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A Convergence Analysis of Yee's Scheme on Non–uniform Grids

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Subject classifications: AMS(MOS): 65N10, 65N15, 35L50

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A CONVERGENCE ANALYSIS OF YEE'S SCHEME ON NON–UNIFORM GRIDS

PETER MONK* AND ENDRE SÜLI†

Abstract. The Yee scheme is the principle finite difference method used in computing time domain solutions of Maxwell's equations. On a uniform grid the method is easily seen to be second order accurate in space. In this paper we show that the Yee scheme is also second order accurate on a non–uniform mesh despite the fact that the local truncation error is (nodally) only first order.

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1. Introduction. The Yee scheme is the principle finite difference method used in the electromagnetism community and has been developed and extended extensively (cf. for example [9]). In this paper we shall analyze its order of accuracy for approximating the Maxwell system with simple boundary conditions [11]. We shall concentrate on studying the order of convergence in space (the time discretization is quite standard and is a second order conditionally stable leap–frog scheme), our main concern being the effect of mesh non-uniformity on the accuracy of the scheme, and in particular the sensitivity of the scheme to mesh stretching and compressing in the coordinate directions. It is easy to see that the Yee scheme is second order accurate in space on a uniform grid (cf. [7]). However, if the grid is non–uniform (but still orthogonal), the local truncation error is only first order at the meshpoints. Nevertheless, we are able to show that Yee's scheme is second order accurate regardless of mesh non–uniformity. The phenomenon whereby the global error of the finite difference scheme is of higher order than the truncation error is usually referred to as *supraconvergence* and has recently been the subject of intensive research [5, 6, 3, 1, 2].

The technique of analysis we use is motivated by the work of Süli [8] where the accuracy of finite volume approximations of Laplace's equation is analyzed on non-uniform meshes. Our approach here is based on the fact that the Yee scheme is also a finite volume scheme (i.e. it arises from the integral formulation of Maxwell's equations) and hence the truncation error of the scheme is of the special form $T_h = \nabla_h \times \eta + \psi$, where ∇_h is a suitable discrete gradient and the functions η and ψ in this decomposition are $O(h^2)$ as h, the maximum gridsize, approaches 0. By performing a duality argument which amounts to manipulating the truncation error in a discrete negative Sobolev norm, we show that the method is second order accurate. In fact, since the proof of this result does not require any hypothesis on the regularity of the mesh, we deduce that the accuracy of the Yee scheme is insensitive to mesh stretching and compressing

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in the coordinate directions. The main result of this paper is encapsulated in Theorem 3.1.

2. Derivation of the Yee scheme. The original Yee scheme was constructed on a uniform grid. The method can be extended to non–uniform grids and we describe next the extension presented by Weiland [10] which is based on the integral form of Maxwell's equations.

First let us state the problem to be approximated. For simplicity we start by considering a rectangular parallelepiped cavity $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ containing an isotropic, linear dielectric (extensions to more exotic geometries and materials will be discussed later). We suppose that a sufficiently smooth vector function $\boldsymbol{J}(\boldsymbol{x}, t)$ is known which specifies the current density in Ω at position $\boldsymbol{x}$ and time t. We desire to compute the resulting electric and magnetic fields $\boldsymbol{E} \equiv \boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{H} \equiv \boldsymbol{H}(\boldsymbol{x}, t)$ which satisfy the Maxwell system in Ω:

$$\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \Omega, \quad t > 0, \tag{1a}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \quad \text{in } \Omega, \quad t > 0. \tag{1b}$$

We assume that the field satisfies a perfectly conducting boundary condition on the boundary of Ω (denoted Γ) so that

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma, \quad t > 0. \tag{2}$$

To complete the specification of the electromagnetic field, we suppose that initial fields $\boldsymbol{E}_0 \equiv \boldsymbol{E}_0(\boldsymbol{x})$ and $\boldsymbol{H}_0 \equiv \boldsymbol{H}_0(\boldsymbol{x})$ are given such that

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega. \tag{3}$$

It is well–known that, for suitably smooth data, equations (1)–(3) have a unique solution for all time [4]. We wish to analyze the use of finite differences to approximate $\boldsymbol{E}$ and $\boldsymbol{H}$.

Let us consider an arbitrary tensor-product grid on Ω, defined as the Cartesian product of the following one-dimensional meshes:

$$\begin{aligned}
\bar{\Omega}_x^h &= \left\{x_i, i = 0, 1, \ldots, N_x : x_0 = 0,\ x_{i+1} - x_i = h_i^x > 0,\ x_{N_x} = L_x\right\}, \\
\bar{\Omega}_y^h &= \left\{y_j, j = 0, 1, \ldots, N_y : y_0 = 0,\ y_{j+1} - y_j = h_j^y > 0,\ y_{N_y} = L_y\right\}, \\
\bar{\Omega}_z^h &= \left\{z_k, k = 0, 1, \ldots, N_z : z_0 = 0,\ z_{k+1} = z_k = h_k^z > 0,\ z_{N_z} = L_z\right\}.
\end{aligned}$$

The mesh on Ω, denoted by $\bar{\Omega}^h$, is therefore $\bar{\Omega}^h = \bar{\Omega}_x^h \times \bar{\Omega}_y^h \times \bar{\Omega}_z^h$. We further define

$$x_{i+\frac{1}{2}} = x_i + h_i^x/2, \quad y_{j+\frac{1}{2}} = y_j + h_j^y/2 \quad \text{and} \quad z_{k+\frac{1}{2}} = z_k + h_k^z/2.$$

It will also be convenient to introduce $h_{-1}^x = h_{-1}^y = h_{-1}^z = h_{N_x}^x = h_{N_y}^y = h_{N_z}^z = 0$ and to define the averaged mesh sizes

$$\hbar_i^x = \frac{h_i^x + h_{i-1}^x}{2}, \quad \hbar_j^y = \frac{h_j^x + h_{j-1}^x}{2} \quad \text{and} \quad \hbar_k^z = \frac{h_k^z + h_{k-1}^z}{2}. \tag{4}$$

In keeping with the Yee scheme we shall associate each electric field degree of freedom (or unknown) with the mid–point of an edge in the mesh, and associate each degree of freedom for the magnetic field with the centroid of a face in the mesh. Thus the electric field is approximated as follows:

$$E_{i+\frac{1}{2},j,k}(t) \simeq E_1(x_{i+\frac{1}{2}}, y_j, z_k, t) \begin{cases} 0 \le i \le N_x - 1 \\ 0 \le j \le N_y \\ 0 \le k \le N_z, \end{cases} \tag{5a}$$

$$E_{i,j+\frac{1}{2},k}(t) \simeq E_2(x_i, y_{j+\frac{1}{2}}, z_k, t) \begin{cases} 0 \le i \le N_x \\ 0 \le j \le N_y - 1 \\ 0 \le k \le N_z, \end{cases} \tag{5b}$$

$$E_{i,j,k+\frac{1}{2}}(t) \simeq E_3(x_i, y_j, z_{k+\frac{1}{2}}, t) \begin{cases} 0 \le i \le N_x \\ 0 \le j \le N_y \\ 0 \le k \le N_z - 1. \end{cases} \tag{5c}$$

The magnetic field unknowns are

$$H_{i,j+\frac{1}{2},k+\frac{1}{2}}(t) \simeq H_1(x_i, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}}, t) \begin{cases} 0 \le i \le N_x \\ 0 \le j \le N_y - 1 \\ 0 \le k \le N_z - 1, \end{cases} \tag{6a}$$

$$H_{i+\frac{1}{2},j,k+\frac{1}{2}}(t) \simeq H_2(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}}, t) \begin{cases} 0 \le i \le N_x - 1 \\ 0 \le j \le N_y \\ 0 \le k \le N_z - 1, \end{cases} \tag{6b}$$

$$H_{i+\frac{1}{2},j+\frac{1}{2},k}(t) \simeq H_3(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k, t) \begin{cases} 0 \le i \le N_x - 1 \\ 0 \le j \le N_y - 1 \\ 0 \le k \le N_z. \end{cases} \tag{6c}$$

We shall usually drop the explicit dependence on time of the discrete quantities. The notation and geometry for a single grid cell is shown in Figure 1.

