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ISRAEL JOURNAL OF TECHNOLOGY, Vol. 9, No. 4, 1971, pp. 311–322
Proc. Fifth Israel Ann. Conf. MECHANICAL ENGINEERING, 1971

A Contribution to the Engineering Design of Machine Elements Involving Contrashaped Contacts

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Received April 7, 1971

ABSTRACT

In this study a simple method is developed which leads to formulae defining the minimum crowning radius required for compensation of edge phenomena and misalignment. First it is shown that any contrashaped contact can be reduced to that between a plane surface and a crowned roller with a minor radius of curvature r and a major radius of curvature R. By introducing a crowning ratio $\omega = R/r$, the well known Hertz formulae are substantially simplified and new practical formulae are derived. The entire field of contact situations, as defined by the crowning ratio ω, is divided into ranges, namely $\omega < 25$ and $\omega > 25$. In each of these two ranges Hertz formulae including elliptical integral functions, can be approximated by simple exponential expressions. These formulae are simple and attractive especially for crowned cylinders in the range $\omega > 25$.

NOTATION

a	—	semi major axis of the contact ellipse (mm)
b	—	semi minor axis of the contact ellipse (mm)
$e = a/b$	—	axis ratio of the contact ellipse
B	—	modulus of contact width (kgf/mm^2)
E	—	modulus of elasticity (kgf/mm^2)
E_0	—	reduced modulus of elasticity (kgf/mm^2)
$F(\rho)$	—	function of curvatures
G	—	shear modulus of elasticity (kgf/mm^2)
f	—	a measure in the test stand (mm)
q	—	a measure in the test stand (mm)
K	—	function containing elliptic integral
k	—	Stribeck's value (kgf/mm^2)
l	—	roller length and contact length (mm)
m	—	Poisson's ratio
h	—	height of segment (mm)
Q	—	force (load) (kgf/mm^2)
R	—	crowning radius (mm)
r	—	roller radius (mm)
s	—	half roller length (mm)
C	—	influence coefficient
β	—	angle of misalignment (radians)
δ	—	relative approach of remote points (mm)
μ, ν	—	functions, each containing elliptic integral
σ	—	normal stress (kgf/mm^2)
ω	—	crowning ratio
ρ	—	curvature (mm^{-1})

Subscripts

B	—	refers to modulus of contact width
δ	—	refers to relative approach
ω	—	refers to crowning ratio
l	—	refers to technical line contact
H	—	refers to maximum normal stress (Hertz)
D_1, D_{01}, D_2, D_{02}	—	refers to permissible normal stresses and Stribeck's values
c	—	refers to perfectly rectangular line contact
p	—	refers to point contact

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1. SOME ASPECTS OF HERTZ'S THEORY

In 1881 Hertz published a theory enabling stresses and deformations, occurring on the pressing together contrashaped surfaces of two elastic bodies, to be calculated. He assumed that both bodies, when unloaded, touch each other at a point that, in the case of loading, turns into a flat elliptical area of contact. He made the following assumptions:

1) the contact area is relatively small compared with the dimensions of the bodies, and consequently with the radii of curvature of the body surfaces,

2) the material of the bodies is homogeneous and isotropic,

3) only normal stresses occur in the contact area, and

4) the proportional limit is not exceeded.

At the point of contact each of the two body surfaces may be characterised by two main curvatures in perpendicular planes. The point contact turns into an infinitely long line contact, if the planes of the two main curvatures coincide and the corresponding curvatures have zero value.

Hertz derived formulae for the half-axis lengths of the elliptical contact area, for the stress in the contact area, and for the relative approach of remote points. The formulae include variable factors depending on the principal radii of curvature of the body surfaces. For the line contact, which can also be conceived as an infinitely long contact ellipse, the theory gives an infinitely high value for the approach. Hertz remarks that the principal assumptions have obviously not been fulfilled and that the real approach is defined by the total shape of the two bodies. Although most practical cases of contrashaped contacts, as with gearwheels, cams and rollers, and rolling bearings, answer the theoretical main conditions only to a small degree, Hertz's formulae appear useful in dimensioning contrashaped contacts.

2. THE KNOWN FORMULAE FOR POINT CONTACTS AND LINE CONTACTS

In the technical literature formulae are given that deviate only slightly from the original ones (*Hertz*, 1895).

For point contacts with elliptical contact areas the following formulae apply:

semi-major axis $$a = \mu \sqrt[3]{\frac{3Q}{E_0 \Sigma\rho}} \tag{2.1}$$

semi-minor axis $$b = \nu \sqrt[3]{\frac{3Q}{E_0 \Sigma\rho}} \tag{2.2}$$

The compressive stress in the contact area varies according to half an ellipsoid (Fig. 1) with a maximum of:

$$\sigma_H = \frac{3Q}{2\pi ab} \tag{2.3}$$

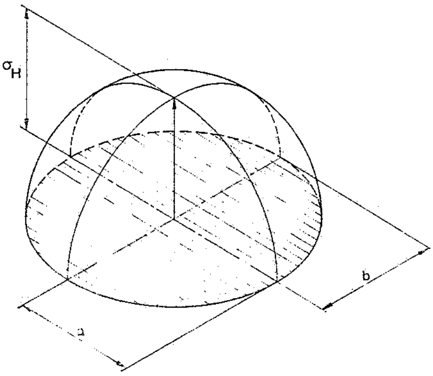

Fig. 1. Elliptical contact area and stress ditribution.

From (2.1), (2.2) and (2.3) follows:

$$\sigma_H = \frac{1.5}{\pi\mu\nu} \sqrt[3]{\frac{E_0^2}{9} Q (\Sigma\rho)^2} \tag{2.4}$$

The approach between two remote points of the two bodies is:

$$\delta = \frac{3KQ}{\pi a E_0} = \frac{K}{\pi\mu} \sqrt[3]{\frac{9Q^2 \Sigma\rho}{E_0^2}} \tag{2.5}$$

where

$$\frac{1}{E_0} = \frac{1}{2E_{01}} + \frac{1}{2E_{02}} \tag{2.6}$$

$$E_{01} = \frac{2m_1}{m_1 - 1} \quad G_1 = \frac{m_1^2}{m_1^2 - 1} E_1$$

and (2.7)

$$E_{02} = \frac{2m_2}{m_2 - 1} \quad G_2 = \frac{m_2^2}{m_2^2 - 1} E_2$$

The indices 1 and 2 are related to the corresponding bodies. The variable factors μ, ν and K are given functions of $F(\rho)$. They include elliptical integrals and are generally given in tables as functions of $F(\rho)$.

