J. Math. Soc. Japan
Vol. 26, No. 1, 1974

A convergence theorem of nonlinear semigroups and its application to first order quasilinear equations

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(Received Oct. 24, 1972)
(Revised April 2, 1973)

Introduction.

This paper deals with the convergence of nonlinear semigroups and the difference approximation for the Cauchy problem, (CP), for the scalar quasilinear equation

$$(DE) \qquad u_t+\sum_{i=1}^{d}(\phi_i(u))_{x_i}=0 \qquad \text{for } t>0,\ x=(x_1, x_2, \cdots, x_d)\in R^d$$

from the viewpoint of the approximation theory for nonlinear semigroups.

This investigation is motivated by the works of Kružkov [**13, 14**], Crandall [**5**] and Kojima [**9, 10**]. Kružkov treats this problem over the space $L^\infty(R^d)$ and discusses the existence and uniqueness of the generalized solution of (CP) under the assumption that $\phi_i\in C^1(R^1)$ for all i. His proof is based on the so-called method of vanishing viscosity and generalizes (DE) to allow the ϕ_i to depend on t and x as well as u. Kojima treats this problem by employing the finite-difference method and shows that the solution of the difference scheme formulated for (DE) converges in the topology of $L^1_{\mathrm{loc}}(R^d)$ to the generalized solution of (CP) in the sense of Kružkov. On the other hand, Crandall succeeded to treat this problem in $L^1(R^d)$ via the theory of nonlinear semigroups. He constructs a semigroup $\{T(t)\,;\,t\geqq 0\}$ of nonlinear contractions on $L^1(R^d)$ such that $T(t)u$ gives the generalized solution of (CP) in the sense of Kružkov provided that u belongs to $L^1(R^d)\cap L^\infty(R^d)$. In order to construct such a semigroup, it is needed to find a dissipative operator A in $L^1(R^d)$ such that $-\sum_{i=1}^{d}(\phi_i(u))_{x_i}$ is a representative function of Au for sufficiently many u and then to verify that the A satisfies the following condition:

$$(R) \qquad R(I-\lambda A)\supset D(A) \qquad \text{for } \lambda>0\,,$$

where $R(I-\lambda A)$ and $D(A)$ denote the range of $I-\lambda A$ and the domain of A respectively. This condition is called the *range condition* and Crandall proves this (R) by applying a certain perturbation theorem due to Brézis [**1**].

Here we shall exhibit how the convergence of difference approximation for (CP) may be interpreted in $L^1(R^d)\cap L^\infty(R^d)$ via the approximation theory for nonlinear semigroups. Our results furnish a semigroup theoretic approach to the difference approximation for (CP) and at the same time present another method for constructing the semigroup solution of (CP). Our objects are roughly stated as follows:

In this paper we shall employ the Friedrichs type difference approximation to (DE). This type of difference scheme is formulated according to initial data, since the mesh ratio must be determined depending upon the L^∞-norm of the initial-value u. Accordingly, the convergence theorems given for instance in Brezis-Pazy [2] and Miyadera-Oharu [15] can not directly be applied to the problem of convergence of this approximation. Our first purpose is to modify those convergence theorems to some appropriate forms which are applicable to our arguments. Here we shall consider a monotone increasing sequence of closed convex sets $\{X_m\}$ and make some assumptions which yield that

(S) There exists a family of semigroups $\{T_{m,n}(t);\ t\geqq 0\}$ of contractions on X_m, m, $n=1, 2, \cdots$, such that $T_{m,n}(t)u$ is strongly continuously differentiable in $t\geqq 0$ for $u\in X_m$ and such that the infinitesimal generator $A_{m,n}$ is a dissipative operator on X_m.

With this setting, we want to obtain the following type of convergence under some additional assumptions:

There exists a semigroup $\{T(t);\ t\geqq 0\}$ of nonlinear contractions on $X_0=\bigcup_{m\geqq 1} X_m$ and for each m,

$$\lim_{n\to\infty} T_{m,n}(t)u=T(t)u \qquad \text{for } t\geqq 0 \text{ and } u\in X_m.$$

The additional assumptions are stated as follows:

(C) There exists a pseudo-resolvent $\{J_\lambda;\ \lambda>0\}$ of contractions from X_0 into itself and for each $\lambda>0$ and m, the sequence of resolvents $\{(I-\lambda A_{m,n})^{-1}\}$ converges to J_λ on X_m.

(C_1) There exists a single-valued operator A_1 in X and a set $D\subset X_0$ such that $A_{m,n}u$ converges to $A_1u\in X_0$ as $n\to\infty$ for $u\in X_m\cap D$ and $m\geqq 1$.

We wish to consider the difference approximation to (DE) in the space $L^1(R^d)$. Our second purpose is to prove the L^1-convergence of the difference approximation by applying the above-mentioned modified convergence theorems. We let X_m be the closed convex set $\{u\in L^1(R^d)\cap L^\infty(R^d);\ \|u\|_\infty\leqq m\}$. By taking appropriate sequences $h_{m,n}\downarrow 0$ and $l_{m,n}\downarrow 0$ as the mesh sizes of time- and space-differences respectively, we define on each X_m a difference operator $C_{m,n}$ associated with the Friedrichs scheme and then construct a sequence of approximate semigroups $\{T_{m,n}(t);\ t\geqq 0\}$ determined by $A_{m,n}=$

$h_{m,n}^{-1}(C_{m,n}-I)$: The difference scheme to (DE) can be written as

$$h_{m,n}^{-1}[u_{m,n}(t+h_{m,n})-u_{m,n}(t)]=A_{m,n}u_{m,n}(t)\,, \qquad u_{m,n}(0)=u_0\in X_m$$

and $T_{m,n}(t)u_0$ gives the solution of

$$(d/dt)u_{m,n}(t)=A_{m,n}u_{m,n}(t)\,, \qquad u_{m,n}(0)=u_0\,.$$

In this setting, (S) is referred to the *stability condition.* (C_1) is a nonlinear analogue of the consistency condition which is treated in Lax's equivalence theorem (Richtmyer-Morton [**18**; Section 3.5]). In our case, the combination of (C) and (C_1) is referred to the *consistency condition* for the difference approximation. $\{J_\lambda;\ \lambda>0\}$ determines a dissipative operator A satisfying $R(I-\lambda A)=X_0\supset D(A)$, $\lambda>0$; hence if condition (C) is proved, then it turns out that a dissipative operator which satisfies (R) is automatically obtained and the operator is a restriction of the operator: $u\mapsto -\sum_{i=1}^{d}(\phi(u))_{x_i}$. By proving (S), (C) and (C_1) we can show not only the convergence of $\{T_{m,n}(t);\ t\geq 0\}$ but also the L^1-convergence of $C_{m,n}^{\nu}u_0$ to the generalized solution of (CP) through the approximation theory for nonlinear semigroups. Moreover, the Friedrichs type scheme is of the purely explicit form. This type of explicit difference scheme is changed to the implicit form by considering the resolvents $(I-\lambda A_{m,n})^{-1}$; such an implicit scheme satisfies the stability condition. In this way, the covergence theorem of semigroups can be applied to treat the so-called approximation-solvability of a Cauchy problem. For a similar type of treatment, see Konishi [**11**].

This limit of the double sequence of approximate semigroups $\{T_{m,n}(t);\ t\geq 0\}$ is obtained as an L^1-contractive semigroup on $L^1(R^d)\cap L^\infty(R^d)$. Our third purpose is to investigate the relationship between this semigroup and that of Crandall. We shall show by using the convergence of the approximate semigroups that our semigroup gives the generalized solutions of Kružkov's type and coincides with the restriction of the semigroup of Crandall to $L^1(R^d)\cap L^\infty(R^d)$. Also, among others, we shall mention that the properties of the semigroup of Crandall are derived through the difference approximation.

This paper consists of six sections. Section 1 contains some special notations used in this paper, some basic notions and the fundamental facts concerning those notions. Section 2 deals with the convergence and approximation of semigroups of nonlinear contractions. In Section 3, the approximating difference scheme formulated for (CP) is introduced and the main results are given. The proofs are given in the successive two sections. In Section 4, we shall discuss the convergence of the approximation introduced in Section 3. Section 5 is concerned with the relationships among our results obtained in Section 3 and the works of Kružkov, Crandall and others. Finally, in Section 6 we shall give a variety of observations on our results.

§ 1. Preliminaries.

In this section we list some special notations, basic notions and some of their fundamental properties. For further explanations on them, we refer to Oharu [**16**].

Let X be a Banach space with elements $u, v, w, \cdots$ and with the norm $\|\cdot\|$. By an *operator* A in X we mean a (possibly multi-valued) operator with the *domain* $D(A)$ and the *range* $R(A)$ in X, that is, A assigns to each $u \in X$ a subset Au of X; $D(A)$ is the set $\{u \in X;\ Au \neq \emptyset\}$ and $R(A) = \bigcup_{u \in X} Au$. Note that a single-valued operator is a special case of a multi-valued operator in which Au, $u \in D(A)$, denotes the value of A at u or the singleton set consisting of this element, and Au is the empty set if $u \notin D(A)$.

Let $S \subset X$. We write $A[S]$ for $\bigcup_{u \in S} Au$. By a *restriction* of A to S, denoted by $A|S$, we mean an operator such that $D(A|S) = D(A) \cap S$ and $(A|S)u = Au$ for $u \in D(A) \cap S$. $\bar{S}$ denotes the *closure* of S in X.

Let A and B be operators in X. Then B is called the *closure* of A if $G(B) = \overline{G(A)}$ in $X \times X$; we write $B = \bar{A}$, where $G(\cdot)$ denotes the graph of the operator. Also, we say that B is an *extension* of A, and A is a *restriction* of B (denoted by $B \supset A$ or $A \subset B$), if $D(A) \subset D(B)$ and $Au \subset Bu$ for $u \in D(A)$. For the notations of addition, scalar multiplication and composition of operators in X, we use the same notations as in Oharu [**16**; Section 0]. We write $\gamma + \lambda A$ for the operator $\gamma I + \lambda A$, where I denotes the *identity operator* in X. Also, we denote by A^{-1} the *inverse operator* of an operator A in X. Note that $G(A^{-1}) = \{(v, u);\ (u, v) \in G(A)\}$.

Let A be a single-valued operator in X such that $R(A) \subset D(A)$. Then for any positive integer i, we can define the *iteration* A^i on $D(A)$ by $A^i u = A(A^{i-1}u)$; we write $A^0 = I$.

Let $C \subset X$ and let T be a single-valued operator in X. T is called a *contraction on* C if $\|Tu - Tv\| \leqq \|u - v\|$ for $u, v \in C$.

An operator A in X is said to be *dissipative* if for every $u, v \in D(A)$ and $u' \in Au$, $v' \in Av$, there exists an $f \in F(u - v)$ such that

$$\operatorname{Re} \langle u' - v', f \rangle \leqq 0 ,$$

where F denotes the *duality mapping* from X into its dual X^* defined by

$$F(u) = \{f \in X^*;\ \operatorname{Re} \langle u, f \rangle = \|u\|^2 = \|f\|^2\} , \qquad u \in X .$$

It is well-known (Kato [**8**; Lemma 1.1]) that A is dissipative if and only if

$$\|u_1 - u_2\| \leqq \|(u_1 - \lambda v_1) - (u_2 - \lambda v_2)\| \tag{1.1}$$

for $u_i \in D(A)$, $v_i \in Au_i$, $i=1, 2$, and $\lambda > 0$. Note that (1.1) implies that for every $\lambda > 0$, $(I-\lambda A)^{-1}$ exists as a contraction on $R(I-\lambda A)$. Also, if $-A$ is dissipative then A is said to be *accretive*.

If A is a dissipative operator such that $R(I-\lambda A)=X$, then we say that A is *m-dissipative*. If A is dissipative in X, then so is $\bar{A}$. If A is a dissipative operator such that $\overline{R(I-\lambda_0 A)}=X$ for some $\lambda_0 > 0$, then $\bar{A}$ is m-dissipative.

A one-parameter family $\{J_\lambda ; \lambda > 0\}$ of contractions in X is called a *pseudo-resolvent* (*of contractions*) if for every λ, $\mu > 0$ and $u \in D(J_\lambda)$, $R\left(\frac{\mu}{\lambda}+\left(1-\frac{\mu}{\lambda}\right)J_\lambda\right) \subset D(J_\mu)$ and

$$J_\lambda u = J_\mu\left[\frac{\mu}{\lambda}u+\left(1-\frac{\mu}{\lambda}\right)J_\lambda u\right] \qquad \text{for } u \in D(J_\lambda). \tag{1.2}$$

PROPOSITION 1.1. (i) *Let A be a dissipative operator in X and let $J_\lambda = (I-\lambda A)^{-1}$ for $\lambda > 0$. Then $\{J_\lambda ; \lambda > 0\}$ forms a pseudo-resolvent of contractions in X. If in addition, A is single-valued, then each J_λ is injective.*

(ii) *Let $\{J_\lambda ; \lambda > 0\}$ be a pseudo-resolvent of contractions in X. Then $R(J_\lambda)$ is constant with respect to $\lambda > 0$ and there is a dissipative operator A, defined on $D \equiv R(J_\lambda)$, such that $J_\lambda = (I-\lambda A)^{-1}$ for every $\lambda > 0$. If in addition, some J_{λ_0} is injective, then the associated A is single-valued.*

For a proof of Proposition 1.1, see Oharu [**16**; Propositions 3.1 and 3.2].

Let $C \subset X$. By $\mathrm{Cont}(C)$ we mean the set of all contractions on C into itself. A one-parameter family $\{T(t) ; t \geqq 0\} \subset \mathrm{Cont}(C)$ is called a *semigroup* (*of nonlinear contractions*) *on* C if it has the following properties:

$$T(0)=I|C, \qquad T(t+s)=T(t)T(s) \qquad \text{for } t,\ s \geqq 0; \tag{1.3}$$

$$\text{for each } u \in C,\ T(t)u \text{ is strongly continuous in } t \geqq 0. \tag{1.4}$$

It is clear that if $T \in \mathrm{Cont}(C)$ then $\bar{T} \in \mathrm{Cont}(\bar{C})$. Hence, if $\{T(t) ; t \geqq 0\}$ is a semigroup on C, then $\{\overline{T(t)} ; t \geqq 0\}$ forms a semigroup on $\bar{C}$.

For the generation of the semigroup of nonlinear contractions, the following theorem due to Crandall-Liggett [**6**; Theorem I] is fundamental:

THEOREM 1.2. *Let A be a dissipative operator in X satisfying the range condition (R): $R(I-\lambda A) \supset D(A)$ for $\lambda > 0$. Then there exists a semigroup $\{T(t) ; t \geqq 0\}$ on $\overline{D(A)}$ such that*

$$\|T(t)u-(I-hA)^{-[t/h]}u\| \leqq 2\sqrt{th}\,|||Au||| \qquad \text{for } t \geqq 0 \text{ and } u \in D(A),$$

where $[\cdot]$ denotes the Gaussian blacket and $|||Au||| = \inf\{\|v\| ; v \in Au\}$. Therefore,

$$T(t)u = \lim_{h \to +0}(I-hA)^{-[t/h]}u = \lim_{k \to \infty}\left(I-\frac{t}{k}A\right)^{-k}u$$

for $t \geqq 0$ and $u \in D(A)$.