To discretize (1) we consider each component in turn, and proceed in the following heuristic fashion. First let us consider (1b). For any suitably smooth surface S with boundary ∂S, Stokes theorem applied to (1b) shows that

$$\int_S \frac{\partial \boldsymbol{H}}{\partial t} \cdot d\boldsymbol{A} = -\oint_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{S}.$$

Now we pick S to be a face in the mesh. For example

$$S = \{x = x_i, \quad y_j < y < y_{j+1}, \quad z_k < z < z_{k+1}\}.$$

With this choice, we approximate $\int_S \partial \boldsymbol{H}/\partial t \cdot d\boldsymbol{A}$ by quadrature using a single quadrature point at the centroid of the face, and approximate $\int_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{S}$ by mid–point quadrature on each straight segment of ∂S. Finally we use the approximate, finite difference, values in the quadrature (see (5), (6)) to obtain

$$h_j^y h_k^z \frac{d}{dt} H_{i,j+\frac{1}{2},k+\frac{1}{2}} + h_k^z (E_{i,j+1,k+\frac{1}{2}} - E_{i,j,k+\frac{1}{2}}) + h_j^y (E_{i,j+\frac{1}{2},k} - E_{i,j+\frac{1}{2},k+1}) = 0 \tag{7}$$

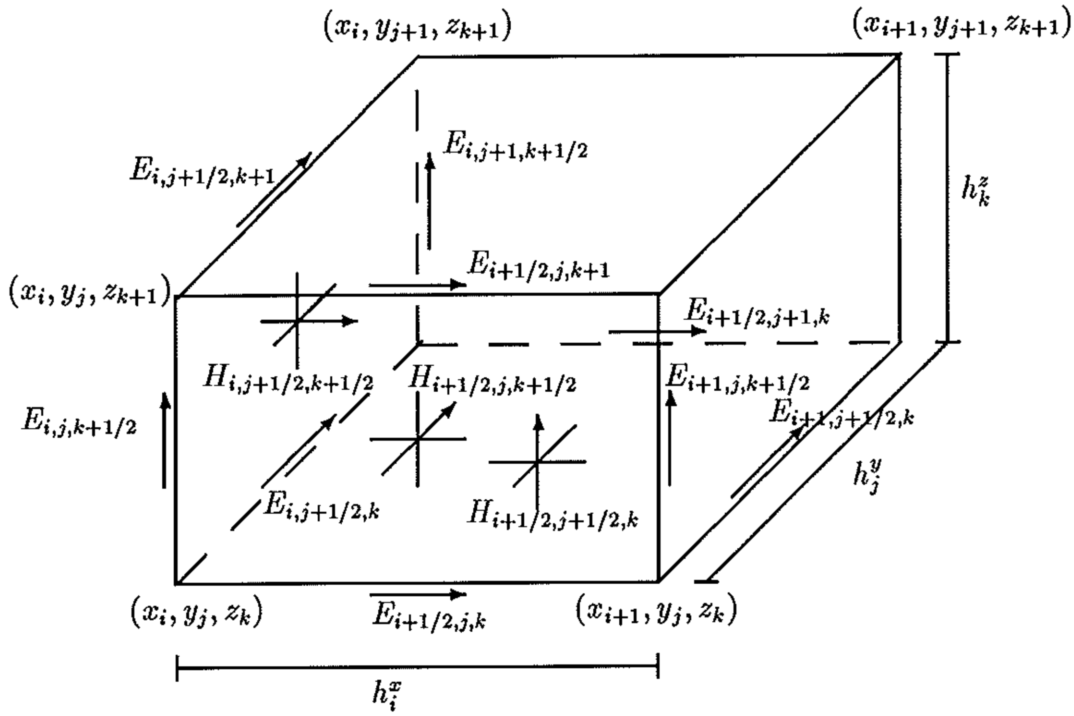


FIG. 1. *A parallelepiped in the mesh showing the geometry of the unknowns in the finite difference grid. The magnetic field unknowns ($H_{i,j+1/2,k+1/2}$ etc) are associated with the centroids of faces of the element. Electric field unknowns are associated with mid-points of the edges of the element. For simplicity we have only shown the degrees of freedom entering into the equations for $H_{i,j+1/2,k+1/2}$, $H_{i+1/2,j,k+1/2}$ and $H_{i+1/2,j+1/2,k}$.*

which holds for $0 \le i \le N_x$, $0 \le j \le N_y - 1$ and $0 \le k \le N_z - 1$.

Proceeding similarly, using successively

$$S = \{x_i < x < x_{i+1},\ y = y_j,\ z_k < z < z_{k+1}\}$$

and

$$S = \{x_i < x < x_{i+1},\ y_j < y < y_{j+1},\ z = z_k\},$$

we obtain the remaining equations for the magnetic field

$$(8)\qquad h_i^x h_k^z \frac{d}{dt} H_{i+\frac{1}{2},j,k+\frac{1}{2}} + h_k^z (E_{i,j,k+\frac{1}{2}} - E_{i+1,j,k+\frac{1}{2}}) + h_i^x (E_{i+\frac{1}{2},j,k+1} - E_{i+\frac{1}{2},j,k}) = 0$$

for $0 \le i \le N_x - 1$, $0 \le j \le N_y$, $0 \le k \le N_z - 1$, and

$$(9)\qquad h_i^x h_j^y \frac{d}{dt} H_{i+\frac{1}{2},j+\frac{1}{2},k} + h_j^y (E_{i+1,j+\frac{1}{2},k} - E_{i,j,+\frac{1}{2},k}) + h_i^x (E_{i+\frac{1}{2},j,k} - E_{i+\frac{1}{2},j+1,k}) = 0$$

for $0 \le i \le N_x - 1$, $0 \le j \le N_y - 1$ and $0 \le k \le N_z$.

A glance at (7)–(9) shows that these equations are nothing more than standard centered finite difference approximation to (1b). Thus for these equations the local truncation error is second order (see Section 3).

To discretize the electric field equations we use the integral form of (1a):

$$(10)\qquad \int_S \left(\frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{J} \right) \cdot d\boldsymbol{A} = \oint_{\partial S} \boldsymbol{H} \cdot d\boldsymbol{S}.$$

For the electric field, we must use the "dual grid" formed by connecting centroids of elements in $\bar{\Omega}^h$. To discretize the equation for the first component of $\boldsymbol{E}$, we pick

$$S = \left\{ x = x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}} < y < y_{j+\frac{1}{2}}, z_{k-\frac{1}{2}} < z < z_{k+\frac{1}{2}} \right\}.$$

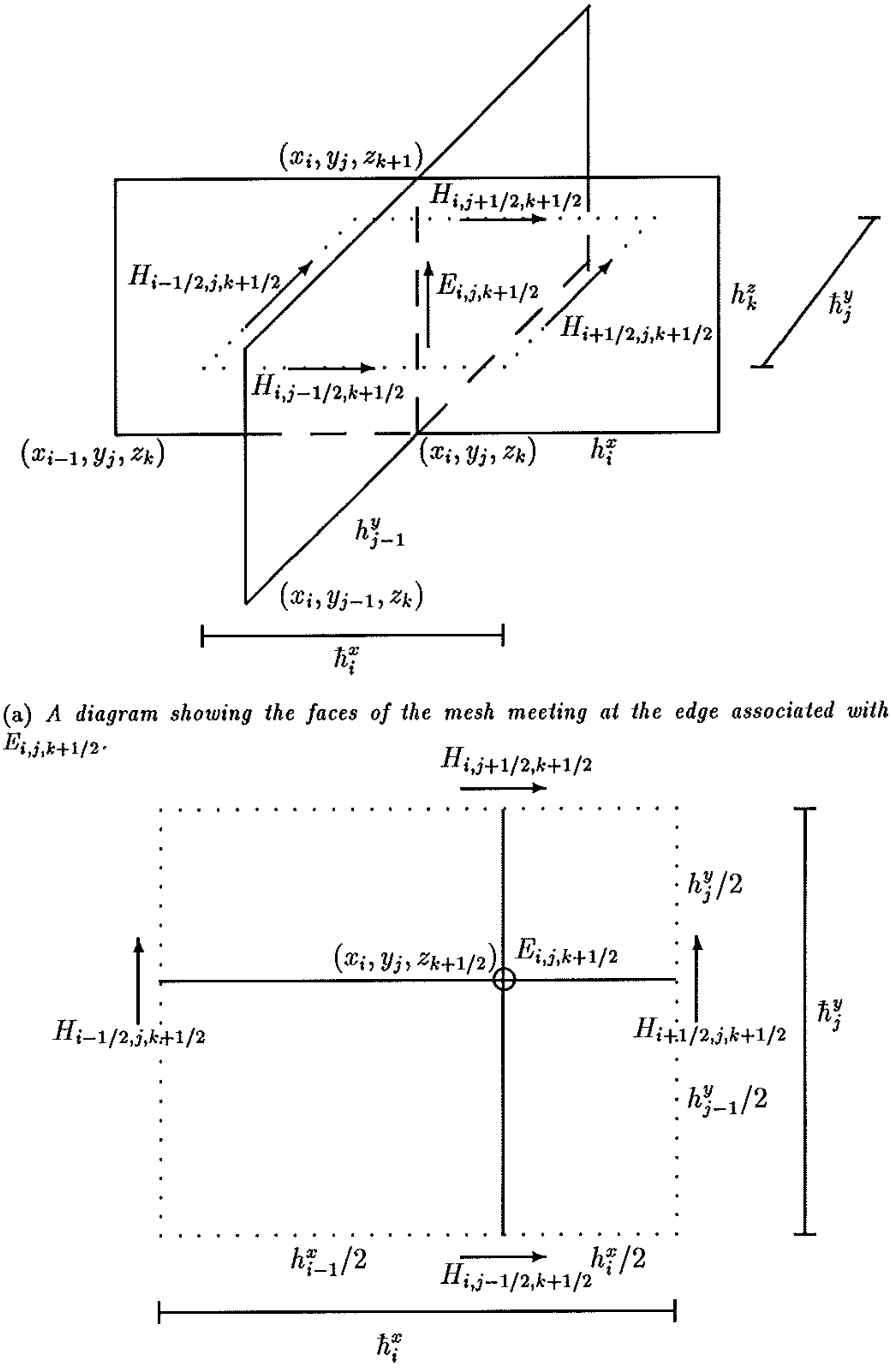
Again, we use one point quadrature formulae to approximate the integrals in (10) (but these quadratures are no longer mid-point quadratures since the unknowns are generally no longer at the centroid of S or the mid-points of its edges). See Figure 2 for a typical geometry in this case (actually the picture pertains to the derivation of equation (12) below). We obtain

