1}q^{\gamma}(\omega_t)$	$(\omega_t,\omega_1,\dots,\omega_{t-1})$	$a-e(\omega_t)$

Table 1. s is the index of the vertex of τ to be replaced

Now consider the case that the T-complete facet of τ opposite vertex w^s, $1 \leq s \leq t+1$, is not a facet of another t-simplex in $G(\gamma)$.

Lemma 2.4. Let $\tau(w^1,\omega)$ be a t-simplex in $G(\gamma)$. The facet of τ opposite vertex w^s, $1 \leq s \leq t+1$ lies on the boundary of $A(\gamma)$ iff

a) $s = 1$: $\omega_1 = Z^0$ and $b = d_1-1$; the T-complete (t-1)-simplex $\bar{\tau}(w^2,\dots,w^{t+1})$ lies in $S_{i,h}$, $(i,h) \notin T$, and is therefore complete;

b) $1 < s < t+1$: $\omega_{s-1} = (j,k^j_{i-1})$ and $\omega_s = (j,k^j_i)$ for certain $j \in I_N$, $1 < i \leq z(j)$, while $a(\omega_{s-1}) = a(\omega_s)$

or

$\omega_{s-1} = Z^0$ and $\omega_s = (j,k^j_1)$ for certain $j \in I_N$ while $b = a(\omega_s)$; the facet of τ opposite vertex w^s is a facet of the T-complete t-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$, with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1,\dots,\omega_{s-2},\omega_s,\omega_{s-1},\dots,\omega_t)$ and $\bar{a} = a$, and τ lies in $G(\gamma)$ where γ is given by

$$\bar{\gamma}_h = \begin{cases} ((j,k^j_0),\dots,(j,k^j_{i-2}),(j,k^j_i),(j,k^j_{i-1}),\dots,(j,k^j_{z(j)})) & , h = j \\ \gamma_h & , h \neq j, \end{cases}$$

c) $s = t+1$: $\omega_t = (j,k^j_{z(j)})$ for certain $j \in I_N$ and $a(\omega_t) = 0$; the T-complete $(t-1)$-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$, with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1, \ldots,\omega_{t-1})$ and $\bar{a} = a$, lies in $G(\bar{\gamma})$ where $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} ((j,k^j_0),\ldots,(j,k^j_{z(j)-1})) & , h = j \\ \gamma_h & , h \neq j. \end{cases}$$

Furthermore we have the following lemma concerning the increase of dimension of a t-simplex $\tau(w^1,\omega)$ in $G(\gamma)$, $t < n$.

Lemma 2.5. Let $\tau(w^1,\omega)$ be a $T \cup \{(j,k)\}$-complete t-simplex in $G(\gamma)$, for some $(j,k) \in I \setminus T$, with γ_j a permutation of the elements of T_j, $j \in I_N$ and $t < n$. Then τ is a facet of the $(t+1)$-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$, with $\bar{w}^1 = w^1$, $\bar{\omega} = (\omega_1,\ldots,\omega_t,(j,k))$ and $\bar{a} = a$, in $G(\bar{\gamma})$ where $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} ((j,k^j_0),\ldots,(j,k^j_{z(j)}),(j,k)) & , h = j \\ \gamma_h & , h \neq j. \end{cases}$$

We will now give the steps of the algorithm, omitting the initialization step, in order to either generate a path of adjacent simplices from the 0-dimensional simplex $\tau(v)$ to a complete simplex, say τ^0, or to generate such a path from a complete simplex τ^1, $\tau^1 \neq \tau^0$, to another complete simplex τ^2. The number $\bar{s}$ is the index of the vertex of τ whose label $\ell(w^s)$ has to be calculated.

Step 1: Calculate $\ell(w^{\bar{s}})$. Perform a pivot step by bringing $\ell(w^{\bar{s}})$ in the linear system

$$\Sigma^{t+1}_{\substack{i=1\\ i\neq\bar{s}}}\lambda_i\binom{\ell(w^i)}{1} + \Sigma_{(i,h)\notin T}\mu_{i,h}\binom{e(i,h)}{0} - \Sigma^N_{j=1}\beta_j\binom{\bar{e}(j)}{0} = \binom{\bar{0}}{1}.$$

Either $\mu_{i,h}$, $(i,h) \notin T$, becomes zero, then go to step 3, or λ_s becomes zero for exactly one vertex $w^s \neq w^{\bar{s}}$. The facet opposite vertex w^s is T-complete.

Step 2: If $s = 1$, $\omega_1 = Z^0$ and $b = d_1 - 1$ then the facet of τ opposite vertex w^1 is a complete $(t-1)$-simplex and the algorithm terminates.

If $1 < s < t+1$, and if $\omega_{s-1} = (j, k^j_{i-1})$, $\omega_s = (j, k^j_i)$, $a(\omega_{s-1}) = a(\omega_s)$ for certain $j \in I_N$, $1 < i \leq z(j)$, or if $\omega_{s-1} = Z^0$, $\omega_s = (j, k^j_1)$, $b = a(\omega_s)$ for certain $j \in I_N$, then τ and γ are adapted according to lemma 2.4 (b).

If $s = t+1$, $\omega_t = (j, k^j_{z(j)})$ for certain $j \in I_N$ and $a(\omega_t) = 0$, then the dimension is decreased; set $t = t-1$, $T = T \setminus \{(j, k^j_{z(j)})\}$ and $p = (j, k^j_{z(j)})$ while τ and γ are adapted according to lemma 2.4 (c) and go to step 4. In all other cases $\tau(w^1, \omega)$ and a are adapted according to table 1.

Return to step 1 with $\bar{s}$ the index of the new vertex of τ.

Step 3: If $t = n$, then $\tau(w^1, \omega)$ is a complete n-simplex and the algorithm terminates; otherwise $\tau(w^1, \omega)$ and γ are adapted according to lemma 2.5, set $t = t+1$ and $T = T \cup \{(i,h)\}$. Return to step 1 with $\bar{s}$ the index of the new vertex of τ.

Step 4: Perform a pivot step by bringing $e(p)$ in the linear system

$$\Sigma_{i=1}^{t+1} \lambda_i \binom{\ell(w^i)}{1} + \Sigma_{\substack{(i,h) \notin T \\ (i,h) \neq p}} \mu_{i,h} \binom{e(i,h)}{0} - \Sigma_{j=1}^{N} \beta_j \binom{\bar{e}(j)}{0} = \binom{0}{1}.$$

If for some $(i,h) \notin T$, $\mu_{i,h}$ becomes zero go to step 3, otherwise return to step 2 with s the index of the vertex whose label $\ell(w^s)$ is eliminated.

We can distinguish the following three initializations of the algorithm described above:

1) with the 0-dimensional simplex $\tau(v)$
2) for some $T \subset I$ with a complete T-complete $(t-1)$-simplex $\tilde{\tau}(\tilde{w}^1, \tilde{\omega})$ of $G(\gamma)$ in $A(\gamma) \cap (\bigcap_{(i,h) \notin T} S_{i,h})$, where $\tilde{\omega}$ is a permutation of the elements in Z, with basic solution $\lambda^*_i > 0, i = 1, \ldots, t$, $\mu^*_{i,h} > 0$, $(i,h) \notin T$ and β^*_j, $j \in I_N$

3) and for some γ with a complete I-complete n-simplex $\tilde{\tau}(\tilde{w}^1,\tilde{\omega})$ in $A(\gamma)$, where $\tilde{\omega}$ is a permutation of Z^0 and the elements of Z, with basic solution λ_i^*, $i = 1,\dots,n+1$ and β_j^*, $j \in I_N$.

In the first case the algorithm is initialized with the Z^0-complete 1-simplex $\tau(w^1,\omega)$, $Z^0 = \cup_{j=1}^N \{(j,k_0^j)\}$ where the index (j,k_0^j) is such that $\ell_{j,k_0^j}(v) = \max_k \ell_{j,k}(v)$, $j \in I_N$, $w^1 = v$, $\omega = (Z^0)$, $a(i,h) = 0$, $(i,h) \in I$, and with basic solution $\lambda_1 = 1$, $\mu_{i,h} = \ell_{i,k_0^i}(v) - \ell_{i,h}(v)$, $(i,h) \notin Z^0$, and $\beta_j = \ell_{j,k_0^j}(v)$, $j \in I_N$. The index $\bar{s}$ is set equal to 2 and the algorithm starts with step 1.

In the second case the algorithm is initialized with the T-complete t-simplex $\tau(w^1,\omega)$ with $w^1 = \tilde{w}^1 - d_1^{-1}q^\gamma(Z^0)$, $\omega = (Z^0,\tilde{\omega}_1,\dots,\tilde{\omega}_{t-1})$, $a = \tilde{a} - e(Z^0)$ where $\tilde{a}$ induces $\tilde{w}^1$, and the basic solution $\lambda_i = \lambda_{i-1}^*$, $i = 2,\dots,t+1$, $\mu_{i,h} = \mu_{i,h}^*$, $(i,h) \notin T$ and $\beta_j = \beta_j^*$, $j \in I_N$. The index $\bar{s}$ is set equal to 1 and the algorithm starts with step 1.

In the third case the algorithm is initialized with the I-complete n-simplex $\tau(w^1,\omega)$ with $w^1 = \tilde{w}^1$, $\omega = \tilde{\omega}$ and $a = \tilde{a}$. Let (i,h) be the unique index in $Z^0 \cup Z$ not in I then we set $T = I\setminus\{(i,h)\}$ and $p = (i,h)$ while the basic solution of τ is given by $\lambda_i = \lambda_i^*$, $i = 1,\dots,n+1$, $\beta_j = \beta_j^*$, $j \in I_N$. Now the algorithm starts with step 4.

In this section we have discussed the variable dimension restart algorithm which generates the path of adjacent simplices from some point v in S to a complete simplex τ^0, from another complete simplex to a third complete simplex, etc. The continuous deformation algorithm on $S \times [1,\infty)$ to be described in section 5 is initialized by applying the variable dimension algorithm (on level 1) to generate on level one a path from v to a complete simplex τ^0. Then the continuous deformation algorithm continues in $S \times [1,\infty)$ with a path of adjacent complete simplices starting with the complete simplex $co(\tau^0 \times \{1\}, \{v(\tau^0)\} \times \{2\})$ where $v(\tau^0)$ denotes a specific grid point in τ^0. If the continuous deformation algorithm generates a complete simplex, say τ^1, on level 1 then $\tau^1 \neq \tau^0$ and we again apply the variable dimension algorithm on level 1 generating a path of adjacent simplices from the complete simplex τ^1 to another complete simplex τ^2, $\tau^2 \neq \tau^0, \tau^1$. Then the continuous deformation algorithm again moves into the set $S \times [1,\infty)$, etc.

In the following section we will describe a triangulation of $S \times [1,\infty)$ by describing the triangulation of $S \times [m,m+1]$, $m = 1,2,\ldots$, where for some sequence of increasing positive integers d_m, $m = 1,2,\ldots$, on each level m S is simplicially subdivided according to the V-triangulation with grid size d_m^{-1}.