We shall say that the semigroup $\{T(t);\ t\geqq 0\}$ is *generated by* A (or, A *generates* $\{T(t);\ t\geqq 0\}$).

R^d denotes the d-dimensional euclidean space. $L^1(R^d)$ and $L^\infty(R^d)$ denote the ordinary Lebesgue spaces. Also, $C_0^1(R^d)$ and $C_0^\infty(R^d)$ have the usual meaning. In this paper we write simply L^1 and L^∞ for the spaces $L^1(R^d)$ and $L^\infty(R^d)$, respectively; we denote the L^1-norm and L^∞-norm by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. We write $u\in L^1$ (or $u\in L^\infty$) if a real-valued function $u(x)$ defined on R^d is a representative function of an element of L^1 (or L^∞). Conversely, let $u\in L^1$ (or $u\in L^\infty$). Then we sometimes write $[u](x)$ for a representative function of u. Let $u\in L^1$. We denote the integral of u over R^d with respect to the Lebesgue measure on R^d by

$$\int_{R^d} u(x)dx\,.$$

We denote by $\langle u, f\rangle$ the pairing between $u\in L^1$ and $f\in L^\infty$. Accordingly, for every $u\in L^1$ and $f\in C_0^\infty(R^d)$, we write $\langle u, f\rangle$ for the integral

$$\int_{R^d} u(x)f(x)dx\,.$$

We shall use the following notations:

$$\text{sign}\,(s)=\begin{cases} 1 & \text{if } s>0 \\ 0 & \text{if } s=0 \\ -1 & \text{if } s<0\,. \end{cases} \tag{1.5}$$

In particular, we shall frequently treat the composite function $\text{sign}\,(u(x)-k)$ in later arguments, where $k\in R^1$ and $u(x)$ is a measurable function on R^d. In order to approximate such a function, Crandall [5; Lemma 1.1] gives the following functions:

$$\Phi_j(s)=\begin{cases} -s & \text{if } s<-1/j \\ (j/2)s^2+(1/2j) & \text{if } |s|\leqq 1/j \\ s & \text{if } s>1/j\,, \qquad j=1, 2, 3, \cdots\,. \end{cases} \tag{1.6}$$

Clearly, $\Phi_j(s)$ has the derivative $\Phi_j''(s)$ which is piecewise continuous and non-negative, and has a compact support $[-1/j, 1/j]$. Moreover, it is easily seen that

$$\lim_{j\to\infty}\int_k^{s_0}\Phi_j''(s-k)\phi(s)ds=\text{sign}\,(s_0-k)\phi(k) \tag{1.7}$$

$$\lim_{j\to\infty}\Phi_j'(s_0-k)=\text{sign}\,(s_0-k) \tag{1.8}$$

for s_0, $k\in R^1$ and every function $\phi\in C^1(R^1)$.

Let F be the duality mapping of L^1. Then it is proved ([5; Section 4]) that $\|u\|_1 \operatorname{sign}(u(\cdot)) \in F(u)$ for $u \in L^1$.

Finally, we introduce some notations concerning the difference approximation. Let $l>0$ and let $u(x)$ be a real-valued function defined on R^d. Then we write

$$
(1.9)\qquad
\begin{aligned}
&[D_i^0 u](x) = (2l)^{-1}[u(x+le_i)-u(x-le_i)]\,,\\
&[D_i^+ u](x) = l^{-1}[u(x+le_i)-u(x)]\,,\\
&[D_i^- u](x) = l^{-1}[u(x)-u(x-le_i)]\,, \qquad x \in R^d\,,
\end{aligned}
$$

for the central, forward and backward differences of u, respectively, where e_i denotes the unit vector whose i-th element is 1. Accordingly, we can write as

$$[D_i^- D_i^+ u](x) = l^{-2}[u(x+le_i)-2u(x)+u(x-le_i)]\,.$$

We shall apply these difference operations to $u(x)$ which is a representative function of $u \in L^1 \cap L^\infty$. D_i^0, D_i^+ as well as D_i^- depend on l. But there will be no confusions in later arguments, even if l is not specified in these symbols. Also, these are regarded as bounded linear operators on L^1 in a natural way, since they can be represented as linear combinations of translation operators. It is easily seen that if $u \in L^1$ and $f \in C_0^\infty(R^d)$ then

$$
\begin{aligned}
&\langle D_i^0 u, f\rangle = -\langle u, D_i^0 f\rangle\,,\\
&\langle D_i^- D_i^+ u, f\rangle = \langle u, D_i^- D_i^+ f\rangle
\end{aligned}
$$

and so on.

§ 2. Convergence Theorems.

In this section we treat some convergence theorems for semigroups of nonlinear contractions.

Let $\{X_m\}_{m=1,2,\cdots}$ be a monotone increasing sequence of closed subsets of X, $X_0 = \bigcup_{m\geq 1} X_m$, and let us consider a family $\{A_{m,n}\}_{m,n=1,2,\cdots}$ of single-valued, dissipative operators in X such that $D(A_{m,n}) = X_m$ for m, $n \geqq 1$ and such that $R(I-\lambda A_{m,n}) \supset X_m$ for $\lambda>0$ and m, $n \geqq 1$. Then by Theorem 1.2, each $A_{m,n}$ generates a semigroup $\{T_{m,n}(t);\ t\geqq 0\} \subset \mathrm{Cont}(X_m)$ such that

$$(2.1)\qquad \left\| T_{m,n}(t)u - \left(I-\frac{t}{k}A_{m,n}\right)^{-k} u \right\| \leqq (2t/\sqrt{k})\|A_{m,n}u\| \quad \text{for } t\geqq 0 \text{ and } u \in X_m\,.$$

We start with a theorem on convergence of $\{T_{m,n}(t);\ t\geqq 0\}$.

THEOREM 2.1. *Let $\{X_m\}$ and $\{A_{m,n}\}$ be as above. If*

(C) *there exists a pseudo-resolvent $\{J_\lambda;\ \lambda>0\} \subset \mathrm{Cont}(X_0)$ and*

$$J_\lambda u = \lim_{n\to\infty} (I-\lambda A_{m,n})^{-1} u \qquad \textit{for } \lambda>0 \textit{ and } u \in X_m\,,$$

then we have:

(i) *There exists a dissipative operator A in X such that $R(I-\lambda A)=X_0\supset D(A)$ for $\lambda>0$ and such that $J_\lambda=(I-\lambda A)^{-1}$ for $\lambda>0$;*

(ii) *A generates a semigroup $\{T(t);\ t\geqq 0\}$ on $\overline{D(A)}$ such that*

$$T(t)[X_0\cap\overline{D(A)}]\subset X_0\cap\overline{D(A)}\qquad for\ t\geqq 0;$$

(iii) *$\lim_{n\to\infty} T_{m,n}(t)u=T(t)u$ for $t\geqq 0$ and $u\in X_m\cap\overline{D(A)}$, where the convergence is uniform with respect to t in every bounded subinterval of $[0,\infty)$.*

PROOF. Since $\{J_\lambda;\ \lambda>0\}$ is a pseudo-resolvent of contractions from X_0 into itself, (i) follows from Proposition 1.1 (ii).

By virtue of (i) and Theorem 1.2, A generates a semigroup $\{T(t);\ t\geqq 0\}$ on $\overline{D(A)}$. Since $(I-\lambda A_{m,n})^{-1}$ maps X_m into itself for all m and n and since X_m is closed in X, we see that each X_m is invariant under J_λ, $\lambda>0$. Thus, $X_m\cap\overline{D(A)}$, $m=1,2,\cdots$, are invariant under $T(t)$, $t\geqq 0$. This proves (ii).

(iii) is proved in a similar way to Brezis-Pazy [**2**; Theorem 3.1]: Let $u\in X_m\cap D(A)$ and $\lambda>0$. Since $J_\mu[X_m]\subset X_m$ for $\mu>0$, we see that $J_{t/k}^q J_\lambda u\in X_m\cap D(A)$ for all k and q. Hence, we have the following estimates:

$$\|T_{m,n}(t)J_\lambda u-T_{m,n}(t)(I-\lambda A_{m,n})^{-1}u\|\leqq\|J_\lambda u-(I-\lambda A_{m,n})^{-1}u\|\,, \tag{2.2}$$

$$\left\|T_{m,n}(t)(I-\lambda A_{m,n})^{-1}u-\left(I-\frac{t}{k}A_{m,n}\right)^{-k}(I-\lambda A_{m,n})^{-1}u\right\| \tag{2.3}$$
$$\leqq(2t/\sqrt{k}\lambda)\|(I-\lambda A_{m,n})^{-1}u-u\|\qquad(\text{by }(2.1))\,,$$

$$\left\|\left(I-\frac{t}{k}A_{m,n}\right)^{-k}(I-\lambda A_{m,n})^{-1}u-J_{t/k}^k J_\lambda u\right\| \tag{2.4}$$
$$\leqq\|(I-\lambda A_{m,n})^{-1}u-J_\lambda u\|$$
$$+\sum_{q=0}^{k-1}\left\|\left(I-\frac{t}{k}A_{m,n}\right)^{-k+q}J_{t/k}^q J_\lambda u-\left(I-\frac{t}{k}A_{m,n}\right)^{-k+q+1}J_{t/k}^{q+1}J_\lambda u\right\|$$
$$\leqq\|(I-\lambda A_{m,n})^{-1}u-J_\lambda u\|$$
$$+\sum_{q=0}^{k-1}\left\|\left(I-\frac{t}{k}A_{m,n}\right)^{-1}J_{t/k}^q J_\lambda u-J_{t/k}J_{t/k}^q J_\lambda u\right\|,$$

$$\|J_{t/k}^k J_\lambda u-T(t)J_\lambda u\|\leqq(2t/\sqrt{k}\lambda)\|J_\lambda u-u\|\,. \tag{2.5}$$

Combining (2.2)-(2.5) with condition (C), we obtain

$$\lim_{n\to\infty}\|T_{m,n}(t)J_\lambda u-T(t)J_\lambda u\|=0\,.$$

Since $\|J_\lambda u-u\|\to 0$ as $\lambda\to 0$ and since $T_{m,n}(t)$ and $T(t)$ are contractions, we have the assertion (iii). Q. E. D.

COROLLARY 2.2. *Let (C) be satisfied for $\{X_m\}$ and $\{A_{m,n}\}$. Suppose that*

X_0 is a linear manifold in X and that

(C_1) *There exist a single-valued operator A_1 in X and a set $D\subset X_0$ such that*

$$\lim_{n\to\infty} A_{m,n}u = A_1u \in X_0 \qquad \text{for } u\in X_m\cap D \text{ and } m\geqq 1\,.$$

Then $D\subset D(A)$ and $A_1u\in Au$ for $u\in D$. If in addition, D is dense in X_0, then the convergence (iii) *of Theorem* 2.1 *holds on X_m, instead of $X_m\cap\overline{D(A)}$.*

PROOF. Let $u\in D$ and $\lambda>0$. Since X_0 is a linear manifold, there exists an integer m such that $u\in X_m$ and such that $u-\lambda A_1u=v\in X_m$. Let $v_n=(I-\lambda A_{m,n})u$. Then, (C_1) implies that $v_n\to v$ as $n\to\infty$. Since $(I-\lambda A_{m,n})^{-1}v_n=u$, we have

$$\|u-J_\lambda v\|\leqq\|v_n-v\|+\|(I-\lambda A_{m,n})^{-1}v-J_\lambda v\|\,.$$

By virtue of conditions (C) and (C_1), the right side goes to 0 as $n\to\infty$. Hence, $u=J_\lambda v$ or $v\in(I-\lambda A)u$, from which it follows that $A_1u\in Au$. Q.E.D.

REMARKS. (1) The convergence theorems given in Brezis-Pazy [2] are nonlinear analogues of Trotter-Kato's theorem. Theorem 2.1 is a modification of Brezis-Pazy's result and this type of modification seems to be proper to the nonlinear setting.

(2) If X_0 is a linear manifold in X, then condition (C) can be regarded as the combination of the following two conditions:

(C') $\lim_{n\to\infty}(I-\lambda A_{m,n})^{-1}u$ exists for $\lambda>0$, $u\in X_m$ and $m\geqq 1$;

(C'') There exists a dissipative operator B in X such that $v_{m,\lambda}=\lim_{n\to\infty}(I-\lambda A_{m,n})^{-1}u\in D(B)$ and $\lim_{n\to\infty}A_{m,n}(I-\lambda A_{m,n})^{-1}u\in Bv_{m,\lambda}$ for $\lambda>0$, $u\in X_m$ and $m\geqq 1$.

In fact, suppose that (C) holds. Then (C') is trivially satisfied and in virtue of Theorem 2.1 (i), we see that (C'') also holds for B replaced by A. Conversely, assume that (C') and (C'') are satisfied. Then, $\lim_{n\to\infty}A_{m,n}(I-\lambda A_{m,n})^{-1}u=\lambda^{-1}(v_{m,\lambda}-u)\in Bv_{m,\lambda}$ for $\lambda>0$, $u\in X_m$ and $m\geqq 1$, and so, we see that $u\in R(I-\lambda B)$ and $v_{m,\lambda}=(I-\lambda B)^{-1}u$. This means that $v_{m,\lambda}$ depends only on λ and u. We set $J_\lambda u=v_{m,\lambda}$ for $\lambda>0$, $u\in X_m$ and $m\geqq 1$. Then J_λ is defined on X_0 as a single-valued operator. Since $(I-\lambda A_{m,n})^{-1}$ satisfies the resolvent formula (1.2) by Proposition 1.1 (i) and since X_0 is a linear manifold, it follows from (C') that $\{J_\lambda;\ \lambda>0\}$ forms a pseudo-resolvent of contractions belonging to Cont(X_0). Moreover, let A be a dissipative operator associated with this pseudo-resolvent through Proposition 1.1 (ii). Then $A\subset B$. See also Remark (2) after Theorem 5.3.

(3) Condition (C_1) is referred to the consistency condition in the finite-difference method. Condition (C) might be also called the consistency condition in a generalized sense (cf. Takahashi-Oharu [**19**; Section 2]). (C_1) does not necessarily imply (C) (see Remark after Theorem 4.5). However, if

$D(A_1)\subset X_0\subset R(I-\lambda A_1)$ for $\lambda>0$ and if (C_1) is satisfied for the set D replaced by $D(A_1)$, then it is proved by the same way as in Brezis-Pazy [2; Theorem 4.1] that condition (C) holds for the A_1.

(4) If we do not assume that $\{X_m\}$ is monotone increasing, then we consider conditions (C) and (C_1) on $X_0=\liminf_{m\to\infty} X_m$ and the above-mentioned results are extended to more general cases. However, we shall not give any in this paper, since the theorems mentioned above are sufficient for the applications treated in later sections.

(5) As is seen from Theorem 2.1, $\{T(t)|X_0\cap\overline{D(A)};\, t\geqq 0\}$ forms a semigroup on $X_0\cap\overline{D(A)}$. Hence, precisely saying, the limit semigroup of $\{T_{m,n}(t);\, t\geqq 0\}$ is $\{T(t)|X_0\cap\overline{D(A)};\, t\geqq 0\}$ (and $\{T(t)|X_0;\, t\geqq 0\}$ in Corollary 2.2).

Next, we state a result on convergence of iterations of difference operators.

