# A Convergence Proof for Adaptive Finite Elements without Lower Bound

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## Outline





## Motivation

Most convergence results for adaptive finite elements rely on

- Energy minimization
  - symmetric elliptic operators
  - $p$-Laplacian
  - obstacle problems
  - convex minimization

  Can be relaxed to disturbed Galerkin Orthogonality.

- Special properties of the estimators
  - Discrete local lower bound

- Dörfler marking: Given $\theta \in (0, 1]$

$$\text{Select } \mathcal{M} \subset \mathcal{T}: \qquad \theta \mathcal{E}_{\mathcal{T}}(\mathcal{T}) \le \mathcal{E}_{\mathcal{T}}(\mathcal{M})$$

- Special refinement of selected elements.

Optimality up to now only for symmetric elliptic operators.



## Motivation

Convergence and optimality of adaptive finite elements is observed for

- A larger class of problems
  - convection-diffusion,
  - saddle point problems,
  - ...

- Efficient estimators, where only a continuous lower bound is available.

- Other marking strategies
  - Maximum strategy
  - Equidistribution Strategy
  - ...

- Minimal refinement.

Convergence in a rather general setting by Morin, S., Veeser '08.

Optimality in this general setting completely open.

## Motivation

### Setting of the Basic Convergence Result

Formulation of only few and basic assumptions that lead to convergence. These assumptions should be "necessary" – at least reasonable – and "easy to verify" for many problems.

### Main Focus in this Talk: Discrete Lower Bound

Previous convergence proofs rely on a discrete local lower bound:

1. Discrete lower bounds may be more difficult to obtain than continuous ones;
2. For more complex problems estimators may not be efficient, but still we may want to prove convergence.

Reliability of an estimator should be the key property for convergence. Overestimation should not forestall convergence:

1. Overestimation is a problem for efficiently stopping;
2. Overestimation is a problem for optimal complexity.



## Problem

### Problem

*Variational formulation of a linear, elliptic PDE in a domain $\Omega \subset \mathbb{R}^d$:*

$$u \in \mathbb{V}: \qquad \mathcal{B}[u,\, v] = \langle f,\, v\rangle \qquad \forall v \in \mathbb{V}, \tag{P}$$

*where*

- *$\mathbb{V}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{V}}$, induced norm $\|\cdot\|_{\mathbb{V}}$;*
- *$\mathcal{B}\colon \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ is a continuous bilinear form;*
- *$f \in \mathbb{V}^*$.*

### Theorem (Niremberg, Nečas, Babuška, Brezzi)

*Problem (P) admits for any $f \in \mathbb{V}^*$ a unique solution, if and only if $\mathcal{B}$ fulfills an inf-sup condition.*

- Coercive forms $\mathcal{B}$ satisfy the inf-sup condition:

$$\mathcal{B}[v,\, v] \ge c_{\mathcal{B}} \|v\|_{\mathbb{V}}^2 \qquad \forall v \in \mathbb{V}.$$



## Examples

### Example (Poisson Problem in $\mathbb{R}^d$)

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \partial\Omega.$$

Variational formulation in $\mathbb{V} = H_0^1(\Omega)$:

$$\mathcal{B}[u,\, v] = \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f\, v\, dx.$$

- $\mathcal{B}$ is continuous and coercive.
- Discretization with continuous Lagrange elements of order $p \ge 1$.
- Global upper bound for the residual estimator build from

$$\mathcal{E}_{\mathcal{T}}^2(T) := h_T^2 \| -\Delta U_{\mathcal{T}} - f\|_{2;T}^2 + h_T \| \,[\![ U_{\mathcal{T}} ]\!]\, \|_{2;\partial T\cap\Omega}^2.$$

- Continuous and discrete local lower bounds.



## Examples

### Continous Local Lower Bound

$$\mathcal{E}_{\mathcal{T}}(T) \lesssim \|U_{\mathcal{T}} - u\|_{\mathbb{V}(\omega(T))} + \operatorname{osc}_{\mathcal{T}}(\omega(T))$$

with $\operatorname{osc}_{\mathcal{T}}(T) = h_T \|f - f_{\mathcal{T}}\|_{2;T}$.