$$(11)\qquad \begin{aligned} &\hbar_j^y \hbar_k^z \frac{d}{dt} E_{i+\frac{1}{2},j,k} - \hbar_k^z (H_{i+\frac{1}{2},j+\frac{1}{2},k} - H_{i+\frac{1}{2},j-\frac{1}{2},k}) - \hbar_j^y (H_{i+\frac{1}{2},j,k-\frac{1}{2}} - H_{i+\frac{1}{2},j,k+\frac{1}{2}}) \\ &\quad = \hbar_j^y \hbar_k^z J_{i+\frac{1}{2},j,k} \end{aligned}$$

where $J_{i+\frac{1}{2},j,k} = J_1(x_{i+\frac{1}{2}}, y_j, z_k, t)$, and (11) holds at interior edges so that $0 \le i \le N_x - 1$, $1 \le j \le N_y - 1$, and $1 \le k \le N_z - 1$.

Similarly taking S in (10) to be

$$S = \left\{ x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}},\quad y = y_{j+\frac{1}{2}},\quad z_{k-\frac{1}{2}} < z < z_{k+\frac{1}{2}} \right\}$$



(a) *A diagram showing the faces of the mesh meeting at the edge associated with* $E_{i,j,k+1/2}$.

(b) *A projection of the face in the "dual" mesh associated with* $E_{i,j,k+1/2}$ *down the z axis. Solid lines represent edges in the mesh, and dotted lines are edges in the dual mesh.*

FIG. 2. *These figures show the geometry of the unknowns in the equation for* $E_{i,j,k+1/2}$. *The geometry is similar for other electric field variables. The direction of the electric field variable is normal to a face in the "dual" grid.*

we can derive the approximate equation for E_2 given by

$$\hbar_i^x \hbar_k^z \frac{d}{dt} E_{i,j+\frac{1}{2},k} - \hbar_i^x (H_{i,j+\frac{1}{2},k+\frac{1}{2}} - H_{i,j+\frac{1}{2},k-\frac{1}{2}}) - \hbar_k^z (H_{i-\frac{1}{2},j+\frac{1}{2},k} - H_{i+\frac{1}{2},j+\frac{1}{2},k}) \tag{12}$$
$$= \hbar_i^x \hbar_k^z J_{i,j+\frac{1}{2},k}$$

where $J_{i,j+\frac{1}{2},k} = J_2(x_i, y_{j+\frac{1}{2}}, z_k, t)$ for $1 \le i \le N_x - 1$, $0 \le j \le N_y - 1$ and $1 \le k \le N_z - 1$.

Finally, choosing

$$S = \left\{ x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \quad y_{j-\frac{1}{2}} < y < y_{j+\frac{1}{2}}, \quad z = z_{k+\frac{1}{2}} \right\}$$

we obtain

$$\hbar_i^x \hbar_j^y \frac{d}{dt} E_{i,j,k+\frac{1}{2}} - \hbar_j^y (H_{i+\frac{1}{2},j,k+\frac{1}{2}} - H_{i-\frac{1}{2},j,k+\frac{1}{2}}) - \hbar_i^x (H_{i,j-\frac{1}{2},k+\frac{1}{2}} - H_{i,j+\frac{1}{2},k+\frac{1}{2}}) \tag{13}$$
$$= \hbar_i^x \hbar_j^y J_{i,j,k+\frac{1}{2}}$$

where $J_{i,j,k+\frac{1}{2}} = J_3(x_i, y_j, z_{k+\frac{1}{2}}, t)$ for $1 \le i \le N_x - 1$, $1 \le j \le N_y - 1$, and $0 \le k \le N_z - 1$. In Section 3 we shall show that (11)–(13) are first order approximations to (1a).

Equations (7)–(9) and (11)–(13) approximate equations (1). The boundary condition (2) is easily satisfied by selecting suitable unknowns to be zero. We choose all degrees of freedom for $\boldsymbol{E}$ associated with edges on Γ to be zero:

$$E_{i+\frac{1}{2},j,k} = 0 \quad \text{if} \begin{cases} j = 0 \text{ or } j = N_y & \text{and} \\ 0 \le i \le N_x - 1 & \text{and } 0 \le k \le N_z \\ \qquad \text{or} & \\ k = 0 \text{ or } k = N_z & \text{and} \\ 0 \le i \le N_x - 1 & \text{and } 0 \le j \le N_y \end{cases} \tag{14a}$$

$$E_{i,j+\frac{1}{2},k} = 0 \quad \text{if} \begin{cases} i = 0 \text{ or } i = N_x & \text{and} \\ 0 \le j \le N_y - 1 & \text{and } 0 \le k \le N_z \\ \qquad \text{or} & \\ k = 0 \text{ or } k = N_z & \text{and} \\ 0 \le j \le N_y - 1 & \text{and } 0 \le i \le N_x \end{cases} \tag{14b}$$

$$E_{i,j,k+\frac{1}{2}} = 0 \quad \text{if} \begin{cases} i = 0 \text{ or } i = N_x & \text{and} \\ 0 \le j \le N_y & \text{and } 0 \le k \le N_z - 1 \\ \qquad \text{or} & \\ j = 0 \text{ or } j = N_y & \text{and} \\ 0 \le i \le N_x & \text{and } 0 \le k \le N_z - 1. \end{cases} \tag{14c}$$

Finally, the initial data (3) is imposed by requiring that (5) is satisfied exactly at time $t = 0$.

To obtain a fully discrete scheme, (7)–(9) and (11)–(13) must be discretized in time. This is usually done using the leap–frog scheme [11], but we shall ignore this step here since we wish to analyze spatial accuracy.

3. Error Analysis. This section is devoted to proving the error estimate in Theorem 3.1 (stated below) which shows that the scheme outlined in the previous section is second order convergent regardless of mesh non–uniformity. We define the following mesh dependent error norms

$$\begin{aligned}
\|\boldsymbol{E}-\boldsymbol{E}^h\|_E^2 \;=\; & \sum_{k=1}^{N_z-1}\sum_{j=1}^{N_y-1}\sum_{i=0}^{N_x-1} h_i^x \hbar_j^y \hbar_k^z (E_1(x_{i+\frac12}, y_j, z_k) - E_{i+\frac12,j,k})^2 \\
& + \sum_{k=1}^{N_z-1}\sum_{j=0}^{N_y-1}\sum_{i=1}^{N_x-1} \hbar_i^x h_j^y \hbar_k^z (E_2(x_i, y_{j+\frac12}, z_k) - E_{i,j+\frac12,k})^2 \\
& + \sum_{k=0}^{N_z-1}\sum_{j=1}^{N_y-1}\sum_{i=1}^{N_x-1} \hbar_i^x \hbar_j^y h_k^z (E_3(x_i, y_j, z_{k+\frac12}) - E_{i,j,k+\frac12})^2
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\|\boldsymbol{H}-\boldsymbol{H}^h\|_H^2 \;=\; & \sum_{k=0}^{N_z-1}\sum_{j=0}^{N_y-1}\sum_{i=0}^{N_x} \hbar_i^x h_j^y h_k^z (H_1(x_i, y_{j+\frac12}, z_{k+\frac12}) - H_{i,j+\frac12,k+\frac12})^2 \\
& + \sum_{k=0}^{N_z-1}\sum_{j=0}^{N_y}\sum_{i=0}^{N_x-1} h_i^x \hbar_j^y h_k^z (H_2(x_{i+\frac12}, y_j, z_{k+\frac12}) - H_{i+\frac12,j,k+\frac12})^2 \\
& + \sum_{k=0}^{N_z}\sum_{j=0}^{N_y-1}\sum_{i=0}^{N_x-1} h_i^x h_j^y \hbar_k^z (H_3(x_{i+\frac12}, y_{j+\frac12}, z_k) - H_{i+\frac12,j+\frac12,k})^2
\end{aligned} \tag{16}$$

THEOREM 3.1. *Suppose that $\boldsymbol{E}$ and $\boldsymbol{H}$ are three times continuously differentiable on $\bar{\Omega}$, that $\boldsymbol{H}_t$ is twice continuously differentiable on $\bar{\Omega}$ and that all the previously mentioned derivatives are continuous in time. Then for any fixed $T > 0$ there is a constant C depending on T such that*

$$\|\boldsymbol{E}-\boldsymbol{E}^h\|_E + \|\boldsymbol{H}-\boldsymbol{H}^h\|_H \le Ch^2.$$

In order to prove this theorem we shall first establish a sequence of preliminary results which are stated in Lemmas 3.1-3.8.