Let $\{X_m\}_{m\geqq 1}$ be a monotone increasing sequence of closed convex subsets of X and $\{C_{m,n}\}_{m,n\geqq 1}$ be a family of operators in X such that

$$\{C_{m,n}\}_{n\geqq 1}\subset \mathrm{Cont}\,(X_m) \qquad \text{for each } m\geqq 1. \tag{2.6}$$

Let $\{h_{m,n}\}_{m,n\geqq 1}$ be a double sequence of positive numbers such that $h_{m,n}\to 0$ as $n\to\infty$ for each $m\geqq 1$. We then set

$$A_{m,n}=h_{m,n}^{-1}(C_{m,n}-I) \qquad \text{for } m,\ n=1, 2, 3, \cdots. \tag{2.7}$$

Then, (2.6) yields that each $A_{m,n}$ is a dissipative operator on X_m and satisfies the range condition

$$R(I-\lambda A_{m,n})\supset X_m=D(A_{m,n}) \qquad \text{for } \lambda>0 \text{ and } m,\ n\geqq 1. \tag{2.8}$$

For the proof, we refer to Brezis-Pazy [2; Lemma 2.2]. Also, each $A_{m,n}$ is continuous on X_m. Hence, each $A_{m,n}$ generates a semigroup $\{T_{m,n}(t);\, t\geqq 0\}$ on X_m such that $T_{m,n}(t)u\in C^1([0,\infty);X)$ for $u\in X_m$ and

$$\begin{cases} (d/dt)T_{m,n}(t)u=A_{m,n}T_{m,n}(t)u\,, \\ T_{m,n}(t)u=\lim_{\varepsilon\to+0}(I-\varepsilon A_{m,n})^{-[t/\varepsilon]}u\,, \\ A_{m,n}T_{m,n}(t)u=\lim_{\varepsilon\to+0}A_{m,n}(I-\varepsilon A_{m,n})^{-[t/\varepsilon]}u \qquad \text{for } t\geqq 0 \text{ and } u\in X_m\,. \end{cases} \tag{2.9}$$

THEOREM 2.3. (a) *If* $\{A_{m,n}\}$ *defined by* (2.7) *satisfies condition* (C), *then we have the following convergence*

$$\lim_{\substack{n\to\infty\\ \nu h_{m,n}\to t}} C_{m,n}^{\nu}u=T(t)u \qquad \textit{for } t\geqq 0,\ u\in X_m\cap\overline{D(A)} \textit{ and } m\geqq 1, \tag{iv}$$

together with (i)-(iii) *of Theorem* 2.1.

(b) *Suppose that the* $\{A_{m,n}\}$ *satisfies both* (C) *and* (C_1). *If* X_0 *is a linear manifold in* X *and if the set* D *in* (C_1) *is dense in* X_0, *then* $D\subset D(A)$, $A_1|D\subset$

$A|D$ and furthermore, (iv) *holds on X_m, instead of $X_m \cap \overline{D(A)}$.*

REMARK. Later, the operators $C_{m,n}$ will be referred to the difference operators which are induced from the Friedrichs scheme for (DE). The assertion (a) states that under condition (C), the limit of iterations $\{C^{\nu}_{m,n}\}$ is the semigroup $\{T(t)|X_0 \cap \overline{D(A)};\ t \geqq 0\}$ and (b) states that the limit is the semigroup $\{T(t)|X_0;\ t \geqq 0\}$.

PROOF OF THEOREM 2.3. Applying a result of Miyadera-Oharu [**15**; Appendix, Lemma 4] to $\{T_{m,n}(t);\ t \geqq 0\}$ and $\{C_{m,n}\}$ and then estimating (2.9), we obtain

$$\|T_{m,n}(t)u - C^{\nu}_{m,n}u\| \leqq (|t-\nu h_{m,n}| + \sqrt{\tau h_{m,n}})\|A_{m,n}u\|$$

for $u \in X_m$, t, $\nu h_{m,n} \in [0, \tau]$, $\tau > 0$ and m, $n \geqq 1$.
Now (iv) is proved in a similar way to Brezis-Pazy [2; Theorem 3.2]: Let $\{T(t);\ t \geqq 0\}$ be the semigroup determined by A through Theorem 2.1 (ii). Let $u \in X_m \cap D(A)$, $\tau > 0$, $0 \leqq t \leqq \tau$ and let $\lambda > 0$. Then $J_\lambda u \in X_m$ and

$$\begin{aligned} &\|T(t)J_\lambda u - C^{\nu}_{m,n}J_\lambda u\| \\ &\quad \leqq \|T(t)J_\lambda u - T_{m,n}(t)J_\lambda u\| + \|T_{m,n}(t)J_\lambda u - T_{m,n}(t)(I-\lambda A_{m,n})^{-1}u\| \\ &\qquad + \|T_{m,n}(t)(I-\lambda A_{m,n})^{-1}u - C^{\nu}_{m,n}(I-\lambda A_{m,n})^{-1}u\| \\ &\qquad + \|C^{\nu}_{m,n}(I-\lambda A_{m,n})^{-1}u - C^{\nu}_{m,n}J_\lambda u\| \\ &\quad \leqq \|T(t)J_\lambda u - T_{m,n}(t)J_\lambda u\| + 2\|(I-\lambda A_{m,n})^{-1} - J_\lambda u\| \\ &\qquad + (|t-\nu h_{m,n}| + \sqrt{\tau h_{m,n}})\lambda^{-1}\|(I-\lambda A_{m,n})^{-1}u - u\|. \end{aligned}$$

Combining this estimate with Theorem 2.1 (iii) and with condition (C), we see that

$$\|T(t)J_\lambda u - C^{\nu}_{m,n}J_\lambda u\| \longrightarrow 0 \qquad \text{as } n \to \infty \text{ and } \nu h_{m,n} \to t.$$

Since $J_\lambda u \to u$ as $\lambda \to 0$ and since $T(t)$ and $C^{\nu}_{m,n}$ are contractions, we have the assertion (a). (b) is now evident from (a) and Corollary 2.2. Q.E.D.

§ 3. Difference approximation for (CP).

In this section we discuss the difference approximation to

$$(DE) \qquad u_t + \sum_{i=1}^{d} (\phi_i(u))_{x_i} = 0, \qquad t > 0,\ x = (x_1, x_2, \cdots, x_d) \in R^d$$

and then state our main theorem together with some comments on it. The proof will be given in later sections.

Throughout the remainder of this paper, we assume that $\phi_i \in C^1(R^1)$ and $\phi_i(0) = 0$ for all i and treat the Cauchy problem (CP) for the above-mentioned (DE) over the space $L^1 = L^1(R^d)$.

As is well-known, we can not expect the exact solution of (CP) for a general initial-value $u_0 \in L^1$. Here we define the generalized solution of (CP) as follows:

DEFINITION 3.1. Given a $u_0 \in L^1 \cap L^\infty$, $L^1 \cap L^\infty$-valued function $u(t)=u(t, \cdot)$ defined on $[0, \infty)$ is called a *generalized solution* of (CP) *with the initial-value* u_0 if it satisfies the following conditions:

$(G.1)$ $u(t)$ is continuous with respect to L^1-norm and $\|u(t)\|_\infty$ is uniformly bounded on $[0, \infty)$.

$(G.2)$ For every $k \in R^1$ and every nonnegative function $f \in C_0^\infty((0, \infty) \times R^d)$,

$$\int_0^\infty \int_{R^d} \{|u(t, x)-k| f_t(t, x) + \operatorname{sign}(u(t, x)-k) \sum_{i=1}^d [\phi_i(u(t, x))-\phi_i(k)] f_{x_i}(t, x)\} \, dx dt \geqq 0 .$$

$(G.3)$ $$\lim_{t \to +0} \int_{R^d} |u(t, x)-u_0(x)| \, dx = 0 .$$

REMARKS. (1) The above definition of solution of (CP) is a modified version of that proposed by Kružkov in the sense that the generalized solution of Kružkov (which moves in L^∞) is also contained in L^1. Hence, the meaning of our solution is more strict than that of Kružkov. The generalized solution in the sense of Kružkov is always unique ([**14**; Theorem 2]); hence we see that our solution is also unique.

(2) Crandall shows that the generalized solution of Definition 3.1 exists if the initial-value u belongs to $L^1 \cap L^\infty$ and that such solutions are represented by an L^1-contractive semigroup on L^1. He also generalizes (DE) to allow the ϕ_i to be of class $C^0(R^1)$ and obtains a semigroup solution of (CP) ([**5**; Corollary 2.2]). It is possible to extend our results in this direction. For details, we shall mention it in Section 6.

(3) The solution in the sense of Kružkov is the limit u of the (exact) solutions u^ε, $\varepsilon > 0$, of the Cauchy problems

$$(CP)_\varepsilon \qquad \begin{cases} u_t^\varepsilon + \sum_{i=1}^d (\phi_i(u^\varepsilon))_{x_i} = \varepsilon \Delta u^\varepsilon , & t > 0, \ x \in R^d \\ u^\varepsilon|_{t=0} = u_0(x) , \end{cases}$$

and condition $(G.2)$ is derived through the convergence $u^\varepsilon \to u$ as $\varepsilon \to 0$. This fact suggests that it is natural to employ the Friedrichs scheme to approximate (DE). Also, as in indicated in Kružkov [**14**; Section 2], $(G.2)$ might be regarded as the "*entropy condition*" in the case of several space variables.

The Friedrichs scheme approximating (CP) for (DE) is written as

$$
(DS) \quad \begin{cases} h^{-1}[u^{\nu+1}(x)-(2d)^{-1}\sum_{i=1}^{d}(u^{\nu}(x+le_i)+u^{\nu}(x-le_i))] \\ \qquad +(2l)^{-1}\sum_{i=1}^{d}[\phi_i(u^{\nu}(x+le_i))-\phi_i(u^{\nu}(x-le_i))]=0\,, \\ \qquad\qquad h,\ l>0,\ x\in R^d;\ \nu=0,1,2,\cdots \\ u^0(x)=u_0(x)\,. \end{cases}
$$

In this scheme, the mesh sizes of space differences are all taken to be equal. However, as is seen from later arguments, we can generalize (DS) to allow the mesh sizes to be distinct. (DS) approximates (CP) as $h,\ l\to 0$ if the mesh ratio l/h lies in a compact interval contained in $(0,\infty)$. Observe that (DS) is also written as

$$
h^{-1}[u^{\nu+1}(x)-u^{\nu}(x)]-(l^2/2dh)\sum_{i=1}^{d}[D_i^- D_i^+ u^{\nu}](x)+\sum_{i=1}^{d}[D_i^0\phi_i(u^{\nu})](x)=0\,,
$$

where D_i^0, D_i^+ and D_i^- mean the difference operators defined by (1.9). Hence if $l^2/2dh=\varepsilon$, then (DS) approximates $(CP)_\varepsilon$ as $h,\ l\to 0$ (cf. Oleinik [**17**; Section 5] and Remark (3) after Definition 3.1).

The difference operators $C_{h,l}$, $h,\ l>0$, are defined by

$$
(3.1) \qquad [C_{h,l}u](x)=(2d)^{-1}\sum_{i=1}^{d}(u(x+le_i)+u(x-le_i))-h\sum_{i=1}^{d}[D_i^0\phi_i(u)](x)
$$

for $u\in L^1$, whenever $C_{h,l}u\in L^1$. Accordingly, (DS) is written as

$$
(DS) \qquad u^{\nu}=C_{h,l}u^{\nu-1}=C_{h,l}^{\nu}u_0\,, \qquad \nu=1,2,3,\cdots.
$$

Using these $C_{h,l}$, we define the operators $C_{m,n}$ which are of the type of Theorem 2.3 as follows: Let

$$
X_m=\{u\in L^1\cap L^\infty;\ \|u\|_\infty\leqq m\}\,, \qquad m=1,2,3,\cdots.
$$

Then

$$
X_0=\bigcup_{m\geqq 1}X_m=L^1\cap L^\infty.
$$

Let

$$
M_m=\max_{1\leqq i\leqq d}\sup_{|s|\leqq m}|\phi_i'(s)|
$$

and let $\{\delta_m\}_{m\geqq 1}$ be a sequence of positive numbers such that $\delta_m\leqq 1/dM_m$ for $m\geqq 1$. For these $\{M_m\}$ and $\{\delta_m\}$, we choose two sequences $\{h_{m,n}\}_{m,n\geqq 1}$ and $\{l_{m,n}\}_{m,n\geqq 1}$ of positive numbers such that for each m, $h_{m,n},\ l_{m,n}\to 0$ as $n\to\infty$ and

$$
(3.2) \qquad \delta_m\leqq h_{m,n}/l_{m,n}\leqq 1/dM_m \qquad \text{for } n\geqq 1\,.
$$

Then $C_{h_{m,n},l_{m,n}}$ are well-defined on X_m. We set

$$
(3.3) \qquad C_{m,n}=C_{h_{m,n}l_{m,n}}|X_m \quad \text{and} \quad A_{m,n}=h_{m,n}^{-1}(C_{m,n}-I)\,.
$$

Now, our main theorem is stated as follows:

THEOREM 3.2. *Let* $\{C_{m,n}\}$ *and* $\{A_{m,n}\}$ *are the operators defined by* (3.3). *Then we have:*

(i) *There exists a single-valued dissipative operator* A *in* L^1 *such that* $C_0^1(R^d)\subset D(A)\subset L^1\cap L^\infty$ *and such that*

$$Au=-\sum_{i=1}^{d}(\phi_i(u))_{x_i} \qquad \text{for } u\in D(A)\,,$$

where the differentiation is taken in the sense of distributions.

(ii) *For every* $\lambda>0$ *and* $u\in X_m$, $(I-\lambda A)^{-1}u=\lim_{n\to\infty}(I-\lambda A_{m,n})^{-1}u$ *and* $(I-\lambda A)^{-1}u$ *is a solution of the equation*

$$v+\lambda\sum_{i=1}^{d}(\phi_i(v))_{x_i}=u\,, \qquad v\in D(A)\,.$$

Therefore, $R(I-\lambda A)=L^1\cap L^\infty$ *for* $\lambda>0$.

(iii) *There is an* L^1*-contractive semigroup* $\{T(t)\,;\,t\geqq 0\}$ *on* $L^1\cap L^\infty$ *such that*

$$T(t)u=\lim_{n\to\infty}\Big(I-\frac{t}{n}A\Big)^{-n}u \qquad \text{for } t\geqq 0 \text{ and } u\in L^1\cap L^\infty$$

and such that for each $u\in L^1\cap L^\infty$, $u(t,x)=[T(t)u](x)$ *gives a generalized solution of* (CP) *with the initial-function* $u(x)$.

(iv) *For every* $u\in X_m$ *and* $t\geqq 0$, $C_{m,n}^{\nu}u$ *converges strongly to* $T(t)u$ *as* $n\to\infty$ *and* $\nu h_{m,n}\to t$.

REMARKS. (1) When we apply Theorem 2.3 in the proof of this theorem, we see that semigroup $\{T_{m,n}(t)\,;\,t\geqq 0\}$ generated by $A_{m,n}$ converges to $\{T(t)\,;\,t\geqq 0\}$ in the form of Theorem 2.1. This means that the solution of the semi-discrete approximation

$$\begin{cases} (d/dt)u_{m,n}(t)=A_{m,n}u_{m,n}(t)\,, & t\geqq 0 \\ u_{m,n}(0)=u\in X_m\,, & n=1,2,3,\cdots\,;\ m=1,2,3,\cdots \end{cases}$$

converges to the generalized solution of (CP).