3. Triangulation of $S \times [1,\infty)$

Let $d_1, d_2, \ldots$ be a sequence of increasing integers such that $d_{m+1} = k_m d_m$, $m = 1,2\ldots$, with k_m an arbitrary integer larger than one. In the following we triangulate for each m the subset $S \times [m,m+1]$ of $S \times [1,\infty)$ such that all its grid points are points of $S \times \{m\}$ or of $S \times \{m+1\}$. Combining the triangulations of $S \times [m,m+1]$, $m = 1,2,\ldots$, we obtain a triangulation of $S \times [1,\infty)$. For $m = 1,2,\ldots$ $S \times \{m\}$ is triangulated by the V-triangulation with grid size equal to d_m^{-1} as described in section 2. Now for $j \in I_N$ let $\gamma_j = ((j,k_0^j),\ldots,(j,k_{z(j)}^j))$, $z(j) = n_j$, be a permutation of the n_j+1 elements of $I(j)$, $Z_j^0 = \{(j,k_0^j)\}$, $Z_j = I(j)\setminus Z_j^0$, and let $\gamma = (\gamma_1,\ldots,\gamma_N)$, $Z^0 = \cup_{j=1}^N Z_j^0$ and $Z = \cup_{j=1}^N Z_j$. Then for $j_0 \in I_N$

$$A(\gamma,j_0) = \{x \in S \mid x = v + bq^{\gamma}(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)q^{\gamma}(i,h), \text{ where}$$

$$0 \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_1^j) \leq b \leq 1,\ j \in I_N \text{ and } a(j_0,k_{z(j_0)}^{j_0}) = 0\}.$$

In fact $A(\gamma,j_0)$ is equal to $A(\bar{\gamma})$, with $\bar{\gamma}_h = \gamma_h$, $h \neq j_0$ and $\bar{\gamma}_{j_0} = ((j_0,k_0^{j_0}),\ldots,(j_0,k_{z(j_0)-1}^{j_0}))$. The set $G_m(\gamma,j_0)$ is the collection of n-simplices $\tau(w^1,\omega)$ with vertices $w^1,\ldots,w^{n+1}$ such that

i) $w^1 = v + bd_m^{-1}q^{\gamma}(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)d_m^{-1}q^{\gamma}(i,h)$ for nonnegative integers b and $a(i,h),(i,h) \in Z$ such that $0 \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_1^j) \leq b \leq d_m-1$, $j \in I_N$ and $a(j_0,k_{z(j_0)}^{j_0}) = 0$;

ii) $\omega = (\omega_1,\ldots,\omega_{n+1})$ is a permutation of n+1 elements consisting of Z^0 and the n elements of Z such that $\omega_{n+1} = (j_0,k_{z(j_0)}^{j_0})$ and for all $i = 1,\ldots,z(j)$: $s > s'$ if $a(j,k_i^j) = a(j,k_{i-1}^j)$ where $\omega_s = (j,k_i^j)$ and $\omega_{s'} = (j,k_{i-1}^j)$, $j \in I_N$. In the case $i = 1$, $a(j,k_0^j) = b$ and $\omega_{s'} = Z^0$;

iii) $w^{i+1} = w^i + d_m^{-1}q^{\gamma}(\omega_i)$, $i = 1,\ldots,n+1$, with the convention $i+1 = 1$ in the case $i = n+1$.

Observe that $G_m(\gamma,j_0)$ is equivalent to $G(\bar{\gamma})$ with grid size d_m^{-1} so that the union of $G_m(\gamma,j_0)$ over all permutations γ of I and indices $j_0 \in I_N$ is the V-triangulation of S with grid size d_m^{-1}. This union will be denoted by V_m.

We will now introduce a function s^m from the grid points in $A(\gamma,j_0)$ to I_{n+1} such that each n-simplex $\tau(w^1,\omega)$ in $G_m(\gamma,j_0)$ is completely labelled, i.e., $\{s^m(w^i)\,|\,i = 1,\ldots,n+1\} = I_{n+1}$. The function s^m is given by

$$s^m(x) = 1 + (b+\Sigma_{(i,h)\in Z}a(i,h)) \bmod (n+1),$$

where $x = v + bd_m^{-1}q^{\gamma}(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)d_m^{-1}q^{\gamma}(i,h)$, satisfying i), is a grid point of V_m.

We are now ready to triangulate $S \times [m,m+1]$ for some given $m \geq 1$. First we choose nonnegative integers $\theta_1^m,\ldots,\theta_{n+1}^m$ with sum equal to $k_m = d_{m+1}/d_m$. For any n-simplex $\tau(w^1,\omega)$ of V_m we call the point

$$v(\tau) = \Sigma_{i=1}^{n+1}\delta_i k_m^{-1}w^i ,$$

with $\delta_i = \theta^m_{s^m(w^i)}$, $i = 1,\ldots,n+1$, the centrepoint of τ. Observe that $v(\tau)$ is a grid point of V_{m+1}. It will appear that the triangulation of $S \times [m,m+1]$ is completely determined by the numbers $\theta_1^m,\ldots,\theta_{n+1}^m$. To triangulate $S \times [m,m+1]$ we first triangulate each $\tau(w^1,\omega) \times [m,m+1]$ and then we prove that the union of the triangulations of $\tau(w^1,\omega) \times [m,m+1]$ over all n-simplices τ of V_m is a triangulation of $S \times [m,m+1]$. The triangulation of $S \times [m,m+1]$ will be such that for all τ of V_m, the (n+1)-simplex $co(\tau \times \{m\},\{v(\tau)\} \times \{m+1\})$ is a simplex of this triangulation.

To triangulate $\tau(w^1,\omega) \times [m,m+1]$, τ in V_m, we define for any proper subset T of $\{\omega_1,\ldots,\omega_{n+1}\}$ the regions $\mathring{A}(T,\tau)$ in τ by

$$\mathring{A}(T,\tau) = \{x \in \tau \,|\, x = v(\tau) + \Sigma_{j\in T}\alpha_j q^{\gamma}(j),\ \alpha_j > 0,\ j \in T\}.$$

Let $A(T,\tau)$ be the closure of $\mathring{A}(T,\tau)$, then on level $m+1$ $A(T,\tau)$ is triangulated by V_{m+1} in t-simplices $\sigma(y^1,\pi(T))$ with vertices $y^1,\ldots,y^{t+1}$ in τ such that

1) $y^1 = v(\tau) + \sum_{h=1}^{n+1} R_{\omega_h} d_{m+1}^{-1} q^{\gamma}(\omega_h)$, $R_{\omega_h} > 0$, $\omega_h \in T$, $R_{\omega_h} = 0$, $\omega_h \notin T$;

2) $\pi(T) = (\pi_1, \ldots, \pi_t)$ is a permutation of the t elements in T;

and

3) $y^{i+1} = y^i + d_{m+1}^{-1} q^{\gamma}(\pi_i)$, $i = 1, \ldots, t$.

In the sequel the (N+n)-vector R is defined by

$$R_{j,k} = \begin{cases} R_{\omega_h} & \omega_h = (j,k) \in Z \\ R_{Z^0} & (j,k) \in Z^0 \\ 0 & (j,k) \notin Z^0 \cup Z. \end{cases}$$

Now $\tau \times [m,m+1]$ is triangulated by (n+1)-simplices ψ^{γ} where for some $T \subset \{\omega_1, \ldots, \omega_{n+1}\}$ and $\sigma(y^1, \pi(T))$ in $A(T,\tau)$

$$\psi^{\gamma} = \mathrm{co}(\mathrm{co}(\{w^i \mid \omega_i \notin T\}) \times \{m\}, \sigma(y^1, \pi(T)) \times \{m+1\}).$$

Lemma 3.1. Let $\tau(w^1, \omega)$ be an n-simplex in $G_m(\gamma, j_0)$ with centrepoint $v(\tau)$. If all the grid points x of V_{m+1} in $\mathring{A}(T,\tau)$ on level m+1, for proper subsets T of $\{\omega_1, \ldots, \omega_{n+1}\}$, are connected with the vertices w^i, $\omega_i \notin T$, on level m, the (n+1)-simplices obtained in this way induce a triangulation of $\tau \times [m,m+1]$.

Proof. This lemma is a straightforward generalization of the theorem on the unit simplex, which proof can be found in van der Laan and Talman [9].

To prove that the union of the triangulation of $\tau(w^1, \omega) \times [m, m+1]$ over all τ in V_m is a triangulation of $S \times [m,m+1]$ we need to know how $\tau(w^1, \omega)$ and $v(\tau)$ change when we move from τ to an adjacent simplex

$\bar{\tau}$. So, let $\tau(w^1,\omega)$ be an n-simplex in $G_m(\gamma,j_0)$ and let $\bar{\tau}(\bar{w}^1,\bar{\omega})$ be the n-simplex in V_m sharing with τ the facet opposite w^s. If $\bar{\tau}$ lies in $G_m(\gamma,j_0)$, then $\bar{\tau}$ is obtained from τ as given in table 2.

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s = 1$	$w^1+d_m^{-1}q^{\gamma}(\omega_1)$	$(\omega_2,\ldots,\omega_n,\omega_1,\omega_{n+1})$	$a+e(\omega_1)$
$1 < s < n+1$	w^1	$(\omega_1,\ldots,\omega_{s-2},\omega_s,\omega_{s-1},\ldots,\omega_{n+1})$	a
$s = n+1$	$w^1-d_m^{-1}q^{\gamma}(\omega_n)$	$(\omega_n,\omega_1,\ldots,\omega_{n-1},\omega_{n+1})$	$a-e(\omega_n)$

Table 2. s is the index of the vertex of $\tau(w^1,\omega)$ to be replaced

The centrepoint of $\bar{\tau}$ is in this case adapted as given in table 3.

	$v(\bar{\tau})$	$\bar{\delta}$
$s = 1$	$v(\tau) + \delta_1 k_m^{-1}(\bar{w}^{n+1}-w^1)$	$(\delta_2,\ldots,\delta_{n+1},\delta_1)$
$1 < s < n+1$	$v(\tau) + \delta_s k_m^{-1}(\bar{w}^s-w^s)$	$(\delta_1,\ldots,\delta_{n+1})$
$s = n+1$	$v(\tau) + \delta_{n+1} k_m^{-1}(\bar{w}^1-w^{n+1})$	$(\delta_{n+1},\delta_1,\ldots,\delta_n)$

Table 3. s is the index of the vertex of τ to be replaced

We will now consider the cases that the facet of τ opposite vertex w^s is either a facet of an n-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$ in $G(\bar{\gamma},j_0)$ with $\bar{\gamma} \neq \gamma$ or a facet of an n-simplex $\bar{\tau}(\bar{w}^1,\omega)$ in $G(\gamma,\bar{j}_0)$ with $\bar{j}_0 \neq j_0$. The first case occurs iff

$1 < s \leq n+1$, $\omega_{s-1} = (j,k^j_{i-1})$, $\omega_s = (j,k^j_i)$ and $a(\omega_{s-1}) = a(\omega_s)$ for certain $j \in I_N$.

In this case the parameters $\bar{w}^1$, $\bar{\omega}$ and $\bar{a}$ of $\bar{\tau}$ are given by

$$\bar{w}^1 = w^1,\ \bar{\omega} = (\omega_1,\ldots,\omega_{s-2},\omega_s,\omega_{s-1},\ldots,\omega_{n+1}) \text{ and } \bar{a} = a \tag{3.1}$$

and $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} ((j,k^j_0),\ldots,(j,k^j_{i-2}),(j,k^j_i),(j,k^j_{i-1}),\ldots,(j,k^j_{z(j)})) & h = j \\ \gamma_h & h \neq j. \end{cases} \tag{3.2}$$

The second case occurs iff

$$s = n+1,\ \omega_n = (j,k^j_{z(j)}) \text{ for certain } j \in I_N \text{ and } a(\omega_n) = 0. \tag{3.3}$$

In this case the parameters $\bar{w}^1$, $\bar{\omega}$ and $\bar{a}$ of $\bar{\tau}$ are given by

$$\bar{w}^1 = w^1,\ \bar{\omega} = (\omega_1,\ldots,\omega_{n-1},\omega_{n+1},\omega_n) \text{ and } \bar{a} = a \tag{3.4}$$

and $\bar{j}_0$ is given by $\bar{j}_0 = j$. In both cases the centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_s k_m^{-1}(\bar{w}^s - w^s). \tag{3.5}$$

<u>Theorem 3.2</u>. The union of the triangulations of $\tau(w^1,\omega) \times [m,m+1]$ over all n-simplices τ of V_m triangulates $S \times [m,m+1]$.

<u>Proof</u>. The triangulation of $\tau \times [m,m+1]$ is well defined for all simplices τ of V_m. Let $\tau(w^1,\omega)$ and $\bar{\tau}(\bar{w}^1,\bar{\omega})$ be two adjacent simplices of V_m and let x be a grid point of V_{m+1} in the common facet. Then it is sufficient to prove that if in the triangulation of $\tau \times [m,m+1]$, x is connected with a vertex w of $\tau \cap \bar{\tau}$, x is also connected with w in the triangulation of $\bar{\tau} \times [m,m+1]$. Suppose that τ lies in $G(\gamma,j_0)$ for some permutation γ and index j_0, then we have to consider the following three cases:

a) $\bar{\tau}(\bar{w}^{1},\bar{\omega})$ lies in $G(\gamma,j_0)$, b) $\bar{\tau}(\bar{w}^{1},\bar{\omega})$ lies in $G(\bar{\gamma},j_0)$ where $\bar{\gamma}$ is given as in (3.2) or c) $\bar{\tau}(\bar{w}^{1},\bar{\omega})$ lies in $G(\gamma,j)$, $j \neq j_0$, with j given as in (3.3).

<u>case a)</u>: $\bar{\tau}(\bar{w}^{1},\bar{\omega})$ is an n-simplex in $G(\gamma,j_0)$ with a common facet with τ opposite vertex w^s, $1 \leq s \leq n+1$.

