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A CONTINUOUS VERSION OF THE BORSUK-ULAM THEOREM

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ABSTRACT. Let $p: E \to B$ be an n-sphere bundle, $q: V \to B$ be an $\mathbf{R}^n$-bundle and $f: E \to V$ be a fibre preserving map over a paracompact space B. Let $\bar{p}: \bar{E} \to B$ be the projectivized bundle obtained from p by the antipodal identification and let $\bar{A}_f$ be the subset of $\bar{E}$ consisting of pairs $\{e, -e\}$ such that $fe = f(-e)$. If the cohomology dimension d of B is finite then the map $(\bar{p}|\bar{A}_f)^*: H^d(B; \mathbf{Z}_2) \to H^d(\bar{A}_f; \mathbf{Z}_2)$ is injective for a continuous cohomology theory H^*. Moreover, if the jth Stiefel-Whitney class of q is zero for $1 < j < r$ then $(\bar{p}|\bar{A}_f)^*$ is injective in degrees $i > d - r$. If all the Stiefel-Whitney classes of q are zero then $(\bar{p}|\bar{A}_f)^*$ is injective in every degree.

Introduction. The Borsuk-Ulam theorem [1] says that if $f: S^n \to \mathbf{R}^n$ is a map then the set A_f of points $x \in S^n$ such that $fx = f(-x)$ is nonempty. Because A_f is symmetric with respect to the antipodal involution, it is more convenient to consider the subset $\bar{A}_f$ of the real projective n-space P^n corresponding to A_f under the antipodal identification.

If a single S^n and an $\mathbf{R}^n$ are replaced by continuous families $E \to B$ with fibre S^n and $V \to B$ with fibre $\mathbf{R}^n$ over a space B, and if f is replaced by a fibre preserving map $f: E \to V$, one may expect the existence of a cross-section of sorts in the set $\bar{A}_f$ of pairs $\{e, -e\}$ such that $e \in E$ and $fe = f(-e)$, at least on an algebraic level.

A result in this direction in the case when E is the product bundle $E = S^k \times S^n$ and V is a single $\mathbf{R}^n$ follows from a theorem proved by J. E. Connett [2]. In this note we are going to consider this question for fibre preserving maps $E \to V$ where E is an n-sphere bundle and V is an n-dimensional real vector space bundle over a paracompact space B. If B is a point, then the theorem proved below reduces to the classical Borsuk-Ulam theorem.

Main result. If X is a space with an involution $t: X \to X$, we denote by $\bar{X}$ the orbit space X/t of t. If $p: E \to B$ is a fibre bundle with a fibre preserving involution $t: E \to E$, we write $\bar{p}: \bar{E} \to B$ for the bundle $p/t: E/t \to B$; its fibre is $\bar{X}$, where X is the fibre of p. Thus if $p: E \to B$ is an n-sphere bundle, then $\bar{p}: \bar{E} \to B$ is the associated real projective n-space bundle.

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If E is any space with an involution t: $E \to E$ and f: $E \to V$ is a map of E into some space V, let A_f denote the set of points $e \in E$ such that $fe = fte$ and let $\bar{A}_f$ be the image of A_f in $\bar{E}$.

We are going to use the Alexander-Spanier cohomology theory H^* mod 2. The coefficient group $\mathbf{Z}_2$ will be suppressed from the notation. If Z is a space, A is a subset of Z and i: $A \to Z$ is the inclusion map, then the image of a cohomology class $z \in H^*(Z)$ under the induced homomorphism i^*: $H^*(Z) \to H^*(A)$ will sometimes be denoted by $z|A$ and called the restriction of z to A. We denote by dim Z the covering dimension of Z and by $d(Z)$ its cohomology dimension, that is, $d(Z) = \operatorname{Sup}\{m$: $H^m(Z) \neq 0\}$. We have $d(Z) \leqslant \dim Z$ if Z is paracompact. If q: $V \to B$ is a vector space bundle over B then the jth Stiefel-Whitney class of q is denoted by $w_j(q)$.

We will assume throughtout the paper that B is a paracompact space.

THEOREM. *Let $p: E \to B$ be an n-sphere bundle with the antipodal involution, let q: $V \to B$ be an $\mathbf{R}^n$-bundle and let f: $E \to V$ be a fibre preserving map over B. If $d(B) \leqslant d$ and $w_j(q) = 0$ for $1 \leqslant j \leqslant r$ then the map $(\bar{p}|\bar{A}_f)^*$: $H^i(B) \to H^i(\bar{A}_f)$ is injective for $i \geqslant d - r$.*

In the following corollaries we specify particular cases of this theorem to illustrate its significance.

COROLLARY 1. *If f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involution into an $\mathbf{R}^n$-bundle q: $V \to B$ and if $d(B) = d < \infty$, then the map $(\bar{p}|\bar{A}_f)^*$: $H^d(B) \to H^d(\bar{A}_f)$ is injective.*

COROLLARY 2. *If f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involution into an $\mathbf{R}^n$-bundle q: $V \to B$ and if all the Stiefel-Whitney classes of q are zero then the map $(\bar{p}|\bar{A}_f)^*$: $H^i(B) \to H^i(\bar{A}_f)$ is injective for every i.*

COROLLARY 3. *If B is closed manifold and f: $E \to V$ is a fibre preserving map of an n-sphere bundle p: $E \to B$ with the antipodal involtion into an $\mathbf{R}^n$-bundle q: $V \to B$ then* $\dim A_f = \dim \bar{A}_f \geqslant \dim B$.

In Corollary 3, we have $d = d(B) = \dim B$ and $H^d(B) \neq 0$. On the other hand, $\dim \bar{A}_f = \dim A_f$ since the orbit map $A_f \to \bar{A}_f$ is a double covering.

