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A CONTINUOUS REVIEW INVENTORY MODEL WITH COMPOUND POISSON DEMAND PROCESS AND STOCHASTIC LEAD TIME °

by

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S U M M A R Y

Using Markov renewal theory, analytic expressions for the expected average cost associated with (s,S) - policies are derived for a continuous review inventory model with a compound Poisson demand process and stochastic lead time under the (restrictive) assumption that only one order can be outstanding.

N O T A T I O N

Z : integers

N : non-negative integers

R : reals

R_+ : nonnegative real numbers

1. A CONTINUOUS REVIEW INVENTORY MODEL.

The demand process is of the compound Poisson type, i.e., the probability of a cumulative demand k, in a time period of length t is given by

$$\sum_{l=0}^{k} e^{-\lambda t} \frac{(\lambda t)^{l}}{l!} v_k^{(l)} ,$$

where $\lambda > 0$ and $v_k^{(l)}$ is the l-fold convolution of the demand-size distribution at k; $v_k^{(1)} = v_k$, with $v_0 = 0$, $v_k > 0$ $k = 1, 2, \ldots,$ denotes, of course, the probability that, once a request is made on the inventory system, k items are demanded.[1]

The ordering process is also assumed to be stochastic. When an order is placed, the probability that the order is delivered within a time period of length t is given by the lead-time distribution function G(t), which is assumed to be continuous and Riemann integrable and to have a finite mean. (Note that G(t) is assumed to be independent of the order size.)

As to the cost structure we consider :

(i) an ordering cost of the form $C_0 + c_0k$, when k items are ordered,

(ii) an inventory carrying cost of c_1 per unit time,

(iii) a backorder cost of c_2 per unit time.

Furthermore, it will be assumed that only inventory policies of the (s,S)- type will be utilized.

The optimization criterion will be the expected average cost criterion.

So, we want to solve the following optimization problem : Find a (s,S)- policy that minimizes the expected average cost over an infinite planning

1 The assumption that $v_k > 0$ for all $k \geq 1$ is made for analytical convenience and can easily be relaxed

horizon of the inventory problem with the demand and ordering process and the cost structure described above.[1]

To achieve this task, we will employ techniques of Markov renewal theory (see Cinlar [1], [2] and Schellhaas [4]) to obtain an analytic expression for the expected average cost associated with a given (s,S)- policy (Sections 2 and 3) to reduce the optimization problem to a straightforward search procedure .

The proposed development will, unfortunately, necessitate the following restrictive

Assumption : No orders can overlap.

In other words, if an order is outstanding, the next order can only be placed after the arrival of the outstanding order. It should be stressed that the tools of Markov renewal theory can be employed without this assumption, however, the difficulties arise when one attempts to derive analytical expressions for the cost rate associated with (s,S)- policies.[2]

To contrast this model with some existing work, we can make reference to Gross and Harris [3] and the references mentioned there, and to Schellhaas [4] . Gross and Harris consider a model where the lead-time is state-dependent, i.e., orders can be procured according to different lead-time processes with changing inventory levels. Schellhaas, in the part of the paper devoted to inventory theory , derives an expression for the average cost in a model without overlapping assumption but with a Poisson demand process.

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1. The subsequent analysis can be adjusted to incorporate the expected discounted cost criterion.
2. A manageable problem results when (S-1,S) policies are employed.

2. THE INVENTORY PROCESS AS A REGENERATIVE STOCHASTIC PROCESS.

In this section we introduce some results from Markov renewal theory in the spirit of Cinlar [1] to apply them to the inventory model described above.

Consider a probability space $(\Omega, \mathcal{F}, P)$ and the random variables

$X_n : \Omega \to Z$,

$T_n : \Omega \to R^+$,

such that $T_0 = 0$, $T_n \leqslant T_{n+1}$ for all $n \in N$ and, finally, $T_n \to \infty$ a.s..

Then

<u>Definition 1</u> : $\{ (X_n, T_n) ; n \in N \}$ forms a <u>Markov renewal process</u> with state space Z if

$$P(X_{n+1} = j, T_{n+1} - T_n \leqslant t / X_0, \ldots, X_n; T_0, \ldots, T_n)$$

$$= P(X_{n+1} = j, T_{n+1} - T_n \leqslant t / X_n, T_n) \text{ for all } j \in Z \quad \text{and } t \in [0, \infty) .$$

<u>Definition 2</u> : Let $\{ X(t); t \in R_+ \}$ be any stochastic process such that $X(t) = X_n$ for $t = T_n$, then, $\{ X(t) ; t \in R_+ \}$ is said to be a <u>regenerative process</u> with respect to a Markov renewal process $\{ (X_n, T_n) ; n \in N \}$ if

$$P(X(t) = j / X_0, \ldots, X_n \ T_0, \ldots, T_n; X(\tau) \text{ for all } \tau \leqslant T_n)$$

$$= P(X(t) = j / X_n, T_n) \text{ for all } t \geqslant T_n, n \in N.$$

Now we show how the inventory problem of Section 1 can be described in terminology of Markov renewal theory.