Principal idea by Verfürth: Construct $\phi_T \in \mathbb{V}$ with $\|\phi_T\|_{\mathbb{V}} = 1$, $\operatorname{supp} \phi_T \subset \omega(T)$ such that

$$\mathcal{E}_{\mathcal{T}}(T) \lesssim \langle \mathcal{R}(U_{\mathcal{T}}),\, \phi_T\rangle := \mathcal{B}[U_{\mathcal{T}} - u,\, \phi_T] \le \|\mathcal{B}\| \|U_{\mathcal{T}} - u\|_{\mathbb{V}(\omega(T))}$$

### Construction of $\phi_T$

1. Changing to a computable error indicator leads to potential overestimation.
   - Projection to a finite dimensional space; leads to oscillation.
2. Localization by a suitable continuous cut-off function $\lambda_T$.

## Examples

**Discrete Local Lower Bound**

Let $\mathcal{T}'$ be a refinement of $\mathcal{T}$ with sufficient refinement around $T \in \mathcal{T}$

$$\mathcal{E}_{\mathcal{T}}(T) \lesssim \|U_{\mathcal{T}} - U_{\mathcal{T}'}\|_{\mathbb{V}(\omega(T))} + \operatorname{osc}_{\mathcal{T}}(\omega(T))$$

with $\operatorname{osc}_{\mathcal{T}}(T) = h_T \|f - f_{\mathcal{T}}\|_{2;T}$.

Principal idea by Dörfler and Morin, Nochetto, S.: Construct $\Phi_T \in \mathbb{V}(\mathcal{T}')$ with $\|\Phi_T\|_{\mathbb{V}} = 1$, $\operatorname{supp}\Phi_T \subset \omega(T)$ such that

$$\mathcal{E}_{\mathcal{T}}(T) \lesssim \langle \mathcal{R}(U_{\mathcal{T}}),\, \Phi_T \rangle = \mathcal{B}[U_{\mathcal{T}} - U_{\mathcal{T}'},\, \Phi_T] \le \|\mathcal{B}\| \|U_{\mathcal{T}} - U_{\mathcal{T}'}\|_{\mathbb{V}(\omega(T))}$$

**Construction of $\Phi_T$**

1. Projection to a finite dimensional space; leads to oscillation.
2. Localization by a suitable discrete cut-off function $\Lambda_T$.

- Projection is limited by the degree of the FE space and the discrete cut-off function.
- Utilizing a discrete cut-off function is not always possible: A localized function has to be contructed explicetely.



## Examples

**Example ($H(\operatorname{div};\Omega)$ Elliptic Operator in $\mathbb{R}^d$, $d = 2, 3$)**

$$-\nabla \operatorname{div} \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

Variational formulation in $\mathbb{V} = H_0(\operatorname{div};\Omega)$:

$$\mathcal{B}[\boldsymbol{u},\, \boldsymbol{v}] := \int_\Omega \operatorname{div}\boldsymbol{u} \operatorname{div}\boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}\, dx = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \mathbb{V}.$$

- $\mathcal{B}$ is continuous and coercive;
- Discretization by Raviart-Thomas or Brezzi-Douglas-Marini Elements of any order $p$;
- Global upper bound for any order;
- Continuous and discrete local lower bound for any order:
  - the projection in the discrete lower bound for Raviart-Thomas Elements of order $p \ge 2$ is sub-optimal.



## Examples

**Example (Eddy Current Equations in $\mathbb{R}^3$)**

$$\operatorname{curl}\operatorname{curl}\boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} \wedge \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

Variational formulation in $\mathbb{V} = H_0(\operatorname{curl};\Omega)$:

$$\mathcal{B}[\boldsymbol{u},\, \boldsymbol{v}] := \int_\Omega \operatorname{curl}\boldsymbol{u} \operatorname{curl}\boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}\, dx = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle \qquad \forall \boldsymbol{v} \in \mathbb{V}.$$

- $\mathcal{B}$ is continuous and coercive;
- Discretization by Nedelec Elements of any order $p$;
- Global upper bound for any order;
- Continuous local lower bound for any order:
- Discrete local lower bound available only for lowest order, i. e., for the Whitney Elements.