Proof of the theorem. If X is any space with a free involution t: $X \to X$, let $u(X)$ denote its characteristic class. It is an element $u(X) \in H^1(\bar{X})$, where $\bar{X}$ is, as usual, the orbit space of t. In other words, $u(X)$ is the Stiefel-Whitney class of the double covering $X \to \bar{X}$. The class $u(S^n)$ of the antipodal involution generates the polynomial ring $H^*(P^n)$ of height n.

Let $b \in B$. Then the fibre of $\bar{p}$ over b is $\bar{p}^{-1}b \cong P^n$ and the polynomial ring $H^*(\bar{p}^{-1}b)$ is generated by $u(p^{-1}b) \in H^1(\bar{p}^{-1}b)$. The fibre inclusion $p^{-1}b \to E$ is an equivariant map. By the naturality of u, the restriction of $u(E) \in H^1(\bar{E})$ to the fibre $\bar{p}^{-1}b$ is equal to $u(p^{-1}b)$. By the Leray-Dold-Hirsch theorem [3, p. 229], $H^*(\bar{E})$ is an $H^*(B)$-module freely generated by the powers $1, u(E), \ldots, u^n(E)$, with

$H^*(B)$ acting on $H^*(\bar{E})$ via the cup product. In other words, the map

$$\bigoplus_{i=0}^{n} H^{m+i}(B) \to H^{m+n}(\bar{E}),$$

$$(x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto \sum_{i=0}^{n} (\bar{p}^* x_{m+i}) \cup u^{n-i}(E)$$

is an isomorphism. This map restricted to $H^m(B)$ gives a monomorphism

$$\iota\colon H^m(B) \to H^{m+n}(\bar{E}), \qquad x \mapsto (\bar{p}^* x) \cup u^n(E).$$

Let 0 be the zero section in V and $V_0 = V - 0$. Then the antipodal map is a free involution in V_0 and the fibre of the bundle $q_0 = q|V_0$: $V_0 \to B$ is $\mathbf{R}_0^n = \mathbf{R}^n - (0)$. The bundle q_0 is fibre homotopy equivalent to its S^{n-1}-bundle and hence $H^*(\bar{V}_0)$ is an $H^*(B)$-module freely generated by 1, $u(V_0), \ldots, u^{n-1}(V_0)$. Moreover, $u^n(V_0) = \Sigma_{j=1}^n(\bar{q}_0^* w_j) \cup u^{n-j}(V_0)$, where the coefficient $w_j = w_j(q)$ is the jth Stiefel-Whitney class of q [3, p. 232].

Let g: $E \to V$ be defined by $ge = fe - f(-e)$. Then g is equivariant, $g(-e) = -ge$, $A_f = A_g = g^{-1}0$ and the restriction of g to $E_0 = E - A_f$ defines an equivariant map g_0: $E_0 \to V_0$. By the naturality of u, we have $\bar{g}_0^* u(V_0) = u(E_0)$, where $\bar{g}_0$: $\bar{E}_0 \to \bar{V}_0$ is the map of the orbit bundles induced by g_0, and $u(E_0) = u(E)|\bar{E}_0$. It follows that

$$u^n(E)|\bar{E}_0 = \bar{g}_0^* u^n(V_0) = \sum_{j=1}^{n} \left[(\bar{p}^* w_j)|\bar{E}_0\right] \cup \left[u^{n-j}(E)|\bar{E}_0\right]$$

$$= \left[\sum_{j=1}^{n} (\bar{p}^* w_j) \cup u^{n-j}(E)\right]|\bar{E}_0.$$

To show that $(\bar{p}|\bar{A}_f)^*$ is a monomorphism in the degrees specified in the theorem, suppose that $x \in H^i(B)$ with $i > d - r$ and $(\bar{p}|\bar{A}_f)^* x = 0$, i.e., $(\bar{p}^* x)|\bar{A}_f = 0$. By the continuity of H^*, there is a neighborhood U of A_f in E such that $(\bar{p}^* x)|\bar{U} = 0$ ($\bar{U}$ denotes, as usual, the image of U in $\bar{E}$). Let e: $\bar{E} \to (\bar{E}, \bar{U})$ and k: $\bar{E} \to (\bar{E}, \bar{E}_0)$ be the inclusion maps. Since $(\bar{p}^* x)|\bar{U} = 0$, then $\bar{p}^* x = e^* y$, for some $y \in H^i(\bar{E}, \bar{U})$. Let $v = u^n(E) - \Sigma_{j=1}^n(\bar{p}^* w_j) \cup u^{n-j}(E)$. Then $v|\bar{E}_0 = 0$; hence $v = k^* z$, for some $z \in H^n(\bar{E}, \bar{E}_0)$. Since $(\bar{E}; \bar{U}, \bar{E}_0)$ is an excisive triad, $e^* y \cup k^* z = y \cup z = 0$; hence $0 = (\bar{p}^* x) \cup u^n(E) - (\bar{p}^* x) \cup [\Sigma_{j=1}^n(\bar{p}^* w_j) \cup u^{n-j}(E)]$. Therefore

$$(\bar{p}^* x) \cup u^n(E) = \sum_{j=1}^{n} \bar{p}^*(x \cup w_j) \cup u^{n-j}(E).$$

Now if $j < r$ then $w_j = 0$ by the assumption. If $j > r$ then $\deg(x \cup w_j) = i + j > i + r > d \geqslant d(B)$ since $i > d - r$. Therefore all the coefficients in this polynomial are zero. Hence $(\bar{p}^* x) \cup u^n(E) = 0$. But $(\bar{p}^* x) \cup u^n(E) = \iota x$ and ι is a monomorphism. Therefore $x = 0$ and thus $(\bar{p}|\bar{A}_f)^*$ is a monomorphism. Q.E.D.

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