Further

$$\Sigma\rho = \rho_{11} + \rho_{12} + \rho_{21} + \rho_{22} \tag{2.8}$$

$$F(\rho) = \frac{\sqrt{(\rho_{11}-\rho_{12})^2+(\rho_{21}-\rho_{22})^2+2(\rho_{11}-\rho_{12})(\rho_{21}-\rho_{22})\cos 2\Omega}}{\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22}} \tag{2.9}$$

The first index is related to the bodies, the second to the planes of the main curvatures. A main curvature plane of body 1 makes an angle Ω with the main curvature plane of body 2. In most technical applications Ω is either 0 or π. Hence, $\cos 2\Omega = 1$, and $F(\rho)$ becomes:

$$F(\rho) = \frac{\rho_{11}-\rho_{12}+\rho_{21}-\rho_{22}}{\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22}} \tag{2.10}$$

The curvature ρ always has a positive value when the curvature centre lies within the body concerned, and a negative value when the curvature centre lies out of it.

For line contacts with strip-shaped contact areas and length l the half width is:

$$b = \sqrt{\frac{8Q}{\pi E_0 l(\rho_1+\rho_2)}} \tag{2.11}$$

The compressive stress of this contact area varies according to half an elliptical cylinder (Fig. 2) with a maximum of:

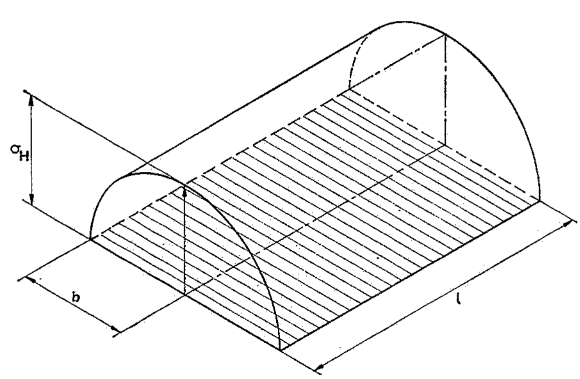

Fig. 2. Strip-shaped contact area and stress distribution.

$$\sigma_H = \frac{2Q}{\pi b l} \tag{2.12}$$

From (2.11) and (2.12) follows:

$$\sigma_H = \sqrt{\frac{E_0 Q(\rho_1+\rho_2)}{2\pi l}} \tag{2.13}$$

Two of the four curvatures have zero value.

The theory gives here an infinitely high value for the approach. Palmgren (1959) offers the following empirical formulae for the contact between a cylindrical roller of the usual shape and a vast flat plane:

$$\delta_l = \frac{2.54}{E_0^{0.9}}\,\frac{Q^{0.9}}{l^{0.8}} \tag{2.14}$$

3. THE REDUCED ROLLER RADIUS AND THE CROWNING RATIO

In technical calculations of line contacts it is usual to introduce the reduced roller radius as follows:

$$\rho_1+\rho_2 = \frac{1}{r_1}+\frac{1}{r_2} = \frac{1}{r} \tag{3.1}$$

We will now investigate whether in the case of point contact, $\Sigma\rho$ and $F(\rho)$ may be expressed analogously in reduced curvature radii. We start from the situation with coinciding main curvature planes, in which $\rho_{11} = 1/r_{11}$ and $\rho_{21} = 1/r_{21}$ lie in the main curvature plane 1 and $\rho_{12} = 1/r_{12}$ and $\rho_{22} = 1/r_{22}$ lie in the main curvature plane 2.

We rewrite (2.9) as follows:

$$F(\rho) = \frac{\rho_{11}-\rho_{12}+\rho_{21}-\rho_{22}}{\rho_{11}+\rho_{12}+\rho_{21}+\rho_{22}} = \frac{(\rho_{11}+\rho_{21})-(\rho_{12}+\rho_{22})}{(\rho_{11}+\rho_{21})+(\rho_{12}+\rho_{22})} \tag{3.2}$$

and, moreover, define according to (3.1): $\rho_{11}+\rho_{21} = 1/r_{11}+1/r_{21} = 1/r$ and $\rho_{12}+\rho_{22} = 1/r_{12}+1/r_{22} = 1/R$ where r is the reduced roller radius and R the reduced crowning radius. Furthermore we postulate

$$\omega = \frac{R}{r} > 1 \tag{3.3}$$

and call ω the crowning ratio.

From (2.8) and (3.1) follows:

$$\Sigma\rho = \frac{1}{r}+\frac{1}{R} = \frac{1+\omega}{\omega r} \tag{3.4}$$

and from (3.1) and (3.2.):

$$F(\rho) = \frac{\omega - 1}{\omega + 1} \tag{3.5}$$

In the case of coinciding main curvature planes, r and ω follow directly from the principal radii of curvature. In the rather rarely occurring case of non-coinciding main curvature planes, $\Sigma\rho$ and $F(\rho)$ have to be calculated from the four principal radii of curvature according to (2.8) and (2.9), from which then ω and r can be calculated:

$$\omega = \frac{1 + F(\rho)}{1 - F(\rho)} \tag{3.6}$$

$$r = \frac{1 + \omega}{\omega \Sigma\rho} \tag{3.7}$$

As a consequence, the form of the contact between two contrashaped body surfaces in the case of an elliptical contact area can always be characterised by a reduced roller radius r and a crowning ratio ω. So the contact can always be replaced by the contact between a crowned cylindrical roller with radius r and crowning radius R, and a flat plane.

It seems logical not to translate the variable factors μ, ν and K from Hertz's formulae as functions of $F(\rho)$, but as functions of the crowning ratio ω.