## Examples

**Example (The Stokes Problem)**

Variational formulation in $\mathbb{V} = H_0^1(\Omega;\mathbb{R}^d) \times L_0^2(\Omega)$:

$$\mathcal{B}[(\boldsymbol{u}, p),\, (\boldsymbol{v}, q)] := \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, dx - \int_\Omega p\, \nabla \cdot \boldsymbol{v}\, dx - \int_\Omega \nabla \cdot \boldsymbol{u}\, q\, dx = \langle \boldsymbol{f},\, \boldsymbol{v} \rangle$$

for all $(\boldsymbol{v}, q) \in \mathbb{V}$.

- $\mathcal{B}$ is continuous and fulfills the inf-sup condition.
- Discretization by the Taylor-Hood Elements of order $p \ge 2$.
- Global upper bound for

  $$\mathcal{E}^2_{\mathcal{T}}(T) := h_T^2 \|{-\Delta \boldsymbol{U}_{\mathcal{T}}} + \nabla P_{\mathcal{T}} - \boldsymbol{f}\|^2_{2;T} + h_T \| [\![ \boldsymbol{U}_{\mathcal{T}} ]\!] \|^2_{2;\partial T \cap \Omega} + \| \operatorname{div} \boldsymbol{U}_{\mathcal{T}} \|^2_{2;T}$$

  and

  $$\mathcal{E}^2_{\mathcal{T}}(T) := h_T^2 \| - \Delta \boldsymbol{U}_{\mathcal{T}} + \nabla P_{\mathcal{T}} - \boldsymbol{f} \|^2_{2;T} + h_T \| [\![ \boldsymbol{U}_{\mathcal{T}} ]\!] \|^2_{2;\partial T \cap \Omega}.$$
- Continuous local lower bound for both variants.
- Discrete local lower bound available only for the second variant.

## Examples



**Example (The Biharmonic Equation in $\mathbb{R}^2$)**

$$\Delta^2 u \quad \text{in } \Omega, \qquad u = \nabla u \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

Variational formulation in $\mathbb{V} = H_0^2(\Omega)$:

$$\mathcal{B}[u,\,v] := \int_\Omega \Delta u \Delta v \, dx = \langle f,\,v\rangle \qquad \forall v \in \mathbb{V}.$$

- $\mathcal{B}$ is continuous and coercive.
- Discretization by the Argyris Triangle: piecewise $\mathbb{P}_5$ and $H^2$ conforming.
- Global upper bound.
- Continuous local lower bound.
- No discrete local lower bound available, seems to be tough.

## Adaptive Loop and Basic Assumptions



Starting with an initial, conforming triangulation $\mathcal{T}_0$ of $\Omega$, the standard adaptive loop SEMR

SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE

produces a sequence

$$\big\{\mathcal{T}_k,\ \mathbb{V}_k,\ U_k,\ \{\mathcal{E}_k(T)\}_{T\in\mathcal{T}_k},\ \mathcal{M}_k\big\}_k,$$

where

- $\mathcal{T}_k$ is a conforming triangulation produced by refinement of $\mathcal{T}_{k-1},\ldots,\mathcal{T}_0$;
- $\mathbb{V}_k = \mathbb{V}(\mathcal{T}_k)$ is a finite element space over $\mathcal{T}_k$;
- $U_k \in \mathbb{V}_k$ is the unique Ritz-Galerkin solution:

  $$U_k \in \mathbb{V}_k: \qquad \mathcal{B}[U_k,\,V] = \langle f,\,V\rangle \qquad \forall V \in \mathbb{V}_k, \qquad (\mathsf{P}_k)$$

  which requires a discrete inf-sup conditon;
- $\mathcal{E}_k(T)$ is an error indicator assosiated with an element $T \in \mathcal{T}_k$;
- $\mathcal{M}_k \subset \mathcal{T}_k$ is the set of selected elements for refinement.