The simplex $\bar{\tau}$ is given as in table 2 and the centrepoint of $\bar{\tau}$ is given by (see table 3)

$$v(\bar{\tau}) = v(\tau) + \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\gamma}(\omega_{s-1})], \tag{3.6}$$

with the convention $s-1 = n+1$ if $s = 1$. Now suppose that x lies in $\mathring{A}(T,\tau)$ for some subset T of $\{\omega_1,\ldots,\omega_{n+1}\}$, then

$$x = v(\tau) + \Sigma_{j \in T}\alpha_j d_{m+1}^{-1} q^{\gamma}(j), \tag{3.7}$$

for positive integers α_j, $j \in T$.
Since the point x lies in the facet of τ opposite vertex w^s (3.7) gives us

$$\delta_s + \alpha_{\omega_{s-1}} - \alpha_{\omega_s} = 0. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) yields

$$\begin{aligned}
x &= v(\tau) + \Sigma_{j \in T}\alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= v(\bar{\tau}) - \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\gamma}(\omega_{s-1})] + \Sigma_{j \in T}\alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\
&= v(\bar{\tau}) + \Sigma_{j \in T\setminus\{\omega_{s-1},\omega_s\}}\alpha_j d_{m+1}^{-1} q^{\gamma}(j) + (\alpha_{\omega_{s-1}}+\delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_{s-1}) + \\
&\quad (\alpha_{\omega_s}-\delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_s) \\
&= v(\bar{\tau}) + \Sigma_{j \in T}\bar{\alpha}_j d_{m+1}^{-1} q^{\gamma}(j)
\end{aligned}$$

with the coefficients $\bar{\alpha}_j$, $j \in T$ given by

$$\bar{\alpha}_j = \begin{cases} \alpha_j & , j \in T\setminus\{\omega_{s-1}, \omega_s\} \\ \alpha_{\omega_s} & , j = \omega_{s-1} \\ \alpha_{\omega_{s-1}} & , j = \omega_s . \end{cases} \tag{3.9}$$

The point x therefore also lies in $\mathring{A}(\bar{T}, \bar{\tau})$ with $\bar{T}$ given by

$$\bar{T} = \begin{cases} T & , \omega_{s-1}, \omega_s \in T \text{ or } \omega_{s-1}, \omega_s \notin T \\ T\setminus\{\omega_{s-1}\} \cup \{\omega_s\} & , \omega_{s-1} \in T,\ \omega_s \notin T \\ T\setminus\{\omega_s\} \cup \{\omega_{s-1}\} & , \omega_s \in T,\ \omega_{s-1} \notin T, \end{cases} \tag{3.10}$$

which proves the theorem for case a.

<u>Case b)</u>: $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an n-simplex in $G(\bar{\gamma}, j_0)$ with $\bar{\gamma}$ given as in (3.2) with a common facet with τ opposite vertex w^s, $1 < s \leq n+1$.

The simplex $\bar{\tau}$ is given as in (3.1) and the centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\bar{\gamma}}(\omega_{s-1})]. \tag{3.11}$$

Combining (3.11), (3.7) and (3.8) yields the following

$$\begin{aligned} x &= v(\tau) + \Sigma_{j \in T}\alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\ &= v(\bar{\tau}) - \delta_s d_{m+1}^{-1}[q^{\gamma}(\omega_s) - q^{\bar{\gamma}}(\omega_{s-1})] + \Sigma_{j \in T}\alpha_j d_{m+1}^{-1} q^{\gamma}(j) \\ &= v(\bar{\tau}) + \Sigma_{j \in T\setminus\{\omega_{s-1}, \omega_s\}}\alpha_j d_{m+1}^{-1} q^{\bar{\gamma}}(j) + \delta_s d_{m+1}^{-1} q^{\bar{\gamma}}(\omega_{s-1}) + \\ &\quad \alpha_{\omega_{s-1}} d_{m+1}^{-1} q^{\gamma}(\omega_{s-1}) + (\alpha_{\omega_s} - \delta_s) d_{m+1}^{-1} q^{\gamma}(\omega_s) \end{aligned}$$

$$= v(\bar{\tau}) + \Sigma_{j \in T\setminus\{\omega_{s-1},\omega_s\}}\alpha_j d_{m+1}^{-1} q^{\bar{\gamma}}(j) + \delta_s d_{m+1}^{-1} q^{\bar{\gamma}}(\omega_{s-1}) +$$

$$\alpha_{\omega_{s-1}} d_{m+1}^{-1}[q^{\bar{\gamma}}(\omega_{s-1}) + q^{\bar{\gamma}}(\omega_s)]$$

$$= v(\bar{\tau}) + \Sigma_{j \in T}\bar{\alpha}_j d_{m+1}^{-1} q^{\bar{\gamma}}(j),$$

with the coefficients $\bar{\alpha}_j$, $j \in T$ given by (3.9). The point x then also lies in $\mathring{A}(\bar{T},\bar{\tau})$ with $\bar{T}$ given by (3.10), which proves the theorem for case b.

<u>Case c):</u> $\bar{\tau}(\bar{w}^1,\bar{\omega})$ is an n-simplex in $G(\gamma,j)$, where j is given as in (3.3), having a common facet with τ opposite vertex w^{n+1}.

The simplex $\bar{\tau}$ is given as in (3.4) and the centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_{n+1} d_{m+1}^{-1}[q^{\gamma}(\omega_{n+1}) - q^{\gamma}(\omega_n)].$$

This case is similar to case a for s = n+1 and yields the same $\bar{T}$ and $\bar{\alpha}_j$, $j \in T$.
□

We have now shown that we can triangulate $S \times [m,m+1]$ for m = 1,2,... with on each level m the V-triangulation with grid size d_m^{-1} as the underlying triangulation. The (n+1)-simplices ψ^{γ} are given by

$$\psi^{\gamma} = \text{co}(\text{co}(\{w^i \mid \omega_i \notin T\}) \times \{m\},\ \sigma(y^1,\pi(T)) \times \{m+1\})$$

with $\tau(w^1,\omega)$ an n-simplex of $G_m(\gamma,j_0)$ and $\sigma(y^1,\pi(T))$ a t-simplex in $A(T,\tau)$ on level m+1. Combining the triangulations of $S \times [m,m+1]$, m = 1,2,..., we get a triangulation of $S \times [1,\infty)$. In the following section we will describe how this triangulation induces a triangulation of the boundary of $S \times [1,\infty)$ allowing us to use arbitrary labelling rules.

4. Triangulation of the boundary of $S \times [1,\infty)$

Let U be a subset of I such that $|U_j| < n_j$, $j \in I_N$. The set $S(U)$ is given by $S(U) = \{x \in S \,|\, x_{j,k} = 0,\ (j,k) \in U\}$. Let $\gamma_j = ((j,k_0^j),\ldots,(j,k_{z(j)}^j))$ denote a permutation of the elements in $I(j)\backslash U_j$ where $z(j) = n_j - u_j$ and $u_j = |U_j|$, let $Z_j^0 = \{(j,k_0^j)\}$, $Z_j = \{(j,k_1^j),\ldots,(j,k_{z(j)}^j)\}$, $j \in I_N$, $\gamma = (\gamma_1,\ldots,\gamma_N)$, $Z^0 = \cup_j Z_j^0$ and $Z = \cup_j Z_j$.

Definition 4.1. For each $j \in I_N$, let U_j be a proper subset of $I(j)$. For $j_0 \in I_N$, the set $A(\gamma,j_0)$ is given by

$$A(\gamma,j_0) = \{x \in S \,|\, x = v(U) + bq^{\gamma}(Z^0) + \Sigma_{(i,h) \in Z}\, a(i,h)q^{\gamma}(i,h),$$

$$\text{where } 0 \leq a(j,k_{z(j)}^j) \leq \ldots \leq a(j,k_1^j) \leq b \leq 1,\ j \in I_N$$

$$\text{and } a(j_0,k_{z(j_0)}^{j_0}) = 0\}$$

where

$$v_{i,h}(U) = \begin{cases} v_{i,h}\left(\sum_{(i,k) \notin U_i} v_{i,k}\right)^{-1} & ,\ (i,h) \notin U_i \\ 0 & ,\ (i,h) \in U_i \end{cases} \qquad i \in I_N$$

and

$$q_j^{\gamma}(Z^0) = p_j(Z_j^0) - p_j(Z_j^0 \cup Z_j)\ ,\ j \in I_N$$

and where the $(N+n)$-vector $q^{\gamma}(j,k_i^j)$, $i = 1,\ldots,z(j)$, $j \in I_N$, is given by $q_h^{\gamma}(j,k_i^j) = 0$, $h \neq j$ and

$$q_j^{\gamma}(j,k_i^j) = p_j(\{(j,k_0^j),\ldots,(j,k_i^j)\}) - p_j(\{(j,k_0^j),\ldots,(j,k_{i-1}^j)\}).$$

Observe that definition 4.1 coincides with definition 2.1 if U is empty. The set $A(\gamma,j_0)$ is a $\Sigma_{j=1}^N(n_j-u_j)$-dimensional subset of $S(U)$. Let $A(\gamma)$ be the union of $A(\gamma,j_0)$ over all indices $j_0 \in I_N$ and let $u = \Sigma_{j=1}^N u_j$. Recall that $n = \Sigma_{j=1}^N n_j$. Then each set $A(\gamma,j_0)$ is triangulated by $(n-u)$-simplices, induced by V_m defined in section 3.

<u>Definition 4.2</u>. The set $G_m(\gamma,j_0)$ is the collection of $(n-u)$-simplices $\tau(w^1,\omega)$ with vertices $w^1,\ldots,w^{n-u+1}$ such that

1) $w^1 = v(U) + bd_m^{-1}q^{\gamma}(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)d_m^{-1}q^{\gamma}(i,h)$, for nonnegative integers b and $a(i,h),(i,h) \in Z$ such that for all $j \in I_N$, $0 \leq a(j,k^j_{z(j)}) \leq \ldots \leq a(j,k^j_1) \leq b \leq d_m-1$ and $a(j_0,k^{j_0}_{z(j_0)}) = 0$;

2) $\omega = (\omega_1,\ldots,\omega_{n-u+1})$ is a permutation of the elements consisting of Z^0 and the n-u elements of Z such that $\omega_{n-u+1} = (j_0,k^{j_0}_{z(j_0)})$, and for all $i = 1,\ldots,z(j)$; $s > s'$ if $a(j,k^j_i) = a(j,k^j_{i-1})$, where $\omega_s = (j,k^j_i)$ and $\omega_{s'} = (j,k^j_{i-1})$, $j \in I_N$. In the case $i = 1$, $a(j,k^j_0) = b$ and $\omega_{s'} = Z^0$;

3) $w^{i+1} = w^i + d_m^{-1}q^{\gamma}(\omega_i)$, $i = 1,\ldots,n-u+1$, with the convention $i+1 = 1$ in the case $i = n-u+1$.

It is clear that $G_m(\gamma,j_0)$ is a triangulation of $A(\gamma,j_0)$ and that the union $G_m(\gamma)$ of $G_m(\gamma,j_0)$ over all $j_0 \in I_N$ triangulates $A(\gamma)$. Finally we observe that the union $G_m(U)$ of $G_m(\gamma)$ over all permutations γ of the elements in $I\backslash U$ induces a triangulation of $S(U)$. Some sets $A(\gamma,j_0)$ are illustrated in figure 1 when $N = 2$, $n_1 = 1$ and $n_2 = 2$. The arrows on the edges determine the ordering of the vertices in the simplices τ of $G_m(\gamma,j_0)$.