4. NEW FORMULAE FOR THE POINT CONTACT

By introducing (3.4) $\Sigma\rho = (1 + \omega)/\omega r$ into the formulae (2.1), (2.2), (2.4) and (2.5) we obtain:

$$a = \mu \sqrt[3]{\frac{\omega}{1+\omega}} \sqrt[3]{\frac{3Qr}{E_0}} \tag{4.1}$$

$$b = \nu \sqrt[3]{\frac{\omega}{1+\omega}} \sqrt[3]{\frac{3Qr}{E_0}} \tag{4.2}$$

$$\sigma_H = \frac{1.5}{\pi\mu\nu} \sqrt[3]{\frac{(1+\omega)^2}{\omega^2}} \sqrt[3]{\frac{E_0^2 Q}{9r^2}} \tag{4.3}$$

$$\delta = \frac{K}{\pi\mu} \sqrt[3]{\frac{1+\omega}{\omega}} \sqrt[3]{\frac{9Q^2}{E_0^2 r}} \tag{4.4}$$

where for further simplification:

$$\mu_\omega = \mu \sqrt[3]{\frac{\omega}{1+\omega}} \tag{4.5}$$

$$\nu_\omega = \nu \sqrt[3]{\frac{\omega}{1+\omega}} \tag{4.6}$$

A. *New Expressions for the Dimensions of the Elliptical Contact Area*

From (4.2) and (4.3) follows:

$$\frac{\sigma_H}{b} = \frac{1.5}{\pi\mu\nu^2} \frac{1+\omega}{\omega} \frac{E_0}{3r}$$

and in dimensionless form:

$$\frac{b}{r} = \frac{\sigma_H}{B_p} \quad \text{where } B_p = \frac{E_0}{2\pi\mu\nu^2} \frac{1+\omega}{\omega}$$

It now becomes evident (see also Table I) that for $\omega \to \infty$ $\mu\nu^2 \to 2/\pi$. $B_p \to E_0/4$.

Introducing

$$\frac{E_0}{4} = B \tag{4.7}$$

and

$$\frac{B_p}{B} = C_B = \frac{2}{\pi} \frac{1+\omega}{\omega\mu\nu^2} \tag{4.8}$$

we obtain the generally applicable formulae:

$$\frac{b}{r} = \frac{\sigma_H}{C_B B} \tag{4.9}$$

with the following limits:

circular point contact $\omega = 1$ $C_B = \frac{4}{\pi}$

theoretical line contact $\omega = \infty$ $C_B = 1$

For the theoretical line contact we have:

$$b/r = \sigma_H/B \tag{4.10}$$

This leads us to call B the *contact width modulus*.

It appears that using the simple formulae

$$\sigma_H = \frac{3Q}{2\pi ab} \quad [(2.3)]$$

and

$$\frac{b}{r} = \frac{\sigma_H}{C_B B} \quad [(4.9)]$$

any information concerning the semi-axes a and b of the contact ellipse and the maximum compression stress σ_H can be given, if the variable factors C_B and $e = a/b$ are known as a function of the crowning ratio ω.

B. *New Expression for the Approach*

Introduction of (3.4) $\Sigma\rho = (1+\omega)/\omega r$ into Hertz's integral functions yields:

$$\frac{2K}{\pi\mu} = \frac{\nu^2\omega + \mu^2}{1+\omega} \qquad (4.11)$$

From (4.1), (4.2) and (4.4) and taking into account $a/b = \mu/\nu$ follows:

$$\delta = \frac{K}{\pi\mu\nu^2}\,\frac{1+\omega}{\omega}\,\frac{b^2}{r}$$

and after elimination of K with the help of (4.11) we obtain in dimensionless form:

$$\frac{\delta}{r} = \frac{b^2}{r^2}C_\delta \qquad (4.12)$$

where

$$C_\delta = \frac{\omega\nu^2 + \mu^2}{2\omega\nu^2} = \frac{1}{2} + \frac{e^2}{2\omega} \qquad (4.13)$$

or otherwise expressed:

$$\delta = \frac{b^2}{2r} + \frac{a^2}{2\omega r} = \frac{b^2}{2r} + \frac{a^2}{2R} \qquad (4.14)$$

with the following limits:

circular point contact $\omega = 1$ $C_\delta = 1$ $\delta = b^2/r$

theoretical line contact $\omega = \infty$ $C_\delta = \infty$ $\delta = \infty$

So it appears that using the simple formulae (4.14), δ can be determined directly from r, ω, a and b.

For easy calculation the relation (4.12) may also be used. For this purpose, in addition to r and b, the variable factor C_δ too has to be known as a function of the crowning ratio ω.

5. DETERMINATION OF THEORETICALLY EXACT VARIABLE FACTORS

With the help of an adapted computer program it is, starting from the relations of Hertz, relatively easy to calculate very accurately the variable factors as a function of ω. Table I shows the variable factors μ, ν, μ_ω, ν_ω, e, C_B, and C_δ in four decimals, corresponding to values of ω according to a geometrical progression with common ratio $= \sqrt{2}$.

TABLE I

VARIABLE FACTORS AS FUNCTION OF ω

ω	$F(\rho)$	μ	ν	μ_ω	ν_ω	$e = a/b$	C_B	C_δ
1,0000	,0000	1,0000	1,0000	,7937	,7937	1,0000	1,2732	1,0000
1,4142	,1716	1,1261	,8939	,9422	,7480	1,2598	1,2077	1,0611
2,0000	,3333	1,2758	,8045	1,1145	,7028	1,5858	1,1564	1,1287
2,8284	,4776	1,4528	,7288	1,3133	,6588	1,9934	1,1167	1,2024
4,0000	,6000	1,6608	,6641	1,5417	,6165	2,5007	1,0864	1,2817
5,6569	,6996	1,9037	,6083	1,8032	,5762	3,1297	1,0635	1,3658
8,0000	,7778	2,1859	,5596	2,1018	,5380	3,9065	1,0464	1,4538
11,3137	,8376	2,5119	,5166	2,4420	,5022	4,8629	1,0338	1,5451
16,0000	,8824	2,8869	,4782	2,8292	,4687	6,0367	1,0245	1,6388
22,6274	,9154	3,3168	,4438	3,2694	,4374	7,4740	1,0177	1,7344
32,0000	,9394	3,8083	,4126	3,7694	,4084	9,2303	1,0127	1,8312
45,2548	,9568	4,3689	,3842	4,3372	,3814	11,3724	1,0091	1,9289
64,0000	,9692	5,0075	,3582	4,9817	,3563	13,9810	1,0065	2,0271
90,5097	,9781	5,7339	,3343	5,7129	,3330	17,1538	1,0047	2,1255
128,0000	,9845	6,5594	,3122	6,5424	,3114	21,0081	1,0033	2,2240
181,0193	,9890	7,4968	,2919	7,4830	,2913	25,6856	1,0024	2,3223
256,0000	,9922	8,5606	,2730	8,5495	,2726	31,3572	1,0017	2,4205
362,0387	,9945	9,7673	,2555	9,7584	,2553	38,2286	1,0012	2,5183
512,0000	,9961	11,1356	,2392	11,1284	,2391	46,5474	1,0009	2,6159
724,0773	,9972	12,6865	,2241	12,6807	,2240	56,6118	1,0006	2,7131
1024,0000	,9980	14,4438	,2100	14,4391	,2099	68,7807	1,0004	2,8100
1448,1547	,9986	16,4345	,1969	16,4307	,1968	83,4858	1,0003	2,9065
2048,0000	,9990	18,6890	,1846	18,6859	,1846	101,2464	1,0002	3,0026
2896,3094	,9993	21,2415	,1731	21,2391	,1731	122,6869	1,0002	3,0985
4096,0000	,9995	24,1310	,1624	24,1291	,1624	148,5577	1,0001	3,1940
5792,6188	,9997	27,4013	,1524	27,3997	,1524	179,7610	1,0001	3,2892
8192,0000	,9998	31,1017	,1431	31,1004	,1431	217,3806	1,0001	3,3842
11585,2375	,9998	35,2880	,1343	35,2870	,1343	262,7186	1,0000	3,4788
16384,0000	,9999	40,0234	,1261	40,0226	,1261	317,3388	1,0000	3,5732
23170,4750	,9999	45,3788	,1184	45,3781	,1184	383,1191	1,0000	3,6674
32768,0000	,9999	51,4344	,1113	51,4339	,1113	462,3140	1,0000	3,7613
46340,9500	1,0000	58,2808	,1045	58,2804	,1045	557,6299	1,0000	3,8550
65536,0000	1,0000	66,0201	,0982	66,0198	,0982	672,3148	1,0000	3,9485
92681,9000	1,0000	74,7674	,0923	74,7672	,0923	810,2658	1,0000	4,0418
131072,0000	1,0000	84,6528	,0867	84,6525	,0867	976,1593	1,0000	4,1350
185363,8000	1,0000	95,8226	,0815	95,8224	,0815	1175,6043	1,0000	4,2279
262144,0000	1,0000	108,4422	,0766	108,4421	,0766	1415,3283	1,0000	4,3207
370727,6001	1,0000	122,6981	,0720	122,6980	,0720	1703,3994	1,0000	4,4133
524288,0000	1,0000	138,8004	,0677	138,8003	,0677	2049,4906	1,0000	4,5058
741455,2002	1,0000	156,9862	,0637	156,9861	,0637	2465,1994	1,0000	4,5982
1048576,0000	1,0000	177,5228	,0599	177,5227	,0599	2964,4286	1,0000	4,6904