## Adaptive Loop and Basic Assumptions



**Assumptions on Refinement**

Use bisectional refinement and denote by $\mathbb{T}$ the set of all possible, conforming refinements of $\mathcal{T}_0$.

- Refinement can be generalized to more general grids and quasi-regular element subdivisions that generate locally quasi-uniform grids.

**Assumptions on Finite Element Spaces**

The finite element spaces have the following properties:

1. for any $\mathcal{T} \in \mathbb{T}$, $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}$ is a conforming finite dimensional space;
2. the spaces are nested: if $\mathcal{T}'$ is a refinement of $\mathcal{T}$ then $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}(\mathcal{T}')$;
3. the spaces satisfy a uniform discrete inf-sup condition.

- Nesting of spaces follows from properties of refinement in combination with appropriate local function spaces.
- Coercivity of $\mathcal{B}$ implies the uniform inf-sup condition.

## Convergence of Mesh Size Functions



Define the local mesh size function $h_k \in L^\infty(\Omega)$ by

$$h_{k|T} := |T|^{1/d} \approx \operatorname{diam}(T) \qquad \forall T \in \mathcal{T}_k.$$

**Lemma (Morin, S. Veeser '08)**

*For any realization of SEMR there exists a unique $h_\infty \in L^\infty(\Omega)$ such that*

$$\lim_{k\to\infty} \|h_k - h_\infty\|_{\infty;\Omega} = 0.$$

**Idea of the Proof.**

For any $x \in \Omega$ the sequence $\{h_k(x)\}_k$ is monotone and bounded from below:

$$h_\infty(x) := \lim_{k\to\infty} h_k(x) \geq 0 \qquad \text{exists for all } x \in \Omega.$$

Convergence in $L^\infty$ the follows from

$$T \text{ is refined into } T_1,\,T_2 \quad \Longrightarrow \quad |T_1| = |T_2| = \frac{1}{2}\,|T|\,. \qquad \square$$

## Convergence of Mesh Size Functions

In general, $h_\infty \not\equiv 0$ in $\Omega$. If $h_\infty(x) > 0$, then there is an element $T \ni x$ and $K = K(x)$ such that

$$T \in \mathcal{T}_k \qquad \forall k > K.$$

**Splitting of $\mathcal{T}_k$**

1. Set of elements that are not refined anymore

$$\mathcal{T}_k^+ := \{T \in \mathcal{T}_k \mid T \in \mathcal{T}_\ell \ \forall \ell \geq k\};$$

2. Set of elements that are refined at least once

$$\mathcal{T}_k^0 := \mathcal{T}_k \setminus \mathcal{T}_k^+.$$

**Corollary (Morin, S. Veeser '08)**

*The mesh size functions vanish uniformly in $\Omega_k^0 = \Omega(\mathcal{T}_k^0) := \bigcup\{T \colon T \in \mathcal{T}_k^0\}$:*

$$\lim_{k\to\infty} \|h_k\|_{\infty;\Omega_k^0} = 0.$$



## Convergence of Galerkin Solutions

**Lemma (Morin, S. Veeser '08)**

*For any realization of SEMR there exists a unique $u_\infty \in \mathbb{V}$ such that*

$$\lim_{k\to\infty} \|U_k - u_\infty\|_{\mathbb{V}} = 0.$$

**Proof for coercive $\mathcal{B}$.**

The space

$$\mathbb{V}_\infty = \overline{\bigcup_k \mathbb{V}_k}^{\|\cdot\|_{\mathbb{V}}}$$

is a closed subspace of $\mathbb{V}$. The Lax-Milgram theorem then implies the existence of a unique solution $u_\infty$ to

$$u_\infty \in \mathbb{V}_\infty : \qquad \mathcal{B}[u_\infty, v] = \langle f, v\rangle \qquad \forall v \in \mathbb{V}_\infty.$$

Convergence follows from the quasi-best approximation property

$$\|U_k - u_\infty\|_{\mathbb{V}} \leq c_{\mathcal{B}}^{-1}\|\mathcal{B}\| \min_{V\in\mathbb{V}_k} \|V - u_\infty\|_{\mathbb{V}} \to 0 \quad \text{as } k \to \infty$$

by construction of $\mathbb{V}_\infty$. □



## Convergence of Galerkin Solutions

**Consequences for a Convergence Proof**

It suffices to show $u_\infty = u$ since convergence

$$\lim_{k\to\infty} U_k \to u_\infty \qquad \text{in } \mathbb{V}$$

is established for any adaptive iteration SEMR.