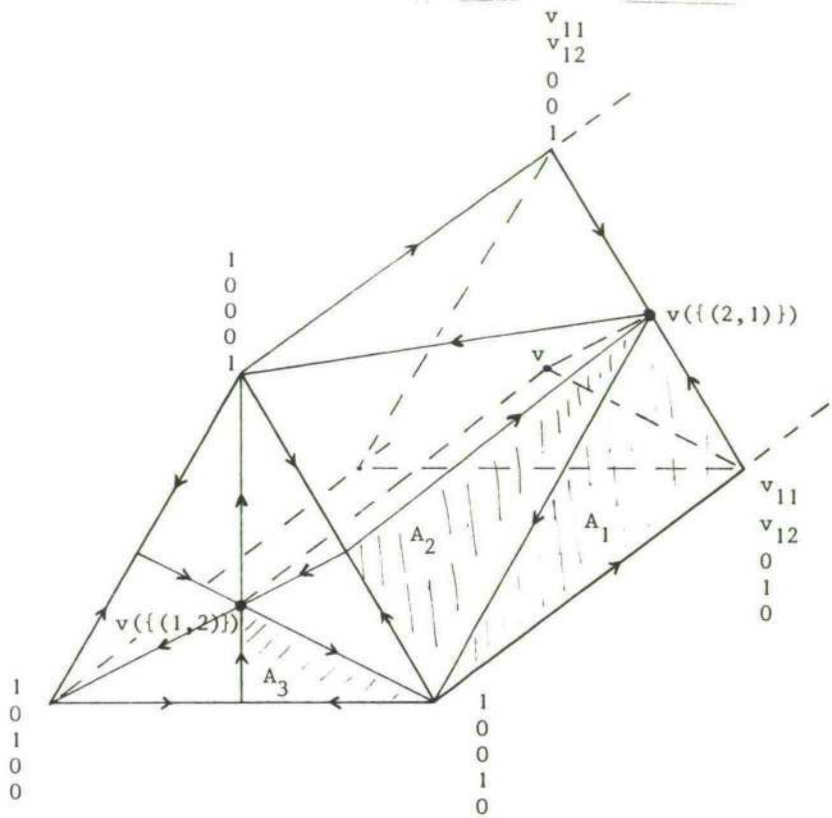

Figure 1. Some regions $A(\gamma, j_0)$; $A_1 = A(\gamma^1, 2)$ with $\gamma^1 = (((1,1),(1,2)),((2,2),(2,3)))$; $A_2 = A(\gamma^1, 1)$ and $A_3 = A(\gamma^2, 2)$ with $\gamma^2 = (((1,1)),((2,2),(2,1),(2,3)))$, N = 2, $n_1 = 1$ and $n_2 = 2$.

As in section 3, given $\theta^m_1, \ldots, \theta^m_{n+1}$, we define for each (n-u)-simplex $\tau(w^1, \omega)$ in $G_m(U)$ a centrepoint $v(\tau)$ of τ in the following way

$$v(\tau) = \Sigma_{i=1}^{n-u+1} \delta_i k_m^{-1} w^i$$

where the vector $\delta = (\delta_1, \ldots, \delta_{n-u+1})$ is given by

$$\delta_i = \begin{cases} \theta^m_{s^m(w^i)} & , \ i \neq r+1 \\ k_m - \Sigma_{\substack{i=1 \\ i \neq r+1}}^{n-u+1} \theta^m_{s^m(w^i)} & , \ i = r+1 \end{cases} \qquad \text{with } \omega_r = Z^0.$$

Observe that $\Sigma_{i=1}^{n-u+1}\delta_i$ is equal to k_m and that $v(\tau)$ is a grid point of V_{m+1} in $\tau(w^1,\omega)$. Furthermore for $U = \emptyset$ the centrepoint coincides with the centrepoint defined in section 3.

Since the algorithm will move from one simplex to an adjacent one we have to describe how the representation of the latter one can be obtained from the representation of the former one, and how the centrepoint changes from one simplex to another adjacent simplex.

So let $\tau(w^1,\omega)$ and $\bar{\tau}(\bar{w}^1,\bar{\omega})$ be in some $G_m(\gamma,j_0)$ with a common facet opposite vertex w^s, $1 \leq s \leq n-u+1$, then $\bar{\tau}$ can be obtained from τ as given in table 4. Furthermore, in tables 5 and 6 we describe how $\bar{\delta}$ and $v(\bar{\tau})$ are obtained from δ and $v(\tau)$.

	$\bar{w}^1$	$\bar{\omega}$	$\bar{a}$
$s=1$	$w^1+d_m^{-1}q^{\gamma}(\omega_1)$	$(\omega_2,\ldots,\omega_{n-u},\omega_1,\omega_{n-u+1})$	$a+e(\omega_1)$
$1<s<n-u+1$	w^1	$(\omega_1,\ldots,\omega_{s-2},\omega_s,\omega_{s-1},\ldots,\omega_{n-u+1})$	a
$s=n-u+1$	$w^1-d_m^{-1}q^{\gamma}(\omega_{n-u})$	$(\omega_{n-u},\omega_1,\ldots,\omega_{n-u-1},\omega_{n-u+1})$	$a-e(\omega_{n-u})$

Table 4. s is the index of the vertex of $\tau(w^1,\omega)$ to be replaced

		$\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_{n-u+1})$
$s=1$	$\omega_1 = Z^0$	$(\theta^m_\ell, \delta_3, \ldots, \delta_{n-u+1}, \delta_1+\delta_2-\theta^m_\ell)$, with $s^m(w^{-1}) = \ell$
	$\omega_1 \neq Z^0$	$(\delta_2, \ldots, \delta_{n-u+1}, \delta_1)$
$1<s<n-u+1$	$\omega_{s-1} = Z^0$	$(\delta_1, \ldots, \delta_{s-1}, \theta^m_\ell, \delta_{s+1}+\delta_s-\theta^m_\ell, \delta_{s+2}, \ldots, \delta_{n-u+1})$, with $s^m(w^{-s}) = \ell$
	$\omega_s = Z^0$	$(\delta_1, \ldots, \delta_{s-1}, \delta_s+\delta_{s+1}-\theta^m_\ell, \theta^m_\ell, \delta_{s+2}, \ldots, \delta_{n-u+1})$, with $s^m(w^{-s+1}) = \ell$
	$\omega_{s-1}, \omega_s \neq Z^0$	$(\delta_1, \ldots, \delta_{n-u+1})$
$s=n-u+1$	$\omega_{n-u} = Z^0$	$(\theta^m_\ell, \delta_{n-u+1}+\delta_1-\theta^m_\ell, \delta_2, \ldots, \delta_{n-u})$, with $s^m(w^{-1}) = \ell$
	$\omega_{n-u} \neq Z^0$	$(\delta_{n-u+1}, \delta_1, \delta_2, \ldots, \delta_{n-u})$

Table 5. s is the index of the vertex of τ to be replaced

	$v(\bar{\tau})$
$s = 1$	$v(\tau) + \delta_1 d^{-1}_{m+1} q^{\gamma}(\omega_1) - \bar{\delta}_{n-u+1} d^{-1}_{m+1} q^{\gamma}(\omega_{n-u+1})$
$1 < s < n-u+1$	$v(\tau) + \delta_s d^{-1}_{m+1} q^{\gamma}(\omega_s) - \bar{\delta}_s d^{-1}_{m+1} q^{\gamma}(\omega_{s-1})$
$s = n-u+1$	$v(\tau) + \delta_{n-u+1} d^{-1}_{m+1} q^{\gamma}(\omega_{n-u+1}) - \bar{\delta}_1 d^{-1}_{m+1} q^{\gamma}(\omega_{n-u})$

Table 6. s is the index of the vertex of τ to be replaced

Lemma 4.3. Let $\tau(w^1,\omega)$ be an (n-u)-simplex in $G_m(\gamma,j_0)$, then the facet opposite vertex w^s, $1 \leq s \leq n-u+1$, lies on the boundary of $A(\gamma,j_0)$ iff

a) $s = 1$: $\omega_1 = Z^0$ and $a(\omega_1) = d_m - 1$

b) $1 < s \leq n-u+1$: $\omega_{s-1} = (j,k^j_{i-1})$, $\omega_s = (j,k^j_i)$ for certain $1<i\leq z(j)$, $j \in I_N$ and $a(\omega_{s-1}) = a(\omega_s)$
or
$\omega_{s-1} = Z^0$, $\omega_s = (j,k^j_1)$ for certain $j \in I_N$ and $a(\omega_s) = b$

c) $s = n-u+1$: $\omega_{n-u} = (j,k^j_{z(j)})$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$.

The lemma follows immediately from the definitions of $G_m(\gamma,j_0)$ and $A(\gamma,j_0)$. If the facet of $\tau(w^1,\omega)$ opposite vertex w^s, $1 \leq s \leq n-u+1$, lies on the boundary of $A(\gamma,j_0)$ then either this facet is an (n-u-1)-simplex in $G_{\underline{m}}(\bar{\gamma},\bar{j}_0)$ or it is a facet of another (n-u)-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$ with either $\bar{\tau}$ in $G_m(\bar{\gamma},j_0)$, $\bar{\gamma} \neq \gamma$, or $\bar{\tau}$ in $G(\gamma,\bar{j}_0)$, $\bar{j}_0 \neq j_0$.

Lemma 4.4. Let $\tau(w^1,\omega)$ be an (n-u)-simplex in $G_m(\gamma,j_0)$ with $\bar{\tau}$ the facet opposite vertex w^1 on the boundary of $A(\gamma,j_0)$, then $\bar{\tau}(\bar{w}^1,\bar{\omega})$ is an (n-u-1)-simplex in $A(\bar{\gamma},\bar{j}_0)$, where

$$\bar{\gamma}_j = \begin{cases} \gamma_j & , j \neq j_0 \\ ((j_0,k^{j_0}_0),\ldots,(j_0,k^{j_0}_{z(j_0)-1})) & , j = j_0 \end{cases}$$

and

$$\bar{Z}_{j_0} = Z_{j_0} \setminus \{(j_0,k^{j_0}_{z(j_0)})\} \text{ and } \bar{Z}_j = Z_j,\ j \neq j_0.$$

Furthermore let J be the index set such that $j \in J$ if $j \neq j_0$ and $z(j) \geq 1$ and $j_0 \in J$ if $z(j_0) \geq 2$. Let $r(j)$, $j \in J$, be such that $\omega_{r(j)} = (j,k^j_{z(j)})$, $j \neq j_0$ and if $j_0 \in J$ $\omega_{r(j_0)} = (j_0,k^{j_0}_{z(j_0)-1})$. The index r is now given by

$$r = r(\bar{j}_0) = \max\{r(j) \mid a(\omega_{r(j)}) = \min\{a(\omega_{r(i)}) \mid i \in J\}\}.$$

Then we have

$$\bar{w}^1 = w^{r+1}, \quad \bar{\omega} = (\omega_{r+1}, \ldots, \omega_{n-u}, \omega_1, \ldots, \omega_r),$$

$\bar{b} = d_m - a(\omega_r) - 1$ and $\bar{a}(j,k_i^j)$ for (j,k_i^j) in $\bar{Z}$ is given by

$$\bar{a}(j,k_i^j) = \begin{cases} a(j,k_i^j) - a(\omega_r) - 1 & , (j,k_i^j) \in \{\omega_{r+1}, \ldots, \omega_{n-u}\} \\ a(j,k_i^j) - a(\omega_r) & , (j,k_i^j) \notin \{\omega_{r+1}, \ldots, \omega_{n-u}\}. \end{cases}$$

Observe that $\bar{a}(\omega_r) = 0$ and from the construction it is clear that $\bar{\tau}(\bar{w}^1, \bar{\omega})$ is an $(n-u-1)$-simplex in $G_m(\bar{\gamma}, \bar{j}_0)$. The centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_1 d_{m+1}^{-1} q^{\gamma}(Z^0),$$

and the vector $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_{n-u})$ is given by

$$\bar{\delta} = (\delta_{r+1}, \ldots, \delta_{n-u+1}, \delta_1 + \delta_2, \delta_3, \ldots, \delta_r).$$

<u>Lemma 4.5</u>. Let $\tau(w^1, \omega)$ be an $(n-u)$-simplex in $G_m(\gamma, j_0)$ with the facet opposite w^s, $1 < s \leq n-u+1$, on the boundary of $A(\gamma, j_0)$. Then we have one of the following two cases

(1) $1 < s \leq n-u-1$,

$\omega_{s-1} = (j,k_{i-1}^j)$, $\omega_s = (j,k_i^j)$ for certain $1 \leq i \leq z(j)$, $j \in I_N$ and $a(\omega_{s-1}) = a(\omega_s)$. In case $i = 1$, $\omega_{s-1} = Z^0$ and $a(\omega_{s-1}) = b$;

or

(2) $s = n-u+1$, $\omega_{n-u} = (j,k_{z(j)}^j)$, for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$.

In case (1), we have that the facet of τ opposite vertex w^s, $1 < s \leq n-u+1$, is a facet of an $(n-u)$-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_m(\bar{\gamma}, j_0)$ where

$$\bar{\gamma}_h = \begin{cases} \gamma_h & , h \neq j \\ ((j,k_0^j),\ldots,(j,k_{i-2}^j),(j,k_i^j),(j,k_{i-1}^j),\ldots,(j,k_{z(j)}^j)) & , h = j \end{cases}$$

$$\bar{w}^1 = w^1 \ , \ \bar{a} = a,$$

and

$$\bar{\omega} = (\omega_1,\ldots,\omega_{s-2},\omega_s,\omega_{s-1},\omega_{s+1},\ldots,\omega_{n-u+1}).$$

The centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_s d_{m+1}^{-1} q^{\gamma}(\omega_s) - \bar{\delta}_s d_{m+1}^{-1} q^{\bar{\gamma}}(\omega_{s-1})$$

with the vector $\bar{\delta} = (\bar{\delta}_1,\ldots,\bar{\delta}_{n-u+1})$ given in table 7.