In a graph with logarithmic scales (Fig. 3) the factors μ_ω, ν_ω, e, C_B and C_δ can be found as a function of ω having values of 1 to 1000. On closer consideration, C_B appears to converge very rapidly towards one: for $\omega = 4$ it is already $C_B \simeq 1.1$; for $\omega = 20$, $C_B \simeq 1.02$ and for $\omega = 100$, $C_B \simeq 1.005$. This means that in practical applications for $\omega > 20$, we may postulate $C_B = 1$, while for very global calculations this is even permissible for $\omega = 4$.

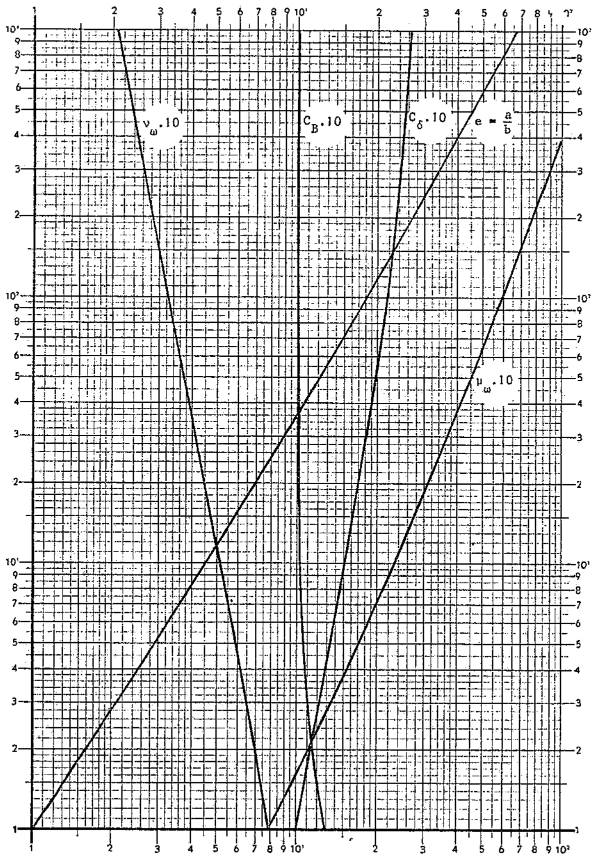

Fig. 3. Graph of variable factors.

It further appears that the factor $\nu_\omega = \nu\sqrt[3]{\dfrac{\omega}{1+\omega}}$ may be approximated very well by one straight line, whereas for the other factors two straight lines are necessary.

6. THE VARIABLE FACTORS AS EXPONENTIAL FUNCTIONS OF THE CROWNING RATIO

It will be evident from the preceding that, when dividing the total range into two parts, it must be possible to represent the variable factors with fair approximation as exponential functions of ω. We now choose the following ranges: $10^5 > \omega > 25$ (20) long elliptical point contact and $1 \leqq \omega < 25$ (30) short elliptical point contact. The values in parentheses are meant for cases of overlapping.

For the first range $C_B = 1$ and ν_ω is an exponential function of ω. For the second range, in addition to ν_ω, a good approximation can be found for μ_ω as an exponential function of ω. All other factors can now be derived directly from the two known ones for both ranges. These approximations have been optimalised with the help of computer programs. The results show a satisfactory correspondence with the theoretical values for practical applications; the deviation is as a rule no more than 2%. Simple fractions were chosen for the exponents. The final results are as follows:

$1 \leqq \omega < 25$ (30) *short elliptical point contact*

$$\nu_\omega = 0.794\omega^{-4/21} \tag{6.1}$$

$$\mu_\omega = 0.794\omega^{11/24} \tag{6.2}$$

$$e = \omega^{11/17} \tag{6.3}$$

$$C_B = 1.275\omega^{-1/13} \tag{6.4}$$

$$C_\delta = \omega^{3/17} \tag{6.5}$$

(20) $25 < \omega < 10^5$ *long elliptical point contact*

$$\nu_\omega = 0.794\omega^{-4/21} \tag{6.6}$$

$$\mu_\omega = 1.015\omega^{8/21} \tag{6.7}$$

$$e = 1.28\omega^{4/7} \tag{6.8}$$

$$C_B = 1 \tag{6.9}$$

$$C_\delta = 1.25\omega^{4/35} \tag{6.10}$$

7. MODIFICATION OF THE LINE CONTACT BY CROWNING

The formulae for the theoretical line contact do not take into account secondary phenomena occurring in practice when applying cylinders with finite lengths. In the case of perfectly parallel axes increased edge stresses are hardly to be prevented, while misalignment entails an increase in stress at one end. When calculating, these facts should be duly considered by means of an addition to the theoretically admissible compressive stress. In the case of considerable misalignment and the use of scarcely plastically deformable materials, such as hardened steel, serious damage or even fracture may occur. In case of line contact between two solid bodies it is therefore often advantageous and sometimes even necessary to crown at least one of the two surfaces. As a matter of fact, in this way a modified line contact is formed without increased edge stresses. For optimal load capacity, however, the degree of crowning has to be so chosen that in the given circumstances edge stresses just do not arise.