**The residual $\mathcal{R}(u_\infty)$ and $u_\infty = u$**

Using the residual $\mathcal{R}(w) \in \mathbb{V}^*$ defined by

$$\mathcal{R}(w) := \mathcal{B}[w - u, v] = \mathcal{B}[w, v] - \langle f, v\rangle \qquad \forall v, w \in \mathbb{V}.$$

we reformulate

$$u_\infty = u \qquad \Longleftrightarrow \qquad \mathcal{R}(u_\infty) = 0 \quad \text{in } \mathbb{V}^*$$

1. In case $\mathbb{V}_\infty = \mathbb{V}$ definition of $u_\infty$ implies $\mathcal{R}(u_\infty) = 0$.
2. In case $\mathbb{V}_\infty \neq \mathbb{V}$ properties of ESTIMATE and MARK have to yield $\mathcal{R}(u_\infty) = 0$.



## Density

**Local Approximation Property of the Finite Element Spaces**

Let $\mathbb{W} \subset \mathbb{V}$ be dense, $q > 0$. Assume that for any $\mathcal{T} \in \mathbb{T}$ there exists an interpolation operator $I_{\mathcal{T}} \colon \mathbb{W} \to \mathbb{V}(\mathcal{T})$ such that for all $w \in \mathbb{W}$

$$\|w - I_{\mathcal{T}} w\|_{\mathbb{V}(T)} \lesssim \|h_{\mathcal{T}}^q\|_{\infty;T} \|w\|_{\mathbb{W}(T)} \qquad \forall T \in \mathcal{T}.$$

**Claim**

$$\mathbb{V}_\infty = \mathbb{V} \qquad \Longleftrightarrow \qquad h_\infty \equiv 0 \quad \text{in } \Omega$$

1. $h_\infty \neq 0$: Then $\mathcal{T}_k^+ \neq$ for $k \geq K$ which implies $\mathbb{V} \not\subset \mathbb{V}_\infty$.
2. $h_\infty \equiv 0$: Use density of finite element spaces: for $v \in \mathbb{V}$ and $w \in \mathbb{W}$ estimate

$$\begin{aligned} \|v - I_k w\|_{\mathbb{V}(\Omega)} &\leq \|v - w\|_{\mathbb{V}(\Omega)} + \|w - I_k w\|_{\mathbb{V}(\Omega)} \\ &\lesssim \|v - w\|_{\mathbb{V}(\Omega)} + \|h_k\|_{\infty;\Omega} \|w\|_{\mathbb{W}(\Omega)} \overset{!}{\leq} \varepsilon \end{aligned}$$

by first choosing $w$ close to $v$ and then $k$ large.

## Density

For $h_\infty \not\equiv 0$ we still obtain for any $v \in \mathbb{V}$ and $w \in \mathbb{W}$

$$\|v - I_k w\|_{\mathbb{V}(\Omega_k^0)} \le \|v - w\|_{\mathbb{V}(\Omega_k^0)} + \|w - I_k w\|_{\mathbb{V}(\Omega_k^0)}$$
$$\lesssim \|v - w\|_{\mathbb{V}(\Omega)} + \|h_k\|_{\infty;\Omega_k^0} \|w\|_{\mathbb{W}(\Omega)} \overset{!}{\le} \varepsilon$$

by first choosing $w$ close to $v$ and then $k$ large, thanks to

$$\|h_k\|_{\infty;\Omega_k^0} \to 0 \qquad \text{as } k \to \infty.$$

### Remarks

1. This local density property we are going to use explicitely in the convergence proof. It replaces a (discrete) local lower bound.
   - Needs a way to build in local features via the upper bound!
2. The local density property is already implicitely used in all other convergence proofs.