Our first lemma shows that the electric field equations (11)–(13) have a first order local truncation error but with a special structure. Let us define

$$e^E_{\alpha,\beta,\gamma}(t) = E_\ell(x_\alpha, y_\beta, z_\gamma, t) - E_{\alpha,\beta,\gamma}(t) \tag{17}$$

for all valid choices of subscripts ℓ, α, β and γ (see equations (5)) and define

$$e^H_{\alpha,\beta,\gamma}(t) = H_\ell(x_\alpha, y_\beta, z_\gamma, t) - H_{\alpha,\beta,\gamma}(t) \tag{18}$$

for all valid choices of subscript in (6). Let us also define

$$\beta_{i,j+\frac12,k+\frac12} = \frac18\left[(h_j^y)^2 H_{1yy}(x_i, y_{j+\frac12}, z_{k+\frac12}) + (h_k^z)^2 H_{1zz}(x_i, y_{j+\frac12}, z_{k+\frac12})\right], \tag{19a}$$

$$\beta_{i+\frac12,j,k+\frac12} = \frac18\left[(h_i^x)^2 H_{2xx}(x_{i+\frac12}, y_j, z_{k+\frac12}) + (h_k^z)^2 H_{2zz}(x_{i+\frac12}, y_j, z_{k+\frac12})\right], \tag{19b}$$

$$\beta_{i+\frac12,j+\frac12,k} = \frac18\left[(h_j^y)^2 H_{3yy}(x_{i+\frac12}, y_{j+\frac12}, z_k) + (h_i^x)^2 H_{3xx}(x_{i+\frac12}, y_{j+\frac12}, z_k)\right], \tag{19c}$$

for all valid choices of i, j and k given in (6). In writing (19) we have used the notation $H_{1xx} \equiv \left(\frac{\partial}{\partial x}\right)^2 H_1$ etc. and have suppressed dependence on time.

We shall only provide details of the local truncation error estimates for the first component of (1a) since the remaining components are similar. With the above definitions we can state and prove the following lemma:

LEMMA 3.2. *Suppose $\boldsymbol{H}$ and $\boldsymbol{E}$ are smooth enough, so that all indicated derivatives exist (see the conditions of Theorem 3.1) then*

$$\begin{aligned}(20)\quad & h_i^x \hbar_j^y \hbar_k^z \frac{d}{dt} e^E_{i+\frac{1}{2},j,k} - h_i^x \hbar_k^z (e^H_{i+\frac{1}{2},j+\frac{1}{2},k} - e^H_{i+\frac{1}{2},j-\frac{1}{2},k}) - h_i^x \hbar_j^y (e^H_{i+\frac{1}{2},j,k-\frac{1}{2}} - e^H_{i+\frac{1}{2},j,k+\frac{1}{2}}) \\ = & -h_i^x \hbar_k^z (\beta_{i+\frac{1}{2},j+\frac{1}{2},k} - \beta_{i+\frac{1}{2},j-\frac{1}{2},k}) + h_i^x \hbar_j^y (\beta_{i+\frac{1}{2},j,k+\frac{1}{2}} - \beta_{i+\frac{1}{2},j,k-\frac{1}{2}}) + h_i^x \hbar_j^y \hbar_k^z \gamma_{i+\frac{1}{2},j,k}\end{aligned}$$

for $0 \le i \le N_x - 1$, $1 \le j \le N_y - 1$ and $1 \le k \le N_z - 1$ where

$$\begin{aligned}\gamma_{i+\frac{1}{2},j,k} = & -\left(\frac{1}{\hbar_j^y}\right)\Bigg\{\frac{(h_j^y)^2}{8}\left[H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k)\right] \\ & -\frac{(h_{j-1}^y)^2}{8}\left[H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k)\right]\Bigg\} \\ & +\left(\frac{1}{\hbar_k^z}\right)\Bigg\{\frac{(h_k^z)^2}{8}\left[H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{2zz}(x_{i+\frac{1}{2}}, y_i, z_{k+\frac{1}{2}})\right] \\ & -\frac{(h_{k-1}^z)^2}{8}\left[H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}})\right]\Bigg\} \\ & +\frac{(h_i^x)^2}{8\hbar_j^y}\left[H_{3xx}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) - H_{3xx}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k)\right] \\ & -\frac{(h_i^x)^2}{8\hbar_k^z}\left[H_{2xx}(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}}) - H_{2xx}(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}})\right] \\ & +\frac{1}{48\hbar_j^y}\left[(h_j^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y+}, z_k) + (h_{j-1}^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y-}, z_k)\right] \\ (21)\quad & +\frac{1}{48\hbar_k^z}\left[(h_k^z)^3 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z+}) + (h_{k-1}^z)^3 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z-})\right].\end{aligned}$$

Here $y_{j-1} < \xi_j^{y-} < \xi_j^{y+} < y_j$ and $z_{k-1} < \xi_k^{z-} < \xi_k^{z+} < z_k$.

Proof. The proof of Lemma 3.2 is easy but tedious. We use the fact that if u is a suitably smooth function (three times continuously differentiable on $[a, b]$ is sufficient) and $a < c < b$ then

$$(22)\qquad u(b) - u(a) = \hbar u'(c) + \frac{h_+^2 - h_-^2}{8} u''(c) + \frac{1}{48}\left[h_+^3 u'''(\xi^+) + h_-^3 u'''(\xi^-)\right]$$

where $h_+ = 2(b - c)$, $h_- = 2(c - a)$, $\hbar = (h_+ + h_-)/2$ and $a < \xi^- < c < \xi^+ < b$. Estimate (22) is proved using standard Taylor series expansion.

Let $R^x_{i+\frac{1}{2},j,k}$ denote the left-hand side of (20), then expanding e^E and e^H on the left-hand side of (20) and using (11) we obtain

$$\begin{aligned}(h_i^x)^{-1} R^z_{i+\frac{1}{2},j,k} = & \hbar_j^y \hbar_k^z \frac{\partial E_1}{\partial t}(x_{i+\frac{1}{2}}, y_j, z_k) - \hbar_k^z (H_3(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) - H_3(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k)) \\ & -\hbar_k^z (H_2(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}}) - H_2(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}})) - \hbar_j^y \hbar_k^z J_{i+\frac{1}{2},j,k}.\end{aligned}$$

Now using (22) first with $u(y) = H_3(x_{i+\frac{1}{2}}, y, z_k)$ and $a = y_{j-\frac{1}{2}}, c = y$, and $b = y_{j+\frac{1}{2}}$, and then with $u(z) = H_2(x_{i+\frac{1}{2}}, y_j, z), a = z_{k-\frac{1}{2}}, c = z_k, b = z_{k+\frac{1}{2}}$, and using the equation for E_1 (first component of (1a)) we obtain

$$
\begin{aligned}
(h_i^x)^{-1} R^x_{i+\frac{1}{2},j,k} \;=\; & -\hbar_k^z \Bigg(\frac{(h_j^y)^2 - (h_{j-1}^y)^2}{8} H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) \\
& + \frac{1}{48}\left[(h_j^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y+}, z_k) + (h_{j-1}^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y-}, z_k) \right] \Bigg) \\
& + \hbar_j^y \Bigg(\frac{(h_k^z)^2 - (h_{k-1}^z)^2}{8} H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k) \\
& + \frac{1}{48}\left[(h_k^z)^3 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z+}) + (h_{k-1}^z)^3 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z-}) \right] \Bigg).
\end{aligned}
\tag{23}
$$