To do so, define [1] :

(1) X_n as a random variable taking values in $J = \{S, S-1, \dots\}$, denoting the inventory level at time period T_n ; we assume $(X_0, T_0) = (k, 0)$ with $k \leq S$,

(2) if $X_{n-1} = i$, $n \geq 1$ then T_n is the arrival time of the quantity ordered at T_{n-1} if $i \leq s$ and is the next order point if $s < i \leq S$.

Furthermore, in order to characterize the distribution of the length of a regeneration interval, denote by $B_j(t)$ the probability that the next regeneration point occurs in a time less than or equal to $t \in R^+$ for any $j \in J$.

Lemma 1 : For any $j \in J$, $t \in R^+$

$$B_j(t) = \begin{cases} G(t) & \text{if } j \leq s, \\ \sum_{k=-\infty}^{s} A(j,k,t) & \text{if } j > s, \end{cases}$$

$$\text{with } A(j,k,t) = \sum_{l=0}^{j-k} e^{-\lambda t} \frac{(\lambda t)^l}{l!} v_{j-k}^{(l)} ,$$

Proof : If $j \leq s$, quantity $S-j$ is ordered, hence $B_j(t) = G(t)$; when $j > s$, observe that $B_j(t) = \sum_{k=-\infty}^{s} P(D_t = j-k)$, where D_t denotes the cumulative demand in a time period of length t.

1 From now on we will assume that a particular (s,S)- policy is given, this to avoid cumbersome notation.

<u>Proposition 1</u> : If X_n and T_n are defined by (1) and (2), then $\{(X_n, T_n); n\varepsilon N\}$ is a Markov renewal process.

<u>Proof</u> : For any $n\varepsilon N$, $X_n = i$, $k\varepsilon J$ consider

<u>case (i)</u> : <u>$i > s$</u>

$$P(X_{n+1} = k, T_{n+1} - T_n \leq t \,/\, X_0, \ldots, X_n = i; T_0, T_1, \ldots, T_n)$$

$$= \begin{cases} 0 & \text{if } k > s, \\ \int_0^t P(i - D_\tau = k \,/\, X_0, \ldots, X_n = i; T_0, T_1, \ldots, T_n)\, dB_i(\tau) & \text{if } k \leq s, \end{cases}$$

$$= \begin{cases} 0 & \text{if } k > s, \\ \int_0^t P(i - D_\tau = k \,/\, X_n = i, T_n)\, dB_i(\tau) & \text{if } k \leq s. \end{cases}$$

<u>case (ii)</u> : <u>$i \leq s$</u>

$$P(X_{n+1} = k, T_{n+1} - T_n \leq t \,/\, X_0, \ldots, X_n = i; T_0, T_1, \ldots, T_n)$$

$$= \int_0^t P(S - D_\tau = k \,/\, X_0, \ldots, X_n = i; T_0, T_1, \ldots, T_n)\, dG(\tau)$$

$$= \int_0^t P(S - D_\tau = k)\, dG(\tau).$$

The following proposition is then immediate.

Proposition 2 : Let $X(t)$ be a random variable indicating the inventory level at time t, then $\{X(t);\ t \geq 0\}$ is a regenerative process with respect to the Markov renewal process defined by (1) - (2).

Now that it is established that $\{(X_n, T_n);\ n \in N\}$ is a Markov renewal process, it is, of course, well-known that $\{X_n;\ n \in N\}$ is a Markov chain. To characterize the transition structure of this imbedded Markov chain, let

$$p_{ij} = P\,(X_{n+1} = j \,/\, X_n = i)$$

so that, for any pair $i, j \in J$

$$(3) \quad p_{ij} = \begin{cases} 0 & \text{if } i > s,\ j > s, \\ \int_0^\infty A(i,j,\tau)\, dB_i(\tau) & \text{if } i > s,\ j \leq s, \\ \int_0^\infty A(S,j,\tau)\, dG(\tau) & \text{if } i \leq s. \end{cases}$$

In fact,

Lemma 2 : The Markov chain $\{ X_n;\ n \in N \}$ is ergodic.