## Prior Results

### Directly Related Convergence Results

- Babuška, Vogelius '86: $u'' = f$ in 1d, convergence
- Dörfler '96: Poisson problem in 2d, convergence into tolerance
- Morin, Nochetto, S. '00, '02: constant coefficient matrix, convergence
- Veeser '02: $p$-Laplacian
- S. Veeser '06: obstacle problem
- S. Veeser '06: convergence for the equidistribution strategy
- Morin, S. Veeser '08: general convergence with discrete lower bound

### Convergence and Optimality Results

- Binev, Dahmen, DeVore '02: MNS with coarsening
- Stevenson '06: Modification of Dörfler
- Cascon, Kreuzer, Nochetto, S. '08: Plain SEMR
- Chen, Holst, Xu '08: Mixed formulation of Poisson problem



## Prior Results

Convergence proof without lower bound for symmetric elliptic problems:

$$\mathcal{B}[v,\,w] := \int_\Omega \nabla v^T \boldsymbol{A} \nabla w + c\, v\, w\, dx \qquad v, w \in \mathbb{V} := H_0^1(\Omega)$$

with the residual estimator

$$\mathcal{E}_\mathcal{T}^2(T) := \|h_\mathcal{T}(-\operatorname{div}(\boldsymbol{A}\nabla U_\mathcal{T} + c\,U_\mathcal{T} - f)\|_{2;T}^2 + \|h_\mathcal{T}^{1/2} [\![\boldsymbol{A}\nabla U_\mathcal{T}]\!]\,\|_{2;\partial T}^2$$

and Dörfler marking with $0 < \theta \le 1$

$$\text{Choose } \mathcal{M} \subset \mathcal{T}: \qquad \theta \mathcal{E}_\mathcal{T}(\mathcal{T}) \le \mathcal{E}_\mathcal{T}(\mathcal{M}).$$

### Theorem (Cascon, Kreuzer, Nochetto, S. '08)

*SEMR is a contraction, i. e., there exists $0 < \alpha < 1$ and $\beta > 0$ such that*

$$\|U_k - u\|_\Omega^2 + \beta \mathcal{E}_k(\mathcal{T}_k) \le \alpha\big(\|U_{k-1} - u\|_\Omega^2 + \beta \mathcal{E}_{k-1}(\mathcal{T}_{k-1})\big).$$

*If, in addition, $\theta$ is sufficiently small and $\mathcal{M}_k$ minimal, then SEMR is quasi-optimal in terms of DOFs.*

- Optimality proof utilizes the global continuous lower bound.



## Error Estimation and Marking

### Assumptions on the Estimator

1. We assume an upper bound with the following build-in localization: For any subset $\mathcal{S} \subset \mathcal{T}$ holds:
$$|\langle \mathcal{R}(U_\mathcal{T}),\, v\rangle| \lesssim \mathcal{E}_\mathcal{T}(\mathcal{S})\|v\|_{\Omega(\mathcal{S})} + \mathcal{E}_\mathcal{T}(\mathcal{T} \setminus \mathcal{S})\|v\|_{\Omega(\mathcal{T}\setminus\mathcal{S})} \qquad \forall v \in \mathbb{V}.$$
2. We assume stability of the indicators: there exists $D \in L^2(\Omega)$ such that
$$\mathcal{E}_\mathcal{T}(T) \lesssim \|U_\mathcal{T}\|_{\mathbb{V}(T)} + \|D\|_{2;T} \qquad \forall T \in \mathcal{T}$$

### Remarks

1. The continuous inf-sup condition and the upper bound for $\mathcal{S} = \mathcal{T}$ imply
$$\|U_\mathcal{T} - u\|_\mathbb{V} \lesssim \|\mathcal{R}(U_\mathcal{T})\|_{\mathbb{V}^*} = \sup_{\|v\|_\mathbb{V}=1} |\langle \mathcal{R}(U_\mathcal{T}),\, v\rangle| \lesssim \mathcal{E}_\mathcal{T}(\mathcal{T}).$$
2. Boundedness of $\{U_k\}_k$ and stability of the indicators yield
$$\sup_k \mathcal{E}_k(\mathcal{T}_k) \lesssim 1.$$