		$\bar{\delta}$
$1<s<n-u+1$	$\omega_{s-1} = z^0$	$(\delta_1,\ldots,\delta_{s-1},\theta_\ell^m,\delta_s+\delta_{s+1}-\theta_\ell^m,\delta_{s+2},\ldots,\delta_{n-u+1})$, with $s^m(\bar{w}^s) = \ell$
	$\omega_{s-1} \neq z^0$	$(\delta_1,\ldots,\delta_{n-u+1})$
$s = n-u+1$	$\omega_{n-u} = z^0$	$(\theta_\ell^m,\delta_1+\delta_{n-u+1}-\theta_\ell^m,\delta_2,\ldots,\delta_{n-u})$, with $s^m(\bar{w}^{n-u+1}) = \ell$
	$\omega_{n-u} \neq z^0$	$(\delta_1,\ldots,\delta_{n-u+1})$

Table 7. s is the index of the vertex of τ to be replaced

In case (2) we have that the facet of τ opposite vertex w^{n-u+1} is a facet of an (n-u)-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$ in $G_m(\gamma,\bar{j}_0)$ with $\bar{j}_0 = j$, and with

$$\bar{w}^1 = w^1 \ , \ \bar{\omega} = (\omega_1,\ldots,\omega_{n-u-1},\omega_{n-u+1},\omega_{n-u}) \text{ and } \bar{a} = a.$$

The centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) + \delta_{n-u+1} d_{m+1}^{-1} [q^{\gamma}(\omega_{n-u+1}) - q^{\gamma}(\omega_{n-u})],$$

and $\bar{\delta} = \delta$.

Consider now the case that $\tau(w^1,\omega)$ is an $(n-u)$-simplex in $G_m(\gamma,j_0)$ with $u \geq 1$, then τ is a facet of only one $(n-u+1)$-simplex in $S(U \setminus \{(j,k)\}), (j,k) \in U$.

<u>Lemma 4.6</u>. Let $\tau(w^1,\omega)$ be an $(n-u)$-simplex in $G_m(\gamma,j_0)$ with $u \geq 1$, and let (j,k) be an element in U, then τ is a facet of exactly one $(n-u+1)$-simplex $\bar{\tau}$ in $G_m(U \setminus \{(j,k)\})$. More precisely, $\bar{\tau}$ lies in $G_m(\bar{\gamma},j)$ with

$$\bar{\gamma}_h = \begin{cases} ((j,k_0^j),\ldots,(j,k_{z(j)}^j),(j,k)) & , h = j \\ \gamma_h & , h \neq j. \end{cases}$$

The parameters of the simplex $\bar{\tau}$ are given by

$$\bar{w}^1 = w^{r+1} - d_m^{-1} q^{\bar{\gamma}}(Z^0) \text{ where } r \text{ is the index such that } \omega_r = Z^0,$$

$$\bar{\omega} = (\omega_r,\ldots,\omega_{n-u+1},\omega_1,\ldots,\omega_{r-1},(j,k)),$$

$$\bar{b} = d_m - 1, \ \bar{a}(j,k) = 0,$$

and the coefficients $\bar{a}(i,h), (i,h) \in Z$, are given by

$$\bar{a}(i,h) = \begin{cases} d_m - b + a(i,h) - 1 & , (i,h) \in \{\omega_{r+1},\ldots,\omega_{n-u+1}\} \\ d_m - b + a(i,h) & , (i,h) \notin \{\omega_{r+1},\ldots,\omega_{n-u+1}\}. \end{cases}$$

The centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) - \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0)$$

where the vector $\bar{\delta} = (\bar{\delta}_1,\ldots,\bar{\delta}_{n-u+2})$ is given by

$$\bar{\delta} = (\theta^m_\ell, \delta_{r+1}-\theta^m_\ell, \delta_{r+2},\ldots,\delta_{n-u+1},\delta_1,\ldots,\delta_r), \text{ with } \ell = s^m(\bar{w}^1)$$

The triangulation of $S(U) \times [m,m+1]$ for proper subsets U of I is obtained as follows. Let $\tau(w^1,\omega)$ be an $(n-u)$-simplex in $G_m(\gamma,j_0)$ and let $\mathring{A}(T,\tau)$ for proper subsets T of $\{\omega_1,\ldots,\omega_{n-u+1}\}$ be defined as in section 3. Then the triangulation of $\tau \times [m,m+1]$ is induced by connecting all the grid points x of V_{m+1} in $\mathring{A}(T,\tau)$ on level $m+1$ with the vertices w^i, $\omega_i \notin T$, of τ on level m. An $(n-u+1)$-simplex ψ^γ of this triangulation is given by

$$\psi^\gamma = \text{co}(\text{co}(\{w^i \mid \omega_i \notin T\}) \times \{m\},\ \sigma(y^1,\pi(T)) \times \{m+1\}). \tag{4.1}$$

We will now show that an $(n-u+1)$-simplex ψ^γ of the triangulation of $S(U) \times [m,m+1]$ for nonempty U is a facet of just one $(n-u+2)$-simplex of the triangulation of $S(U\setminus\{(j,k)\}) \times [m,m+1]$, for any $(j,k) \in U$.

Lemma 4.7. Let ψ^γ be an $(n-u+1)$-simplex of the triangulation of $S(U) \times [m,m+1]$ with U nonempty. Let (j,k) be an element of U, then ψ^γ is a facet of the $(n-u+2)$-simplex $\bar{\psi}^{\bar{\gamma}}$ of the triangulation of $S(U\setminus\{(j,k)\}) \times [m,m+1]$ where $\bar{\gamma}$ is given by

$$\bar{\gamma}_h = \begin{cases} ((j,k^j_0),\ldots,(j,k^j_{z(j)}),(j,k)) & , h = j \\ \gamma_h & , h \neq j \end{cases}$$

and for some $\bar{T}$, $\bar{\psi}^{\bar{\gamma}}$ is given by

$$\bar{\psi}^{\bar{\gamma}} = \text{co}(\text{co}(\{\bar{w}^i \mid \bar{\omega}_i \notin \bar{T}\}) \times \{m\},\ \bar{\sigma}(\bar{y}^1,\bar{\pi}(\bar{T})) \times \{m+1\})$$

where $\bar{\tau}(\bar{w}^1,\bar{\omega})$ is an $(n-u+1)$-simplex of $G_m(\bar{\gamma},j)$ and $\bar{\sigma}(\bar{y}^1,\bar{\pi}(\bar{T}))$ is a $\bar{t}$-simplex, $\bar{t} = |\bar{T}|$, in $A(\bar{T},\bar{\tau})$. The $(n-u+1)$-simplex $\bar{\tau}(\bar{w}^1,\bar{\omega})$ is described in lemma 4.6. We will now construct the simplex $\bar{\sigma}$. Recall from lemma 4.6 that the index r is such that $\omega_r = Z^0$ and that the centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = v(\tau) - \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0).$$

We consider the two cases (i) $\omega_r \notin T$ and (ii) $\omega_r \in T$. In the case i) $\omega_r \notin T$ we have for the vertex y^1 of σ

$$y^1 = v(\tau) + \Sigma_{h \in T} R_h d_{m+1}^{-1} q^{\gamma}(h)$$

$$= v(\bar{\tau}) + \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0) + \Sigma_{h \in T} R_h d_{m+1}^{-1} q^{\bar{\gamma}}(h).$$

If $\bar{\delta}_1$ is positive, then $\bar{\sigma}(\bar{y}^1, \bar{\pi}(\bar{T}))$ is a (t+1)-simplex in $A(\bar{T}, \bar{\tau})$ with

$$\bar{y}^1 = y^1 - d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0),$$

$$\bar{T} = T \cup \{\omega_r\},$$

$$\bar{\pi}(\bar{T}) = (Z^0, \pi_1, \ldots, \pi_t),$$

and

$$\bar{R} = R + (\bar{\delta}_1 - 1)e(Z^0).$$

If $\bar{\delta}_1 = 0$, then $\bar{\sigma} = \sigma$ is a t-simplex in $A(T, \bar{\tau})$.

In the case ii) $\omega_r \in T$ there is an index s, $1 \leq s \leq t$, such that $\pi_s = \omega_r$. We have for the vertex y^1

$$y^1 = v(\tau) + \Sigma_{h \in T} R_h d_{m+1}^{-1} q^{\gamma}(h)$$

$$= v(\bar{\tau}) + \bar{\delta}_1 d_{m+1}^{-1} q^{\bar{\gamma}}(Z^0) + \Sigma_{h \in T \setminus \{\omega_r\}} R_h d_{m+1}^{-1} q^{\bar{\gamma}}(h)$$

$$+ R_{\omega_r} d_{m+1}^{-1} [q^{\bar{\gamma}}(Z^0) + q^{\bar{\gamma}}(j,k)].$$

The (t+1)-simplex $\bar{\sigma}(\bar{y}^1,\bar{\pi}(\bar{T}))$ is now given by

$$\bar{y}^1 = y^1,$$

$$T = T \cup \{(j,k)\},$$

$$\bar{\pi}(\bar{T}) = (\pi_1,\ldots,\pi_{s-1},(j,k),\pi_s,\ldots,\pi_t),$$

and

$$\bar{R} = R + \bar{\delta}_1 e(Z^0) + R_{\omega_r} e(j,k).$$

From the construction it is clear that $\bar{\sigma}$ lies in $A(\bar{T},\bar{\tau})$. In all cases the (n-u+1)-simplex ψ^γ is a facet of the (n-u+2)-simplex $\bar{\psi}^{\bar{\gamma}}$. More precisely, if $\omega_r \notin T$ and $\bar{\delta}_1 > 0$, then ψ^γ is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{y}^1,m+1)$; if $\omega_r \notin T$ and $\bar{\delta}_1 = 0$, then ψ^γ is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{w}^1,m)$ and if $\omega_r \in T$, then ψ^γ is the facet of $\bar{\psi}^{\bar{\gamma}}$ opposite vertex $(\bar{y}^{s+1},m+1)$ where $\pi_s = \omega_r$.

By extending this lemma it can easily be shown that for any extension $\bar{\gamma}$ of γ each (n-u+1)-simplex ψ^γ of the triangulation of $S(U) \times [m,m+1]$ is a face of just one (n+1)-simplex $\bar{\psi}^{\bar{\gamma}}$ of the triangulation of $S \times [m,m+1]$ described in section 3.

5. The steps of the algorithm

Let z be a continuous function from S into R^{N+n} such that for all p in S

$$\Sigma_{k=1}^{n_j+1} p_{j,k} z_{j,k}(p) = 0 \ , \ j \in I_N$$

holds. The problem is to find a p^* in S such that $z(p^*) \leq 0$. We call this problem the nonlinear complementarity problem on S. To solve this problem we embed S in $S \times [1,\infty)$ and triangulate $S \times [1,\infty)$ as described in section 3, inducing also a triangulation of $S(U) \times [1,\infty)$ for proper subsets U of I. Each point $x = (p,t)$ in $S \times [1,\infty)$ is labelled with the $(N+n)$-vector $\ell(x) = z(p)$.

<u>Definition 5.1</u>. Let U be a proper subset of I, γ_j a permutation of the elements in $I(j)\setminus U_j$, $j \in I_N$, $\gamma = (\gamma_1,\ldots,\gamma_N)$ and let ψ^γ be a k-simplex, $k = n-u, n-u+1$, in $S(U) \times [m,m+1]$ for certain $m = 1,2,\ldots$. The simplex $\psi^\gamma(x^1,\ldots,x^{k+1})$ with $x^i = (p^i,t^i)$, $p^i \in S$ and $t^i \in \{m,m+1\}$ is complete if the system of linear equations

$$\Sigma_{i=1}^{k+1}\lambda_i\binom{\ell(x^i)}{1} + \Sigma_{(i,h)\in U}\mu_{i,h}\binom{e(i,h)}{0} - \Sigma_{j=1}^{N}\beta_j\binom{\bar{e}(j)}{0} = \binom{0}{1} \qquad (5.1)$$

has a solution $\lambda_i^* \geq 0$, $i = 1,\ldots,k+1$, $\mu_{i,h}^* \geq 0$, $(i,h) \in U$, and β_j^*, $j \in I_N$.