Proof : Aperiodicity and irreducibility follow from (3). To show positive recurrence, we first exhibit an explicit solution to the following system :

$$(4) \qquad \Pi_i = \sum_j p_{ji}\, \Pi_j \qquad i \in J,$$

$$(5) \qquad \Pi_i \geq 0 \qquad i \in J,$$

(6) $$\sum_{i \in J} \Pi_i = 1$$

with p_{ij} as in (3).

For all $i \leq s$, we rewrite (4) as

(7) $$\Pi_i = \sum_{j=s+1}^{S} p_{ji} \Pi_j + a_i \sum_{j=-\infty}^{s} \Pi_j$$

with $a_i = p_{ji}$ for all $j \leq s$.

For $s < i \leq S$, we obtain for (4)

(8) $$\Pi_i = a_i \sum_{j=-\infty}^{s} \Pi_j .$$

Define

(9) $$\Pi_i' = \begin{cases} a_i K & \text{if } s < i \leq S, \\ b_i K & \text{if } i \leq s, \end{cases}$$

with $b_i = \sum_{j=s+1}^{S} p_{ji} a_j + a_i$

and

$$K = \left(\sum_{i=s+1}^{S} a_i + 1 \right)^{-1} .$$

The $\{\Pi_i'\}$ defined in (9) solve the system (5) - (8); the uniqueness of this solution follows trivially from observing that $\left(\sum_{s+1}^{S} \Pi_i \right) \left(\sum_{-\infty}^{s} \Pi_j \right)^{-1}$ has to be a constant in order to solve (5) - (8).

Lemma 2 will not only prove to be useful in subsequent theoretical developments, but also provides us with an <u>explicit solution</u> (see (9)) for the stationary distribution of the Markov chain $\{X_n\}$.

In order to be able to invoke some results of Schellhaas [4] we need some definitions.

Let

$$\Phi_{jk}(t,u) = P(X(t) = k / (X(0) = j,\ T_1 = u) \quad \text{for} \quad 0 \leq t < u,$$

and $\Psi_{jk}(t) = P(X(t) = k,\ t < T_1 < \infty \ / \ X(0) = j),$

so that

$$\Psi_{jk}(t) = \int_{t^+}^{\infty} \Phi_{jk}(t,u)\, dB_j(u).$$

Hence

$$(10) \quad \Psi_{jk}(t) = \begin{cases} 0 & \text{for } j > s,\ k > j \text{ or } k \leq s, \\ \int_{t^+}^{\infty} A(j,k,t)\, dB_j(u) & \text{for } j > s,\ s < k \leq j, \\ \int_{t^+}^{\infty} A(j,k,t)\, dG(u) & \text{for } j \leq s,\ k \leq j. \end{cases}$$

The expected length of a regeneration interval is denoted by m_j if j is the initial state, i.e.,

$$(11) \qquad m_j = \int_0^{\infty} t\, dB_j(t), \quad j \in J.$$

If $P_{jk}(t) = P(X(t) = k \ / \ X(0) = j)$, then

Theorem 1 : For any $k \epsilon J$

(12) $$P^{*}_{k} = \lim_{t\to\infty} P_{jk}(t) = \frac{\sum_{J} \Pi_i \int_0^{\infty} \Psi_{ik}(t)\, dt}{\sum_{J} \Pi_i m_i}.$$

Proof : To apply a result of Schellhaas [4 , Korrolar 1.1., p. 12] , it suffices to observe that the imbedded Markov chain is ergodic (Lemma 2), that $B_j(t)$ is continuous and $\Psi_{jk}(t)$ is Riemann-integrable.

If $V_j(t)$ denotes the expected cost associated with a particular (s,S)-policy during a time interval $[0,t]$ if $X(0) = j$, our interest lies in the limiting behavior of $t^{-1} V_j(t)$. In fact, we have the following

Theorem 2 : The limit of $t^{-1} V_j(t)$ exists for all $j \epsilon J$, in fact

$$g_j \equiv \lim_{t\to\infty} \frac{V_j(t)}{t} = g,$$

with $g = g^0 + g^1 + g^2$

and

(13) $$g^0 = \frac{C_0 + c_0 \left[S - \sum_{j \leqslant s} j \frac{\Pi_j}{\sum_{j \leqslant s} \Pi_j} \right] \cdot \sum_{j \leqslant s} \Pi_j}{\sum_{j=s+1}^{S} \Pi_j (m_j - m') + m'},$$

(14) $$g^1 = c_1 \sum_{j=0}^{S} j P^{*}_{j},$$

(15) $$g^2 = c_2 \sum_{j=-\infty}^{0} j P^{*}_{j},$$

where the Π_j's are the solution of (4) - (6), the m_j's are defined by (11), the P_j^*'s are given in (12)

and

$$m' = \int_0^\infty t dG(t).$$

<u>Remark</u> : The fact that $g = g^0 + g^1 + g^2$, reflects the additive cost structure : the average cost associated with the ordering process, the inventory costs and the backorder costs are added up to obtain the overall average costs. Note the intuitive interpretation of (13) - (15).