On a uniform mesh the terms $(h_j^y)^2 - (h_{j-1}^y)^2$ and $(h_k^z)^2 - (h_{k-1}^z)^2$ vanish, but on a non–uniform mesh these terms give rise to a first order local truncation error. However, as we shall see, this local truncation error has a special form. We exploit this to rewrite the right-hand side of (23) so that the first order error term appears as the discrete curl (in the sense of equations (11)–(13)) of a second order quantity. We rewrite the term $((h_j^y)^2 - (h_{j-1}^y)^2) H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k)$ as follows

$$
\begin{aligned}
& ((h_j^y)^2 - (h_{j-1}^y)^2) H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) \\
= \;& (h_j^y)^2 H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) - (h_{j-1}^y)^2 H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) \\
& + (h_j^y)^2 \left[H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) \right] \\
& - (h_{j-1}^y)^2 \left[H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k) \right].
\end{aligned}
\tag{24}
$$

Similarly we rewrite $((h_k^z)^2 - (h_{k-1}^z)^2) H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k)$ as a difference of two second order quantities. Using these expansions in (23) shows that

$$
\begin{aligned}
(h_i^x)^{-1} R^x_{i+\frac{1}{2},j,k} \;=\; & -\hbar_k \left(\frac{(h_j^y)^2}{8} H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) - \frac{(h_{j-1}^y)^2}{8} H_{3yy}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k) \right) \\
& + \hbar_j^y \left(\frac{(h_k^z)^2}{8} H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}}) - \frac{(h_{k-1}^z)^2}{8} H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}}) \right) \\
& - \frac{\hbar_k}{8} (h_j^y)^2 [H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k)] \\
& + \frac{\hbar_k}{8} (h_{j-1}^y)^2 \left[H_{3yy}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{3yy}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k) \right] \\
& + \frac{\hbar_j^y}{8} (h_k^z)^2 \left[H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}}) \right] \\
& - \frac{\hbar_j^y}{8} (h_{k-1}^z)^2 \left[H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_k) - H_{2zz}(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}}) \right] \\
& + \frac{1}{48}\left[(h_j^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y+}, z_k) + (h_{j-1}^y)^3 H_{3yyy}(x_{i+\frac{1}{2}}, \xi_j^{y-}, z_k) \right] \\
& + \frac{1}{48}\left[(h_k^z)^3 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z+}) + (h_{k-1}^z)^2 H_{2zzz}(x_{i+\frac{1}{2}}, y_j, \xi_k^{z-}) \right].
\end{aligned}
\tag{25}
$$

Equation (25) still does not contain all the terms necessary, so we add and subtract the term

$$
\begin{aligned}
&-\bar{h}_k^z \frac{(h_i^x)^2}{8}\left[H_{3xx}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_k) - H_{3xx}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z_k)\right] \\
&+\bar{h}_j^y \frac{(h_i^x)^2}{8}\left[H_{2xx}(x_{i+\frac{1}{2}}, y_j, z_{k+\frac{1}{2}}) - H_{2xx}(x_{i+\frac{1}{2}}, y_j, z_{k-\frac{1}{2}})\right].
\end{aligned}
$$

The resulting equation is exactly (20). □

Lemma 3.2 gives the desired decomposition of the local truncation error, but we need an estimate of the remainder terms. This is supplied in the next lemma where we have used the notation $||f||_\infty = \max_{x\in\bar{\Omega}} |f(x)|$.

LEMMA 3.3. *Suppose $\boldsymbol{E}$ and $\boldsymbol{H}$ are three times continuously differentiable on $\bar{\Omega}$, suppose that $\boldsymbol{H}_t$ is twice continuously differentiable on $\bar{\Omega}$, and suppose all previously mentioned derivatives are continuous in time, then*

$$
\begin{aligned}
|\gamma_{i+\frac{1}{2},j,k}| &\le h^2\left[\frac{1}{3}||H_{3yyy}||_\infty + \frac{1}{3}||H_{2zzz}||_\infty + \frac{1}{8}||H_{3xxy}||_\infty + \frac{1}{8}||H_{2xxz}||_\infty\right] \\
&\le h^2 M_3^H \qquad (26)
\end{aligned}
$$

where $M_p^H = \max_{\substack{1\le \ell\le 3, i+j+k=p\\ i\ge 0, j\ge 0, k\ge 0}} ||\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j\left(\frac{\partial}{\partial z}\right)^k H_\ell||_\infty$.

$$
|\beta_{i,j+\frac{1}{2},k+\frac{1}{2}}| \le h^2\left[\frac{1}{8}||H_{1yy}||_\infty + \frac{1}{8}||H_{1zz}||_\infty\right] \le \frac{h^2 M_2^H}{4}, \qquad (27a)
$$

$$
|\beta_{i+\frac{1}{2},j,k+\frac{1}{2}}| \le h^2\left[\frac{1}{8}||H_{2xx}||_\infty + \frac{1}{8}||H_{2zz}||_\infty\right] \le \frac{h^2 M_2^H}{4}, \qquad (27b)
$$

$$
|\beta_{i+\frac{1}{2},j+\frac{1}{2},k}| \le h^2\left[\frac{1}{8}||H_{3yy}||_\infty + \frac{1}{8}||H_{3xx}||_\infty\right] \le \frac{h^2 M_2^H}{4}. \qquad (27c)
$$

Finally

$$
|\frac{d}{dt}\beta_{i,j+\frac{1}{2},k+\frac{1}{2}}| \le \frac{h^2 M_{2t}^H}{4}, \qquad (28a)
$$

$$
|\frac{d}{dt}\beta_{i+\frac{1}{2},j,k+\frac{1}{2}}| \le \frac{h^2 M_{2t}^H}{4}, \qquad (28b)
$$

$$
|\frac{d}{dt}\beta_{i+\frac{1}{2},j,k+\frac{1}{2}}| \le \frac{h^2 M_{2t}^H}{4}, \qquad (28c)
$$

where $M_{pt}^H = \max_{\substack{1\le \ell\le 3, i+j+k=p\\ i\ge 0, j\ge 0, k\ge 0}} ||\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j\left(\frac{\partial}{\partial z}\right)^k\left(\frac{\partial}{\partial t}\right) H_\ell||_\infty$.

Remark: This lemma shows that the error term $\gamma_{i+\frac{1}{2},j,k}$ in (20) is second order but that the term involving β is only first order.

Proof. This result follows trivially from the mean value theorem. □

We now state, without proof, the analogous results to (20) for the remaining components of (1a).

LEMMA 3.4. *Suppose* $\boldsymbol{H}$ *and* $\boldsymbol{E}$ *are smooth enough (see Lemma 3.3) then*

$$\begin{aligned}(29)\quad & \hbar^x_i h^y_j \hbar^z_k \frac{d}{dt} e^E_{i,j+\frac12,k} - \hbar^x_i h^y_j (e^H_{i,j+\frac12,k+\frac12} - e^H_{i,j+\frac12,k-\frac12}) - h^y_j \hbar^z_k (e^H_{i-\frac12,j+\frac12,k} - e^H_{i+\frac12,j+\frac12,k}) \\ = \quad & -\hbar^x_i h^y_j (\beta_{i,j+\frac12,k+\frac12} - \beta_{i,j+\frac12,k-\frac12}) - h^y_j \hbar^z_k (\beta_{i-\frac12,j+\frac12,k} - \beta_{i+\frac12,j+\frac12,k}) + \hbar^x_i h^y_j \hbar^z_k \gamma_{i,j+\frac12,k}\end{aligned}$$

for $1 \le i \le N_x - 1$, $0 \le j \le N_y - 1$ *and* $1 \le k \le N_z - 1$ *where the remainder term* $\gamma_{i,j+\frac12,k}$ *is a complicated expression similar to that for* $\gamma_{i+\frac12,j,k}$ *in (25) and satisfying the estimate* $|\gamma_{i,k+\frac12,k}| \le h^2 M^H_3$.

In addition

$$\begin{aligned}(30)\quad & \hbar^x_i \hbar^y_j h^z_k \frac{d}{dt} e^E_{i,j,k+\frac12} - \hbar^y_j h^z_k (e^H_{i+\frac12,j,k+\frac12} - e^H_{i-\frac12,j,k+\frac12}) - \hbar^x_i h^z_k (e^H_{i,j-\frac12,k+\frac12} - e^H_{i,j+\frac12,k+\frac12}) \\ = \quad & -\hbar^y_j h^z_k (\beta_{i+\frac12,j,k+\frac12} - \beta_{i-\frac12,j,k+\frac12}) - \hbar^x_i h^z_k (\beta_{i,j-\frac12,k+\frac12} - \beta_{i,j+\frac12,k+\frac12}) + \hbar^x_i \hbar^y_j h^z_k \gamma_{i,j,k+\frac12}\end{aligned}$$

for $1 \le i \le N_x - 1$, $1 \le j \le N_y - 1$ *and* $0 \le k \le N_z - 1$ *where* $\gamma_{i,j,k+\frac12}$ *satisfies the estimate* $|\gamma_{i,j,k+\frac12}| \le h^2 M^H_3$.

Our next lemma gives the local truncation error for the discrete approximation to the first component of (18) by (7).