A solution λ_i^*, $i = 1,\ldots,k+1$, $\mu_{i,h}^*$, $(i,h) \in U$, β_j^*, $j \in I_N$ will be denoted by $(\lambda^*,\mu^*,\beta^*)$.

<u>Nondegeneracy assumption</u>. If ψ^γ is a complete k-simplex in $S(U) \times [m,m+1]$ then the system (5.1) has a unique solution $(\lambda^*,\mu^*,\beta^*)$ for $k = n-u$ with $\lambda_i^* > 0$, $i = 1,\ldots,n-u+1$, $\mu_{i,h}^* > 0$, $(i,h) \in U$, and for $k = n-u+1$ at most one variable of (λ^*,μ^*) is equal to zero.

By this nondegeneracy assumption, a complete (n-u+1)-simplex ψ^{γ} contains a line segment of solutions with two end points. Each of the end points is characterized by a solution with exactly one variable in (λ^*,μ^*) equal to zero. All other variables in (λ^*,μ^*) are positive. We call the solution at an end point of such a line segment a basic solution. To each solution of (5.1) there corresponds a point $x = \Sigma_{i=1}^{n-u+2} \lambda_i^* x^i$ in ψ^{γ}. In particular, when at a basic solution one of the λ_i^*'s, say $\lambda_{\bar{s}}^*$, is equal to zero, the corresponding x lies in the interior of the facet of ψ^{γ} opposite vertex $x^{\bar{s}}$. This facet is then also complete. If at a basic solution $\mu_{i,h}^* = 0$ for some (i,h) in U, then the corresponding x lies in the interior of ψ^{γ}. Each line segment of solutions to (5.1) induces by this way a line segment of points x in ψ^{γ} with two end points. This line segment of points can be followed by making a linear programming step in the system (5.1).

Let g be the function from $S \times [1,\infty)$ into R^{N+n} given by $g(p,t) = z(p)$, $(p,t) \in S \times [1,\infty)$ and let $\bar{g}$ be the piecewise linear approximation of g with respect to the underlying triangulation of $S \times [1,\infty)$. Observe that for all $(p,t) \in S \times [1,\infty)$ we have $p_j^T \bar{g}_j(p,t) = 0$, $j \in I_N$. We will now show that a complete simplex induces an approximate solution to the nonlinear complementarity problem on S.

<u>Theorem 5.2</u>. Let $\varepsilon > 0$ and let δ be such that

$$\max\{\,|z_{i,h}(p)-z_{i,h}(q)|\;|(i,h)\in I,\ p,q\in S,\ |p_{i,h}-q_{i,h}| < \delta,\ (i,h)\in I\} < \varepsilon.$$

Let m be such that mesh $V_m < \delta$ and let $\psi^{\gamma}(x^1,\ldots,x^{k+1})$, $x^i = (p^i,t^i) \in S \times \{m,m+1\}$, $i = 1,\ldots,k+1$, be a complete k-simplex, $k = n-u$ or $k = n-u+1$, with solution $(\lambda^*,\mu^*,\beta^*)$, in $S(U) \times [m,m+1]$. Furthermore let $\bar{p}$ be given by $\bar{p} = \Sigma_{i=1}^{k+1}\lambda_i^* p^i$, then

(1) $$-\varepsilon < \beta_j^* < \varepsilon \qquad , j \in I_N$$

(2) $$\beta_i^* - \varepsilon < z_{i,h}(\bar{p}) < \beta_i^* + \varepsilon \qquad , (i,h) \notin U$$

(3) $$z_{i,h}(\bar{p}) < \beta_i^* + \varepsilon \qquad , (i,h) \in U.$$

<u>Proof</u>. Let $\psi^{\gamma}(x^1,\ldots,x^{k+1})$ be a complete k-simplex in $S(U) \times [m,m+1]$ with solution $(\lambda^*,\mu^*,\beta^*)$, then we have for all $j \in I_N$,

$$\bar{g}_{j,h}(\bar{x}) = \beta^*_j \qquad \text{if } (j,h) \notin U_j$$

and (5.2)

$$\bar{g}_{j,h}(\bar{x}) \leq \beta^*_j \qquad \text{if } (j,h) \in U_j$$

where $\bar{x} = \Sigma_{i=1}^{k+1}\lambda^*_i x^i$.

Therefore

$$|\beta^*_j| = |\bar{p}^T_j \bar{g}_j(\bar{p},\bar{t})| = |\bar{p}^T_j(\bar{g}_j(\bar{p},\bar{t}) - g_j(\bar{p},\bar{t}))|$$

$$= |\bar{p}^T_j(\Sigma_{i=1}^{k+1}\lambda^*_i[g_j(p^i,t^i) - g_j(\bar{p},\bar{t})])|$$

$$= |\bar{p}^T_j(\Sigma_{i=1}^{k+1}\lambda^*_i[z_j(p^i) - z_j(\bar{p})])| < \varepsilon. \tag{5.3}$$

Observe that $\max_{(j,h)} |p^i_{j,h} - \bar{p}_{j,h}| \leq \delta$, $i = 1,\ldots,k+1$, since mesh $V_m \leq \delta$. Furthermore, for all $(j,h) \in I(j)$

$$|\bar{g}_{j,h}(\bar{x}) - z_{j,h}(\bar{p})| = |\Sigma_{i=1}^{k+1}\lambda^*_i(g_{j,h}(p^i,t^i) - g_{j,h}(\bar{p},\bar{t}))|$$

$$= |\Sigma_{i=1}^{k+1}\lambda^*_i(z_{j,h}(p^i) - z_{j,h}(\bar{p}))| < \varepsilon. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4) proves the theorem.

$\square$

Now let m be a fixed integer and let U be a proper subset of I. The complete (n-u+1)-simplices in $S(U) \times [m,m+1]$ determine paths of adjacent simplices with complete facets such that each path is either a loop or has two end points. An end point is either (1) a complete facet on level m, (2) a complete facet on level m+1, (3) a complete facet in $bd(S(U)) \times [m,m+1]$ or (4) if U is nonempty, a complete (n-u+1)-simplex in $S(U\setminus\{(i,h)\}) \times [m,m+1]$ with $\mu^*_{i,h} = 0$, for some (i,h) in U. In case

(3), the facet is an end point of a path of complete $(n-u)$-simplices in $S(U\cup\{(j,k)\}) \times [m,m+1]$ for certain (j,k) not in U, and in case (4) the simplex is a facet of a complete $(n-u+2)$-simplex in $S(U\setminus\{(i,h)\}) \times [m,m+1]$ which is an end point of a path of adjacent complete $(n-u+2)$-simplices in $S(U\setminus\{(i,h)\}) \times [m,m+1]$. Linking the paths of complete simplices in $S(U) \times [m,m+1]$ in this way together for varying U we obtain paths of adjacent complete simplices of varying dimension in $S \times [m,m+1]$. Again each path is either a loop or has two end points. An end point is now either a complete simplex in $S(U) \times \{m\}$ for some $U \subset I$ or a complete simplex in $S(U') \times \{m+1\}$, for some $U' \subset I$. In the former case, if $m > 1$, the complete simplex is an end point of a path of adjacent complete simplices in $S(U) \times [m-1,m]$, and in the latter case it is an end point of a path of adjacent complete simplices in $S(U') \times [m+1,m+2]$.

For varying m, $m \geq 1$, the complete simplices therefore yield paths of adjacent complete simplices in $S \times [1,\infty)$. Each path has 0, 1 or 2 end points. Each end point is a complete simplex in $S \times \{1\}$. A path with two end points connects therefore two complete simplices in $S \times \{1\}$, while a path with one end point has a complete simplex in $S \times \{1\}$ and must exceed each level $S \times \{m\}$, $m = 2,3,\ldots$ since the number of simplices in $S \times [1,m]$ is finite for each m. A path with no end points is either a loop and remains in $S \times [m_0,m_1]$ for certain $1 \leq m_0 < m_1 < \infty$, or there is an m_0, $m_0 \geq 1$, for which the path exceeds each level $S \times \{m\}$, $m > m_0$ with at least two different complete simplices.

The algorithm described in this section starts on level one with the variable dimension algorithm described in section 2, starting in the point v yielding within a finite number of steps a complete simplex τ^0 in $S \times \{1\}$. Then the algorithm continues by following the path of adjacent complete simplices in $S \times [1,\infty)$ starting with the unique complete simplex ψ in $S \times [1,2]$ containing τ^0 as a facet. The algorithm can be terminated when the accuracy of an approximate solution is sufficient, i.e., when β^* is small enough. If the path returns to $S \times \{1\}$ with a complete simplex τ^1, then we again apply the variable dimension restart algorithm of section 2 starting with the complete simplex τ^1. This yields within a finite number of steps another complete simplex τ^2 in $S \times \{1\}$. Observe that both τ^1 and τ^2 differ from τ^0. The algorithm continues with the path in $S \times [1,\infty)$ starting with the unique complete simplex $\bar{\psi}$ in $S \times [1,2]$ containing τ^2 as a facet, etc. The steps of the algorithm follow from the description of the replacement steps given in section 4 and are described below. Therefore let $\psi^{\gamma}(x^{1},\ldots,x^{n-u+2})$ be a complete (n-u+1)-simplex in $S(U) \times [m,m+1]$. By making a linear programming pivot step in (5.1) we can follow the line segment of solutions (λ,μ,β) with respect to ψ^{γ}. Then either a $\lambda_{\bar{s}}$ becomes zero for some $\bar{s}$ in $\{1,\ldots,n-u+2\}$ or a $\mu_{i,h}$ becomes zero for some $(i,h) \in U$. If $\lambda_{\bar{s}}$ becomes zero the facet of ψ^{γ} opposite the vertex $x^{\bar{s}}$ is also complete and yields a new adjacent complete simplex. We then have to consider two cases, either $x^{\bar{s}}$ lies on level m (case I) or $x^{\bar{s}}$ lies on level m+1 (case II). If $\mu_{i,h}$ becomes zero then ψ^{γ} is a facet of a unique (n-u+2)-simplex in $S(U\setminus\{(i,h)\}) \times [m,m+1]$ which case is described in case III. In the following $\bar{w}$ will denote the new vertex of τ and $\bar{y}$ the new vertex of σ.

<u>Case I</u>. The point $x^{\bar{s}}$ lies on level m, i.e., $x^{\bar{s}} = (w^{s},m)$ for some s, $1 \leq s \leq n-u+1$.

a) Suppose that (w^{s},m) is not the only vertex of ψ^{γ} on level m.

Then the points $y^{1},\ldots,y^{t+1}$ lie in the facet of τ opposite vertex w^{s} iff

$$\omega_{s-1} \notin T \text{ and } \delta_{s} = 0. \tag{5.5}$$

First suppose that (5.5) holds then we have the following 4 cases.

(1) If $s = 1$, $\omega_{1} = Z^{0}$ and $a(\omega_{1}) = d_{m} - 1$, then γ, j_{0}, $\tau(w^{1},\omega)$, a and δ are adapted according to lemma 4.4 and T, $\sigma(y^{1},\pi(T))$ and R do not change. A pivot step is made with $(e^{\top}(j_{0},k^{j_{0}}_{z(j_{0})}),0)^{\top}$.

(2) If $1 < s \leq n-u+1$, $\omega_{s-1} = (j,k^{j}_{i-1})$ or Z^{0} when $i = 1$, $\omega_{s} = (j,k^{j}_{i})$ for certain $j \in I_{N}$, $1 \leq i \leq z(j)$ and $a(\omega_{s-1}) = a(\omega_{s})$, then γ and $\tau(w^{1},\omega)$ are adapted according to lemma 4.5 (1) and T, $\sigma(y^{1},\pi(T))$ and R do not change. A pivot step is made with $(\ell^{\top}(\bar{w},m),1)^{\top}$.

(3) If $s = n-u+1$, $\omega_{n-u} = (j,k^j_{z(j)})$ for certain $j \in I_N$, and $a(\omega_{n-u}) = 0$, then j_0 and $\tau(w^1,\omega)$ are adapted according to lemma 4.5 (2) and T, $\sigma(y^1,\pi(T))$ and R do not change. A pivot step is made with $(\ell^{\mathsf{T}}(\bar{w},m),1)^{\mathsf{T}}$.

(4) In all other cases γ and j_0 do not change and $\tau(w^1,\omega)$, a and δ are adapted according to tables 4 and 5.