(2) As is shown in Remark (1) after Theorem 5.3, $\bar{A}$ is m-dissipative. Hence, the convergence $\lim_{n\to\infty}\Big(I-\frac{t}{n}\bar{A}\Big)^{-n}u=\bar{T}(t)u$ holds for $t\geqq 0$ and $u\in L^1$, where $\{\bar{T}(t)\,;\,t\geqq 0\}$ is the semigroup on L^1 which is obtained by extending $\{T(t)\,;\,t\geqq 0\}$ in (iii) onto L^1. This extended semigroup coincides with that of Crandall.

(3) We can also define the difference operators $C_{m,n}$ on the set $X_m=\{u\in L^\infty\,;\,\|u\|_\infty\leqq m\}$ in the same way as in (3.3). Kojima [**10**] shows that for every $u\in X_m$ and $t\geqq 0$, $C_{m,n}^{\nu}u$ converges to a generalized solution in the sense of Kružkov of (CP) in the topology of $L_{\mathrm{loc}}^1(R^d)$. He proves this convergence by employing Kružkov's uniqueness theorem. We note the Theorem 3.2 (iv) is proved without the theorem.

§ 4. Convergence of difference approximation.

In this section we show that the conditions imposed in Theorem 2.3 are satisfied for the operators $\{C_{m,n}\}$ and $\{A_{m,n}\}$ which are defined by (3.3).

Throughout this and later sections, we write simply h and l for $h_{m,n}$ and $l_{m,n}$, respectively. We use these abbreviations just for brevity in notation. The m and n to be specified will be indicated by the subscripts associated with the operators $C_{m,n}$, $A_{m,n}$ and $(I-\lambda A_{m,n})^{-1}$, $\lambda>0$.

We start with the following result which implies the stability condition for (DS).

LEMMA 4.1. *For each* $m\geqq 1$, $\{C_{m,n}\}_{n\geqq 1}\subset \mathrm{Cont}(X_m)$. *Moreover,*

$$\|C_{m,n}u\|_p\leqq \|u\|_p \qquad \text{for } u\in X_m,\ m,\ n\geqq 1 \text{ and } p=1,\ \infty .$$

PROOF. Let u, $v\in X_m$. Then by (3.1) and (3.3), we have

$$
\begin{aligned}
(4.1)\qquad & \int_{R^d}|[C_{m,n}u](x)-[C_{m,n}v](x)|\,dx \\
&\leqq \sum_{i=1}^{d}\int_{R^d}|(2d)^{-1}[u(x+le_i)-v(x+le_i)] \\
&\qquad\qquad -(h/2l)[\phi_i(u(x+le_i))-\phi_i(v(x+le_i))]|\,dx \\
&\quad +\sum_{i=1}^{d}\int_{R^d}|(2d)^{-1}[u(x-le_i)-v(x-le_i)] \\
&\qquad\qquad +(h/2l)[\phi_i(u(x-le_i))-\phi_i(v(x-le_i))]|\,dx \\
&=\sum_{i=1}^{d}\int_{R^d}|(2d)^{-1}[u(x)-v(x)]-(h/2l)[\phi_i(u(x))-\phi_i(v(x))]|\,dx \\
&\quad +\sum_{i=1}^{d}\int_{R^d}|(2d)^{-1}[u(x)-v(x)]+(h/2l)[\phi_i(u(x))-\phi_i(v(x))]|\,dx .
\end{aligned}
$$

Applying the mean value theorem, we have

$$
\begin{aligned}
&\sum_{i=1}^{d}|(2d)^{-1}[u(x)-v(x)]-(h/2l)[\phi_i(u(x))-\phi_i(v(x))]| \\
&\quad +\sum_{i=1}^{d}|(2d)^{-1}[u(x)-v(x)]+(h/2l)[\phi_i(u(x))-\phi_i(v(x))]| \\
&=\sum_{i=1}^{d}\{[(2d)^{-1}-(h/2l)\phi_i'(\theta_i(x))] \\
&\qquad\qquad +[(2d)^{-1}+(h/2l)\phi_i'(\theta_i(x))]\}\,|u(x)-v(x)| \\
&=|u(x)-v(x)|,
\end{aligned}
$$

for almost all $x\in R^d$, where $\theta_i(x)$ are certain values between $u(x)$ and $v(x)$; note that $(2d)^{-1}\pm(h/2l)\phi_i'(s)\geqq 0$ for $|s|\leqq m$ and for all i by condition (3.2). Thus, we obtain

$$\|C_{m,n}u-C_{m,n}v\|_1\leqq \|u-v\|_1 \qquad \text{for } u,\ v\in X_m \text{ and } m,\ n\geqq 1 .$$

Next let $u \in X_m$. Then, in view of (3.1) and the mean value theorem,

$$[C_{m,n}u](x) = \sum_{i=1}^{d} [(2d)^{-1} - (h/2l)\phi_i'(\theta_i(x))]u(x+le_i)$$

$$+ \sum_{i=1}^{d} [(2d)^{-1} + (h/2l)\phi_i'(\theta_i(x))]u(x-le_i),$$

where $\theta_i(x)$ are certain values between $u(x+le_i)$ and $u(x-le_i)$. Therefore,

$$|[C_{m,n}u](x)| \leqq \sum_{i=1}^{d} [(2d)^{-1} - (h/2l)\phi_i'(\theta_i(x))]\|u\|_\infty$$

$$+ \sum_{i=1}^{d} [(2d)^{-1} + (h/2l)\phi_i'(\theta_i(x))]\|u\|_\infty$$

$$= \|u\|_\infty,$$

for almost all $x \in R^d$, and hence $\|C_{m,n}u\|_\infty \leqq \|u\|_\infty$. This also means that each $C_{m,n}$ maps X_m into itself.

Finally, observing that $C_{m,n}0 = 0$ for $m, n \geqq 1$, we have that $\|C_{m,n}u\|_1 \leqq \|u\|_1$. Q. E. D.

REMARK. In the above proof we used only the condition that $h_{m,n}/l_{m,n} \leqq 1/dM_m$ for $m, n \geqq 1$. Since $\{M_m\}$ is monotone increasing, we see that $C_{p,n}|X_m \in \mathrm{Cont}(X_m)$ for $p > m$ and $n \geqq 1$.

From Lemma 4.1 it follows that each $A_{m,n}$ is dissipative on the closed convex set X_m and hence, in the same way as in (2.8), we have that $R(I-\lambda A_{m,n}) \supset X_m$ for $\lambda > 0$. Thus, by Theorem 1.2, $A_{m,n}$ generates a semigroup $\{T_{m,n}(t);\ t \geqq 0\}$ on X_m.

Next, we give a technical lemma which plays a central role in our arguments.

LEMMA 4.2. *Let $u \in X_m$. Then for every constant k with $|k| \leqq m$ and every nonnegative $f \in C_0^\infty(R^d)$,*

$$\langle \mathrm{sign}\,(u-k)A_{m,n}u, f\rangle \leqq (2d)^{-1} \sum_{i=1}^{d} \langle |u-k|, (l^2/h)D_i^- D_i^+ f\rangle + \sum_{i=1}^{d} \langle \mathrm{sign}\,(u-k)(\phi_i(u)-\phi_i(k)), D_i^0 f\rangle, \quad n \geqq 1. \tag{4.2}$$

PROOF. Let k be a constant such that $|k| \leqq m$ and let $u \in X_m$. Then we can write as

$$\mathrm{sign}\,(u(x)-k)[A_{m,n}u](x)$$

$$= \mathrm{sign}\,(u(x)-k)\{(l^2/2dh) \sum_{i=1}^{d} [D_i^- D_i^+ u](x) - \sum_{i=1}^{d} [D_i^0 \phi_i(u)](x)\}$$

$$= \operatorname{sign}(u(x)-k)\{(l^2/2dh)\sum_{i=1}^{d}[D_i^- D_i^+(u-k)](x)$$
$$- \sum_{i=1}^{d}[D_i^0(\phi_i(u)-\phi_i(k))](x)\} .$$

Fix an i and let $x \in R^d$ be such that $|u(x \pm le_i)| \leqq m$. Then

$$\phi_i(u(x\pm le_i)) - \phi_i(k) = \phi_i'(\theta_i^\pm(x))(u(x\pm le_i)-k)$$

by the mean value theorem. Hence, noting that

$$(2d)^{-1} \mp (h/2l)\phi_i'(\theta_i^\pm(x)) \geqq 0 \qquad \text{(by (3.2))}$$

and

$$[\operatorname{sign}(u(x\pm le_i)-k) - \operatorname{sign}(u(x)-k)](u(x\pm le_i)-k) \geqq 0 ,$$

we obtain

$$(2d)^{-1}[|u(x\pm le_i)-k| - \operatorname{sign}(u(x)-k)(u(x\pm le_i)-k)]$$
$$\mp (h/2l)[\operatorname{sign}(u(x\pm le_i)-k) - \operatorname{sign}(u(x)-k)][\phi_i(u(x\pm le_i)) - \phi_i(k)]$$
$$= [(2d)^{-1} \mp (h/2l)\phi_i'(\theta_i^\pm(x))][\operatorname{sign}(u(x\pm le_i)-k)$$
$$- \operatorname{sign}(u(x)-k)](u(x\pm le_i)-k) \geqq 0$$

for almost all $x \in R^d$, from which it follows that

$$\operatorname{sign}(u(x)-k)(l^2/2dh)[D_i^- D_i^+(u-k)](x)$$
$$- \operatorname{sign}(u(x)-k)[D_i^0(\phi_i(u)-\phi_i(k))](x)$$
$$\leqq (l^2/2dh)[D_i^- D_i^+|u-k|](x) - [D_i^0(\operatorname{sign}(u-k)(\phi_i(u)-\phi_i(k)))](x)$$

for almost all $x \in R^d$ and all i. Therefore,

$$\operatorname{sign}(u(x)-k)[A_{m,n}u](x)$$
$$\leqq (l^2/2dh)\sum_{i=1}^{d}[D_i^- D_i^+|u-k|](x) - \sum_{i=1}^{d}[D_i^0(\operatorname{sign}(u-k)(\phi_i(u)-\phi_i(k)))](x)$$

for almost all $x \in R^d$. Multiplying both sides by a nonnegative function $f \in C_0^\infty(R^d)$ and integrating them over R^d, we have

$$\langle \operatorname{sign}(u-k)A_{m,n}u, f\rangle$$
$$\leqq (l^2/2dh)\sum_{i=1}^{d}\langle D_i^- D_i^+|u-k|, f\rangle - \sum_{i=1}^{d}\langle D_i^0(\operatorname{sign}(u-k)(\phi_i(u)-\phi_i(k))), f\rangle$$
$$= (l^2/2dh)\sum_{i=1}^{d}\langle |u-k|, D_i^- D_i^+ f\rangle + \sum_{i=1}^{d}\langle \operatorname{sign}(u-k)(\phi_i(u)-\phi_i(k)), D_i^0 f\rangle .$$

Q. E. D.

REMARK. Let $v \in X_m$ and let f be a nonnegative C^∞-function on R^d such that f and f_{x_i}, $i=1, 2, \cdots, d$, are uniformly bounded on R^d. Then we see letting $k=0$ in the above proof and using (3.2) that

$$
\begin{aligned}
(4.3)\qquad & \langle \operatorname{sign}(v)A_{m,n}v, f\rangle \\
&\leqq (l/dh)\sum_{i=1}^{d}\langle |v|, (l/2)D_i^-D_i^+f\rangle+\sum_{i=1}^{d}\langle \operatorname{sign}(v)\phi_i(v), D_i^0f\rangle \\
&\leqq (1/d\delta_m)\sum_{i=1}^{d}(\sup_x|f_{x_i}(x)|)\int_{R^d}|v(x)|dx+\sum_{i=1}^{d}M_m(\sup_x|f_{x_i}(x)|)\int_{R^d}|v(x)|dx \\
&=[(1/d\delta_m)+M_m](\sum_{i=1}^{d}\sup_x|f_{x_i}(x)|)\|v\|_1 .
\end{aligned}
$$

In the remainder of this section we proceed with the proof of condition (C) of Theorem 2.1.

LEMMA 4.3. *Let $u\in X_m$, $\lambda>0$, and let $v_n=(I-\lambda A_{m,n})^{-1}u$, $n=1, 2, \cdots$. Then we have the following estimates:*

(i) $\|v_n\|_p\leqq\|u\|_p$ *for* $n\geqq 1$ *and* $p=1, \infty$;

(ii) $\int_{R^d}|v_n(x+y)-v_n(x)|dx\leqq\int_{R^d}|u(x+y)-u(x)|dx$ *for* $y\in R^d$ *and* $n\geqq 1$;

(iii) $\int_{|x|>\rho}|v_n(x)|dx\longrightarrow 0$ *as* $\rho\to+\infty$, *uniformly in* n.

Consequently, $\{(I-\lambda A_{m,n})^{-1}u ; n\geqq 1\}$ *is conditionally compact in* L^1 *for each* $u\in X_m$ *and* $\lambda>0$.

PROOF. Let $u\in X_m$ and $\lambda>0$. Since $A_{m,n}0=0$ for $m, n\geqq 1$, Lemma 4.1 states that

$$\|v_n\|_p=\|(I-\lambda A_{m,n})^{-1}u\|_p\leqq\|u\|_p \qquad \text{for } n\geqq 1 \text{ and } p=1, \infty ;$$

hence (i) is obtained.

Since each $A_{m,n}$ commutes with translations, so does $(I-\lambda A_{m,n})^{-1}$. Therefore, again by Lemma 4.1,

$$
\begin{aligned}
&\int_{R^d}|v_n(x+y)-v_n(x)|dx \\
&=\int_{R^d}|[(I-\lambda A_{m,n})^{-1}u(\cdot+y)](x)-[(I-\lambda A_{m,n})^{-1}u](x)|dx \\
&\leqq\int_{R^d}|u(x+y)-u(x)|dx, \qquad y\in R^d,\ n\geqq 1,
\end{aligned}
$$

which proves (ii).

Next, to prove (iii), let f be a nonnegative C^∞-function on R^d such that f and f_{x_i}, $i=1, 2, \cdots, d$, are uniformly bounded on R^d and put $v=v_n$ in (4.3). Then, using the relations $A_{m,n}v_n=\lambda^{-1}(v_n-u)$, $n=1, 2, \cdots$, we obtain

$$\int_{R^d} |v_n(x)| f(x) dx - \int_{R^d} |u(x)| f(x) dx \leq \lambda[(1/d\delta_m)+M_m]\Big(\sum_{i=1}^{d} \sup |f_{x_i}(x)|\Big)\|u\|_1 . \tag{4.4}$$

Now, we choose a family $\{\delta_{r,\rho};\ \rho>r>0\}$ of C^∞-functions defined on R^1 such that $0 \leq \delta_{r,\rho}(s) \leq 1$ for $s \in R^1$,

$$\delta_{r,\rho}(s)=0 \qquad \text{for } |s| \leq r ,$$

$$\delta_{r,\rho}(s)=1 \qquad \text{for } |s| \geq \rho$$

and such that $\sup_s |\delta'_{r,\rho}(s)| \to 0$ as $\rho \to +\infty$ for each fixed $r>0$. Set

$$f_{r,\rho}(x)=\prod_{i=1}^{d} \delta_{r,\rho}(x_i) \qquad \text{for } x \in R^d .$$

Then it follows from (4.4) that

$$\int_{|x|>\rho} |v_n(x)| dx \leq \int_{|x|>r} |u(x)| dx + \lambda[(1/d\delta_m)+M_m] \sum_{i=1}^{d} \|\delta'_{r,\rho}(x_i)\|_\infty \|u\|_1 .$$

This estimate implies (iii).