In (13), for instance, $\sum_{j \leqslant s} j \frac{\Pi_j}{\sum_{j \leqslant s} \Pi_j}$ is the expected inventory level <u>given</u> that levels below s are considered.

<u>Proof</u> : The theorem is an easy consequence of a result of Schellhaas [4 , Korrolar 2.1., p.21] , which, in turn, can be established by standard renewal theoretic techniques, cf. Cinlar [2] . Schellhaas proved that, if the embedded Markov chain is ergodic and for a cost structure which has a finite expected value on finite intervals

(16) $$g \equiv \lim_{t \to \infty} \frac{V_j(t)}{t} = \frac{\sum_J \Pi_j \rho_j}{\sum_J \Pi_j m_j},$$

where ρ_j is the expected cost in a regeneration interval with initial state j.

This result will be applied to ordering, inventory carrying and backorder costs.

For the ordering costs we have

$$\rho_j^0 = \begin{cases} 0 & \text{if } j > s, \\ \sum_{k=-\infty}^{S} C_0 + c_0 (S-j) p_{jk} & \text{if } j \leqslant s. \end{cases}$$

Hence

$$\rho_j^0 = \begin{cases} 0 & \text{if } j > s, \\ C_0 + c_0(S - j) & \text{if } j \leq s. \end{cases}$$

So

$$g^0 = \frac{\sum_{j \leq s} \Pi_j \, (C_0 + c_0 \, (S - j))}{\sum_{j=s+1}^{S} \Pi_j \, m_j + \sum_{j=-\infty}^{s} \Pi_j \, m_1}$$

and (13) obtains.

Observe that for the case of the inventory carrying costs,

$$\rho_j^1 = \begin{cases} 0 & \text{if } j \leq 0, \\ \sum_{k \in J} c_1 \int_0^\infty \Psi_{jk}(t) \, dt & \text{if } 0 < j \leq S. \end{cases}$$

Substitution in (16) and changing the order of summation gives then (14). The same argument applies to the backorder costs.

Theorem 1 and 2 give us an explicit method to compute the expected average cost of a particular (s,S)-policy, however, we will see in the next section how these results can be simplified from a computational point of view.

3. SOME ANALYTIC RESULTS.

In this section we exploit the properties of the compound Poisson process in order to obtain manageable analytic expressions for the 'cost rates' (13) - (15).

In view of (9), it is clear that an analytic expression for the stationary distribution can be obtained if the transition probabilities (see (3)) can be explicitly computed.

So

Lemma 3 : For any given (s,S)- policy

$$
p_{ij} = \begin{cases}
0 & \text{if } S \geqslant i > s,\ j > s, \\
\sum_{l=0}^{s+1-j} v_{i-j}^{(l)} \; 2^{-(l+1)} & \text{if } i = s+1,\ j \leqslant s, \\
\sum_{l=0}^{i-j} v_{i-j}^{(l)} \; 2^{-(l+1)} - \sum_{k=1}^{i-s-1} \sum_{n=1}^{k} \frac{v_k^{(m)}}{(n-1)!} \sum_{l=0}^{i-j} v_{i-j}^{(l)} \frac{(l+n-1)!}{l!\,2^{l+n}} \left(1 - \frac{l+n}{2n}\right. & \\
 & \text{if } S \geqslant i \geqslant s+1,\ j \leqslant s, \\
\sum_{l=0}^{S-j} v_{S-j}^{(l)} \int_0^{\infty} e^{-\lambda t} \frac{(\lambda t)^l}{l!} \, dG(t) & \text{if } i \leqslant s .
\end{cases}
$$

Proof : Consider the case where $S \geqslant i \geqslant s+1$ and $j \leqslant s$.