Formulae will now be derived for the minimum crowning desired for a cylindrical roller contacting a vast flat plane. The axis of the roller may be either parallel or not parallel. In the former case, we speak of a two-sided or perfectly rectangular line contact, in the latter of a one-sided rectangular line contact. Any other contact situation may be reduced to these standard cases by reducing the radii; the shorter reduced radius r is understood as roller radius and the longer, R, as crowning radius.

A. *The Perfectly Rectangular Line Contact*

When a symmetrically crowned cylindrical roller of sufficient length is pressed on to a flat plane, the contact area will be elliptical. Keeping the axis parallel to the plane and the load constant, and reducing the roller length $l(=2s)$ symmetrically, the following situations will occur successively (Fig. 4):

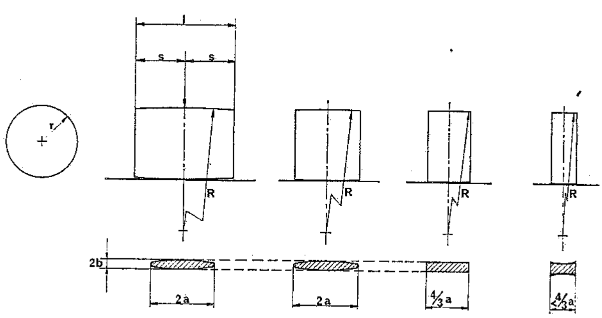

Fig. 4. Several contact situations.

$s=a$; the contact ellipse is still undisturbed, and its extremities just touch the circular end planes of the roller. If the roller length is further reduced, a modified line contact will be formed. The contact ellipse becomes wider at the extremities, while with a sufficiently high value of $e=a/b$ the values σ_H, b and δ do not change appreciably;

$s=2/3\ a$; the contact ellipse has changed to a rectangular contact area with a virtually constant σ_H over the whole length if e is sufficiently high. This optimum situation we call *perfectly rectangular line contact*. Lundberg and Palmgren (1947) were the first to point out this phenomenon.

On continued reduction of the roller length σ_H, b and δ will increase, entailing a greater increase of compression stress and contact width at the extremities than in the centre.

The rule of Palmgren concerning the perfectly rectangular line contact may be provable theoretically, but can be made plausible as follows: For the ellipse we have: $Q=(2/3)\pi ab\sigma'_H$; For the rectangle: $Q=\pi s b_l \sigma''_H$. If $\sigma'_H=\sigma''_H$ and $b=b_l$: $s=2/3\ a$.

The perfectly rectangular line contact is more exact than the technical line contact and corresponds better to the theoretical one as in Eq. (4.9), $b/r=\sigma_H/C_B B$, the factor C_B approaches the limit one more closely. So it may be expected that in the range of the long elliptical point contact ($\omega>25$) Eq. (4.10) is very suitable, and hence the formulae of the theoretical line contact may be used without reserve. Even for $\omega>4$ roughly usable results may be expected.

For (20) $25<\omega<10^5$ or (7) $8<e<925$, it follows from $s=2/3\ a$, $b/r=\sigma_H/B$ [Eq. (4.10)], and $e=a/b=1.28\omega^{4/7}$ [Eq.(6.8)], that

$$\frac{3s}{2r}=1.28\ \frac{\sigma_H}{B}\omega^{4/7} \tag{7.1}$$

For the approach δ_c in the case of perfectly rectangular line contact it follows from $2s=l$, $\delta/r=b^2/r^2\ C_\delta$ [Eq. (4.12)] and $C_\delta=1.25\omega^{4/35}$ [Eq. (6.10)], that

$$\frac{\delta_c}{r}=1.15\left(\frac{\sigma_H}{B}\right)^{1.8}\left(\frac{l}{r}\right)^{0.2} \tag{7.2}$$

For the approach δ_l in the case of technical line contact between an uncrowned cylindrical roller and a vast flat plane it follows from

$$\delta_l=\frac{2.54}{E_0^{0.9}}\cdot\frac{Q^{0.9}}{l^{0.8}}, \tag{2.14}$$

$$b_l=\sqrt{\frac{8Qr}{\pi E_0 l}}\ (2.11)\ \text{and}\ \frac{b}{r}=\frac{\sigma_H}{B} \tag{4.10}$$

that

$$\frac{\delta_l}{r}=1.1\left(\frac{\sigma_H}{B}\right)^{1.8}\left(\frac{l}{r}\right)^{0.1} \tag{7.3}$$

The correspondence between the formulae (7.2) and (7.3) is striking. In fact, with equal values of σ_H, B, l and r, $\delta_c/\delta_l = (1.15/1.1)\ (l/r)^{0.1}$, hence for $l/r = 1$, $\delta_c/\delta_l = 1.045$, and for $l/r = 20$, $\delta_c/\delta_l = 1.41$.

Finally, the height of the arc of the symmetrical crowning, is (approximately):

$$h = \frac{l^2}{8R} = \frac{l^2}{8\omega r} \tag{7.4}$$

B. *The One-Sided Rectangular Line Contact*

If the axis of the crowned roller makes an angle β with the flat plane, the rule of Palmgren may be applied unilaterally. If the symmetrically crowned roller has a length $l = 2s$ and β is relatively small, then according to Fig. 5 we have:

$$s_\beta = s - \beta R = s - \beta\omega r \tag{7.5}$$

for the distance from the centre of the ellipse to the end plane of the roller. When the roller length is now reduced, the load remaining constant, the contact ellipse will stay unchanged until: $s - \beta\omega r = a$.