LEMMA 3.5. *Assuming that* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *are sufficiently smooth (see Lemma 3.3).*

$$\begin{aligned} & \hbar^x_j h^y_j h^z_k \frac{d}{dt} e^H_{i,j+\frac12,k+\frac12} + \hbar^x_i h^z_k (e^E_{i,j+1,k+\frac12} - e^E_{i,j,k+\frac12}) + \hbar^x_i h^y_j (e^E_{i,j+\frac12,k} - e^E_{i,j+\frac12,k+1}) \\ (31)\quad = \quad & \hbar^x_i h^y_j h^z_k \alpha_{i,j+\frac12,k+1}\end{aligned}$$

for $0 \le i \le N_x$, $0 \le j \le N_y$ *and* $0 \le k \le N_x - 1$ *where the remainder term is given by*

$$(32)\qquad \alpha_{i+\frac12,j,k+\frac12} = \frac{(h^y_j)^2}{24} E_{3yyy}(x_i, \eta^y_j, z_{k+\frac12}) - \frac{(h^z_k)^2}{24} E_{2zzz}(x_i, y_{j+\frac12}, \eta^z_k).$$

Here $y_j < \eta^y_j < y_{j+1}$ *and* $z_k < \eta^z_k < z_{k+1}$. *Hence*

$$(33)\qquad |\alpha_{i+\frac12,j,k+\frac12}| \le \frac{h^2}{12} M^E_3$$

where $M^E_p = \max\limits_{\substack{1 \le \ell \le 3, i+j+k=p \\ i \ge 0, j \ge 0, k \ge 0}} \| \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial z}\right)^k E_\ell \|_\infty$.

Proof. This is a simple use of Taylor series since (7) is a centered difference approximation to the first component of (1b). □

We now summarize the truncation error results for (8) and (9)

LEMMA 3.6. *Assuming that* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *are sufficiently smooth (see Lemma 3.5)*

$$\begin{aligned} & h^x_i \hbar^y_j h^z_k \frac{d}{dt} e^H_{i+\frac12,j,k+\frac12} + \hbar^y_j h^z_k (e^E_{i,j,k+\frac12} - e^E_{i+1,j,k+\frac12}) + h^x_i \hbar^z_j (e^E_{i+\frac12,j,k+1} - e^E_{i+\frac12,j,k}) \\ (34)\quad & = h^x_j \hbar^y_j h^z_k \alpha_{i+\frac12,j,k+\frac12} \\ & h^x_i h^y_j \hbar^z_k \frac{d}{dt} e^H_{i+\frac12,j+\frac12,k} + h^y_j \hbar^z_k (e^E_{i+1,j+\frac12,k} - e^E_{i,j+\frac12,k}) + h^x_i \hbar^z_j (e^E_{i+\frac12,j,k} - e^E_{i+\frac12,j+1,k}) \\ (35)\quad & = h^x_j h^y_j \hbar^z_k \alpha_{i+\frac12,j+\frac12,k}\end{aligned}$$

where $|\alpha_{i+\frac{1}{2},j,k+\frac{1}{2}}| \leq h^2 M_3^E/12$ *and* $|\alpha_{i+\frac{1}{2},j+\frac{1}{2},k}| \leq h^2 M_3^E/12$.

Lemmas 3.2–3.6 express the behavior of the local truncation error. Next we need to analyze how this local error contributes to the global error. This is done by deriving a discrete energy estimate for the error equations. First we summarize the discrete error equations and analyze the form of these equations.

Let us enumerate the electric and magnetic unknowns (i.e. form long vectors of electric and magnetic unknowns) then (20), (29) and (30) may be written

$$M_{EE}\frac{d}{dt}\vec{e}^{\,E} - C_{EH}\,\vec{e}^{\,H} = C_{EH}\,\vec{\beta} + M_{EE}\,\vec{\gamma} \tag{36}$$

where $\vec{e}^{\,E}$ is the electric error vector, $\vec{e}^{\,H}$ the magnetic error vector, $\vec{\beta}$ the vector of β values (see (19)) enumerated like the magnetic field and $\vec{\gamma}$ the vector of γ values (see (21) and Lemma 3.3) enumerated like the electric field. The matrix M_{EE} is a diagonal matrix with diagonal entries $\bar{h}_i^x h_j^y h_k^z$, $h_i^x \bar{h}_j^y h_k^z$ or $h_i^x h_j^y \bar{h}_k^z$ depending on which of (7), (8) or (9) is relevant. The matrix C_{EH} is a sparse matrix corresponding to the discrete curl in (7)–(9). The choice of $\vec{\beta}$ was dictated (see Lemma 3.2) by the need for $C_{EH}\,\vec{\beta}$ to appear on the right-hand side of (36). Let us note that

$$\vec{u}^{\,T} M_{EE}\,\vec{u} = \|\,\vec{u}\,\|_E^2 \tag{37}$$

where the discrete norm $\|\,\vec{u}\,\|_E$ is defined in (15).

Using the same enumeration of unknowns we may write (31), (34) and (35) as

$$M_{HH}\frac{d}{dt}\vec{e}^{\,H} + C_{HE}\,\vec{e}^{\,E} = M_{HH}\,\vec{\alpha} \tag{38}$$

where $\vec{\alpha}$ is the vector of α values (see (32) and Lemma 3.6) enumerated like the magnetic unknowns. M_{HH} is a diagonal matrix and C_{HE} corresponds to the discrete curl in (31), (34) and (35).

Note that

$$\vec{v}^{\,T} M_{HH}\,\vec{v} = \|\,\vec{v}\,\|_H^2 \tag{39}$$

where $\|\,\vec{v}\,\|_H$ is defined in (16).

An important point is that the extension of the Yee scheme to a non–uniform grid has preserved the relationship between C_{HE} and C_{EH} which is present for a uniform grid. The next lemma shows that

$$C_{HE} = (C_{EH})^T$$

hence we may write $C = C_{HE}$ and rewrite (36) and (38) as

$$M_{EE}\frac{d}{dt}\vec{e}^{\,E} - C^T\,\vec{e}^{\,H} = C^T\,\vec{\beta} + M_{EE}\,\vec{\gamma} \tag{40a}$$

$$M_{HH}\frac{d}{dt}\vec{e}^{\,H} + C\,\vec{e}^{\,E} = M_{HH}\,\vec{\alpha} \tag{40b}$$

LEMMA 3.7. *Suppose that the discrete function $(v_{i+\frac{1}{2},j,k}, v_{i,j+\frac{1}{2},k}, v_{i,j,k+\frac{1}{2}})$ is defined for all values of the subscripts in (5) and that it satisfies the boundary conditions (14). Suppose another discrete function $(u_{i,j+\frac{1}{2},k+\frac{1}{2}}, u_{i+\frac{1}{2},j,k+\frac{1}{2}}, u_{i+\frac{1}{2},j+\frac{1}{2},k})$ is defined for all values of the subscripts in (6) then*

$$
\begin{aligned}
&\sum_{k=0}^{N_z-1}\sum_{j=0}^{N_y-1}\sum_{i=0}^{N_x}\hbar_i^x u_{i,j+\frac{1}{2},k+\frac{1}{2}}\left\{h_k^z(v_{i,j,k+\frac{1}{2}}-v_{i,j+1,k+\frac{1}{2}})+h_j^y(v_{i,j+\frac{1}{2},k+1}-v_{i,j+\frac{1}{2},k})\right\}\\
&+\sum_{k=0}^{N_z-1}\sum_{j=0}^{N_y}\sum_{i=0}^{N_x-1}\hbar_j^y u_{i+\frac{1}{2},j,k+\frac{1}{2}}\left\{h_k^z(v_{i+1,j,k+\frac{1}{2}}-v_{i,j,k+\frac{1}{2}})+h_i^x(v_{i+\frac{1}{2},j,k}-v_{i+\frac{1}{2},j,k+1})\right\}\\
&+\sum_{k=0}^{N_z}\sum_{j=0}^{N_y-1}\sum_{i=0}^{N_x-1}\hbar_k^z u_{i+\frac{1}{2},j+\frac{1}{2},k}\left\{h_j^y(v_{i,j+\frac{1}{2},k}-v_{i+1,j+\frac{1}{2},k})+h_i^x(v_{i+\frac{1}{2},j+1,k}-v_{i+\frac{1}{2},j,k})\right\}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&=\sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1}\sum_{k=1}^{N_z-1} h_i^x v_{i+\frac{1}{2},j,k}\left\{\hbar_k^z(u_{i+\frac{1}{2},j-\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k})\right.\\
&\qquad\left.-\hbar_j^y(u_{i+\frac{1}{2},j,k-\frac{1}{2}}-u_{i+\frac{1}{2},j,k+\frac{1}{2}})\right\}\\
&+\sum_{j=0}^{N_y-1}\sum_{i=1}^{N_x-1}\sum_{k=1}^{N_z-1} h_j^y v_{i,j+\frac{1}{2},k}\left\{\hbar_i^x(u_{i,j+\frac{1}{2},k-\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})\right.\\
&\qquad\left.-\hbar_k^z(u_{i-\frac{1}{2},j+\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k})\right\}\\
&+\sum_{k=0}^{N_z-1}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} h_k^z v_{i,j,k+\frac{1}{2}}\left\{\hbar_j^y(u_{i-\frac{1}{2},j,k+\frac{1}{2}}-u_{i-\frac{1}{2},j,k+\frac{1}{2}})\right.\\
&\qquad\left.-\hbar_i^x(u_{i,j-\frac{1}{2},k+\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})\right\}.
\end{aligned}
$$

Remark: We may rewrite (41) compactly using matrix notation as

$$\vec{u}^T\,[C_{HE}\,\vec{v}]=\vec{v}^T\,[C_{EH}\,\vec{u}]$$

which implies that $C_{EH}=C_{HE}^T$.

