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Erratum

A Contract and Balancing Mechanism for Sharing Capacity in a Communication Network

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In a recent paper, the first three authors proposed a method for determining how much to charge users of a communication network when they share bandwidth, and studied the existence and form of Nash equilibria for players' choices of capacity. However, the proof of one of the propositions in that paper contained a flaw. In this note, we prove that the original proposition is true under an additional condition, and provide two examples to show that this condition is necessary.

Key words: capacity contracts; congestion pricing; Nash equilibrium
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1. Introduction

In a recent paper of the first three authors (Anderson et al. 2006), a contract and balancing mechanism was proposed as a method for sharing capacity in a communication network. It was shown that, with $n \geq 2$ players contracting for capacity on a single link over a time period $[0, 1]$, there is a unique Nash equilibrium for the contract quantities:

Proposition 2 (Anderson et al. 2006). *Under price complementarity, and assuming that all players follow a price-taking policy, there is a unique Nash equilibrium for the contract quantities y_i, $i = 1, 2, \ldots, n$. At the Nash equilibrium, the time-averaged expected price is equal to the cost per unit of capacity,*

$$\int_0^1 \mathbb{E}[p(t)]\,\mathrm{d}t = c, \tag{1}$$

and player i's optimal choice of contract quantity y_i satisfies the following equation:

$$y_i = \frac{\int_0^1 \mathbb{E}[p(t)D_i(t, p(t))]\,\mathrm{d}t}{\int_0^1 \mathbb{E}[p(t)]\,\mathrm{d}t}. \tag{2}$$

Here, $D_i(t, p(t))$ is the price-dependent demand function for player i. Later, a stylized network model was discussed, with a set of links J, and each player associated with a route r which is a subset of J. The following partial generalization of Proposition 2 was stated:

Proposition 3 (Anderson et al. 2006). *If y_r, $r \in R$, is a Nash equilibrium at which $y_r > 0$, $r \in R$, then y_r satisfies the equation*

$$y_r = \frac{\int_0^1 \mathbb{E}[w_r(t)D_r(t, p_r(t))]\,\mathrm{d}t}{\int_0^1 \mathbb{E}[w_r(t)]\,\mathrm{d}t}, \tag{3}$$

where $w_r(t) = \partial p_r(t)/\partial y_r$. Further, the time-averaged expected price on link j is equal to c_j, the cost per unit of capacity on link j, i.e.,

$$\int_0^1 \mathbb{E}[p_j(t)]\,\mathrm{d}t = c_j. \tag{4}$$

However, as found by the fourth author of this note, a crucial step in the proof of Proposition 3 is flawed, and in fact the result does not hold in general. The purpose of this note is to prove that, when demand is deterministic and does not vary with time, Proposition 3 does hold. Furthermore, we present two counterexamples to the original Proposition 3 to show that these two additional assumptions are necessary.

We recall some results and assumptions in Anderson et al. (2006). The set of routes R includes $\{j\}$ for each $j \in J$, which ensures that the link-route incidence matrix has rank J. The demand on each link $j \in J$ cannot exceed the total capacity, that is,

$$\sum_{r:\, j\in r} D_r(t, p_r(t)) \le \sum_{r:\, j\in r} y_r, \tag{5}$$

and the *price complementarity assumption* asserts that if strict inequality holds in (5), then $p_j = 0$ and $\partial p_j(t)/\partial y_j = 0$. Also, on each route $r \in R$, the price $p_r(t)$ satisfies

$$p_r(t) = \sum_{j\in r} p_j(t). \tag{6}$$

The demand functions $D_r(t, p_r(t))$ for each $r \in R$ are decreasing and continuously differentiable functions of the respective prices $p_r(t)$. Finally, following Anderson et al. (2006), we shall use the approximations that

$$\frac{\partial p_r(t)}{\partial y_r} = \sum_{j\in r} \frac{\partial p_j(t)}{\partial y_j}, \quad r \in R, \tag{7}$$

$$\frac{\partial p_r(t)}{\partial y_j} = \frac{\partial p_j(t)}{\partial y_j}, \quad r \in R,\ j \in r. \tag{8}$$

We will need to make use of an additional property of the derivatives $\partial p_j(t)/\partial y_j$, which did not appear in Anderson et al. (2006) Suppose that for each $j \in J$, equality holds in (5); differentiating with respect to y_j, and noting that $y_j = y_{\{j\}}$ appears once in the sum on the right-hand side of (5), we have

$$\sum_{r:\, j\in r} D'_r(t, p_r(t)) \frac{\partial p_r(t)}{\partial y_j} = 1, \tag{9}$$

and using assumption (8),

$$\sum_{r:\, j\in r} D'_r(t, p_r(t)) \frac{\partial p_j(t)}{\partial y_j} = 1.$$

Because the demand functions D_r are decreasing, this implies that $\partial p_j(t)/\partial y_j < 0$. On the other hand, if equality does *not* hold in (5), then by price complementarity, $\partial p_j(t)/\partial y_j = 0$. It follows that

$$\frac{\partial p_j(t)}{\partial y_j} \le 0, \quad j \in J,\ t \in [0, 1]. \tag{10}$$

Finally, let us explicitly state that on each link $j \in J$, the cost per unit of capacity c_j is strictly positive:

$$c_j > 0, \quad j \in J. \tag{11}$$

2. Two Counterexamples

In the original proof of Proposition 3, the error proceeds that the fact that the fourth displayed equation from the end cannot be written in the form $A^{\mathrm{T}} z = 0$ as claimed. We first present a counterexample to the proposition in the case where demand is time dependent.