Finally, by the Fréchet-Kolmogorov theorem, (i), (ii) and (iii) imply that $\{(I-\lambda A_{m,n})^{-1}u;\ n \geq 1\}$ is conditionally compact in L^1. Q. E. D.

By Lemma 4.3, $\{(I-\lambda A_{m,n})^{-1}u\}_{n\geq 1}$ contains a convergent subsequence. The following lemma proves that such a sequence has only one cluster point. The crucial step of the proof is based on the method proposed by Crandall [**5**; Lemma 2.1] (cf. Kružkov [**14**]).

LEMMA 4.4. *Let $u, v \in X_m$, $\lambda>0$ and $p \geq m$. Let $w_n=(I-\lambda A_{m,n})^{-1}u$ and $z_n=(I-\lambda A_{p,n})^{-1}v$, $n=1, 2, \cdots$. If w is a cluster point of $\{w_n\}$ and z is that of $\{z_n\}$, then*

$$\|w-z\|_1 \leq \|u-v\|_1 .$$

Therefore, for each $\lambda>0$ and $u \in X_m$, $\{(I-\lambda A_{m,n})^{-1}u\}_{n\geq 1}$ is Cauchy in L^1 and the limit is independent of m.

PROOF. For brevity in notation, we denote by the same symbols $\{w_n\}$ and $\{z_n\}$ the subsequences converging in L^1 to w and z respectively, and also by taking their subsequences if necessary, we assume that $w_n(x)$ and $z_n(x)$ converge to $w(x)$ and $z(x)$ almost everywhere on R^d respectively. Also, throughout this proof, we write h' and l' for $h_{p,n}$ and $l_{p,n}$, respectively.

Now, let $f(x, y) \in C_0^\infty(R^d \times R^d)$ and set $u=w_n$, $k=z_n(y)$ and put $f(x)=f(x, y)$ in (4.2). Using the relation $A_{m,n}w_n=\lambda^{-1}(w_n-u)$ and integrating with respect to y, we have

(4.5) $$0 \leqq \int_{R^d \times R^d} \operatorname{sign}(w_n(x)-z_n(y))(u(x)-w_n(x))f(x, y)dxdy$$
$$+(\lambda/2d)\sum_{i=1}^{d}\int_{R^d \times R^d}|w_n(x)-z_n(y)|(l^2/h)[D_i^- D_i^+ f(\cdot, y)](x)dxdy$$
$$+\lambda\sum_{i=1}^{d}\int_{R^d \times R^d}\operatorname{sign}(w_n(x)-z_n(y))[\phi_i(w_n(x))-\phi_i(z_n(y))][D_i^0 f(\cdot, y)](x)dxdy,$$

where $[D_i^0 f(\cdot, y)](x)=(2l)^{-1}[f(x+le_i, y)-f(x-le_i, y)]$ and the others denote similar difference operations. Next, interchanging w_n and z_n, then x and y, we obtain the inequality symmetric to (4.5). Add these two inequalities to find

(4.6) $$0 \leqq \int_{R^d \times R^d}\operatorname{sign}(w_n(x)-z_n(y))(u(x)-v(y)-w_n(x)+z_n(y))f(x, y)dxdy$$
$$+(\lambda/2d)\sum_{i=1}^{d}\int_{R^d \times R^d}|w_n(x)-z_n(y)|\{(l^2/h)[D_i^- D_i^+ f(\cdot, y)](x)$$
$$+(l'^2/h')[D_i^- D_i^+ f(x, \cdot)](y)\}dxdy$$
$$+\lambda\int_{R^d \times R^d}\operatorname{sign}(w_n(x)-z_n(y))\sum_{i=1}^{d}[\phi_i(w_n(x))-\phi_i(z_n(y))]$$
$$\times\{[D_i^0 f(\cdot, y)](x)+[D_i^0 f(x, \cdot)](y)\}dxdy,$$

where $[D_i^0 f(x, \cdot)](y)=(2l')^{-1}[f(x, y+l'e_i)-f(x, y-l'e_i)]$ and the others denote similar difference operations.

Since

$$\operatorname{sign}(w_n(x)-z_n(y)) \longrightarrow \operatorname{sign}(w(x)-z(y))$$
$$\text{a. e. on } \{(x, y)\in R^d \times R^d;\ w(x) \gtrless z(y)\}$$

and

$$\operatorname{sign}(w_n(x)-z_n(y))\sum_{i=1}^{d}[\phi_i(w_n(x))-\phi_i(z_n(y))] \longrightarrow 0$$
$$\text{a. e. on } \{(x, y)\in R^d \times R^d;\ w(x)=z(y)\},$$

the Lebesgue convergence theorem yields that

(4.7) $$0 \leqq \int_{R^d \times R^d}(|u(x)-v(y)|-|w(x)-z(y)|)f(x, y)dxdy$$
$$+\lambda\int_{R^d \times R^d}\operatorname{sign}(w(x)-z(y))\sum_{i=1}^{d}[\phi_i(w(x))-\phi_i(z(y))](f_{x_i}+f_{y_i})dxdy.$$

Now, take a nonnegative function $\sigma \in C_0^\infty(R^1)$ such that $\int_{-\infty}^{\infty}\sigma(s)ds=1$. Let

$$\omega(x)=\prod_{i=1}^{d}\sigma(x_i), \qquad x \in R^d$$

and

$$\omega_\rho(x)=\rho^{-d}\omega\Big(\frac{x}{\rho}\Big), \qquad \rho>0,\ x\in R^d.$$

We then set for a nonnegative function $g\in C_0^\infty(R^d)$

$$f(x, y)=g\Big(\frac{x+y}{2}\Big)\omega_\rho\Big(\frac{x+y}{2}\Big).$$

Substituting this f into (4.7) and using transformations $2\xi=x+y$, $2\eta=x-y\in R^d$, we find

$$(4.8)\qquad 0\leqq\int_{R^d}\Big[\int_{R^d}\{(|u(\xi+\eta)-v(\xi-\eta)|-|w(\xi+\eta)-z(\xi-\eta)|)g(\xi)$$
$$+\lambda\,\mathrm{sign}(w(\xi+\eta)-z(\xi-\eta))\sum_{i=1}^{d}(\phi_i(w(\xi+\eta))-\phi_i(z(\xi-\eta)))g_{\xi_i}(\xi)\}\,d\xi\Big]\omega_\rho(\eta)d\eta.$$

Let us denote the integral in [ ] by $I_g(\eta)$. Then we have

$$(4.9)\qquad 0\leqq\liminf_{\rho\to+0}\int_{R^d}I_g(\eta)\omega_\rho(\eta)d\eta$$
$$\leqq\limsup_{|\eta|\to0}I_g(\eta)$$
$$\leqq\lim_{|\eta|\to0}\int_{R^d}\{(|u(\xi+\eta)-v(\xi-\eta)|-|w(\xi+\eta)-z(\xi-\eta)|)g(\xi)$$
$$+\lambda\sum_{i=1}^{d}|[\phi_i(w(\xi+\eta))-\phi_i(z(\xi-\eta))]g_{\xi_i}(\xi)|\}\,d\xi$$
$$=\int_{R^d}(|u(\xi)-v(\xi)|-|w(\xi)-z(\xi)|)g(\xi)d\xi$$
$$+\lambda\sum_{i=1}^{d}\int_{R^d}|[\phi_i(w(\xi))-\phi_i(z(\xi))]g_{\xi_i}(\xi)|\,d\xi$$

by the Lebesgue convergence theorem. Set $g(\xi)=\kappa(|\xi|/r)$ in (4.9), where $\kappa\in C_0^\infty(R^1)$, $\kappa\geqq0$ and $\kappa(s)=1$ for $|s|\leqq1$, and then let $r\to+\infty$ to conclude that

$$0\leqq\int_{R^d}(|u(\xi)-v(\xi)|-|w(\xi)-z(\xi)|)d\xi,$$

that is,

$$\|w-z\|_1\leqq\|u-v\|_1. \qquad \text{Q.E.D.}$$

We are now in position to prove condition (C_1) and (C) in Theorem 2.3.

THEOREM 4.5. (i) *For every* $u\in C_0^1(R^d)\cap X_m$,

$$\lim_{n\to\infty}A_{m,n}u=-\sum_{i=1}^{d}(\phi_i(u))_{x_i}=-\sum_{i=1}^{d}\phi_i'(u)u_{x_i},$$

where the differentiation is taken in the classical sense.

(ii) *There exists a pseudo-resolvent* $\{J_\lambda;\lambda>0\}\subset\mathrm{Cont}(X_0)$ *such that*

$$J_\lambda u = \lim_{n\to\infty} (I-\lambda A_{m,n})^{-1}u \qquad \text{for } \lambda>0,\ u\in X_m \text{ and } m\geqq 1.$$

Moreover, for every $\lambda>0$ *and* $u\in X_0=L^1\cap L^\infty$, $J_\lambda u$ *gives a solution of the equation*

$$u=v+\lambda\sum_{i=1}^d (\phi_i(v))_{x_i},$$

where the differentiation is taken in the sense of distributions.

PROOF. (i) Let $u\in X_m\cap C_0^1(R^d)$. Then

$$[A_{m,n}u](x)=(l/2dh)\sum_{i=1}^d [lD_i^- D_i^+ u](x) - \sum_{i=1}^d [D_i^0\phi_i(u)](x).$$

In view of (3.2), the first term goes to 0 and the second to $-\sum_{i=1}^d \phi_i'(u)u_{x_i}$ as $n\to\infty$, uniformly on R^d. Hence, we have the assertion (i).

(ii) By Lemma 4.4, we can define the operators J_λ, $\lambda>0$, on X_0 by $J_\lambda u=\lim_{n\to\infty}(I-\lambda A_{m,n})^{-1}u$ for $u\in X_m$. Since each $\{(I-\lambda A_{m,n})^{-1};\ \lambda>0\}$ satisfies the resolvent formula (1.2), $(I-\lambda A_{m,n})^{-1}[X_m]\subset X_m$, and since X_0 is linear, we see that $\{J_\lambda;\ \lambda>0\}$ forms a pseudo-resolvent of contractions from X_0 into itself. Next, let $\lambda>0$ and $u\in X_0$. Noting that $u\in X_m$ for some $m\geqq 1$, we set $v_n=(I-\lambda A_{m,n})^{-1}u$, $n=1,2,3,\cdots$. We first demonstrate that $A_{m,n}v_n$ converges to $-\sum_{i=1}^d(\phi_i(J_\lambda u))_{x_i}$ in the sense of distributions. For any $f\in C_0^\infty(R^d)$, we have

$$\begin{aligned}\langle A_{m,n}v_n, f\rangle &= (l^2/2dh)\sum_{i=1}^d \langle D_i^- D_i^+ v_n, f\rangle - \sum_{i=1}^d \langle D_i^0\phi_i(v_n), f\rangle \\ &= (l^2/2dh)\sum_{i=1}^d \langle v_n, D_i^- D_i^+ f\rangle + \sum_{i=1}^d \langle \phi_i(v_n), D_i^0 f\rangle .\end{aligned}$$

Employing (3.2) and passing to the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\langle A_{m,n}v_n, f\rangle = \sum_{i=1}^d \langle \phi_i(J_\lambda u), f_{x_i}\rangle \qquad \text{for } f\in C_0^\infty(R^d).$$

This implies the desired convergence. On the other hand, $A_{m,n}v_n=\lambda^{-1}(v_n-u)\to\lambda^{-1}(J_\lambda u-u)$ in L^1, from which it follows that $\sum_{i=1}^d(\phi_i(J_\lambda u))_{x_i}$ becomes a function and

$$u=J_\lambda u+\lambda\sum_{i=1}^d(\phi_i(J_\lambda u))_{x_i} \quad \text{in } L^1. \tag{4.10}$$

This proves the last assertion of (ii). Q.E.D.

REMARK. Theorem 4.5 (i) states that (C_1) holds for $X_0=L^1\cap L^\infty$ and $D=C_0^1(R^d)$. Let us define the limit operator A_1 by $A_1u=-\sum_{i=1}^d(\phi_i(u))_{x_i}$, $u\in C_0^1(R^d)$. Though A_1 is densely defined, $\overline{R(I-A_1)}$ does not necessarily coincide with L^1 (Crandall [**5**; Example 3.3]). This means that $R(I-\lambda A_1)\not\supset X_0$ and that even $\bar{A}_1$ does not satisfy the range condition (R). In this way, (C_1) does not necessarily imply (C).

Theorem 4.5 states that all of the assumptions of Theorem 2.3 (b) are satisfied for $X_0 = L^1 \cap L^\infty$ and $D = C_0^1(R^d)$. Therefore, there exists a dissipative operator A such that $C_0^1(R^d) \subset D(A) \subset L^1 \cap L^\infty$ and $J_\lambda = (I - \lambda A)^{-1}$ for $\lambda > 0$. Moreover, A is single-valued. In fact, let $J_\lambda u = J_\lambda v$, then from the relation (4.10) it follows that $u = v$. This means that each J_λ is injective. Hence A is single-valued by Proposition 1.1 (ii) and furthermore, (4.10) implies that $Au = -\sum_{i=1}^{d} (\phi_i(u))_{x_i}$ for $u \in D(A)$. Consequently, the assertions (i) and (ii) of Theorem 3.2 are proved.

Finally, we see from the above-mentioned that A generates a semigroup $\{\bar{T}(t);\ t \geqq 0\}$ on $L^1 = \overline{D(A)}$. Set $T(t) = \bar{T}(t) | X_0$ for $t \geqq 0$. Then by Remark (5) after Corollary 2.2, $\{T(t);\ t \geqq 0\}$ forms an L^1-contractive semigroup on $X_0 = L^1 \cap L^\infty$ such that $T(t)u = \lim_{n \to \infty} \left(I - \frac{t}{n} A\right)^{-n} u$ for $t \geqq 0$ and $u \in L^1 \cap L^\infty$. Therefore, we have the assertion (iv) of Theorem 3.2 by applying Theorem 2.3 (b).

§ 5. Generalized solution of (CP).