## Error Estimation and Marking



### Assumption on Marking

We assume the existence of $g \in C^0(\mathbb{R}_0^+; \mathbb{R}_0^+)$ with $g(0) = 0$ such that the set of marked elements $\mathcal{M}$ satisfies

$$\mathcal{E}_\mathcal{T}(T) \le g\big(\max\{\mathcal{E}_\mathcal{T}(T) \mid T \in \mathcal{M}\}\big) \qquad \forall T \in \mathcal{T} \setminus \mathcal{M}.$$

### Additional Assumption on Refinement

All marked elements are refined at least once.

### Remarks

1. The assumption on marking includes standard marking strategies like Maximum, Equidistribution and Minimal Dörfler marking with $g(s) = s$.
2. Assumption on refinement implies $\mathcal{M}_k \subset \mathcal{T}_k^0$.
3. Convergence of the Galerkin Solutions, stability of the indicators, and assumption on marking and refinement yield

$$\max\{\mathcal{E}_k(T) \mid T \in \mathcal{T}_k\} \to 0 \qquad \text{as } k \to \infty.$$

## Convergence of the Error

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### Theorem (S. '08)

*Assume that the above assumptions on refinement, finite element spaces, estimator, and marking are satisfied. Then SEMR convergece, i. e.,*

$$\lim_{k\to\infty} \|U_k - u\|_\mathbb{V} = 0.$$

### Proof

Since $U_k \to u_\infty$ in $\mathbb{V}$, it remains to show

$$\langle \mathcal{R}(u_\infty), v\rangle = 0 \quad \forall v \in \mathbb{V} \qquad \Longleftrightarrow \qquad \langle \mathcal{R}(u_\infty), w\rangle = 0 \quad \forall w \in \mathbb{W},$$

by density of $\mathbb{W}$ in $\mathbb{V}$. Using continuity of $\mathcal{R}\colon \mathbb{V} \to \mathbb{V}^*$ this reduces to

$$\lim_{k\to\infty} \langle \mathcal{R}(U_k), w\rangle = 0 \quad \forall w \in \mathbb{W},\ \|w\|_\mathbb{W} = 1.$$

The sets $\mathcal{T}_k^+$ are nested, which grants for $k \ge \ell$

$$\mathcal{T}_\ell^+ \subset \mathcal{T}_k^+ \subset \mathcal{T}_k \qquad \text{and} \qquad \Omega_\ell^0 = \Omega(\mathcal{T}_\ell^0) = \Omega(\mathcal{T}_k \setminus \mathcal{T}_\ell^+).$$

## Convergence of the Error

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### Proof (continued)

Use the upper bound with $\mathcal{S} = \mathcal{T}_k \setminus \mathcal{T}_\ell^+$ for $w \in \mathbb{W}$, $\|w\|_\mathbb{W} = 1$

$$\begin{aligned} |\langle \mathcal{R}(U_k), w\rangle| &= |\langle \mathcal{R}(U_k), w - I_k w\rangle| \\ &\lesssim \mathcal{E}_k(\mathcal{T}_k \setminus \mathcal{T}_\ell^+)\|w - I_k w\|_{\mathbb{V}(\Omega_\ell^0)} + \mathcal{E}_k(\mathcal{T}_\ell^+)\|w - I_k w\|_{\mathbb{V}(\Omega_\ell^+)} \\ &\lesssim \|h_k\|_{\infty;\Omega_\ell^0} + \mathcal{E}_k(\mathcal{T}_\ell^+) \overset{!}{\le} \varepsilon \end{aligned}$$

1. Choose $\ell$ sufficiently large such that

$$\|h_k\|_{\infty;\Omega_\ell^0} \le \|h_\ell\|_{\infty;\Omega_\ell^0} \le \frac{\varepsilon}{2}.$$

2. Then choose $k \ge \ell$ such that

$$\mathcal{E}_k(T) \le \frac{\varepsilon}{2}(\#\mathcal{T}_\ell^+)^{-1/2} \qquad \forall T \in \mathcal{T}_\ell^+,$$

which implies

$$\mathcal{E}_k(\mathcal{T}_\ell^+) \le \frac{\varepsilon}{2}.$$

□

## Convergence of the Estimator

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### Remark

The theorem does not imply convergence of the estimator, since it includes non-efficient estimators and allows for strong overestimation!