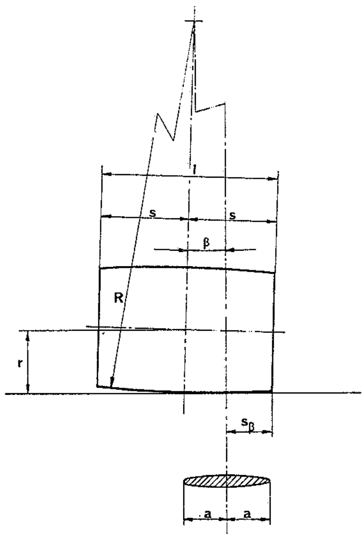

Fig. 5. One-sided line contact.

We may now assume that on further reduction of the roller length the resultant of the load will continue to act on the original centre of the ellipse, whereas the extremity of the contact ellipse will be widened and σ_H, b and δ remain virtually unchanged. If:

$$s - \beta\omega r = 2/3\ a \tag{7.6}$$

one half of the contact ellipse changes into a rectangle with a virtually constant σ_H over the length of the rectangular part. This relatively optimum situation we would call *one-sided rectangular line contact.*

Continuing the reduction of the roller length, the contact becomes less favourable as in the case two-sided rectangular line contact. Here, too, it may be expected that with $\omega > 25$ the usability of the formulae above is very high.

For (20) $25 < \omega < 10^5$ or (7) $8 < e < 925$ it follows from $s - \beta\omega r = 2/3a$, [Eq. (7.6)], $b/r = \sigma_H/B$ [Eq. (4.10)], and $e = a/b = 1.28\omega^{4/7}$[Eq. (6.8)], that

$$\frac{3}{2}\left(\frac{s}{r} - \beta\omega\right) = 1.28\,\frac{\sigma_H}{B}\,\omega^{4/7} \tag{7.7}$$

According to (7.2), for the approach applies:

$$\frac{\delta_c'}{r} = 1.15\left(\frac{\sigma_H}{B}\right)^{1.8}\left(\frac{l}{r} - 2\beta\omega\right)^{0.2} \tag{7.8}$$

and for the load:

$$Q = \frac{2}{3}\pi ab\sigma_H = \pi b\left(\frac{l}{2} - \beta\omega r\right)\sigma_H \tag{7.9}$$

By taking $\beta = 0$ in the given formulae the latter become as those for the prefectly rectangular line contact. This may be considered as an exception to the general case of misalignment β.

8. EXPERIMENTAL VERIFICATION AND ILLUSTRATION

In order to check the usefulness of the formulae derived, a simple test stand was used, in which a crowned cylindrical roller was loaded by a known force between two flat blocks. In Fig. 6 the test stand

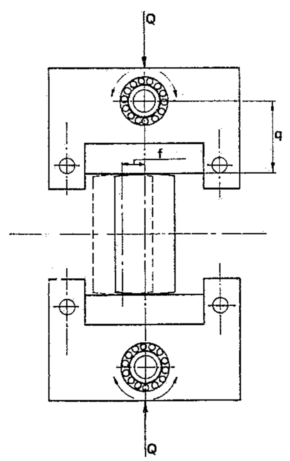

Fig. 6. Test stand.

is reproduced diagrammatically. Roller and blocks were made of polyester (arnite). This material has a high proportional limit and low creep. With the given dimensions, on account of the relatively low modulus of elasticity, distinct contact areas are formed. By thinly coating the crowned roller with paint before loading, the areas are made visible on the flat blocks. To observe the edge phenomena, the flat blocks are fitted in steel holders, pivoting by means of needle bearings on steel shafts. The shafts may be loaded in a line with the axis of the pivots by two equal but opposite forces Q.

If the central cross-section of the symmetrically crowned roller is now placed at a distance f and parallel to the plane through the two pivot axes, the blocks, in the case of loading, will so adjust themselves that:

$$\beta = \frac{f}{R - q} \tag{8.1}$$

where q is the distance from the contact area to the pivot axis. From this it follows that for $f = 0$, $\beta = 0$ and that the configuration is only stable if the crowning radius $R = \omega r > q$.

As the steel holders are provided with pins at the extremities the misalignment β as well as the approach may be simply determined by measuring the changes in the distance between the pins. The test has been carried out with a series of rollers with $r = 20$ and $l = 20$ and a crowning radius $R = \infty$; 1600; 800; 400; 200; 100; 50. The corresponding crowning ratios are $R/r = \omega = \infty$; 80; 40; 20; 10; 5; 2.5. The ratings of arnite are: $E = 300$ kgf/mm^2 and $m = 3$. So $E_0 = 300 \cdot 9/8 = 340$ kgf/mm^2 and consequently $B = 85$ kgf/mm^2. The proportional limit is about: $\sigma_D = 7$ kgf/mm^2.

For security, we loaded the rollers so that according to the theoretical formulae, $\sigma_D = 4$ kgf/mm^2. For a long elliptical point contact we have:

$$\omega \geqq 20;\ e = \frac{a}{b} = 1.28\omega^{4/7};\ Q = 2.68\omega^{4/7} r^2 \frac{\sigma_D^3}{B^2} \tag{8.2}$$

For a short elliptical point contact:

$$\omega < 30;\ e = \frac{a}{b} = \omega^{11/17};\ Q = 1.29\omega^{4/5} r^2 \frac{\sigma_D^3}{B^2} \tag{8.3}$$

For line contact:

$$Q = 1.57 rl \frac{\sigma_D^2}{B} \tag{8.4}$$

For the one-sided rectangular line contact the following criteria applies:

$$\omega \geqq 20: \quad \beta = 0.5 \frac{l}{r}\omega^{-1} - 0.85 \frac{\sigma_D}{B}\omega^{-3/7} \tag{8.5}$$

$$\omega < 30: \quad \beta = 0.5 \frac{l}{r}\omega^{-1} - 0.52 \frac{\sigma_D}{B}\omega^{-5/18} \tag{8.6}$$

A number of tests were carried out, with symmetrical as well as unilateral loading. The prints on the flat blocks have been photographed and are reproduced in Fig. 7. From the measurement of the prints it appears that the measured values of $e = a/b$ are 10 to 15% smaller than those calculated from ω. This is simply explained from an expectable anisotropy of the extruded material. This can also explain the fact that the contact area of the truly cylindrical roller does not show edge stresses and that the calculated perfectly rectangular line contacts do not appear to full advantage.

For the rest, the differences between theory and practice are rather small (5 to 10%), and adequate to justify the usefulness of the formulae derived.