Proof. We use repeatedly the following summation by parts formula. Let the sequence $\{s_i\}_{i=0}^N$ be such that $s_0=s_N=0$ and let $\left\{t_{i+\frac{1}{2}}\right\}_{i=0}^{N-1}$ be another sequence then

$$\sum_{i=0}^{N-1} t_{i+\frac{1}{2}}(s_{i+1}-s_i)=-\sum_{i=1}^{N-1} s_i(t_{i+\frac{1}{2}}-t_{i-\frac{1}{2}}). \tag{42}$$

As noted in the remark following the lemma, the left hand side of (41) is just $\vec{u}^T\,C_{HE}\,\vec{v}$. Applying (42) to each term we obtain

$$\vec{u}^T\,C_{HE}\,\vec{v}\ =\ -\sum_{k=0}^{N_z-1}\sum_{i=0}^{N_x-1}\hbar_i^x\sum_{j=1}^{N_y-1}h_k^z v_{i,j,k+\frac{1}{2}}(u_{i,j-\frac{1}{2},k+\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})$$

$$
\begin{aligned}
&+\sum_{j=0}^{N_y-1}\sum_{i=0}^{N_x}\hbar_i^x\sum_{k=1}^{N_z-1} h_j^y v_{i,j+\frac{1}{2},k}(u_{i,j+\frac{1}{2},k-\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})\\
&+\sum_{k=0}^{N_z-1}\sum_{j=0}^{N_y}\hbar_j^y\sum_{i=0}^{N_x-1} h_k^z v_{i,j,k+\frac{1}{2}}(u_{i-\frac{1}{2},j,k+\frac{1}{2}}-u_{i+\frac{1}{2},j,k+\frac{1}{2}})\\
&-\sum_{j=0}^{N_y}\sum_{i=0}^{N_x-1}\hbar_j^y\sum_{k=1}^{N_z-1} h_i^x v_{i+\frac{1}{2},j,k}(u_{i+\frac{1}{2},j,k-\frac{1}{2}}-u_{i+\frac{1}{2},j,k+\frac{1}{2}})\\
&-\sum_{k=0}^{N_z}\sum_{j=0}^{N_y-1}\hbar_k^z\sum_{i=1}^{N_x-1} h_j^y v_{i,j+\frac{1}{2},k}(u_{i-\frac{1}{2},j+\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k})\\
&+\sum_{k=0}^{N_z}\sum_{i=0}^{N_x-1}\hbar_k^z\sum_{j=1}^{N_y-1} h_i^x v_{i+\frac{1}{2},j,k}(u_{i+\frac{1}{2},j-\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k}).
\end{aligned}
$$

Regrouping the terms we obtain

$$
\begin{aligned}
\vec{u}^T C_{HE}\,\vec{v} \;=\; &\sum_{i=0}^{N_x-1} h_i^x\Big\{\sum_{j=1}^{N_y-1}\sum_{k=0}^{N_z}\hbar_k^z v_{i+\frac{1}{2},j,k}(u_{i+\frac{1}{2},j-\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k})\\
&\quad+\sum_{j=0}^{N_y}\sum_{k=1}^{N_z-1}\hbar_j^y v_{i+\frac{1}{2},j,k}(u_{i+\frac{1}{2},j,k-\frac{1}{2}}-u_{i+\frac{1}{2},j,k+\frac{1}{2}})\Big\}\\
&+\sum_{j=0}^{n_y-1} h_j^y\Big\{\sum_{i=0}^{N_x}\sum_{k=1}^{N_z-1}\hbar_i^x v_{i,j+\frac{1}{2},k}(u_{i,j+\frac{1}{2},k-\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})\\
&\quad-\sum_{k=0}^{N_z}\sum_{i=1}^{N_x-1}\hbar_k^z v_{i,j+\frac{1}{2},k}(u_{i-\frac{1}{2},j+\frac{1}{2},k}-u_{i+\frac{1}{2},j+\frac{1}{2},k})\Big\}\\
&+\sum_{k=0}^{N_z-1} h_k^z\Big\{\sum_{i=1}^{N_x-1}\sum_{j=0}^{N_y}\hbar_j^y v_{i,j,k+\frac{1}{2}}(u_{i-\frac{1}{2},j,k+\frac{1}{2}}-u_{i+\frac{1}{2},j,k+\frac{1}{2}})\\
&\quad-\sum_{i=0}^{N_x-1}\sum_{j=1}^{N_y-1}\hbar_i^x v_{i,j,k+\frac{1}{2}}(u_{i,j-\frac{1}{2},k+\frac{1}{2}}-u_{i,j+\frac{1}{2},k+\frac{1}{2}})\Big\}.
\end{aligned}
$$

Now we use the boundary data (14) (for example $v_{i+\frac{1}{2},j,0}=0$) to modify the limits of summation to obtain the right hand side of (41). □

Next we state and prove a stability result for (40).

LEMMA 3.8. *Suppose $\vec{e}^{\,E}$ and $\vec{e}^{\,H}$ satisfy (40) and that $\vec{e}^{\,E}(0)=0$ and $\vec{e}^{\,H}(0)=0$ (i.e. (5) and (6) are satisfied exactly at $t=0$). Suppose in addition that $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are continuous in time and that $\vec{\beta}$ is continuously differentiable in time then*

$$
\begin{aligned}
\|\vec{e}^{\,E}\|_E+\|\vec{e}^{\,H}\|_H \le 3\Big(&2\max_{0\le s\le t}\|\vec{\beta}(s)\|_H\\
&+\int_0^t \|\vec{\alpha}(s)\|_H+\|\vec{\beta}_t(s)\|_H+\|\vec{\gamma}(s)\|_E\,ds\Big).
\end{aligned}
$$

Proof. If we multiply (40a) by $(\vec{e}^{\,H})^T$ and (40b) by $(\vec{e}^{\,E})^T$ then add the equations and use (37) and (39) we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\| \vec{e}^{\,E} \|_E^2 + \| \vec{e}^{\,H} \|_H^2\right\} = (C\,\vec{e}^{\,E})^T\,\vec{\beta} + \left(\vec{e}^{\,E}\right)^T M_{EE}\,\vec{\gamma} + \left(\vec{e}^{\,H}\right)^T M_{HH}\,\vec{\alpha}\,.$$

But by (40b) $C\,\vec{e}^{\,E} = M_{HH}\,\vec{\alpha} - M_{HH}\,\vec{e}^{\,H}_t$, so

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\| \vec{e}^{\,E} \|_E^2 + \| \vec{e}^{\,H} \|_H^2\right\}\\ &\quad = \left(\vec{\alpha}\right)^T M_{HH}\,\vec{\beta} - \left(\vec{e}^{\,H}_t\right)^T M_{HH}\,\vec{\beta} + \left(\vec{e}^{\,E}\right)^T M_{EE}\,\vec{\gamma} + \left(\vec{e}^{\,H}\right)^T M_{HH}\,\vec{\alpha}\,.\end{aligned}$$

If we integrate this expression from $t = 0$ to $t = t_1$ and use the fact that $\vec{e}^{\,E}(0) = 0$ and $\vec{e}^{\,H}(0) = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\left\{\| \vec{e}^{\,E}(t_1)\|_E^2 + \| \vec{e}^{\,H}(t_1)\|_H^2\right\} &= \int_0^{t_1}\left\{\vec{\alpha}^T M_{HH}\,\vec{\beta} + \left(\vec{e}^{\,E}\right)^T M_{EE}\,\vec{\gamma}\right.\\ &\qquad \left. + \left(\vec{e}^{\,H}\right)^T M_{HH}\,\vec{\alpha} - \left(\vec{e}^{\,H}_t\right)^T M_{HH}\,\vec{\beta}\right\} ds.\end{aligned}$$