### Continuous Lower Bound

Let the indicators satisfy

$$\mathcal{E}_\mathcal{T}(T) \lesssim \|U_\mathcal{T} - u\|_{\mathbb{V}(\omega(T))} + \operatorname{osc}_\mathcal{T}(\omega(T)),$$

where oscillation can be estimated by

$$\operatorname{osc}_\mathcal{T}(T) \lesssim \|h_\mathcal{T}^r\|_{\infty;T}\big(\|U_\mathcal{T}\|_{\mathbb{V}(\omega(T))} + \|D\|_{2;\omega(T)}\big)$$

for some $r > 0$ and $D \in L^2(\Omega)$.

### Corollary (S. '08)

*If, in addition, the estimator satisfies the continuous local lower bound, then SEMR yields*

$$\lim_{k\to\infty} \mathcal{E}_k(\mathcal{T}_k) = 0.$$

## Convergence of the Estimator

**Proof**

As in the previous proof we split for $k \ge \ell$

$$\mathcal{E}_k(\mathcal{T}_k) \lesssim \mathcal{E}_k(\mathcal{T}_k \setminus \mathcal{T}_\ell^0) + \mathcal{E}_k(\mathcal{T}_\ell^+) \lesssim \|U_k - u\|_{\mathbb{V}(\Omega_\ell^0)} + \operatorname{osc}_k(\Omega_\ell^0) + \mathcal{E}_k(\mathcal{T}_\ell^+). \quad (*)$$

1. The error is controlled by the previous theorem:

$$\|U_k - u\|_{\mathbb{V}(\Omega_\ell^0)} \le \|U_k - u\|_{\mathbb{V}} \to 0 \qquad \text{as } k \to \infty.$$

2. Oscillation can be estimated in $\Omega_\ell^0$ by assumption in an a priori way:

$$\operatorname{osc}_k(\Omega_\ell^0) \lesssim \|h_\ell^r\|_{\infty;\Omega_\ell^0}\big(\|U_k\|_{\mathbb{V}} + \|D\|_{L^2(\Omega)}\big) \lesssim \|h_\ell^r\|_{\infty;\Omega_\ell^0} \to 0 \qquad \text{as } \ell \to \infty.$$

3. The remaining part of the estimator can be handeled as before:

$$\mathcal{E}_k(\mathcal{T}_\ell^+) \to 0 \qquad \text{for } \ell \text{ fixed and } k \to \infty.$$

Summarizing: The right hand side of (*) can be made arbitrarily small by first choosing $\ell$ large and then $k \ge \ell$ even larger. □

Convergence of AFEM
K. G. Siebert
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## Remarks

1. General convergence proof for adaptive finite elements with mild assumptions on the ingredients, most easy to verify.
2. Convergence does not need the lower bound, "practical" convergence and convergence into tolerance need efficient estimators:
   - Includes strategies, where the given tolerance enters the selection, like the equidistribution strategy:

$$\mathcal{M} = \{T \in \mathcal{T} \mid \mathcal{E}_{\mathcal{T}}(T) \ge \theta\, \mathsf{TOL}\, (\#\mathcal{T})^{-1/2}\},$$

3. For efficient estimators, the assumption on marking can be generalized such that it is essentially necessary:

$$\text{if } \lim_{k\to\infty} \max\{\mathcal{E}_k(T) \mid T \in \mathcal{M}_k\} = 0 \quad \text{then} \quad \forall T \in \mathcal{T}^+: \quad \lim_{k\to\infty} \mathcal{E}_k(T) = 0,$$

where

$$\mathcal{T}^+ = \bigcup_{k\ge 0} \bigcap_{\ell \ge k} \mathcal{T}_\ell$$

is the set of elements that are not refined.