In this section we discuss the generalized solution of (CP) and give some comments on the semigroup $\{T(t);\ t \geqq 0\}$ constructed in the preceding section. We start with the following theorem which proves the assertion (iii) of Theorem 3.2:

THEOREM 5.1. *Let $\{T(t);\ t \geqq 0\}$ be a semigroup on $L^1 \cap L^\infty$ obtained in the preceding section. Then for any $u \in L^1 \cap L^\infty$, $u(t) = T(t)u$ gives a generalized solution of (CP) with the initial-value u.*

PROOF. We want to show that $(G.1)$, $(G.2)$ and $(G.3)$ stated in Definition 3.1 hold for $u(t, x) = [T(t)u](x)$. First, $(G.1)$ and $(G.3)$ are evident from the property of $\{T(t);\ t \geqq 0\}$ and from the fact that $T(t)[X_m] \subset X_m$ for $t \geqq 0$ and $m \geqq 1$. We then prove $(G.2)$. Let $u \in L^1 \cap L^\infty$. Noting that $u \in X_m$ for some $m \geqq 1$, we set $u_\varepsilon(t) = (I - \varepsilon A_{m,n})^{-[t/\varepsilon]} u$ and $u_\varepsilon(t, x) = [u_\varepsilon(t)](x)$ for $\varepsilon > 0$, $t \geqq 0$ and $x \in R^d$. Then $\lim_{\varepsilon \to +0} u_\varepsilon(t) = T_{m,n}(t)u \in X_m$ holds uniformly for t on bounded subintervals of $[0, \infty)$ by (2.9), and

$$
\begin{aligned}
(5.1) \qquad & \operatorname{sign}(u_\varepsilon(t, x) - k)[A_{m,n} u_\varepsilon(t)](x) \\
&= \varepsilon^{-1} \operatorname{sign}(u_\varepsilon(t, x) - k)[(u_\varepsilon(t, x) - k) - (u_\varepsilon(t - \varepsilon, x) - k)] \\
&\geqq \varepsilon^{-1}(|u_\varepsilon(t, x) - k| - |u_\varepsilon(t - \varepsilon, x) - k|) \quad \text{for } k \in R^1 \text{ and } x \in R^d.
\end{aligned}
$$

Let $|k| \leqq m$ and let $f \in C_0^\infty((0, \infty) \times R^d)$, $f \geqq 0$. Set $u(x) = u_\varepsilon(t, x)$ and $f(x) = f(t, x)$ in (4.2). Then we see using (5.1) and integrating both sides of (4.2) over $\varepsilon \leqq t < \infty$ that

(5.2) $$0 \leqq I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon),$$

where

$$I_1(\varepsilon)=\int_\varepsilon^\infty\int_{R^d}\varepsilon^{-1}(|u_\varepsilon(t-\varepsilon, x)-k|-|u_\varepsilon(t, x)-k|)f(t, x)dxdt$$

$$=\int_0^\infty\int_{R^d}|u_\varepsilon(t, x)-k|\varepsilon^{-1}(f(t+\varepsilon, x)-f(t, x))dxdt$$

$$+\int_0^\varepsilon\int_{R^d}\varepsilon^{-1}|u_\varepsilon(t, x)-k|f(t, x)dxdt,$$

$$I_2(\varepsilon)=\int_\varepsilon^\infty\int_{R^d}|u_\varepsilon(t, x)-k|(l^2/2dh)\sum_{i=1}^d[D_i^-D_i^+f(t, \cdot)](x)dxdt$$

and

$$I_3(\varepsilon)=\int_\varepsilon^\infty\int_{R^d}\mathrm{sign}\,(u_\varepsilon(t, x)-k)\sum_{i=1}^d[\phi_i(u_\varepsilon(t, x))-\phi_i(k)][D_i^0f(t, \cdot)](x)dxdt.$$

Since $f(t, x)$ has a support which is compact in $(0, \infty)\times R^d$, the second term of $I_1(\varepsilon)$ is equal to 0 for sufficiently small $\varepsilon>0$. Hence,

$$\lim_{\varepsilon\to+0} I_1(\varepsilon)=\int_0^\infty\int_{R^d}|[T_{m,n}(t)u](x)-k|f_t(t, x)dxdt.$$

Since $u_\varepsilon(t, \cdot)\in X_m$ and $l/2dh \leqq 1/2d\delta_m$ by (3.2), we obtain

$$\limsup_{\varepsilon\to+0} I_2(\varepsilon)\leqq \mathrm{const}\,(m, f)l.$$

Also, $|\phi_i(u_\varepsilon(t, x))-\phi_i(k)|\leqq 2mM_m$ and $|[D_i^0f(t, \cdot)](x)-f_{x_i}(t, x)| \leqq \mathrm{const}\,(f)l$, where M_m is the constant associated with X_m through (3.2); hence,

$$\limsup_{\varepsilon\to+0} I_3(\varepsilon)$$

$$\leqq \mathrm{const}\,(m, f)l$$

$$+\int_0^\infty\int_{R^d}\mathrm{sign}\,([T_{m,n}(t)u](x)-k)\sum_{i=1}^d[\phi_i([T_{m,n}(t)u](x))-\phi_i(k)]f_{x_i}(t, x)dxdt.$$

Combining these estimates with (5.2), we have

(5.3) $$0\leqq Ml+\int_0^\infty\int_{R^d}\{|[T_{m,n}(t)u](x)-k|f_t$$

$$+\mathrm{sign}\,([T_{m,n}(t)u](x)-k)\sum_{i=1}^d[\phi_i([T_{m,n}(t)u](x))-\phi_i(k)]f_{x_i}\}dxdt$$

for $|k|\leqq m$, $f\in C_0^\infty((0, \infty)\times R^d)$, $f\geqq 0$, and for some constant M depending on m as well as f.

Now we proceed with the same argument as in the proof of Lemma 4.4. First, by Theorem 2.1, we have that $\lim_{n\to\infty} T_{m,n}(t)u=T(t)u$ holds uniformly on bounded t-intervals. Next, by taking a suitable subsequence if necessary, we see that $\operatorname{sign}([T_{m,n}(t)u](x)-k)\sum_{i=1}^{d}[\phi_i([T_{m,n}(t)u](x))-\phi_i(k)]$ converges to $\operatorname{sign}([T(t)u](x)-k)\sum_{i=1}^{d}[\phi_i([T(t)u](x))-\phi_i(k)]$ a.e. on $\{(t, x)\in[0,\infty)\times R^d;\ [T(t)u](x)\gtrless k\}$ and to 0 a.e. on $\{(t, x)\in[0,\infty)\times R^d;\ [T(t)u](x)=k\}$. Hence, passing to the limit as $n\to\infty$ in (5.3), we have

$$0\leqq\int_0^\infty\int_{R^d}\{|[T(t)u](x)-k|f_t$$
$$+\operatorname{sign}([T(t)u](x)-k)\sum_{i=1}^{d}[\phi_i([T(t)u](x))-\phi_i(k)]f_{x_i}\}\,dxdt$$

for $|k|\leqq m$ and $f\in C_0^\infty((0,\infty)\times R^d)$ with $f\geqq 0$. The m can be arbitrarily large, and so, k can be arbitrary in R^1. This means that $u(t, x)=[T(t)u](x)$ satisfies $(G.2)$. Q.E.D.

In the preceding section we obtained a dissipative operator A from the pseudo-resolvent $\{J_\lambda;\ \lambda>0\}$ on X_0. Crandall introduced in [**5**; Definition 1.1] the following operator A_0.

DEFINITION 5.2. $u\in D(A_0)$ and $v\in A_0u$ if $u, v\in L^1$, $\phi_i(u)\in L^1$, $i=1, 2, \cdots, d$, and if

$$(5.4)\qquad \int_{R^d}\operatorname{sign}(u(x)-k)\{\sum_{i=1}^{d}[\phi_i(u(x))-\phi_i(k)]f_{x_i}(x)+v(x)f(x)\}\,dxdt\geqq 0$$

for every $f\in C_0^\infty(R^d)$ with $f\geqq 0$ and every $k\in R^1$.

A_0 should be treated as a multi-valued operator in L^1, but it is easily seen that $A_0|L^\infty$ is single-valued and for each $u\in D(A_0)\cap L^\infty$, $A_0u=\sum_{i=1}^{d}(\phi_i(u))_{x_i}$ in the sense of distributions. See [**5**; Lemma 1.1].

Crandall proves that $-A_0$ is dissipative in L^1 and its closure $-\bar{A}_0$ is m-dissipative. The following shows the relationship between A and A_0; the central part of the proof is based on the method due to Brézis (see [**5**; Appendix]).

THEOREM 5.3. $A\subset -A_0$. *More precisely, if we define* B_0 *by*

$$B_0u=A_0u\qquad \text{for } u\in D(B_0)=\{u\in D(A_0);\ u,\ A_0u\in L^\infty\}\,,$$

then $A=-B_0$.

PROOF. Let $u\in L^1\cap L^\infty$, $\lambda>0$, $k\in R^1$ and let m be such that $\|u\|_\infty\leqq m$ and $|k+1|$, $|k-1|\leqq m$. Let $\Phi_j(s)$ be the functions defined by (1.6) and put $p_j(s)=\Phi_j'(s-k)$ and $v_n=(I-\lambda A_{m,n})^{-1}u$, $n=1, 2, 3, \cdots$, for simplicity. Then $p_j'(s)$ exists almost everywhere on R^1, $p_j'(s)$ is $\geqq 0$ and has a compact support contained in $[k-1, k+1]$. Hence, in view of lemma 4.2,

$$(5.5)\qquad 0\leqq\int_{R^1}p_j'(s)\Big\{\int_{R^d}\mathrm{sign}\,(v_n(x)-s)[(l^2/2dh)\sum_{i=1}^{d}(v_n(x)-s)[D_i^-D_i^+f](x)$$
$$+\sum_{i=1}^{d}[\phi_i(v_n(x))-\phi_i(s)][D_i^0f](x)-[A_{m,n}v_n](x)f(x)]dx\Big\}ds$$

for $f\in C_0^\infty(R^d)$ with $f\geqq 0$, where the integral makes sense since $\mathrm{supp}\,[p_j']\subset[k-1, k+1]\subset[-m, m]$. Put

$$F_n(x)=(l^2/2dh)v_n(x)\sum_{i=1}^{d}[D_i^-D_i^+f](x)$$
$$+\sum_{i=1}^{d}\phi_i(v_n(x))[D_i^0f](x)-[A_{m,n}v_n](x)f(x)\,,$$
$$G_n(x)=(l^2/2dh)s\sum_{i=1}^{d}[D_i^-D_i^+f](x)+\sum_{i=1}^{d}\phi_i(s)[D_i^0f](x)\,,$$

and apply the Fubini's theorem to find

$$(5.6)\qquad \frac{1}{2}\int_{R^1}p_j'(s)\Big(\int_{R^d}\mathrm{sign}\,(v_n(x)-s)F_n(x)dx\Big)ds$$
$$=\frac{1}{2}\int_{R^d}F_n(x)\Big(\int_{R^1}\mathrm{sign}\,(v_n(x)-s)p_j'(s)ds\Big)dx$$
$$=\frac{1}{2}\int_{R^d}F_n(x)\Big(\int_{-\infty}^{v_n(x)}-\int_{v_n(x)}^{\infty}\Big)p_j'(s)dsdx$$
$$=\int_{R^d}p_j(v_n(x))F_n(x)dx$$

and

$$(5.7)\qquad \int_{R^1}p_j'(s)\Big(\int_{R^d}\mathrm{sign}\,(v_n(x)-s)G_n(x)dx\Big)ds$$
$$=\int_{R^d}(l^2/2dh)\sum_{i=1}^{d}[D_i^-D_i^+f](x)\Big(\int_{R^1}\mathrm{sign}\,(v_n(x)-s)sp_j'(s)ds\Big)dx$$
$$+\int_{R^d}\sum_{i=1}^{d}[D_i^0f](x)\Big(\int_{R^1}\mathrm{sign}\,(v_n(x)-s)\phi_i(s)p_j'(s)ds\Big)dx\,.$$

Also, observe that the integral $\int_{R^1}\mathrm{sign}\,(v_n(x)-s)\phi_i(s)p_j'(s)ds$ can be written as $\Big(\int_{-\infty}^{k}+2\int_{k}^{v_n(x)}-\int_{k}^{\infty}\Big)\phi_i(s)p_j'(s)ds$. Therefore, combining (5.5)-(5.7), we have

$$0\leqq\int_{R^1}p_j'(s)\Big\{\int_{R^d}\mathrm{sign}\,(v_n(x)-s)(F_n(x)-G_n(x))dx\Big\}ds$$
$$=2\int_{R^d}p_j(v_n(x))F_n(x)dx$$
$$-\int_{R^d}(l^2/2dh)\sum_{i=1}^{d}[D_i^-D_i^+f](x)\Big(\int_{R^1}\mathrm{sign}\,(v_n(x)-s)sp_j'(s)ds\Big)dx$$
$$-\int_{R^d}\sum_{i=1}^{d}[D_i^0f](x)\Big(\int_{-\infty}^{k}+2\int_{k}^{v_n(x)}-\int_{k}^{\infty}\Big)\phi_i(s)p_j'(s)dsdx\,.$$

Passing to the limit as $n\to\infty$ and using the convergence $A_{m,n}v_n\to -\sum_{i=1}^{d}(\phi_i(v))_{x_i}$ in $\mathcal{D}'$ (Theorem 4.5 (ii)), where $v=J_\lambda u\in X_m$, it follows that

$$0\leqq\int_{R^d}p_j(v(x))[\sum_{i=1}^{d}\phi_i(v(x))f_{x_i}(x)+\sum_{i=1}^{d}(\phi_i(v(x))_{x_i}f(x)]dx \tag{5.8}$$
$$-\frac{1}{2}\sum_{i=1}^{d}\int_{R^d}f_{x_i}(x)\Big(\int_{-\infty}^{k}+2\int_{k}^{v(x)}-\int_{k}^{\infty}\Big)\phi_i(s)p_j'(s)dsdx.$$

Since $\sum_{i=1}^{d}\Big(\int_{R^d}f_{x_i}(x)dx\Big)\Big(\int_{-\infty}^{k}-\int_{k}^{\infty}\Big)\phi_i(s)p_j'(s)ds=0$, we have letting $j\to\infty$ in (5.8) and then using (1.7) and (1.8) that

$$0\leqq\int_{R^d}\operatorname{sign}(v(x)-k)\{\sum_{i=1}^{d}[\phi_i(v(x))-\phi_i(k)]f_{x_i}+\sum_{i=1}^{d}(\phi_i(v(x)))_{x_i}f\}dx. \tag{5.9}$$

This means that $v=J_\lambda u\in D(A_0)$ and $Av=-\sum_{i=1}^{d}(\phi_i(v))_{x_i}\in -A_0v$. Since v, $Av=\lambda^{-1}(J_\lambda u-u)\in L^\infty$, it follows that $A\subset -B_0$. To show the converse, it suffices to prove that $D(B_0)\subset D(A)$. Let $w\in D(B_0)$, $\lambda>0$ and let $u=w+\lambda B_0w$. Then, w and u belong to some X_m. Hence, we see from the above-mentioned proof that $J_\lambda u\in D(B_0)$ and $\sum_{i=1}^{d}(\phi_i(J_\lambda u))_{x_i}=B_0J_\lambda u$. Therefore, in view of Theorem 4.5 (ii), we have that

$$u=J_\lambda u+\lambda\sum_{i=1}^{d}(\phi_i(J_\lambda u))_{x_i}=(I+\lambda B_0)J_\lambda u.$$

From this it follows that $(I+\lambda B_0)w=(I+\lambda B_0)J_\lambda u$. Since $-B_0(\subset -A_0)$ is dissipative, $J_\lambda u=w(=(I+\lambda B_0)^{-1}u)$. This states that $D(B_0)\subset D(A)$ and consequently, $A=-B_0$. Q.E.D.

REMARKS. (1) Combining Theorem 5.3 with Theorem 4.5, it follows that $R(I+\lambda A_0)\supset L^1\cap L^\infty$ for $\lambda>0$. Since $L^1\cap L^\infty$ is dense in L^1, $\bar{A}=-\bar{A}_0$ and $\bar{A}$ is m-dissipative. This means that the semigroup $\{\bar{T}(t);t\geqq 0\}$ on L^1 which is generated by A coincides with the semigroup of Crandall.