We consider a network with two links, $J = \{1, 2\}$, and three routes $R = \{\{1\}, \{2\}, \{1, 2\}\}$; this corresponds to three nodes connected together in a path. For each $r \in R$, we take the demand function to be $D_r(t, p_r(t)) = \alpha_r(t)/p_r(t)$, recalling that $p_{12}(t) = p_1(t) + p_2(t)$, and set up the costs c_1, c_2 and parameters $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_{12}(t)$ as follows:

$$c_1 = 8 \qquad \alpha_1(t) = \begin{cases} 4, & 0 \le t \le \frac{1}{2}, \\ 72, & \frac{1}{2} < t \le 1, \end{cases}$$

$$c_2 = 2 \qquad \alpha_2(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2}, \\ 16, & \frac{1}{2} < t \le 1, \end{cases}$$

$$\alpha_{12}(t) = \begin{cases} 16, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} < t \le 1. \end{cases}$$

The first-order conditions for a Nash equilibrium are

$$\sum_{j\in r} \left(c_j - \int_0^1 p_j(t)\, \mathrm{d}t \right) = \sum_{j\in r} \int_0^1 \left[[y_r - D_r(t, p_r(t))] \frac{\partial p_j(t)}{\partial y_j} \right] \mathrm{d}t, \quad r \in R. \tag{12}$$

The form of the demand functions together with price complementarity ensure that equality holds in relation (5) for $j = 1, 2$. We solve (5) for the prices p_1, p_2 and (9) for the derivatives, giving

$$p_1(t) = \begin{cases} 2, & 0 \le t \le \frac{1}{2}, \\ 12, & \frac{1}{2} < t \le 1, \end{cases} \qquad p_2(t) = \begin{cases} 2, & 0 \le t \le \frac{1}{2}, \\ 4, & \frac{1}{2} < t \le 1, \end{cases}$$

$$\frac{\partial p_1(t)}{\partial y_1} = \begin{cases} -1, & 0 \le t \le \frac{1}{2}, \\ -2, & \frac{1}{2} < t \le 1, \end{cases} \qquad \frac{\partial p_2(t)}{\partial y_2} = \begin{cases} -2, & 0 \le t \le \frac{1}{2}, \\ -1, & \frac{1}{2} < t \le 1, \end{cases}$$

and find that conditions (12) are met with contract quantities

$$y_1 = 4, \qquad y_2 = 2, \qquad y_{12} = 2.$$

However, we can check that the conclusion of Proposition 3, essentially that both sides of (12) are zero for each $r \in R$, is not satisfied.

We next construct a counterexample to Proposition 3 in which demand is stochastic but constant over the time period $[0, 1]$. In this case, the conditions for a Nash equilibrium are

$$\sum_{j\in r}(c_j - \mathbb{E}(p_j)) = \sum_{j\in r}\mathbb{E}\left[[y_r - D_r(p_r)]\frac{\partial p_j}{\partial y_j}\right], \quad r\in R. \tag{13}$$

In fact, all that is needed is to take the previous counterexample and re-interpret t as a uniform random variable on $[0, 1]$; so that, for example, $\mathbb{P}(\alpha_1 = 4) = \mathbb{P}(\alpha_1 = 72) = \frac{1}{2}$. Conditions (13) are then seen to be equivalent to (12), and the demonstration that Proposition 3 does not hold proceeds as before.

3. Stationary Deterministic Demand

In this section, we use relation (10) to demonstrate that, in the absence of time dependence and stochastic effects, the conclusions of Proposition 3 are valid.

Proposition 3A. *Suppose that each player's demand is deterministic and does not vary with time. If y_r, $r\in R$, is a Nash equilibrium at which $y_r > 0$, $r\in R$, then Equations* (3) *and* (4) *hold, and simplify to*

$$y_r = D_r(p_r) \tag{14}$$

and

$$p_j = c_j. \tag{15}$$

Proof. For each $j\in J$, from (5) we have

$$c_j \sum_{r:\, j\in r}[y_r - D_r(p_r)] \geq 0$$

because $c_j \geq 0$, and

$$p_j \sum_{r:\, j\in r}[y_r - D_r(p_r)] = 0$$

by price complementarity. It follows that

$$\sum_{j\in J}\left((c_j - p_j)\sum_{r:\, j\in r}[y_r - D_r(p_r)]\right) \geq 0,$$

which we can rearrange to obtain

$$\sum_{r\in R}\left([y_r - D_r(p_r)]\sum_{j\in r}(c_j - p_j)\right) \geq 0. \tag{16}$$

The first-order conditions for a Nash equilibrium, as derived in Anderson et al. (2006), require that

$$\sum_{j\in r}(c_j - p_j) = [y_r - D_r(p_r)]\frac{\partial p_r}{\partial y_r}, \quad r\in R; \tag{17}$$

substituting this into (16), and applying (7), we deduce that

$$\sum_{r\in R}\sum_{j\in r}[y_r - D_r(p_r)]^2\frac{\partial p_j}{\partial y_j} \geq 0.$$

However, relation (10) means that all the terms in this double sum are nonpositive, and hence must all be zero. This means that for all $r\in R$ and $j\in r$, either $y_r = D_r(p_r)$ or $\partial p_j/\partial y_j = 0$. Because the set R includes $\{j\}$ for each $j\in J$, it follows from Equation (17) that $c_j = p_j$ for each $j\in J$.

It remains to show that $y_r = D_r(p_r)$ for each $r\in R$; this will now follow from (17) provided that $\partial p_r/\partial y_r \neq 0$. Because, for each link $j\in J$, $p_j = c_j$, which by assumption (11) is strictly positive, we can use price complementarity to show that equality holds in (5). This means that $\partial p_j/\partial y_j < 0$ as shown above, and finally, by (7), we have $\partial p_r/\partial y_r < 0$ for each $r\in R$ as required. □

Reference

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