4.1 If $s = 1$ and $\bar{\delta}_{n-u+1} > 0$, then T becomes $T \cup \{\omega_{n-u+1}\}$, y^1 becomes $y^1 - d^{-1}_{m+1}q^{\gamma}(\omega_{n-u+1})$, $\pi(T)$ becomes $(\omega_{n-u+1},\pi_1,\dots,\pi_t)$ and R becomes $R + (\bar{\delta}_{n-u+1}-1)e(\omega_{n-u+1})$.

4.2 If $1 < s < n-u+1$ and $\bar{\delta}_s > 0$, then T becomes $T \cup \{\omega_{s-1}\}$, y^1 becomes $y^1 - d^{-1}_{m+1}q^{\gamma}(\omega_{s-1})$, $\pi(T)$ becomes $(\omega_{s-1},\pi_1,\dots,\pi_t)$ and R becomes $R + (\bar{\delta}_s-1)e(\omega_{s-1})$.

4.3 In all other cases T, $\sigma(y^1,\pi(T))$ and R do not change.

In the cases 4.1 and 4.2 a pivot step is made with $(\ell^{\mathsf{T}}(\bar{y},m+1),1)^{\mathsf{T}}$ and in case 4.3 a pivot step is made with $(\ell^{\mathsf{T}}(\bar{w},m),1)^{\mathsf{T}}$.

Now suppose that (5.5) does not hold, then γ, j_0, $\tau(w^1,\omega)$, a and δ do not change, T becomes $T \cup \{\omega_s\}$, y^1 and R do not change and $\pi(T)$ becomes $(\pi_1,\dots,\pi_t,\omega_s)$. A pivot step is made with $(\ell^{\mathsf{T}}(\bar{y},m+1),1)^{\mathsf{T}}$.

b) The vertex $(w^{\bar{s}},m)$ is the only vertex of ψ^{γ} on level m.

The (n-u)-simplex $\sigma(y^1,\pi(T))$ is a simplex of the triangulation of $G_{m+1}(\gamma,j_0)$, $T \subset \{\omega_1,\dots,\omega_{n-u+1}\}$, $|T| = n-u$. There is exactly one element in the set $\{\omega_1,\dots,\omega_{n-u+1}\}$ not in T, say ω_h. Let ω_h be denoted by π_{n-u+1} and let r be the index such that $\pi_r = (j_0,k^{j_0}_{z(j_0)})$. The centre point of τ can be denoted by

$$v(\tau) = w^1 + \Sigma^{n-u}_{i=1}\alpha_i d^{-1}_{m+1}q^{\gamma}(\omega_i),$$

with $\alpha_i = \Sigma^{n-u+1}_{h=i+1}\delta_h$, $i = 1,\dots,n-u$. Furthermore

$$w^1 = v(U) + bd_m^{-1}q^\gamma(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)d_m^{-1}q^\gamma(i,h)$$

$$= v(U) + bk_m d_{m+1}^{-1}q^\gamma(Z^0) + \Sigma_{(i,h)\in Z}a(i,h)k_m d_{m+1}^{-1}q^\gamma(i,h).$$

Combining these two results yields

$$v(\tau) = v(U) + b'd_{m+1}^{-1}q^\gamma(Z^0) + \Sigma_{(i,h)\in Z}a'(i,h)d_{m+1}^{-1}q^\gamma(i,h)$$

with $b' = bk_m + \alpha_k$, where k is given by $\omega_k = Z^0$ and

$$a'(i,h) = \begin{cases} a(i,h)k_m + \alpha_{h'} & , \ \omega_{h'} = (i,h) \in Z \setminus \{(j_0, k_{z(j_0)}^{j_0})\} \\ 0 & , \ (i,h) = (j_0, k_{z(j_0)}^{j_0}). \end{cases}$$

The parameters of the (n-u)-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ are given by

$$\bar{w}^1 = y^{r+1},$$

$$\bar{\omega} = (\pi_{r+1}, \ldots, \pi_{n-u+1}, \pi_1, \ldots, \pi_r),$$

$$\bar{b} = \begin{cases} b' + R_{Z^0} & \text{if } Z^0 \notin \{\pi_{r+1}, \ldots, \pi_{n-u+1}\} \\ b' + R_{Z^0} - 1 & \text{if } Z^0 \in \{\pi_{r+1}, \ldots, \pi_{n-u+1}\}, \end{cases}$$

and $\bar{a}(i,h), (i,h) \in Z$, are given by

$$\bar{a}(i,h) = \begin{cases} a'(i,h) + R_{i,h} & , \ (i,h) \notin \{\pi_{r+1}, \ldots, \pi_{n-u+1}\} \\ a'(i,h) + R_{i,h} - 1 & , \ (i,h) \in \{\pi_{r+1}, \ldots, \pi_{n-u+1}\}. \end{cases}$$

If it is the first time that we move into $S \times [m+1, m+2]$, we choose an integer $k_{m+1} > 1$ and integers θ_i^{m+1}, $i = 1, \ldots, n+1$, such that $\Sigma_{i=1}^{n+1}\theta_i^{m+1} = k_{m+1}$. In general we should choose θ_i^{m+1}, $i = 1, \ldots, n+1$, in such a way that $v(\bar{\tau})$ lies close to the approximation found on level m+1. The algorithm continues with $y^1 = v(\bar{\tau})$, $T = \emptyset$, $R = 0$, $m = m+1$ and makes a pivot step with $(\ell^T(\bar{y},m+1),1)^T$.

Case II. The point $x^{\bar{s}}$ lies on level m+1, i.e. $x^{\bar{s}} = (y^p,m+1)$ for some p, $1 \leq p \leq t+1$.

a) Suppose that $(y^p,m+1)$ is not the only vertex of ψ^γ on level m+1.

To describe the replacement steps of the algorithm in this case, we need the following lemma.

Lemma 5.3. Let $\tau(w^1,\omega)$ be an (n-u)-simplex in $G_m(\gamma,j_0)$ and $\sigma(y^1,\pi(T))$ a t-simplex, $t = |T|$, where T is a proper subset of $\{\omega_1,\ldots,\omega_{n-u+1}\}$, in $A(T,\tau)$. The facet of $\sigma(y^1,\pi(T))$ opposite vertex y^p, $1 \leq p \leq t+1$, is not a facet of another t-simplex $\bar{\sigma}$ in $A(T,\tau)$ iff one of the following cases holds.

(i) $p = 1$: $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 1$, with $\omega_s = \pi_1$, and $\omega_{s-1} \notin T$. The points $y^2,\ldots,y^{t+1}$ lie in the facet of τ opposite vertex w^s;

(ii) $1 < p < t+1$: $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 0$, with $\omega_s = \pi_p$, and $\omega_{s-1} = \pi_{p-1}$. The points $y^1,\ldots,y^{p-1},y^{p+1},\ldots,y^{t+1}$ lie in the facet of τ opposite vertex w^s;

(iii) $p = t+1$: $R_{\pi_t} = 0$. The points $y^1,\ldots,y^t$ lie in $A(T\setminus\{\pi_t\},\tau)$.

In all other cases the facet of σ opposite vertex y^p is a facet of the t-simplex $\bar{\sigma}(\bar{y}^1,\bar{\pi}(T))$ in $A(T,\tau)$ with the parameters of $\bar{\sigma}$ given in table 8.

	$\bar{y}^1$	$\bar{\pi}(T)$	$\bar{R}$
$p = 1$	$y^1+d_{m+1}^{-1}q^{\gamma}(\pi_1)$	$(\pi_2,\dots,\pi_t,\pi_1)$	$R+e(\pi_1)$
$1 < p < t+1$	y^1	$(\pi_1,\dots,\pi_{p-2},\pi_p,\pi_{p-1},\dots,\pi_t)$	R
$p = t+1$	$y^1-d_{m+1}^{-1}q^{\gamma}(\pi_t)$	$(\pi_t,\pi_1,\dots,\pi_{t-1})$	$R-e(\pi_t)$

Table 8. p is the index of the vertex of σ to be replaced

(i) $p = 1$

First suppose that $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 1$ and $\omega_{s-1} \notin T$, with $\omega_s = \pi_1$, then we have the following 4 cases.

(1) If $s = 1$, $\omega_1 = Z^0$ and $a(\omega_1) = d_m - 1$, then γ, j_0, $\tau(w^1,\omega)$, a and δ are adapted according to lemma 4.4, and T becomes $T\setminus\{\omega_s\}$, y^1 becomes y^2, $\pi(T)$ becomes $(\pi_2,\dots,\pi_t)$ and R becomes $R - R_{\pi_1} e(\pi_1)$. A pivot step is made with $(e^{\top}(j_0,k^{j_0}_{z(j_0)}),0)^{\top}$.

(2) If $1 < s \leq n-u+1$, $\omega_{s-1} = (j,k^j_{i-1})$ or Z^0 when $i = 1$, $\omega_s = (j,k^j_i)$ for certain $j \in I_N$, $1 \leq i \leq z(j)$, and $a(\omega_{s-1}) = a(\omega_s)$, then γ, $\tau(w^1,\omega)$ and δ are adapted according to lemma 4.5 (1).

2.1 If $\bar{\delta}_s = 0$, then T becomes $T\setminus\{\omega_s\}$, y^1 becomes y^2, $\pi(T)$ becomes $(\pi_2,\dots,\pi_t)$ and R becomes $R - R_{\omega_s} e(\omega_s)$. A pivot step is made with $(\ell^{\top}(\bar{w},m),1)^{\top}$.

2.2 If $\bar{\delta}_s > 0$, then T becomes $T\setminus\{\omega_s\} \cup \{\omega_{s-1}\}$, y^1 becomes $y^2 - d_{m+1}^{-1}q^{\gamma}(\omega_{s-1})$, $\pi(T)$ becomes $(\omega_{s-1},\pi_2,\dots,\pi_t)$ and R becomes $R - R_{\omega_s} e(\omega_s) + (\bar{\delta}_s-1)e(\omega_{s-1})$. A pivot step is made with $(\ell^{\top}(\bar{y},m+1),1)^{\top}$.

(3) If $s = n-u+1$, $\omega_{n-u} = (j,k^j_{z(j)})$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$, then j_0, $\tau(w^1,\omega)$ and δ are adapted according to lemma 4.5 (2), and T becomes $T\setminus\{\omega_{n-u+1}\} \cup \{\omega_{n-u}\}$, y^1 becomes $y^2 - d^{-1}_{m+1}q^{\gamma}(\omega_{n-u})$, $\pi(T)$ becomes $(\omega_{n-u},\pi_2,\ldots,\pi_t)$ and R becomes $R - R_{\omega_{n-u+1}}(e(\omega_{n-u+1})-e(\omega_{n-u}))$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

(4) In all other cases γ and j_0 do not change, $\tau(w^1,\omega)$, a and δ are adapted according to tables 4 and 5.

4.1 If $\bar{\delta}(\bar{w}) = 0$, with $\bar{\delta}(\bar{w})$ the coefficient of the new vertex $\bar{w}$ of $\bar{\tau}$, then T becomes $T\setminus\{\omega_s\}$, y^1 becomes y^2, $\pi(T)$ becomes $(\pi_2,\ldots,\pi_t)$ and R becomes $R - R_{\omega_s}e(\omega_s)$. A pivot step is made with $(\ell^T(\bar{w},m),1)^T$.

4.2 If $\bar{\delta}(\bar{w}) > 0$, then T becomes $T\setminus\{\omega_s\} \cup \{\omega_{s-1}\}$, y^1 becomes $y^2 - d^{-1}_{m+1}q^{\gamma}(\omega_{s-1})$, $\pi(T)$ becomes $(\omega_{s-1},\pi_2,\ldots,\pi_t)$ and R becomes $R - R_{\omega_s}e(\omega_s) + (\bar{\delta}(\bar{w})-1)e(\omega_{s-1}))$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

Now suppose that $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} \neq 1$ or $\omega_{s-1} \in T$, then γ, j_0, $\tau(w^1,\omega)$, a, δ and T do not change, and $\sigma(y^1,\pi(T))$ and R are adapted according to table 8. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

(ii) <u>$1 < p < t+1$</u>

First suppose that $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} = 0$ and $\omega_{s-1} = \pi_{p-1}$, with $\omega_s = \pi_p$, then again we have the following 4 cases.

(1) If $s = 1$, $\omega_1 = z^0$ and $a(\omega_1) = d_m - 1$, then γ, j_0, $\tau(w^1,\omega)$, a and δ are adapted according to lemma 4.4, and T becomes $T\setminus\{\pi_{p-1}\}$, y^1 does not change, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{p-2},\pi_p,\ldots,\pi_t)$, and R becomes $R - R_{\omega_{n-u+1}}(e(\omega_{n-u+1})-e(\omega_1))$. A pivot step is made with $(e^T(j_0,k^{j_0}_{z(j_0)}),0)^T$.