Then

$$
p_{ij} = \int_0^{\infty} A(i,j,t) \, d(1 - e^{-\lambda t} - \sum_{k=1}^{j-s-1} \sum_{n=1}^{k} e^{-\lambda t} \frac{(\lambda t)^n}{n!} v_k^{(n)}),
$$

so that

$$p_{ij} = \lambda \int_0^\infty A(i,j,t)\, e^{-\lambda t}\, dt - \lambda \sum_{k=1}^{i-s-1} \sum_{n=1}^{k} v_k^{(n)} \int_0^\infty A(i,j,t) \left(e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} - e^{-\lambda t}\frac{(\lambda t)^n}{n!} \right) dt.$$

Substituting for A(i,j,t), and exploiting the properties of the gamma density gives

$$\int_0^\infty A(i,j,t)\, e^{-\lambda t} \frac{(\lambda t)^n}{(n)!}\, dt = \sum_{l=0}^{j-i} v_{i-j}^{(l)} \frac{(l+n)!}{\lambda l!\, 2^{l+n+1}}$$

Hence the result for $S \geqslant i \geqslant s+1$ and $j \leqslant s$ obtains. The other cases are trivial.

We also have, in view of Lemma 1 and (11)

<u>Lemma 4</u> : For any $i \varepsilon J$

$$m_i = \begin{cases} \int_0^\infty t\,dG(t) & \text{if } i \leqslant s, \\ \frac{1}{\lambda} & \text{if } i = s+1, \\ \frac{1}{\lambda}\left(1 + \sum_{k=1}^{j-s-1} \sum_{n=1}^{k} v_k^{(n)}\right) & \text{if } i > s+1 . \end{cases}$$

Furthermore, and this is useful to compute the values of P_k^* (see Theorem 1), Lemma 5 : For any $i \epsilon J$

$$\int_0^\infty \Psi_{ik}(t)dt = \begin{cases} 0 & i > s,\ k > i \text{ or } k \leqslant s, \\ \sum_{k=1}^{i-s-1} \sum_{n=0}^{k} v_k^{(n)} \left(\sum_{l=0}^{i-k} v_{i-k}^{(l)} \right) \dfrac{(n+l)!}{n!\ l!\ 2^{n+l+1}\lambda} & \text{if } i > s,\ s \leqslant k \leqslant i, \\ \sum_{l=0}^{i-k} v_{i-l}^{(l)} \lambda^{-1} - \sum_{l=0}^{i-k} v_{i-k}^{(l)} \int_0^\infty e^{-\lambda t} \dfrac{(\lambda t)^l}{l!} dG(t) & \text{if } i \leqslant s,\ k \leqslant j. \end{cases}$$

(For simplicity $v_k^{(0)} \equiv 1$.)

The proof of Lemma 4 and 5 is similar to that of Lemma 3.

The results can be used to evaluate (13) - (15). Unfortunately, two infinite series $\left(\sum_{j \leq s} j \Pi_j \left(\sum_{j \leq s} \Pi_j \right)^{-1} \text{ and } \sum_{j=-\infty}^{0} j P_j^* \right)$ are still to be evaluated.

In practical cases these series would have to be approximated; this is, of course, numerically speaking, not an unsurmountable task.

Further simplifications arise if a special form of the lead-time distribution is assumed. For instance, if $G(t) = 1 - e^{-\mu t}$, then, for all $j \leq s$

$$a_i = p_{ji} = \sum_{l=0}^{S-i} \frac{\mu\lambda^l}{(\lambda+\mu)^{l+1}} v_{s-i}^{(l)} ,$$

and, setting $\Theta = \dfrac{\lambda}{\lambda+\mu}$,

$$\Pi_i = \begin{cases} \dfrac{\sum_{l=0}^{S-i} \Theta^l \, v_{S-i}^{(1)}}{\sum_{i=s+1}^{S} \sum_{l=0}^{S-j} \Theta^l \, v_{S-j}^{(1)} + \frac{\lambda+\mu}{\mu}} & \text{if } S \geq i \geq s, \\ \left(\sum_{l=0}^{S-i} \frac{\mu}{\lambda+\mu} \Theta^l \, v_{S-i}^{(1)} + \sum_{j=s+1}^{S} p_{ji} \sum_{l=0}^{S-j} \frac{\mu^l \, \Theta^l}{(\lambda+\mu)} v_{S-j}^{(1)} \right) K & \text{if } i < s, \end{cases}$$

where K is defined as in (9).

Of course, the expressions for p_{ij}, m_i and P_i^* simplify accordingly.

The results for the case when the demand occurs according to a Poisson process can be obtained by simplifying all previous expressions to account for the fact that in this case $v_i^{(n)} = 0$ whenever $i \neq n$.

If we now consider the overall expected average cost as a function of the policy used, i.e., g(s,S), then it follows that g(.,S) is a unimodal function and if we denote $s^*(S)$ as

$$g(s^*(S), S) = \min_{s<S} g(s,S),$$

the optimal solution is found by minimizing $g(s^*(S),S)$ over S. To achieve this task, simple search procedures can be used.

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