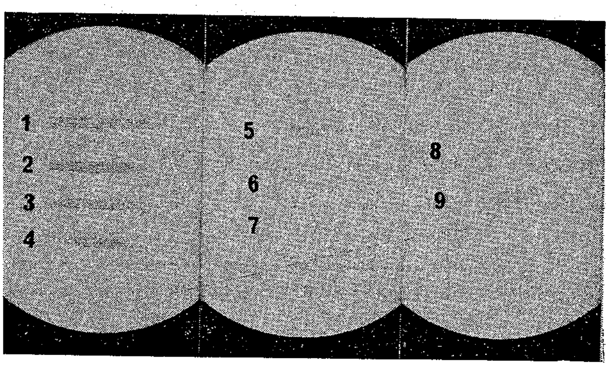

Fig. 7. Contact areas printed with the aid of the test stand. 1. technical line contact; 2. modified line contact $\omega = 80$; 3. long elliptical dash contact $\omega = 40$; 4. long elliptical dash contact $\omega = 20$; 5. short elliptical dash contact $\omega = 10$; 6. short elliptical dash contact $\omega = 5$; 7. short elliptical dash contact $\omega = 2.5$; 8. one-sided rectangular line contact $\omega = 40$; 9. one-sided rectangular-contact $\omega = 20$.

9. SOME PRACTICAL ASPECTS

Some matters that may be of importance to the application of contrashaped contacts will now be treated successively.

A. *The Modulus of Elasticity and the Loading Capacity*

The loading capacity of a cylindrical roller with line contact is often judged with the help of the Stribeck's k value:

$$k = \frac{Q}{2rl} \tag{9.1}$$

The k value is the load divided by the projection of the reduced roller on a flat plane and has the dimension of a tension. Further, see also formula (8.4)

$$k = \pi \frac{\sigma_H^2}{E_0} \tag{9.2}$$

From this it appears that materials with a relatively low E and a high admissible compression stress, such as some thermo-plastics, are especially suitable for applications with line contacts. To show this we compare carbon steel with admissible $\sigma_D = 60$ kgf/mm², $E = 21000$ kgf/mm² and $m = 10/3$, with arnite with admissible $\sigma_D = 7$ kgf/mm², $E = 300$ kgf/mm² and $m = 3$,

For steel on steel: $E_0 = E \cdot m^2/(m^2 - 1) = 21000 \cdot 100/91 = 23100$ kgf/mm², $\sigma_D = 60$ kgf/mm², $k = \pi\sigma_D^2/E_0 = \pi 60^2/23100 = 0.49$ kgf/mm².

For arnite on arnite: $E_0 = 300 \cdot 9/8 = 340$ kgf/mm², $\sigma_D = 7$ kgf/mm², $k = \pi \cdot 7^2/340 = 0.44$ kgf/mm².

For steel on arnite: $1/E_0 = 1/2E_{01} + 1/2E_{02}$, $E_0 = 2E_{01}E_{02}/(E_{01} + E_{02}) = 2 \cdot 340 \cdot 23100/23440 = 670$ kgf/mm², $k = \pi \cdot 7^2/67 = 0.23$ kgf/mm².

Thus it becomes obvious that steel and synthetic materials with low E can compete very well with each other, at least as far as the loading capacity of line contacts without frictional heat is concerned. For point contacts arnite is even more suitable. However, with equal loading capacity the approach is significantly higher for arnite. According to formula (2.14) this is for line contact inversely proportional to $(E_{01}/E_{02})^{0.9}$.

Comparing steel with arnite this factor is:

$$\left(\frac{23100}{340}\right)^{0.9} = 68^{0.9} = 44$$

B. *Admissible Compressive Stresses in Steel*

Table II gives an impression of the admissible compressive stresses and k values in the contrashaped contact between steel bodies, and this for a number of qualities under varying conditions.

The admissible values for the compressive stresses σ_{D1} and σ_{D2} have been derived from gearwheel practice. The compressive stresses applied to rolling bearings are generally higher. This is due to the better controlled quality and machining of the materials, and the statistical character (e.g. 90% life) of the admissible load. The values σ_{D1} cause no notable pitting in case of large numbers of roller rotations (10^7 to 10^8); the values $\sigma_{D2} = \sigma_{D1}\sqrt{2}$ are admissible up to 10^5 to 10^6 roller rotations.

For steel on steel, according to (9.2), we have $\sigma_D = 86\sqrt{k_D}$. The admissible values for the compressive stresses σ_{D01} and σ_{D02} have been derived from rolling-bearing practice, in fact from the criteria of the statistical load capacity (*Palmgren*, 1959). The values $\sigma_{D01} = 2\sigma_{D1}$ cause a hardly noticeable permanent deformation of the contact surfaces. The values $\sigma_{D02} = 2\sigma_{D2}$ cause an approximately 8 times as great a permanent deformation as σ_{D01}, which is

TABLE II

ADMISSIBLE COMPRESSIVE STRESSES

Steel quality	Hardness *HB*	*RC*	σ_{D1} 0.28*HB*	k_{D1}	σ_{D2} 0.4*HB*	k_{D2}	σ_{D01} 0.56*HB*	k_{D01}	σ_{D02} 0.8*HB*	k_{D02}
Optimally hardened	650	62	180	4.5	255	9	360	18	510	36
Adequately hardened	560	55	160	3.5	225	7	320	14	450	28
Optimally alloyed	400	42	110	1.7	155	3.5	220	7	310	14
Adequately alloyed	340	36	90	1.1	130	2.2	180	4.4	260	8.8
High carbon	210		60	0.5	85	1.0	120	2.0	170	4
Low carbon	125		35	0.17	50	0.35	70	0.7	100	1.4

sometimes admissible. σ_{D01} and σ_{D02} are especially important to judging the admissibility of incidental overloading.

C. *Adaptation by Plastic Deformation with Steel*

In case one of the contact surfaces is significantly harder than the other, the softer one is capable of adapting itself to the harder one by plastic deformation. A justifiable criterion of this is that the σ_{D01} of the softer material is about equal to the σ_{D1} of the harder, or, more simply, that a σ_D of the harder material is 1.6 to 2 times as high as the corresponding σ_D of the softer material (see Table II). The admissible value of σ_D is determined by the softer material.

D. *Minimum Depth of Hardening with Steel*

A problem often encountered in practice is the thickness of the hardened layer in case of surface hardening. From Hertz's theory it follows (*Palmgren*, 1959), that in case of line contact pitting is caused by a yield stress occurring at a depth of $0.5b$; this also applies to the long elliptical contact. For short elliptical contacts this depth becomes slightly less, viz. for $\omega = 1$ it is $0.35b$. In case of surface hardening it is therefore recommendable to prescribe a depth of hardening of at least twice $0.5b = b$. For this purpose formula (4.10) $b/r = \sigma_{D/B}$ is very useful.