(2) We mentioned in Remark (2) after Corollary 2.2 that condition (C) is divided into two conditions (C') and (C''). In fact, in the case of the difference approximation (3.3), Theorem 4.5 (ii) proves (C') and Theorem 5.3 shows that (C'') holds for $B=-A_0$.

In the remainder of this section we give some comments on the results mentioned in Section 3.

First, we give a result related to the domain of dependence.

PROPOSITION 5.4. *Let* u, $v\in X_m$ *and* $\tau>0$. *Let* $K_r=\{x\in R^d;\ |x_i|\leqq r,\ i=1,2,\cdots,d\}$ *for* $r>0$. *Then we have:*

(i) $\|C_{m,n}^{\nu}u-C_{m,n}^{\nu}v\|_{L^1(K_r)}\leqq\|u-v\|_{L^1(K_{r+\tau})}$ *for* $\nu l\in[0,\tau]$.

(ii) $\|T(t)u-T(t)v\|_{L^1(K_r)}\leqq\|u-v\|_{L^1(K_{r+\tau})}$ *for* $t\in[0,\delta_m\tau]$.

PROOF. Similarly to the proof of Lemma 4.1, we have

$$\int_{K_r}|[C_{m,n}u](x)-[C_{m,n}v](x)|\,dx \leqq \int_{K_{r+l}}|u(x)-v(x)|\,dx\,.$$

Hence, we can write as $\|C_{m,n}u-C_{m,n}v\|_{L^1(K_r)} \leqq \|u-v\|_{L^1(K_{r+l})}$. Inductively, we have that $\|C^{\nu}_{m,n}u-C^{\nu}_{m,n}v\|_{L^1(K_r)} \leqq \|u-v\|_{L^1(K_{r+\nu l})}$, from which (i) follows. Since $\|w\|_{L^1(K_r)} \leqq \|w\|_1$ for $w \in L^1$ and since $\nu l \leqq \nu h/\delta_m$, (ii) is easily seen from Theorem 3.2 (iv). Q. E. D.

REMARKS. (1) The above proposition together with its proof reflects the shape of the domain of dependence. Lemma 4.1 can be regarded as a special case of this proposition.

(2) Proposition 5.4 (ii) represents a hyperbolic character: For every r, τ and m, $[T(t)u](x)=[T(t)v](x)$ a. e. on K_r if $u, v \in X_m$, $t \in [0, \delta_m\tau]$ and if $u(x)=v(x)$ a. e. on $K_{r+\tau}$.

Next, we introduce two important classes which are invariant under $T(t)$. By $BV \equiv BV(R^d)$ we mean the set of those elements $u \in L^1 \cap L^\infty$ such that for every compact domain Ω, there is a constant $M_\Omega > 0$ and

$$\int_\Omega |u(x+\Delta x)-u(x)|\,dx \leqq M_\Omega |\Delta x| \qquad \text{for } \Delta x \in R^d\,.$$

It is proved (cf. Krickeberg [**12**]) that every element $u \in BV$ is of locally bounded variation in the sense of Tonelli-Cesari. By $UBV \equiv UBV(R^d)$ we denote the set consisting of those elements $u \in L^1 \cap L^\infty$ such that $\|u(\cdot+\Delta x)-u\|_1 \leqq M_u|\Delta x|$ for $\Delta x \in R^d$ and for some constant $M_u > 0$. Observe that $C_0^\infty(R^d) \subset UBV \subset BV$.

For the following two results, we refer to Conway-Smoller [**3**] and Kojima [**9**].

THEOREM 5.5. *BV and UBV are invariant under $T(t)$ as well as $C_{m,n}$.*

PROOF. Let $u \in BV \cap X_m$ and let $\tau, r > 0$. As was mentioned in the proof of Lemma 4.3, $C_{m,n}$ commutes with translations. Hence, we see letting $v(x)=u(x+\Delta x)$ in Proposition 5.4 that

$$\begin{aligned}\|[C^{\nu}_{m,n}u](\cdot+\Delta x)-C^{\nu}_{m,n}u\|_{L^1(K_r)} &= \|C^{\nu}_{m,n}u(\cdot+\Delta x)-C^{\nu}_{m,n}u\|_{L^1(K_r)}\\ &\leqq \|u(\cdot+\Delta x)-u\|_{L^1(K_{r+\nu l})} \leqq \text{const.}|\Delta x| \qquad \text{for } \nu l \in [0, \tau]\,,\end{aligned}$$

where the constant depends only on r, τ and u. Since r is arbitrary, this means that $C^{\nu}_{m,n}u \in BV$. Also, Theorem 3.2 (iv) and Proposition 5.4 (ii) yield that

$$\|[T(t)u](\cdot+\Delta x)-T(t)u\|_{L^1(K_r)} \leqq \text{const.}|\Delta x|$$

for $t \in [0, \delta_m\tau]$ and $\Delta x \in R^d$. Since τ is arbitrary, it follows that $T(t)u \in BV$ for $t \geqq 0$. The invariantness of UBV follows immediately from Lemma 4.1 and Theorem 3.2 (iv). Q. E. D.

REMARK. In view of this theorem, $\{T(t)|BV;\ t\geqq 0\}$ and $\{T(t)|UBV;\ t\geqq 0\}$ form L^1-contractive semigroups on BV and UBV, respectively.

Classes BV and UBV are closely related to the continuity of $T(t)$ in t.

THEOREM 5.6. (i) *Let $u\in BV$. Then for every $r>0$ and $\tau>0$, there is a constant $M_{r,\tau,u}>0$ such that*

$$\|T(t)u-T(s)u\|_{L^1(K_r)}\leqq M_{r,\tau,u}|t-s| \qquad \text{for } t,\ s\in[0,\tau].$$

Therefore, if we define $T(t)u\equiv u$ for $t<0$, then the function $[T(t)u](x)$ belongs to $BV(R^{d+1})$.

(ii) *Let $u\in UBV$. Then $T(t)u$ is uniformly Lipschitz continuous in t with respect to $\|\cdot\|_1$-norm. Therefore, the function $[T(t)u](x)$ belongs to $UBV(R^{d+1})$.*

(iii) *Let $u\in L^1\cap L^\infty$. If u is uniformly Lipschitz continuous on R^d (hence $u\in UBV$), then there is a constant $M_u>0$, depending only on u, such that*

$$\|T(t)u-u\|_\infty\leqq M_u t \qquad \text{for } t\geqq 0.$$

PROOF. (i) Let $u\in BV\cap X_m$, $0<r<\rho$ and let $\tau>0$. Then

$$[C_{m,n}u](x)-u(x)=(2d)^{-1}\sum_{i=1}^{d}[u(x+le_i)-2u(x)+u(x-le_i)]$$
$$-(h/2l)\sum_{i=1}^{d}[\phi_i(u(x+le_i))-\phi_i(u(x-le_i))],$$

whence

$$\int_{K_\rho}|[C_{m,n}u](x)-u(x)|dx$$
$$\leqq(2d)^{-1}\sum_{i=1}^{d}\left\{\int_{K_\rho}|u(x+le_i)-u(x)|dx+\int_{K_\rho}|u(x)-u(x-le_i)|dx\right\}$$
$$+(h/2l)\sum_{i=1}^{d}\int_{K_\rho}|\phi_i(u(x+le_i))-\phi_i(u(x-le_i))|dx$$
$$\leqq(2d)^{-1}\sum_{i=1}^{d}2M_{\rho,u}l+(h/2l)\sum_{i=1}^{d}M_m\int_{K_\rho}|u(x+le_i)-u(x-le_i)|dx$$
$$\leqq M_{\rho,u}(\delta_m^{-1}+dM_m)h,$$

that is,

$$\|C_{m,n}u-u\|_{L^1(K_\rho)}\leqq M_{\rho,u,m}h.$$

Now, let $0\leqq s<t\leqq\tau$ and $\rho=r+\tau\delta_m^{-1}$. Then $r+[t/h]l\leqq\rho$ for $t\in[0,\tau]$ and Proposition 5.4 yields that

$$\|C_{m,n}^{[t/h]}u-C_{m,n}^{[s/h]}u\|_{L^1(K_r)}\leqq\sum_{\nu=[s/h]}^{[t/h]-1}\|C_{m,n}^{\nu+1}u-C_{m,n}^{\nu}u\|_{L^1(K_r)}$$
$$\leqq([t/h]-[s/h])\|C_{m,n}u-u\|_{L^1(K_\rho)}\leqq(t-s+h)M_{\rho,u,m}.$$

Therefore, we have

$$\|T(t)u-T(s)u\|_{L^1(K_T)} \leqq M_{\rho,u,m}|t-s|$$

by Theorem 3.2 (iv). This implies that the function $[T(t)](x)$ belongs to $BV(R^{d+1})$.

(ii) Let $u \in UBV \cap X_m$. Then, in a similar way to the proof of (i), we have

$$\|C_{m,n}u-u\|_1 \leqq M_{u,m}h$$

for some constant $M_{u,m}$. Hence, we have

$$\|T(t)u-T(s)u\|_1 \leqq M_{u,m}|t-s| \qquad \text{for } t,\ s \geqq 0$$

by Lemma 4.1 and Theorem 3.2 (iv). This fact also yields that the function $[T(t)u](x)$ belongs to $UBV(R^{d+1})$.

(iii) Let $u \in X_m$ and let $|u(x)-u(y)| \leqq M|x-y|$ for $x, y \in R^d$ and for some $M>0$. Then, by applying the mean value theorem we have

$$[C_{m,n}u](x)=\sum_{i=1}^{d}[(2d)^{-1}-(h/2l)\phi_i'(\theta_i(x))]u(x+le_i)$$

$$+\sum_{i=1}^{d}[(2d)^{-1}+(h/2l)\phi_i'(\theta_i(x))]u(x-le_i),$$

where $\theta_i(x)$ are certain values between $u(x+le_i)$ and $u(x-le_i)$. This means that $[C_{m,n}u](x)$ is a convex combination of $u(x+jle_i)$, $i=1, 2, \cdots, d$; $j=\pm 1$. Inductively, we see that $[C_{m,n}^{\nu}u](x)$ is a convex combination of $u(x+jle_i)$, where $i=1, 2, \cdots, d$; $j=\pm 1, \pm 3, \cdots, \pm\nu$ if ν is odd and $j=0, \pm 2, \pm 4, \cdots, \pm\nu$ if ν is even. Let us write

$$[C_{m,n}^{\nu}u](x)=\sum\alpha_{ij}(x)u(x+jle_i), \qquad \alpha_{ij}(x) \geqq 0,\ \sum\alpha_{ij}(x)=1$$

for this combination. Then

$$|[C_{m,n}^{\nu}u](x)-u(x)| \leqq \sum\alpha_{ij}(x)|u(x+jle_i)-u(x)|$$

$$\leqq \sum\alpha_{ij}(x)M|j|l \leqq M\delta_m^{-1}\nu h = M_{u,m}\nu h.$$

In view of Theorem 3.2 (iv), we have that

$$|[T(t)u](x)-u(x)| \leqq M_{u,m}t \qquad \text{a.e. on } R^d.$$

Hence,

$$\|T(t)u-u\|_\infty \leqq M_{u,m}t \qquad \text{for } t \geqq 0.$$

Q.E.D.

REMARK. Crandall [4] introduced a notion of generalized domain $\hat{D}(A)$ of a dissipative operator A. In view of Remark (2) after Theorem 3.2, Theorem 5.5 (ii) states that $UBV \subset \hat{D}(\bar{A})$, where $\hat{D}(\bar{A})$ denotes the generalized domain associated with the m-dissipative operator $\bar{A}$.

§ 6. Notes and Remarks.

In this section we give a variety of observations on the results obtained in the preceding sections.

I. First, we review the result of Theorem 4.5 in the case of $d=1$. In this case we can write (DE) as follows:

$$u_t+(\phi(u))_x=0 \qquad \text{for } t>0,\ x\in R^1.$$

Then Theorem 4.5 (ii) states that $(\phi(J_\lambda u))_x\in L^1\cap L^\infty$, $\lambda>0$. Hence, the Radon-Nikodym theorem yields that $\phi(J_\lambda u)$ is absolutely continuous and the derivative of $\phi(J_\lambda u)$ in the ordinary sense coincides with $(\phi(J_\lambda u))_x$ a. e. on R^1. Consequently, $[J_\lambda u](x)$ satisfies the differential equation $v+\lambda(\phi(v))_x=u$ at almost all $x\in R^1$.

Now, let L be the infinitesimal generator of the group $\{e^{tL};\ t\in R^1\}$ of translation operators on L^1 and Φ be the operator defined by

$$[\Phi u](x)=\phi(u(x)) \qquad \text{for } u\in L^1\cap L^\infty.$$

Then, we have

$$A=-L\Phi \quad \text{on} \quad D(A)=\{u\in D(A_0);\ u,\ A_0u\in L^\infty\}, \tag{6.1}$$

where A_0 is the operator introduced in Definition 5.2.

Φ maps $D(L)$ into itself and is Lipschitz continuous on every X_m with Lipschitz constant $M_m=\sup_{|s|\leqq m}|\phi'(s)|$. In fact, let $u\in D(L)$. Then $\|u\|_\infty\leqq\|u_x\|_1$, so $u\in L^\infty$. Hence, $u\in X_m$ for some m. Since $|\phi(u(x))|\leqq M_m|u(x)|$, we see that $\Phi u\in L^1\cap L^\infty$. On the other hand, $\phi\in C^1(R^1)$; hence $\phi(u)$ is absolutely continuous and $(\phi(u))_x=\phi'(u)u_x$ a. e. on R^1. Thus,

$$\int_{R^1}|(\phi(u))_x|\,dx\leqq M_m\int_{R^1}|u_x|\,dx.$$

This means that $\Phi u\in D(L)$. The second assertion is clear from the fact that

$$\|\Phi u-\Phi v\|_1\leqq M_m\|u-v\|_1 \qquad \text{for } u,\ v\in X_m.$$

Therefore, it follows that

$$C_0^1(R^1)\subset D(L)\subset D(L\Phi). \tag{6.2}$$

Moreover, the approximate operators $A_{m,n}$ can be written as

$$A_{m,n}u=h^{-1}(e^{lL}-2I+e^{-lL})u-(2l)^{-1}(e^{lL}-e^{-lL})\Phi u, \tag{6.3}$$

where $u\in X_m$, $0<\delta_m\leqq h/l\leqq 1/M_m$ and $h=h_{m,n}$, $l=l_{m,n}$.

Now we have the following:

THEOREM 6.1. (i) *For every* $u\in D(L)\cap X_m$, $\lim_{n\to\infty}A_{m,n}u=-L\Phi u$.

(ii) *For every* $u\in X_m$, $J_\lambda u=\lim_{n\to\infty}(I-\lambda A_{m,n})^{-1}u$ *gives a unique solution of*

$$v+\lambda L\Phi v=u\,,\qquad v\in D(A),\ \lambda>0\,.$$

(iii) $L\Phi\,|\,D(A)$ *generates an* L^1*-contractive semigroup* $\{T(t)\,;\,t\geqq 0\}$ on $L^1\cap L^\infty$ *and for every* $u\in L^1\cap L^\infty$,

$$T(t)u-u=-L\int_0^t \Phi T(s)uds \qquad \textit{for } t\geqq 0\,.$$

PROOF. (i) follows from (6.2), (6.3) and the fact that L is the infinitesimal generator of $\{e^{tL}\,;\,t\in R^1\}$. (ii) is evident from Theorem 4.5 (ii) and (6.1).