(2) If $1 < s \leq n-u+1$, $\omega_{s-1} = (j,k^j_{i-1})$ or Z^0 when $i = 1$, $\omega_s = (j,k^j_i)$ for certain $j \in I_N$, $1 \leq i \leq z(j)$, and $a(\omega_{s-1}) = a(\omega_s)$, then γ, $\tau(w^1,\omega)$ and δ are adapted according to lemma 4.5 (1), and T and y^1 do not change, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$ and R becomes $R - \delta_s e(\omega_s) + \bar{\delta}_s e(\omega_{s-1})$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

(3) If $s = n-u+1$, $\omega_{n-u} = (j,k^j_{z(j)})$ for certain $j \in I_N$ and $a(\omega_{n-u}) = 0$, then j_0, $\tau(w^1,\omega)$ and δ are adapted according to lemma 4.5 (2), T, y^1 do not change, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$ and R becomes $R - \delta_{n-u+1}(e(\omega_{n-u+1})-e(\omega_{n-u}))$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

(4) In all other cases, γ and j_0 do not change, $\tau(w^1,\omega)$ and δ are adapted according to tables 4 and 5, T and y^1 do not change, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$ and R becomes $R - \delta_s e(\omega_s) + \bar{\delta}(\bar{w})e(\omega_{s-1})$, with $\bar{\delta}(\bar{w})$ the coefficient of the new vertex $\bar{w}$ in $\bar{\tau}$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

Now suppose that $\delta_s - R_{\omega_s} + R_{\omega_{s-1}} \neq 0$ or $\omega_{s-1} \neq \pi_{p-1}$, then γ, j_0, $\tau(w^1,\omega)$, a, δ, T, y^1 and R do not change, and $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\ldots,\pi_t)$. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$.

(iii) <u>$p = t+1$</u>

First suppose that $R_{\pi_t} = 0$, then the points $y^1,\ldots,y^t$ lie in $A(T\setminus\{\pi_t\},\tau)$ and γ, j_0, $\tau(w^1,\omega)$, a and δ do not change, T becomes $T\setminus\{\pi_t\}$, y^1 and R do not change and $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{t-1})$. A pivot step is made with $(\ell^T(w^s,m),1)^T$, where s is given by $\omega_s = \pi_t$.

Now suppose that $R_{\pi_t} \neq 0$, then γ, j_0, $\tau(w^1,\omega)$, a, δ and T do not change, and $\sigma(y^1,\pi(T))$ and R are adapted according to table 8. A pivot step is made with $(\ell^T(\bar{y},m+1),1)^T$, where s is given by $\omega_s = \pi_t$.

b) Suppose that $(y^p, m+1)$ is the only vertex of ψ^γ on level m+1.

In this case we have $T = \emptyset$, $t = 0$ and $p = 1$. We will first consider the case $m > 1$. The (n-u)-simplex $\tau(w^1, \omega)$ is a complete simplex on level m of $G_m(\gamma, j_0)$, all vertices of τ on level m are vertices of ψ^γ and the only vertex of σ on level m+1 has to be replaced. We now have to compute the unique (n-u)-simplex $\bar{\tau}(\bar{w}^1, \bar{\omega})$ in $G_{m-1}(\gamma, j_0)$ and the set $\bar{T}$ such that τ lies in $A(\bar{T}, \bar{\tau})$. The vertex w^1 is given by

$$w^1 = v(U) + bd_m^{-1}q^\gamma(Z^0) + \Sigma_{(i,h) \in Z}a(i,h)d_m^{-1}q^\gamma(i,h). \tag{5.6}$$

Let $a'(i,h), (i,h) \in Z^0 \cup Z$ be given by $a'(i,h) = a(i,h)k_{m-1}^{-1}$ and let $\bar{a}(i,h)$ be the entier of $a'(i,h)$, where entier of x, $x \in R$, is the largest integer less than or equal to x, then $a'(i,h) - \bar{a}(i,h) \geq 0$ for all (i,h) in $Z^0 \cup Z$. Observe that if $a(j,k_{i-1}^j) = a(j,k_i^j)$, then also $\bar{a}(j,k_{i-1}^j) = \bar{a}(j,k_i^j)$. Let $\bar{w}^1$ be given by

$$\bar{w}^1 = v(U) + \bar{b}d_m^{-1}q^\gamma(Z^0) + \Sigma_{(i,h) \in Z}\bar{a}(i,h)d_m^{-1}q^\gamma(i,h), \tag{5.7}$$

where $\bar{b}$ is equal to $\bar{a}(j,k_0^j)$, $j \in I_N$, then we have for all $j \in I_N$

$$0 \leq \bar{a}(j,k_{z(j)}^j) \leq \ldots \leq \bar{a}(j,k_1^j) \leq \bar{b} \leq d_m - 1.$$

Let x be an interior point of τ, i.e., $x = \Sigma_{k=1}^{n-u+1}\lambda_k w^k$, with uniquely determined $\lambda_k > 0$, $k = 1,\ldots,n-u+1$. It is easy to show that

$$x = v(U) + cd_m^{-1}q^\gamma(Z^0) + \Sigma_{(i,h) \in Z}c(i,h)d_m^{-1}q^\gamma(i,h),$$

with $b < c < b+1$, $c(j_0,k_{z(j_0)}^{j_0}) = 0$ and $a(i,h) < c(i,h) < a(i,h)+1$ for all $(i,h) \in Z \setminus \{(j_0,k_{z(j_0)}^{j_0})\}$. Furthermore $c(i,h) \neq c(j,k)$ for all (i,h), (j,k) in Z, $(i,h) \neq (j,k)$. Let $\bar{c}(i,h), (i,h) \in Z \cup Z^0$ be given by

$$\bar{c}(i,h) = c(i,h)k_{m-1}^{-1},$$

with $c(i,h) = c$ for all $(i,h) \in Z^0$, then the entier of $\bar{c}(i,h)$ is equal to $\bar{a}(i,h)$. Then $\bar{\omega} = (\bar{\omega}_1,\dots,\bar{\omega}_{n-u+1})$ is the permutation of Z^0 and the elements of Z such that

$$\bar{c}(\bar{\omega}_1) - \bar{a}(\bar{\omega}_1) < \dots < \bar{c}(\bar{\omega}_{n-u}) - \bar{a}(\bar{\omega}_{n-u}),$$

and $\bar{\omega}_{n-u+1} = (j_0, k^{j_0}_{z(j_0)})$. To complete this case we have to determine the set $\bar{T}$. This set contains n−u elements and is a subset of $\{\omega_1,\dots,\omega_{n-u+1}\}$, i.e., there is exactly one index s with ω_s not in $\bar{T}$. This index is determined as follows. The centrepoint of $\bar{\tau}$ is given by

$$v(\bar{\tau}) = \Sigma_{i=1}^{n-u+1}\delta_i k_{m-1}^{-1}\bar{w}^i = \bar{w}^1 + \Sigma_{i=1}^{n-u}\bar{\alpha}_i d_m^{-1} q^{\gamma}(\bar{\omega}_i)$$

$$= \bar{w}^1 + \Sigma_{i=1}^{n-u}\alpha_i d_m^{-1} q^{\gamma}(\omega_i), \tag{5.8}$$

with $\alpha_i = \bar{\alpha}_j$ when $\omega_i = \bar{\omega}_j$, $i = 1,\dots,n-u$, where $\bar{\alpha}_i = \Sigma_{h=i+1}^{n-u+1}\delta_h$, $i = 1, \dots, n-u$. Let α_{n-u+1} be equal to zero. Equation (5.7) can be expressed in the following way

$$\bar{w}^1 = v(U) + \Sigma_{i=1}^{n-u+1}(\bar{a}(\omega_i)k_{m-1})d_m^{-1}q^{\gamma}(\omega_i). \tag{5.9}$$

Combining (5.8) and (5.9) yields for the centrepoint $v(\bar{\tau})$ and w^1

$$v(\bar{\tau}) = v(U) + \Sigma_{i=1}^{n-u+1}(\bar{a}(\omega_i)k_{m-1}+\alpha_i)d_m^{-1}q^{\gamma}(\omega_i)$$

and

$$w^1 = v(U) + \Sigma_{i=1}^{n-u+1}a(\omega_i)d_m^{-1}q^{\gamma}(\omega_i)$$

$$= v(\bar{\tau}) + \Sigma_{i=1}^{n-u+1}(a(\omega_i)-\bar{a}(\omega_i)k_{m-1}-\alpha_i)d_m^{-1}q^{\gamma}(\omega_i).$$

Let $c(\omega_i)$, $i = 1,\dots,n-u+1$, be given by

$$c(\omega_i) = a(\omega_i) - \bar{a}(\omega_i)k_{m-1} - \alpha_i,$$

and let $\bar{c} = \min\{c(\omega_i) \mid i = 1,\ldots,n-u+1\}$, then $\bar{c}(\omega_i) = c(\omega_i) - \bar{c}$ is non-negative for all i, $1 \leq i \leq n-u+1$. The index s is now given by

$$s = \max\{i \in I_{n-u+1} \mid c(\omega_i) - \bar{c} = 0\}.$$

Then $\bar{T} = \{\omega_1,\ldots,\omega_{s-1},\omega_{s+1},\ldots,\omega_{n-u+1}\}$ and the (n-u)-simplex $\bar{\sigma}(\bar{y}^1,\bar{\pi}(\bar{T}))$ has parameters

$$\bar{y}^1 = w^{s+1},$$

$$\bar{\pi}(\bar{T}) = (\omega_{s+1},\ldots,\omega_{n-u+1},\omega_1,\ldots,\omega_{s-1}).$$

The vector $\bar{R}$ is given by $\bar{R}_{j,k_0^j} = \bar{R}_{Z^0}$, $j = 1,\ldots,N$, with

$$\bar{R}_{Z^0} = \begin{cases} \bar{c}(Z^0) & \text{if } Z^0 \in \{\omega_1,\ldots,\omega_{s-1}\} \\ \bar{c}(Z^0) - 1 & \text{if } Z^0 \in \{\omega_{s+1},\ldots,\omega_{n-u+1}\}, \end{cases}$$

$$\bar{R}_{\omega_i} = \begin{cases} \bar{c}(\omega_i) & , i = 1,\ldots,s \quad , \omega_i \neq Z^0 \\ \bar{c}(\omega_i) - 1 & , i = s+1,\ldots,n-u+1 \ , \omega_i \neq Z^0, \end{cases}$$

and $\bar{R}_{i,h} = 0$, $(i,h) \in U$. From the construction it is clear that $\bar{\sigma}(\bar{y}^1,\bar{\pi}(\bar{T}))$ lies in $A(\bar{T},\bar{\tau})$. Observe that $\bar{R}_{\omega_s}$ is equal to zero and that all other coefficients of $\bar{R}$ are non-negative. Now let $\bar{s}$ be the index such that $\bar{\omega}_{\bar{s}} = \omega_s$, then a pivot step is made with $(\ell^T(\bar{w}^{\bar{s}},m-1),1)^T$.

In the case m = 1 we apply the variable dimension algorithm described in section 2 starting with the complete simplex τ on level 1.

Case III. $\mu_{i,h}$ becomes zero for some $(i,h) \in U$.

This case is described in lemma 4.7 for $(i,h) = (j,k)$. If ψ^γ is the facet of $\bar{\psi}^\gamma$ opposite vertex $(\bar{y}^p, m+1)$, for some p, $1 \leq p \leq t+1$, a pivot step is made with $(\ell^T(\bar{y}^p, m+1), 1)^T$. If ψ^γ is the facet of $\bar{\psi}^\gamma$ opposite vertex $(\bar{w}^1, m)$ then a pivot step is made with $(\ell^T(\bar{w}^1, m), 1)^T$.

The cases above describe the steps of the algorithm to follow a path of complete simplices in $S \times [1,\infty)$ to solve the nonlinear complementarity problem on S with respect to a continuous function z from S to $\Pi_{j=1}^N R^{n_j+1}$. The algorithm can easily be adapted to follow a path of approximating solutions with respect to a continuous function z from $S \times [1,\infty)$ to $\Pi_{j=1}^N R^{n_j+1}$ where t, $t \geq 1$, is interpreted as a time parameter. In this case we can apply the algorithm for a constant grid size on each level by taking k_m equal to one for $m = 1,2,\ldots$.

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