10. EXAMPLES OF CALCULATIONS

Example No. 1

A steel wheel with a crowned V shaped profile runs inside a grooved steel ring, whose cross-section is a straight-legged V (see Fig 8.)

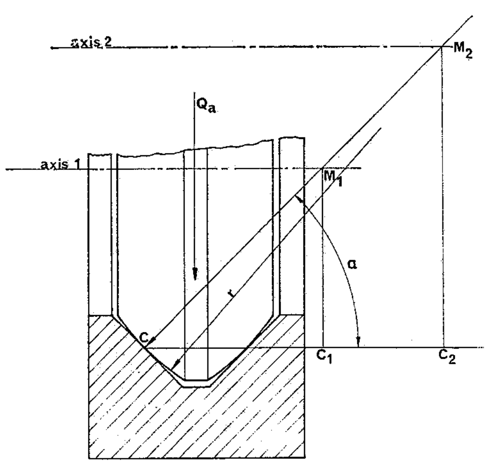

Fig. 8. Example 1.

Given: $M_1C_1 = 30$ mm; $M_2C_2 = 50$ mm; $\alpha = \pi/4$; $\sigma_D = 150$ kgf/mm²; $E_0 = 23100$ kgf/mm²
The point contacts are circular.

Problem.

Calculate: crowning radius r,
radius of contact area b,
load on wheel axis Q,
approach between wheel and ring δ_a.

Solution.

In the plane of design the reduced radius is r.
In the other main plane the reduced radius is R.
According to (3.1): $1/R = 1/r_1 - 1/r_2$

For the circular point contact $r = R$, hence:

$$r = \frac{r_1 r_2}{r_2 - r_1} = \frac{2 \cdot 30 \cdot 50}{20\sqrt{2}} = 75\sqrt{2} = 106 \text{ mm}$$

in which:

$$r_1 = \frac{M_1C_1}{\sin\alpha} = 30\sqrt{2}$$

$$r_2 = \frac{M_2C_2}{\sin\alpha} = 50\sqrt{2}$$

Then:

$$\frac{b}{r} = \frac{\sigma_D}{C_B B} \quad (4.9)$$

Where: $B = E_0/4$ and for circular contact area $C_B = 4/\pi$, hence: $b = \pi r \sigma_D / E_0 = \pi \cdot 106 \cdot 150/23100 = 2.16$. Next, $Q = 2/3\ \pi ab \sigma_D$ (2.3); for circular area, $a = b$ hence, $Q = 2/3\ \pi b^2 \sigma_D$. Further, $Q_a = 2Q \sin\alpha = Q\sqrt{2}$, hence:

$$Q_a = 2/3, \pi b^2 \sigma_D \sqrt{2} = 2/3, \pi 4.65 \cdot 150\sqrt{2} = 2050 \text{ kgf}$$

To find the value of δ we start with (4.14):

$$\delta = \frac{b^2}{2r} + \frac{a^2}{2R}$$

For circular contact area: $a = b$ and $r = R$, hence $\delta = b^2/r$.
Further:

$$\delta_a = \frac{\delta}{\sin\alpha} = \delta\sqrt{2}\text{; hence:}$$

$$\delta_a = \frac{b^2}{r}\sqrt{2} = \frac{2.16^2}{106}\sqrt{2} = 0.062 \text{ mm}$$

Example No. 2

A symmetrical crowned steel foller runs on a steel flat plane and makes a one-sided rectangular line contact (see Fig 9).

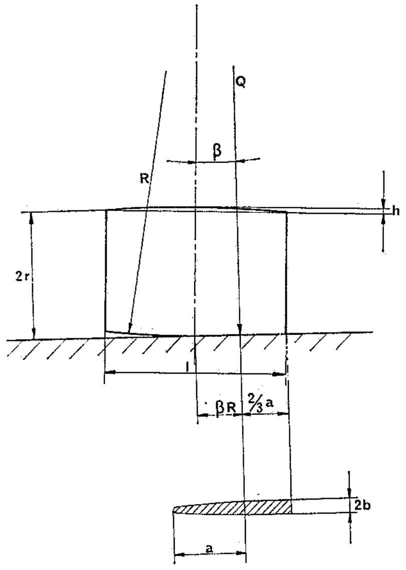

Fig. 9. Example 2.

Given: $r = 20$ mm; $\sigma_D = 100$ kgf/mm^2; $E_0 = 23100$ kgf/mm^2; $\beta = 1/1000$ and $Q = 500$ kgf.

Problem.

Calculate b, a, ω, h, δ.

Solution.

Assume $20 < \omega < 10^5$, then: $b/r = \sigma_D/B$ (4.10)

Where: $B = E_0/4 = 23100/4 = 5775$ kgf/mm^2
Hence, $b = r\ \sigma_D/B = 20 \cdot 100/5775 = 0.346$ mm
Next, $Q = 2/3\ \pi ab\sigma_D$ (2.3) from which: $a = (3/2\pi)\ Q/b\sigma_D = (1.5/\pi)\ 500/0.346 \cdot 100 = 6.9$ mm. Futher, $e = a/b = 1.28\omega^{4/7}$ (6.8). Hence, $\omega = (a/1.28b)^{7/4} = (6.9/1.28 \cdot 0.346)^{7/4} = 15.6^{7/4} = 122$ This satisfies the assumption $\omega > 20$. According to (7.6) $2s = l = (4/3)a + 2\beta\omega r$, we find:

$$l = \frac{4}{3} \cdot 6.9 + 2 \cdot \frac{122}{1000} \cdot 20 = 9.2 + 4.9 = 14.1 \text{ mm}$$

From (7.4) $h = l^2/8\omega r$ we find: $h = 14.1^2/8 \cdot 122 \cdot 20 = 1.02 \cdot 10^{-2}$mm

Finally from (4.12) and (6.10) we find:

$$\delta = \frac{b^2}{r} 1.25\omega^{4/35}$$

$$\delta = \frac{0.346^2}{20} \cdot 1.25 \cdot 122^{4/35} = 7.48 \cdot 10^{-3} \cdot 1.73 = 1.3 \cdot 10^{-2}\text{mm}$$

ACKNOWLEDGEMENT

I am indebted to all who have assisted me in preparing this publication. In particular I would mention my immediate associates, Mr. G. R. W. Kymmell and Mr. M. Verduin, who carried out the experimental investigation and made up the computer programs.

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