Integrating $\left(\vec{e}^{\,H}_t\right)^T M_{HH}\,\vec{\beta}$ by parts we arrive at

$$\begin{aligned}&\frac{1}{2}\left\{\| \vec{e}^{\,E}(t_1)\|_E^2 + \| \vec{e}^{\,H}(t_1)\|_H^2\right\}\\ &= -\left(\vec{e}^{\,H}(t_1)\right)^T M_{HH}\,\vec{\beta}(t_1) + \int_0^{t_1}\left\{\vec{\alpha}^T M_{HH}\,\vec{\beta} + \left(\vec{e}^{\,E}\right)^T M_{EE}\,\vec{\gamma}\right.\\ &\qquad \left. + \left(\vec{e}^{\,H}\right)^T M_{HH}\left(\vec{\alpha} + \vec{\beta}_t\right)\right\} ds.\end{aligned}$$

Now using the Cauchy-Schwarz inequality it is apparent that

$$\begin{aligned}&\frac{1}{2}\left\{\| \vec{e}^{\,E}(t_1)\|_E^2 + \| \vec{e}^{\,H}(t_1)\|_H^2\right\}\\ &\leq \| \vec{e}^{\,H}(t_1)\|_H \| \vec{\beta}(t_1)\|_H + \int_0^{t_1} \| \vec{\alpha} \|_H \|\beta\|_H + \| \vec{e}^{\,E} \|_E \| \vec{\gamma} \|_E\end{aligned}$$

$$+\| \vec{e}^{\,H} \|_H \left(\| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H\right) ds. \tag{43}$$

Suppose that t^* is chosen so that

$$\max_{0\leq s\leq t} \| \vec{e}^{\,E}(s)\|_E + \| \vec{e}^{\,H}(s)\|_H = \| \vec{e}^{\,E}(t^*)\|_E + \| \vec{e}^{\,H}(t^*)\|_H. \tag{44}$$

Then, using (43) with $t_1 = t^*$, the arithmetic-geometric mean inequality, obvious estimates for product terms and (44), we have

$$\frac{1}{3}\left(\| \vec{e}^{\,E}(t^*)\|_E + \| \vec{e}^{\,H}(t^*)\|_H\right)^2 \leq \frac{1}{2}\left(\| \vec{e}^{\,E}(t^*)\|_E^2 + \| \vec{e}^{\,H}(t^*)\|_H^2\right)$$

$$\leq \int_0^{t^*} \| \vec{\alpha} \|_H \| \vec{\beta} \|_H \, ds + \left(\| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H \right) \left(\| \vec{\beta} (t^*) \|_H \right.$$
$$\left. + \int_0^{t^*} \| \vec{\gamma} \|_E + \| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H \, ds \right).$$

But by another application of the arithmetic-geometric mean inequality

$$\frac{1}{3} \left(\| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H \right)^2 \leq \int_0^{t^*} \| \vec{\alpha} \|_H \| \beta \|_H \, ds$$
$$+ \frac{1}{6} \left(\| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H \right)^2$$
$$+ \frac{3}{2} \left(\| \vec{\beta} (t^*) \|_H + \int_0^{t^*} \| \vec{\gamma} \|_E + \| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H \, ds \right)^2.$$

Rearranging this equation and replacing t^* by t on the right-hand side we obtain

$$\frac{1}{6} \left(\| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H \right)^2 \leq$$
$$\frac{3}{2} \left\{ \max_{0 \leq s \leq t} \| \vec{\beta} (s) \|_H \int_0^t \| \vec{\alpha} \|_H \, ds \right.$$
$$\left. + \left(\max_{0 \leq s \leq t} \| \vec{\beta} (s) \|_H + \int_0^t \| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H + \| \vec{\gamma} \|_E \, ds \right)^2 \right\}.$$

In obtaining the above estimate we have also used a crude bound on the overall constant. Taking square roots we obtain

$$\| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H$$
$$\leq 3 \left(2 \max_{0 \leq s \leq t} \| \vec{\beta} (s) \|_H + \int_0^t \| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H + \| \vec{\gamma} \|_E \, ds \right).$$

But $\| \vec{e}^{\,E} (t) \|_E + \| \vec{e}^{\,H} (t) \|_H \leq \| \vec{e}^{\,E} (t^*) \|_E + \| \vec{e}^{\,H} (t^*) \|_H$ so we have proved the desired inequality. □

Proof. [of Theorem 3.1]. By Lemma 3.8,

$$\begin{aligned} & \| \boldsymbol{E} - \boldsymbol{E}^h \|_E + \| \boldsymbol{H} - \boldsymbol{H}^h \|_H \\ \leq \;& 3 \Big(2 \max_{0 \leq s \leq t} \| \vec{\beta} (s) \|_H + \int_0^t \| \vec{\alpha} \|_H + \| \vec{\beta}_t \|_H + \| \vec{\gamma} \|_E \, ds \Big). \end{aligned} \tag{45}$$

But by (27) and (28) of Lemma 3.3 and the definition of $\| \cdot \|_H$,

$$\begin{aligned} \| \vec{\beta} (s) \|_H &\leq \sqrt{3 \operatorname{meas} (\Omega) \frac{h^2 M_2^H (s)}{4}}, \\ \| \vec{\beta}_t (s) \|_H &\leq \sqrt{3 \operatorname{meas} (\Omega) \frac{h^2 M_{2t}^H (s)}{4}}, \end{aligned}$$

where meas(Ω) is the volume of Ω.

By Lemmas 3.5 and 3.6 and the definition of $\|\cdot\|_E$ imply that

$$\|\vec{\gamma}(s)\|_E \leq \sqrt{3\,\mathrm{meas}\,(\Omega)}\frac{h^2 M_3^E(s)}{12}.$$

Finally by Lemmas 3.5 and 3.6

$$\|\vec{\alpha}(s)\|_H \leq \sqrt{3\,\mathrm{meas}\,(\Omega)}\frac{h^2 M_3^E(s)}{12}.$$

In these expressions we have explicitly shown that the bounds M_3^E, M_3^H, M_2^H and M_{2t}^H depend on time. Let $M(T) = \max_{0\leq s\leq T}\left\{M_2^H(s), M_{2t}^H(s), M_3^E(s), M_3^H(s)\right\}$ then (45) can be written

$$\begin{aligned}\|\boldsymbol{E}-\boldsymbol{E}^h\|_E + \|\boldsymbol{H}-\boldsymbol{H}^h\|_H &\leq h^2\sqrt{3\,\mathrm{meas}\,(\Omega)}\left\{\frac{5}{4}M + t\frac{5}{12}M\right\}\\ &\leq h^2\sqrt{3\,\mathrm{meas}\,(\Omega)}\left\{\frac{5}{4}+\frac{5}{12}T\right\}M\end{aligned}$$

as claimed in the theorem. □

Remarks:

1. The theorem is proved for the case when Ω is a rectangular parallelepiped. However the geometry of Ω only enters into the proof of Lemma 3.7. Moreover Lemma 3.7 holds in much greater generality than we have stated (for example it holds for regions made up of the union of finitely many rectangular parallelepipeds). Hence Theorem 3.1 holds on such regions. We only present the proof on a simple rectangular parallelepiped to simplify notation and arguments.
2. The smoothness restrictions on $\boldsymbol{E}$ and $\boldsymbol{H}$ might be reduced (to Sobolev space bounds) if the right-hand side of (11), (12) and (13) is replaced by a suitable integrated current density. For example in (11) we could replace $\hbar_j^y \hbar_k^z J_{i+\frac{1}{2},j,k}$ by

$$\frac{1}{h_i^x}\int_{x_{i-1}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} J_1(\boldsymbol{x},t)\,dV.$$

   Then the Bramble–Hilbert Lemma rather than the Taylor series remainder might be used to bound the error terms (see [8] for this type of argument in other contexts).
3. Our result remains true for the more general problem approximating $\boldsymbol{E}$ and $\boldsymbol{H}$ which satisfy

$$\begin{aligned}\epsilon\boldsymbol{E}_t + \sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} &= \boldsymbol{J} \quad \text{in} \quad \Omega\\ \mu\boldsymbol{H}_t + \nabla\times\boldsymbol{E} &= 0 \quad \text{in} \quad \Omega\end{aligned}$$

   provided ϵ and μ are strictly positive continuous functions on $\bar{\Omega}$ and σ is a non-negative continuous function on $\bar{\Omega}$ (and the required smoothness on $\boldsymbol{E}$ and $\boldsymbol{H}$ is present). For discontinuous ϵ or μ a new formulation of the discrete problem is needed (cf. [9]).

4. Finally, we note that the proof of second-order accuracy in Theorem 3.1 did not require any assumptions on the mesh. In particular the mesh does not have to be quasi-uniform. Thus the accuracy of the Yee scheme is insensitive to mesh stretching and compressing in the coordinate directions.

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