We then demonstrate that (iii) holds. Since $J_\lambda=(I-\lambda A)^{-1}$, $\lambda>0$, and $A\subset -L\Phi$, we have

$$J_\lambda^\nu u-u=\sum_{p=0}^{\nu-1}\lambda AJ_\lambda^p u-\lambda\{Au-AJ_\lambda^\nu u\}$$

for $u\in D(A)$, $\lambda>0$ and $\nu=1,2,3,\cdots$ (cf. Oharu [**16**; p. 543, (6.3)]).
Noting that $J_\lambda^{[s/\lambda]}u=J_\lambda^p u$ for $p\lambda\leqq s<(p+1)\lambda$, we see that $J_\lambda^{[s/\lambda]}u$ and $AJ_\lambda^{[s/\lambda]}u$ are step functions and summable on every finite interval and that

$$\sum_{p=0}^{\nu-1}\lambda AJ_\lambda^p u=\int_0^{\nu\lambda}AJ_\lambda^{[s/\lambda]}uds\,.$$

Now, let $u\in D(A)\cap X_m$. Then $\|AJ_\lambda^p u\|_1\leqq\|Au\|_1$ and $J_\lambda^p u\in X_m$. Hence, we have that

$$\left\|J_\lambda^{[t/\lambda]}u-u-\int_0^t AJ_\lambda^{[s/\lambda]}uds\right\|_1\leqq 3\lambda\|Au\|_1\,.$$

Also,

$$\int_0^t AJ_\lambda^{[s/\lambda]}uds=-L\int_0^t \Phi J_\lambda^{[s/\lambda]}uds\,.$$

Theorem 3.2 (iii) yields that $J_\lambda^{[s/\lambda]}u\to T(s)u$ as $\lambda\to+0$, uniformly on $[0,t]$. Since Φ is Lipschitz continuous on X_m, $\Phi J_\lambda^{[s/\lambda]}u\to\Phi T(s)u$ uniformly on $[0,t]$. Therefore, $\int_0^t\Phi J_\lambda^{[s/\lambda]}uds\to\int_0^t\Phi T(s)uds$ and $-L\int_0^t\Phi J_\lambda^{[s/\lambda]}uds\to T(t)u-u$ as $\lambda\to+0$. Since L is a closed linear operator, $\int_0^t\Phi T(s)uds\in D(L)$ and

$$T(t)u-u=-L\int_0^t\Phi T(s)uds \qquad \text{for } u\in D(A) \text{ and } t\geqq 0\,.$$

Next, let $u\in L^1\cap L^\infty$. Then, by the definition of L, a sequence $\{u_n\}\subset D(L)$ $(\subset D(A))$ can be chosen such that $\{u_n\}$ and u are contained in some X_m and such that $\|u_n-u\|_1\to 0$ as $n\to\infty$. Hence, we see that

$$\int_0^t\Phi T(s)u_nds\longrightarrow\int_0^t\Phi T(s)uds \qquad \text{as } n\to\infty \text{ for } t\geqq 0\,.$$

On the other hand, $-L\int_0^t \Phi T(s)u_n ds \to T(t)u-u$. Consequently, it follows that $\int_0^t \Phi T(s)uds \in D(L)$ and

$$T(t)u-u=-L\int_0^t \Phi T(s)uds\,, \qquad u\in L^1\cap L^\infty\,. \qquad \text{Q.E.D.}$$

REMARKS. (1) Flaschka [7; Section 3.3] proves that if $u\in C_0^\infty(R^1)$ then the solution v of $v+(\phi(v))_x=u$ is of bounded variation and that such a v becomes an "*entropy solution*".

(2) The formula given in (iii) states that $T(t)u$ is a generalized solution of (CP) in the sense that if $\Phi T(s)u\in D(L)$ a. e., then

$$T(t)u-u=\int_0^t (-L\Phi)T(s)uds\,,$$

which means that $T(t)u$ is a strict solution of (CP). From the assertion (iii) we can derive the following integral relation (cf. [7; Theorem 2]):

$$\int_0^\infty \{\langle T(t)u, f'(t)\rangle - \langle \Phi T(t)u, L^*f(t)\rangle\}\, dt=0 \tag{6.4}$$

for every $f(\cdot)\in C_0^1((0,\infty); D(L^*))$, where L^* denotes the dual operator of L. In fact, for every $f\in D(L^*)$, $\langle T(t)u-u, f\rangle = -\left\langle \int_0^t \Phi T(s)uds,\ L^*f \right\rangle = -\int_0^t \langle \Phi T(s)u, L^*f\rangle ds$. Now let $f(\cdot)\in C_0^1((0,\infty); D(L^*))$. Then $\langle T(t+h)u-T(t)u, f(t)\rangle = -\int_t^{t+h}\langle \Phi T(s)u, L^*f(t)\rangle ds$ for h sufficiently small. Since supp (f) is compact in $(0,\infty)$, it follows from the Fubini's theorem that

$$\int_0^\infty \langle T(t+h)u-T(t)u, f(t)\rangle dt = \int_0^\infty \langle T(t)u, f(t-h)-f(t)\rangle dt$$
$$= -\int_0^\infty \int_t^{t+h} \langle \Phi T(s)u, L^*f(t)\rangle dsdt\,.$$

Dividing both integrals by $-h$ and then passing to the limit as $h\to +0$, we obtain

$$\int_0^\infty \langle T(t)u, f'(t)\rangle dt = \int_0^\infty \langle \Phi T(t)u, L^*f(t)\rangle dt\,.$$

The relation (6.4) is an operator theoretic version of the weak solution in the ordinary sense. But it should be noted that this type of solution need not be unique. Theorem 5.1 gives a sharper result than (6.4).

Next, we consider some generalizations of (DE).

II. We can extend Theorem 3.2 to the case in which

(6.5) $\quad \phi_i(0)=0$ and ϕ_i is locally Lipschitz continuous on R^1 for $i=1, 2, \cdots, d$.

First, for each ϕ_i, we choose a sequence $\{\phi_i^{(n)}\} \subset C^1(R^1)$ such that $\phi_i^{(n)}(0)=0$, $\phi_i^{(n)}$ converges to ϕ_i uniformly on bounded intervals of R^1 and such that for each positive integer m,

$$\max_{1\leq i\leq d} \sup_{|s|\leq m} |\phi_i^{(n)\prime}(s)| \leqq M_m = \|\phi_i\|_{\mathrm{Lip}\,[-m,m]},$$

where $\|\phi_i\|_{\mathrm{Lip}\,[-m,m]}$ denotes the smallest Lipschitz constant of ϕ_i on the interval $[-m, m]$. Next we take sequences $\{\delta_m\}$, $\{h_{m,n}\}$ and $\{l_{m,n}\}$ of positive numbers such that $\lim_{n\to\infty} h_{m,n}=0$ for each $m\geqq 1$ and such that (3.2) holds. We then define the difference operators $C_{m,n}$ and $A_{m,n}$, m, $n\geqq 1$, by

$$[C_{m,n}u](x)=(2d)^{-1}\sum_{i=1}^{d}(u(x+le_i)+u(x-le_i))-h\sum_{i=1}^{d}[D_i^0\phi_i^{(n)}(u)](x), \tag{6.6}$$

$$A_{m,n}u=h^{-1}[C_{m,n}-I]u, \qquad \text{for } u\in X_m \text{ and } m,\ n\geqq 1, \tag{6.7}$$

where $h=h_{m,n}$ and $l=l_{m,n}$. Then, we have the same assertion as in Lemma 4.1 and the same form of inequalities as in (4.2) and in (4.3). Hence, we can obtain the estimates in Lemma 4.3. On the other hand, $\phi_i^{(n)}$ converges to ϕ_i uniformly on every bounded interval, so Lemma 4.4 remains true. Thus, the assertion (ii) in Theorem 4.5 can be obtained for the operators $C_{m,n}$ and $A_{m,n}$. By this result we can define a dissipative operator A. We see that $D(A)$ also contains $C_0^1(R^d)$. In fact, let $u\in C_0^1(R^d)\cap X_m$. Then $\phi_i(u)$ is uniformly Lipschitz continuous on R^d. Hence, by the Radon-Nikodym theorem, the derivatives $(\phi_i(u))_{x_i}$ in the sense of distributions exist as elements of $L^1\cap L^\infty$. Moreover, the convergence $D_i^0\phi_i(u(x))\to(\phi_i(u(x))_{x_i}$ holds a.e. on a neighborhood of supp(u). Therefore, by the dominated convergence theorem, $\sum_{i=1}^{d} D_i^0\phi_i(u)\to\sum_{i=1}^{d}(\phi_i(u))_{x_i}$ in L^1. Hence, $A_{m,n}u$ converges to $-\sum_{i=1}^{d}(\phi_i(u))_{x_i}$ in L^1. This is the assertion (i) in Theorem 4.5. Thus, A generates an L^1-contractive semigroup $\{T(t); t\geqq 0\}$ on $L^1\cap L^\infty$. Also, it is proved in the same way as in Theorem 5.1 that for each $u\in L^1\cap L^\infty$, $u(t, x)=[T(t)u](x)$ becomes a generalized solution of (CP) with the initial-value u. Consequently, we obtain under condition (6.5) the following extension of Theorem 3.2.

THEOREM 6.2. *Let* $\{C_{m,n}\}$ *and* $\{A_{m,n}\}$ *be the operators determined by* (6.6) *and* (6.7). *Then we have the same assertions* (i)-(iv) *as in Theorem* 3.2.

III. As mentioned in Remark (2) after Definition 3.1, we can generalize (DE) to allow the ϕ_i to be of class $C^0(R^1)$ and obtain a semigroup solution of (CP). Suppose that

$$\phi_i\in C^0(R^1),\ \phi_i(0)=0 \text{ and } \limsup_{s\to 0}|\phi_i(s)/s|<+\infty \qquad \text{for } i=1, 2, \cdots, d. \tag{6.8}$$

In this case we choose, for each ϕ_i, a sequence $\{\phi_i^{(n)}\}\subset C^1(R^1)$ such that $\phi_i^{(n)}(0)=0$ and $\phi_i^{(n)}$ converges to ϕ_i uniformly on bounded intervals.

Let $\{J_\lambda^{(n)}; \lambda>0\}$ be the pseudo-resolvent associated with $\phi_i^{(n)}$ via Theorem 4.5 (ii). Let $u\in X_0=L^1\cap L^\infty$ and $\lambda>0$. Then for each n, $J_\lambda^{(n)}\in \mathrm{Cont}(X_0)$, $\|J_\lambda^{(n)}u\|_p\leqq\|u\|_p$ for $p=1, \infty$, $J_\lambda^{(n)}$ commutes with translations, and

$$\|[J_\lambda^{(n)}u](\cdot+y)-J_\lambda^{(n)}u\|_1\leqq\|u(\cdot+y)-u\|_1 \qquad \text{for } y\in R^d.$$

Hence, $\{J_\lambda^{(n)}u\}_{n\geqq 1}$ is conditionally compact in $L^1_{\mathrm{loc}}(R^d)$. Let $\{v_n\}$ be a convergent subsequence of $\{J_\lambda^{(n)}u\}$ and v be its limit. Then, $\|v\|_p\leqq\|u\|_p$ for $p=1,\infty$. We wish to prove that $v=(I+\lambda A_0)^{-1}u$, where A_0 is the accretive operator of Definition 5.2 which is associated with the ϕ_i in (6.8). Let $p_j(s)$, $j=1, 2, \cdots$, be the functions treated in the proof of Theorem 5.3. Then for each n, Theorem 5.3 (cf. (5.9)) states that

$$0\leqq\int_{R^1}p_j'(s)\Big\{\int_{R^d}\mathrm{sign}\,(v_n(x)-s)\Big[\sum_{i=1}^d(\phi_i^{(n)}(v_n(x))-\phi_i^{(n)}(s))f_{x_i} + \sum_{i=1}^d(\phi_i^{(n)}(v_n(x)))_{x_i}f\Big]\Big\}dxds. \tag{6.9}$$

Set

$$F_n(x)=\sum_{i=1}^d\phi_i^{(n)}(v_n(x))f_{x_i}(x)+\sum_{i=1}^d(\phi_i^{(n)}(v_n(x)))_{x_i}f(x)$$

and

$$G_n(x)=\sum_{i=1}^d\phi_i^{(n)}(s)f_{x_i}(x)$$

for $n\geqq 1$. Then, in the same way as in (5.6) and (5.7), we get

$$\frac{1}{2}\int_{R^1}p_j'(s)\Big(\int_{R^d}\mathrm{sign}\,(v_n(x)-s)F_n(x)dx\Big)ds=\int_{R^d}p_j(v_n(x))F_n(x)dx$$

and

$$\int_{R^1}p_j'(s)\Big(\int_{R^d}\mathrm{sign}\,(v_n(x)-s)G_n(x)dx\Big)ds$$
$$=\int_{R^d}\sum_{i=1}^d f_{x_i}(x)\Big(\int_{-\infty}^k+2\int_k^{v_n(x)}-\int_k^\infty\Big)\phi_i^{(n)}(s)p_j(s)dsdx.$$

Hence, passing to the limit as $n\to\infty$ in (6.9) and using the convergence $\sum_{i=1}^d(\phi_i^{(n)}(v_n(x)))_{x_i}\to\sum_{i=1}^d(\phi_i(v))_{x_i}$ in $\mathcal{D}'$, we obtain the same inequality as in (5.8). Observe that $\phi_i(v)\in L^1\cap L^\infty$ by (6.8). Thus, we see letting $j\to\infty$ that $v\in D(A_0)$ and $\sum_{i=1}^d(\phi_i(v))_{x_i}=A_0v$. Since $\sum_{i=1}^d(\phi_i(v))_{x_i}$ coincides with the L^1_{loc}-limit of $\sum_{i=1}^d(\phi_i^{(n)}(v_n))_{x_i}=\lambda^{-1}(u-v_n)$ almost everywhere on R^d, it follows that $A_0v=\lambda^{-1}(u-v)$, or $(I+\lambda A_0)v=u$. Thus, $v=(I+\lambda A_0)^{-1}u$. This means that the whole sequence $\{J_\lambda^{(n)}u\}$ converges in $L^1_{\mathrm{loc}}(R^d)$ to $(I+\lambda A_0)^{-1}u$ as $n\to\infty$.

Now, let $J_\lambda=(I+\lambda A_0)^{-1}|X_0$ for $\lambda>0$. Then $\{J_\lambda; \lambda>0\}$ forms a pseudo-resolvent of contractions on X_0 into itself and the dissipative operator A

determined by it is a restriction of $-A_0$. Hence, A generates an L^1-contractive semigroup $\{T(t); t \geqq 0\}$ on $\overline{D(A)}$. Note that $\overline{D(A)}$ depends on the $\{\phi_i\}$ and $\overline{D(A)} \neq L^1$ in general. Also, it is easily seen from a similar argument to the proof of Theorem 5.1 that for each $u \in \overline{D(A)} \cap L^\infty$, $u(t, x) = [T(t)u](x)$ gives the generalized solution of (CP) with the initial-value u.

ACKNOWLEDGEMENT. The authors would like to thank Professor K. Kojima and Mr. N. Sanekata for their kind advices concerning this work.

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